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Research article

k-monogenic functions and Möbius transformations

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Abstract: Clifford analysis is a fundamental framework for extending complex function theory to high-dimensional spaces, where k-monogenic functions (high-order generalizations of monogenic functions) play a pivotal role in geometric function theory and partial differential equations. However, there is a paucity of research findings regarding the Möbius transformations of k-monogenic functions for arbitrary k. In this paper, we first derived that the composite function constructed from a k-monogenic function and a Möbius transformation remains k-monogenic when $k = 2, 3, 4$, and was generalized to the general case. Then, as applications, we proved the Schwarz-Pick-type lemma for harmonic functions using a new method, and we gave a version of the Schwarz-Pick-type lemma for inframonogenic functions. This work fills the gap in the transformation theory of k-monogenic functions, enriches the family of Schwarz-Pick-type lemmas in Clifford analysis, and provides theoretical tools to solve research related to high-dimensional geometric function theory.

Keywords: Möbius transformation; k-monogenic function; Schwarz-Pick-type lemma
Mathematics Subject Classification: 30G35

1. Introduction

The Möbius transformation is a crucial class of transformations in number theory and serves as an important tool for solving numerous problems. The classical framework has difficulty covering high-dimensional transformations, and there remains a research gap in the interaction between Möbius transformations and higher-order monogenic functions within the Clifford analysis framework.

For one complex variable, we define [1]

$$w = T(z) = \frac{az + b}{cz + d}$$

as the Möbius transformation, where $a, b, c, d \in \mathbf{C}$.

In 1985, Ahlfors [2] presented a brief introduction to the applications of Clifford numbers for researching Möbius transformations. The transformation is generalized to high-dimensional spaces.

In 2001, Eriksson, particularly in collaboration with Leutwiler and within the framework of Clifford analysis, established a relationship between monogenic functions and Möbius transformations.

Lemma 1.1. *[3] Let Ω be an open set contained in $\mathbf{R}^n$ and $T:\Omega\to\mathbf{R}^n$ be the Möbius transformation, $T(x)=(ax+b)(cx+d)^{-1}$, where $x=\sum\limits_{i=1}^{n}x_ie_i$, $a,b,c,d\in\mathbf{R}$, and $ad-bc\neq 0$. If f is left monogenic in $T(\Omega)$, then the function F defined by*

$$F(x)=\frac{(cx+d)^{-1}}{|cx+d|^{n-2}}f(T(x)) \tag{1.1}$$

is also left monogenic in Ω.

In 2016, Yonghong Xie et al. [4] considered the composition problem of k-hypergenic functions and the Clifford Möbius transformation. In the same year, Zhongxiang Zhang [5] derived the Schwarz-Pick-type lemma for monogenic functions.

Lemma 1.2. *[5] (Schwarz-Pick-type lemma for monogenic functions) Let $B(0,1)$ be an open unit ball in $\mathbf{R}^n$, $f\in C^1(\overline{B(0,1)},Cl_{0,n})$ be monogenic in $B(0,1)$, $|f(x)|\leq 1$ for all $x\in B(0,1)$, and $f(a)=0$, $|a|<1$. Then for any $x\in B(0,1)$, we have*

$$|f(x)|\leq\frac{(1+|a|)^{n-1}}{\sqrt[n]{2}-1}\frac{|x-a|}{|1-\overline{a}x|^n}. \tag{1.2}$$

In 2021, Haiyan Wang et al. [6] generalized the Möbius transformation and the Schwarz-type lemma to octonionic analysis.

This naturally gives rise to the question: Does a similar relationship exist between k-monogenic functions and Möbius transformations?

The goal of this paper is to extend the classical conclusions to k-monogenic functions. We mainly obtain the result that the composition of a k-monogenic function and a Möbius transformation is k-monogenic when $k=2$, 3, 4. It is generalized to the general case, which extends the functions associated with Lemma 1.1, and we propose a new method to prove the Schwarz-Pick-type lemma for harmonic functions, and obtain the Schwarz-Pick-type lemma for inframonogenic functions. The function classes corresponding to Lemma 1.2 are generalized in this paper.

2. Preliminaries

Let $e_1,e_2,\ldots,e_n$ be a basis of the n-dimensional real liner space $\mathbf{R}^n$ ($n\geq 3, n\in\mathbf{N}^*$). $Cl_{0,n}$ is a 2^n-dimensional Clifford algebra, whose basis is

$$\{e_A\,|\,A=(h_1,h_2,\cdots,h_r)\in PN, 1\leq h_1<h_2<\cdots<h_r\leq n\},$$

where $N=\{1,2,\ldots,n\}$, and PN denotes the set of all subsets of N ordered by the natural numerical order of elements. When $A=\varnothing$, $e_A=1$. Any element $a\in Cl_{0,n}$ can be represented as $a=\sum\limits_{A}a_Ae_A$, $a_A\in\mathbf{R}$. The basis satisfies the following:

$$\begin{cases} e_i^2=-1, & i=1,2,\ldots,n,\\ e_ie_j=-e_je_i, & 1\leq i<j\leq n.\end{cases}$$

We also define conjugate operations as follows:

$$\begin{cases} \overline{e_i} = -e_i, & i = 1, 2, \dots, n, \\ \overline{ab} = \overline{b}\overline{a}, & \forall a, b \in Cl_{0,n}. \end{cases}$$

For $a = \sum_A a_A e_A$, we define the norm $|a|$ of a by $\left(\sum_A a_A^2\right)^{\frac{1}{2}}$. When $a = \sum_{i=1}^n a_i e_i$, we have $a\overline{a} = \overline{a}a = |a|^2$. For additional details, the reader may refer to [7].

Let Ω be an open nonempty connected subset of the whole vector space $\mathbf{R}^n$ and f be a function defined in Ω and valued in $Cl_{0,n}$. Let $B(0, 1)$ be the unit ball in $\mathbf{R}^n$ and $B(0, 1) \subset \Omega$.

In addition, let

$$C^k(\Omega, Cl_{0,n}) = \{f | f : \Omega \to Cl_{0,n}, f = \sum_A f_A e_A, \text{where } f_A \text{ is a } k\text{-time continuously differentiable function in } \Omega\},$$

where $k \in \mathbf{N}^*$, $\mathbf{N}^*$ is the set of positive integers.

For $x \in \mathbf{R}^n$, the left, right Dirac operators and Laplace operator are defined as follows:

$$Df = \sum_{i=1}^n e_i \frac{\partial f}{\partial x_i}; \quad fD = \sum_{i=1}^n \frac{\partial f}{\partial x_i} e_i; \quad \Delta f = \sum_{i=1}^n \frac{\partial^2 f}{\partial x_i^2}.$$

Notice that $\Delta = -D^2$.

Lemma 2.1. *[8] If $f \in C^2(\Omega, Cl_{0,n})$, then $(Df(x))D = D(f(x)D)$.*

We denote $(Df(x))D$ and $D(f(x)D)$ by $Df(x)D$.

Definition 2.1. *If $f \in C^2(\Omega, Cl_{0,n})$ satisfies $Df(x)D = 0$ in Ω, then f is called an inframonogenic function in Ω, or we say that f is inframonogenic in Ω.*

Definition 2.2. *If $f \in C^2(\Omega, Cl_{0,n})$ satisfies $\Delta f(x) = 0$ in Ω, then f is called a harmonic function in Ω, or we say that f is harmonic in Ω.*

Definition 2.3. *[7] If $f \in C^k(\Omega, Cl_{0,n})$ satisfies $D^k f(x) = 0$ ($f(x)D^k = 0$) in Ω, where $D^k = D(D^{k-1})$, then f is called a left (right) k-monogenic function in Ω, or f is called left (right) k-monogenic in Ω. When $k = 1$, the function is called a left monogenic function.*

3. Möbius transformation

In this section, we will introduce Möbius properties associated with k-monogenic functions.

Let Möbius transformation $T_a(x)$ in Clifford analysis be defined as follows:

$$T_a(x) = (x - a)(1 - \bar{a}x)^{-1} = \frac{(x - a)(1 - \bar{x}a)}{|1 - \bar{a}x|^2},$$

where $x = \sum_{i=1}^n x_i e_i$, $a = \sum_{i=1}^n a_i e_i$, $|a| \neq 0, 1$ and $\bar{a}x \neq 1$.

When $|a| \neq 0$, $T_a(x)$ can be rewritten as

$$T_a(x) = -\frac{a}{|a|^2} + \frac{1 - |a|^2}{|a|}\frac{a}{|a|}\left(\frac{a}{|a|} - |a|x\right)^{-1}\frac{a}{|a|}.$$

Furthermore, Möbius transformations can be decomposed into the following four types:

Translations: $T_a(x) = x + a$, $a \in \mathbf{R}^n$.
Dilations: $T_a(x) = \lambda x$, $\lambda \in \mathbf{R}$.
Reflections: $T_a(x) = x^{-1}$, $x \in \mathbf{R}^n$, $|x| \neq 0$.
Rotations: $T_a(x) = axa$, $a \in \mathbf{R}^n$, $|a| = 1$.

Property 3.1. *If $f \in C^2(\Omega, Cl_{0,n})$, then we have*

$$D^2\left(\frac{1}{|x|^{n-2}} f(x^{-1})\right) = \frac{1}{|x|^{n+2}}[(D^2 f)(x^{-1})]. \tag{3.1}$$

Proof. To begin with, we compute $D(\frac{1}{|x|^{n-2}} f(x^{-1}))$.

$$D\left(\frac{1}{|x|^{n-2}} f(x^{-1})\right) = \left(D\frac{1}{|x|^{n-2}}\right) f(x^{-1}) + \sum_{i=1}^{n} e_i \frac{1}{|x|^{n-2}} \frac{\partial f(x^{-1})}{\partial x_i} = \frac{(2-n)x}{|x|^n} f(x^{-1}) + \sum_{i=1}^{n} \frac{-xe_i x}{|x|^{n+2}} \frac{\partial f(x^{-1})}{\partial (x^{-1})_i}. \tag{3.2}$$

Subsequently, applying the D-operation to the first term in Eq (3.2), we have

$$\begin{aligned} D\left(\frac{(2-n)x}{|x|^n} f(x^{-1})\right) =& (2-n)\frac{-n|x|^n - n|x|^{n-2}xx}{|x|^{2n}} f(x^{-1}) + (2-n)\sum_{i=1}^{n} e_i \frac{x}{|x|^n} \frac{\partial f(x^{-1})}{\partial x_i} \\ =& (2-n)\sum_{i=1}^{n} \frac{xe_i}{|x|^{n+2}} \frac{\partial f(x^{-1})}{\partial (x^{-1})_i} = (2-n)\frac{x}{|x|^{n+2}}[(Df)(x^{-1})]. \end{aligned} \tag{3.3}$$

Applying the D-operation to the second term in Eq (3.2), we get

$$\begin{aligned} & D\left(\sum_{i=1}^{n} \frac{-xe_i x}{|x|^{n+2}} \frac{\partial f(x^{-1})}{\partial (x^{-1})_i}\right) \\ =& \frac{(n-2)x}{|x|^{n+2}} \sum_{i=1}^{n} e_i \frac{\partial f(x^{-1})}{\partial x_i} + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{(-|x|^2 e_j + 2x_j x)(-xe_i x)}{|x|^{n+6}} \frac{\partial^2 f(x^{-1})}{\partial (x^{-1})_i \partial (x^{-1})_j} \\ =& \frac{(n-2)x}{|x|^{n+2}}[(Df)(x^{-1})] + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{-xe_j e_i x}{|x|^{n+4}} \frac{\partial^2 f(x^{-1})}{\partial (x^{-1})_i \partial (x^{-1})_j} \\ =& \frac{(n-2)x}{|x|^{n+2}}[(Df)(x^{-1})] + \frac{1}{|x|^{n+2}}[(D^2 f)(x^{-1})]. \end{aligned} \tag{3.4}$$

By Eqs (3.3) and (3.4), we obtain $D^2(\frac{1}{|x|^{n-2}} f(x^{-1})) = \frac{1}{|x|^{n+2}}[(D^2 f)(x^{-1})]$. □

Similar to the proof of Property 3.1, we can prove the following property.

Property 3.2. *If $f \in C^2(\Omega, Cl_{0,n})$, then we have*

$$\left(f(x^{-1})\frac{1}{|x|^{n-2}}\right)D^2 = [(fD^2)(x^{-1})]\left(\frac{1}{|x|^{n+2}}\right). \tag{3.5}$$

Property 3.3. *If $f \in C^3(\Omega, Cl_{0,n})$, then the following equality holds:*

$$D^3\left(\frac{\overline{x}}{|x|^{n-2}} f(x^{-1})\right) = \frac{\overline{x}}{|x|^{n+4}}[(D^3 f)(x^{-1})]. \tag{3.6}$$

Proof. First and foremost, we compute $D(\frac{\overline{x}}{|x|^{n-2}}f(x^{-1}))$.

$$D(\frac{\overline{x}}{|x|^{n-2}}f(x^{-1})) = D(\frac{\overline{x}}{|x|^{n-2}})f(x^{-1}) + \sum_{i=1}^{n} e_i \frac{\overline{x}}{|x|^{n-2}} \frac{\partial f(x^{-1})}{\partial x_i} = \frac{2}{|x|^{n-2}}f(x^{-1}) + \sum_{i=1}^{n} \frac{-x}{|x|^{n}} e_i \frac{\partial f(x^{-1})}{\partial (x^{-1})_i}.$$

Second, we compute $D^2(\frac{\overline{x}}{|x|^{n-2}}f(x^{-1}))$.

By Eq (3.1), we get

$$D^2(\frac{2}{|x|^{n-2}}f(x^{-1})) = \frac{2}{|x|^{n+2}}[(D^2 f)(x^{-1})]. \tag{3.7}$$

Furthermore, performing the D-operation on the second term, we get

$$\begin{aligned} D(\sum_{i=1}^{n} \frac{-x}{|x|^{n}} e_i \frac{\partial f(x^{-1})}{\partial (x^{-1})_i}) =& D(\sum_{i=1}^{n} \frac{-x}{|x|^{n}} e_i) \frac{\partial f(x^{-1})}{\partial (x^{-1})_i} + \sum_{j=1}^{n}\sum_{i=1}^{n} e_j \frac{-x}{|x|^{n}} e_i \frac{\partial(\frac{\partial f(x^{-1})}{\partial (x^{-1})_i})}{\partial x_j} \\ =& \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{-xe_j x(-x)e_i}{|x|^{n+4}} \frac{\partial^2 f(x^{-1})}{\partial (x^{-1})_i \partial (x^{-1})_j} = \sum_{i=1}^{n} \frac{x}{|x|^{n+2}} \frac{\partial^2 f(x^{-1})}{\partial (x^{-1})_i^2}. \end{aligned}$$

Performing the D-operation again, we have

$$\begin{aligned} D(\sum_{i=1}^{n} \frac{x}{|x|^{n+2}} \frac{\partial^2 f(x^{-1})}{\partial (x^{-1})_i^2}) =& D(\sum_{i=1}^{n} \frac{x}{|x|^{n+2}}) \frac{\partial^2 f(x^{-1})}{\partial (x^{-1})_i^2} + \sum_{j=1}^{n}\sum_{i=1}^{n} e_j \frac{x}{|x|^{n+2}} \frac{\partial(\frac{\partial^2 f(x^{-1})}{\partial (x^{-1})_i^2})}{\partial x_j} \\ =& \sum_{i=1}^{n} \frac{2}{|x|^{n+2}} \frac{\partial^2 f(x^{-1})}{\partial (x^{-1})_i^2} + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{xe_j}{|x|^{n+4}} \frac{\partial^3 f(x^{-1})}{\partial (x^{-1})_i^2 \partial (x^{-1})_j}. \end{aligned} \tag{3.8}$$

By Eqs (3.7) and (3.8), we can obtain

$$D^3(\frac{\overline{x}}{|x|^{n-2}}f(x^{-1})) = \frac{\overline{x}}{|x|^{n+4}}[(D^3 f)(x^{-1})].$$

□

Similarly, we can prove the following property.

Property 3.4. *If $f \in C^3(\Omega, Cl_{0,n})$, then we have*

$$(f(x^{-1})\frac{\overline{x}}{|x|^{n-2}})D^3 = [(fD^3)(x^{-1})](\frac{\overline{x}}{|x|^{n+4}}). \tag{3.9}$$

Property 3.5. *If $f \in C^4(\Omega, Cl_{0,n})$, then we have*

$$D^4(\frac{1}{|x|^{n-4}}f(x^{-1})) = \frac{1}{|x|^{n+4}}[(D^4 f)(x^{-1})]. \tag{3.10}$$

Proof. Step 1: We compute $D(\frac{1}{|x|^{n-4}}f(x^{-1}))$.

$$D(\frac{1}{|x|^{n-4}}f(x^{-1})) = D(\frac{1}{|x|^{n-4}})f(x^{-1}) + \sum_{i=1}^{n} e_i \frac{1}{|x|^{n-4}} \frac{\partial f(x^{-1})}{\partial x_i} = \frac{(4-n)x}{|x|^{n-2}} f(x^{-1}) + \sum_{i=1}^{n} \frac{-xe_i x}{|x|^n} \frac{\partial f(x^{-1})}{\partial (x^{-1})_i}. \quad (3.11)$$

Step 2: We compute $D^3(\frac{(4-n)x}{|x|^{n-2}}f(x^{-1}))$.

Combining the first part of Eq (3.11) with Eq (3.6), we have

$$D^3(\frac{(4-n)x}{|x|^{n-2}}f(x^{-1})) = \frac{(4-n)x}{|x|^{n+4}}[(D^3 f)(x^{-1})]. \quad (3.12)$$

Step 3: We compute $D^3(\sum_{i=1}^{n} \frac{-xe_i x}{|x|^n} \frac{\partial f(x^{-1})}{\partial (x^{-1})_i})$.

First we consider $D(\sum_{i=1}^{n} \frac{-xe_i x}{|x|^n} \frac{\partial f(x^{-1})}{\partial (x^{-1})_i})$.

$$\begin{aligned} D(\sum_{i=1}^{n} \frac{-xe_i x}{|x|^n} \frac{\partial f(x^{-1})}{\partial (x^{-1})_i}) =& D(\sum_{i=1}^{n} \frac{-xe_i x}{|x|^n}) \frac{\partial f(x^{-1})}{\partial (x^{-1})_i} + \sum_{j=1}^{n} \sum_{i=1}^{n} e_j \frac{-xe_i x}{|x|^n} \frac{\partial (\frac{\partial f(x^{-1})}{\partial (x^{-1})_i})}{\partial x_j} \\ =& 2\sum_{i=1}^{n} \frac{e_i x}{|x|^n} \frac{\partial f(x^{-1})}{\partial (x^{-1})_i} + (n-2)\sum_{i=1}^{n} \frac{xe_i}{|x|^n} \frac{\partial f(x^{-1})}{\partial (x^{-1})_i} - \sum_{i=1}^{n} \frac{1}{|x|^n} \frac{\partial^2 f(x^{-1})}{\partial (x^{-1})_i^2}, \end{aligned} \quad (3.13)$$

where the last equality holds because

$$\sum_{i=1}^{n} \sum_{j=1}^{n} e_j x e_i e_j = \sum_{i=1}^{n} \sum_{j=1}^{n} (-2x_j - xe_j) e_i e_j = -2\sum_{i=1}^{n} e_i x + (2-n)\sum_{i=1}^{n} xe_i.$$

Then we consider $D^2(\sum_{i=1}^{n} \frac{-xe_i x}{|x|^n} \frac{\partial f(x^{-1})}{\partial (x^{-1})_i})$.

Performing the D-operation for the third time on the three parts of Eq (3.13), we get

$$\begin{aligned} & D(2\sum_{i=1}^{n} \frac{e_i x}{|x|^n} \frac{\partial f(x^{-1})}{\partial (x^{-1})_i}) \\ =& 2\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{e_j e_i e_j}{|x|^n} \frac{\partial f(x^{-1})}{\partial (x^{-1})_i} - 2n\sum_{i=1}^{n} \frac{xe_i x}{|x|^{n+2}} \frac{\partial f(x^{-1})}{\partial (x^{-1})_i} - 2\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{xe_j xe_i x}{|x|^{n+4}} \frac{\partial^2 f(x^{-1})}{\partial (x^{-1})_i \partial (x^{-1})_j} \\ =& 2(n-2)\sum_{i=1}^{n} \frac{e_i}{|x|^n} \frac{\partial f(x^{-1})}{\partial (x^{-1})_i} - 2n\sum_{i=1}^{n} \frac{xe_i x}{|x|^{n+2}} \frac{\partial f(x^{-1})}{\partial (x^{-1})_i} - 2\sum_{i=1}^{n} \sum_{j=1}^{n} \frac{xe_j xe_i x}{|x|^{n+4}} \frac{\partial^2 f(x^{-1})}{\partial (x^{-1})_i \partial (x^{-1})_j}, \end{aligned} \quad (3.14)$$

where the last equality holds because

$$\sum_{i=1}^{n} \sum_{j=1}^{n} e_j e_i e_j = (n-2)\sum_{i=1}^{n} e_i.$$

By means of similar computations, we derive

$$D((n-2)\sum_{i=1}^{n}\frac{xe_i}{|x|^n}\frac{\partial f(x^{-1})}{\partial(x^{-1})_i})=(2-n)\sum_{i=1}^{n}\frac{x}{|x|^{n+2}}\frac{\partial^2 f(x^{-1})}{\partial(x^{-1})_i^2},\tag{3.15}$$

$$D(-\sum_{i=1}^{n}\frac{1}{|x|^n}\frac{\partial^2 f(x^{-1})}{\partial(x^{-1})_i^2})=n\sum_{i=1}^{n}\frac{x}{|x|^{n+2}}\frac{\partial^2 f(x^{-1})}{\partial(x^{-1})_i^2}+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{xe_jx}{|x|^{n+4}}\frac{\partial^3 f(x^{-1})}{\partial(x^{-1})_i^2\partial(x^{-1})_j}.\tag{3.16}$$

By Eqs (3.14)–(3.16), we have

$$\begin{aligned}D^2(\sum_{i=1}^{n}\frac{-xe_ix}{|x|^n}\frac{\partial f(x^{-1})}{\partial(x^{-1})_i})&=2(n-2)\sum_{i=1}^{n}\frac{e_i}{|x|^n}\frac{\partial f(x^{-1})}{\partial(x^{-1})_i}-2n\sum_{i=1}^{n}\frac{xe_ix}{|x|^{n+2}}\frac{\partial f(x^{-1})}{\partial(x^{-1})_i}\\-2\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{xe_jxe_ix}{|x|^{n+4}}\frac{\partial^2 f(x^{-1})}{\partial(x^{-1})_i\partial(x^{-1})_j}&+2\sum_{i=1}^{n}\frac{x}{|x|^{n+2}}\frac{\partial^2 f(x^{-1})}{\partial(x^{-1})_i^2}+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{xe_jx}{|x|^{n+4}}\frac{\partial^3 f(x^{-1})}{\partial(x^{-1})_i^2\partial(x^{-1})_j}.\end{aligned}\tag{3.17}$$

Finally, we perform the D-operation on the five parts of Eq (3.17). After similar calculations, we have

(i)

$$\begin{aligned}&D(2(n-2)\sum_{i=1}^{n}\frac{e_i}{|x|^n}\frac{\partial f(x^{-1})}{\partial(x^{-1})_i})\\=&2(n-2)\sum_{j=1}^{n}\sum_{i=1}^{n}e_j\frac{-n|x|^{n-2}x_je_i}{|x|^{2n}}\frac{\partial f(x^{-1})}{\partial(x^{-1})_i}+2(n-2)\sum_{j=1}^{n}\sum_{i=1}^{n}\frac{-xe_jx}{|x|^4}\frac{e_i}{|x|^n}\frac{\partial^2 f(x^{-1})}{\partial(x^{-1})_i\partial(x^{-1})_j}\\=&2(2-n)n\sum_{i=1}^{n}\frac{xe_i}{|x|^{n+2}}\frac{\partial f(x^{-1})}{\partial(x^{-1})_i}+2(2-n)\sum_{j=1}^{n}\sum_{i=1}^{n}\frac{xe_jxe_i}{|x|^{n+4}}\frac{\partial^2 f(x^{-1})}{\partial(x^{-1})_i\partial(x^{-1})_j},\end{aligned}\tag{3.18}$$

(ii)

$$\begin{aligned}&D(-2n\sum_{i=1}^{n}\frac{xe_ix}{|x|^{n+2}}\frac{\partial f(x^{-1})}{\partial(x^{-1})_i})\\=&2n^2\sum_{i=1}^{n}\frac{e_ix}{|x|^{n+2}}\frac{\partial f(x^{-1})}{\partial(x^{-1})_i}-2n\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{e_jxe_ie_j}{|x|^{n+2}}\frac{\partial f(x^{-1})}{\partial(x^{-1})_i}\\&+2n(n+2)\sum_{i=1}^{n}\frac{xxe_ix}{|x|^{n+4}}\frac{\partial f(x^{-1})}{\partial(x^{-1})_i}-2n\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{xe_je_ix}{|x|^{n+4}}\frac{\partial^2 f(x^{-1})}{\partial(x^{-1})_i\partial(x^{-1})_j}\\=&2n(n-2)\sum_{i=1}^{n}\frac{xe_i}{|x|^{n+2}}\frac{\partial f(x^{-1})}{\partial(x^{-1})_i}-2n\frac{1}{|x|^{n+2}}\sum_{i=1}^{n}\frac{\partial^2 f(x^{-1})}{\partial(x^{-1})_i^2},\end{aligned}\tag{3.19}$$

(iii)

$$\begin{aligned}
&D(-2\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{xe_jxe_ix}{|x|^{n+4}}\frac{\partial^2 f(x^{-1})}{\partial(x^{-1})_i\partial(x^{-1})_j})\\
=&2n\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{e_jxe_ix}{|x|^{n+4}}\frac{\partial^2 f(x^{-1})}{\partial(x^{-1})_i\partial(x^{-1})_j}-2\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{e_kxe_je_ke_ix}{|x|^{n+4}}\frac{\partial^2 f(x^{-1})}{\partial(x^{-1})_i\partial(x^{-1})_j}\\
&-2\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{e_kxe_jxe_ie_k}{|x|^{n+4}}\frac{\partial^2 f(x^{-1})}{\partial(x^{-1})_i\partial(x^{-1})_j}+2(n+4)\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{xxe_jxe_ix}{|x|^{n+6}}\frac{\partial^2 f(x^{-1})}{\partial(x^{-1})_i\partial(x^{-1})_j}\\
&-2\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{xe_ke_jxe_ix}{|x|^{n+6}}\frac{\partial^3 f(x^{-1})}{\partial(x^{-1})_i\partial(x^{-1})_j\partial(x^{-1})_k}\\
=&2(n-2)\sum_{i=1}^{n}\frac{1}{|x|^{n+2}}\frac{\partial^2 f(x^{-1})}{\partial(x^{-1})_i^2}+2(n-2)\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{xe_jxe_i}{|x|^{n+4}}\frac{\partial^2 f(x^{-1})}{\partial(x^{-1})_i\partial(x^{-1})_j}-2\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{e_ix}{|x|^{n+4}}\frac{\partial^3 f(x^{-1})}{\partial(x^{-1})_i\partial(x^{-1})_j^2},
\end{aligned}\tag{3.20}$$

(iv)

$$\begin{aligned}
&D(2\sum_{i=1}^{n}\frac{x}{|x|^{n+2}}\frac{\partial^2 f(x^{-1})}{\partial(x^{-1})_i^2})\\
=&-2n\sum_{i=1}^{n}\frac{1}{|x|^{n+2}}\frac{\partial^2 f(x^{-1})}{\partial(x^{-1})_i^2}-2(n+2)\sum_{i=1}^{n}\frac{xx}{|x|^{n+4}}\frac{\partial^2 f(x^{-1})}{\partial(x^{-1})_i^2}+2\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{xe_j}{|x|^{n+4}}\frac{\partial^3 f(x^{-1})}{\partial(x^{-1})_i^2\partial(x^{-1})_j}\\
=&4\sum_{i=1}^{n}\frac{1}{|x|^{n+2}}\frac{\partial^2 f(x^{-1})}{\partial(x^{-1})_i^2}+2\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{xe_j}{|x|^{n+4}}\frac{\partial^3 f(x^{-1})}{\partial(x^{-1})_i^2\partial(x^{-1})_j},
\end{aligned}\tag{3.21}$$

(v)

$$\begin{aligned}
&D(\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{xe_jx}{|x|^{n+4}}\frac{\partial^3 f(x^{-1})}{\partial(x^{-1})_i^2\partial(x^{-1})_j})\\
=&-n\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{e_jx}{|x|^{n+4}}\frac{\partial^3 f(x^{-1})}{\partial(x^{-1})_i^2\partial(x^{-1})_j}+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{-2e_jx+(2-n)xe_j}{|x|^{n+4}}\frac{\partial^3 f(x^{-1})}{\partial(x^{-1})_i^2\partial(x^{-1})_j}\\
&+(n+4)\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{e_jx}{|x|^{n+4}}\frac{\partial^3 f(x^{-1})}{\partial(x^{-1})_i^2\partial(x^{-1})_j}+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{xe_ke_jx}{|x|^{n+6}}\frac{\partial^4 f(x^{-1})}{\partial(x^{-1})_i^2\partial(x^{-1})_j\partial(x^{-1})_k}\\
=&2\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{e_jx}{|x|^{n+4}}\frac{\partial^3 f(x^{-1})}{\partial(x^{-1})_i^2\partial(x^{-1})_j}+(2-n)\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{xe_j}{|x|^{n+4}}\frac{\partial^3 f(x^{-1})}{\partial(x^{-1})_i^2\partial(x^{-1})_j}+\frac{1}{|x|^{n+4}}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^4 f(x^{-1})}{\partial(x^{-1})_i^2\partial(x^{-1})_j^2}.
\end{aligned}\tag{3.22}$$

By Eqs (3.12), and (3.17)–(3.22), we get

$$D^4(\frac{1}{|x|^{n-4}}f(x^{-1}))=\frac{1}{|x|^{n+4}}[(D^4 f)(x^{-1})].$$

□

Theorem 3.1. *If $f \in C^{2k}(\Omega, Cl_{0,n})$, then we have*

$$\Delta^k(\frac{1}{|x|^{n-2k}}f(x^{-1})) = \frac{1}{|x|^{n+2k}}[(\Delta^k f)(x^{-1})], \tag{3.23}$$

where $k \in \mathbf{N}^*$, and $2k < n$.

Proof. Mathematical induction is employed to prove the theorem.
Step 1: When $k = 1$, by Property 3.1, we have

$$\Delta(\frac{1}{|x|^{n-2}}f(x^{-1})) = \frac{1}{|x|^{n+2}}[(\Delta f)(x^{-1})].$$

Step 2: Suppose that

$$\Delta^k(\frac{1}{|x|^{n-2k}}f(x^{-1})) = \frac{1}{|x|^{n+2k}}[(\Delta^k f)(x^{-1})]. \tag{3.24}$$

Then we only need to prove

$$\Delta^{k+1}(\frac{1}{|x|^{n-2k-2}}f(x^{-1})) = \frac{1}{|x|^{n+2k+2}}[(\Delta^{k+1} f)(x^{-1})]. \tag{3.25}$$

First, we compute $\Delta(\frac{1}{|x|^{n-2k-2}}f(x^{-1}))$.

As

$$\frac{\partial(|x|^2\frac{1}{|x|^{n-2k}}f(x^{-1}))}{\partial x_i} = \frac{2x_i}{|x|^{n-2k}}f(x^{-1}) + |x|^2\frac{\partial(\frac{1}{|x|^{n-2k}}f(x^{-1}))}{\partial x_i},$$

$$\frac{\partial(\frac{2x_i}{|x|^{n-2k}}f(x^{-1}))}{\partial x_i} = \frac{2}{|x|^{n-2k}}f(x^{-1}) + 2x_i\frac{\partial(\frac{1}{|x|^{n-2k}}f(x^{-1}))}{\partial x_i},$$

and

$$\frac{\partial(|x|^2\frac{\partial(\frac{1}{|x|^{n-2k}}f(x^{-1}))}{\partial x_i})}{\partial x_i} = 2x_i\frac{\partial(\frac{1}{|x|^{n-2k}}f(x^{-1}))}{\partial x_i} + |x|^2\frac{\partial^2(\frac{1}{|x|^{n-2k}}f(x^{-1}))}{\partial x_i^2},$$

we have

$$\begin{aligned}\Delta(\frac{1}{|x|^{n-2k-2}}f(x^{-1})) &= \sum_{i=1}^{n}(\frac{2}{|x|^{n-2k}}f(x^{-1}) + 4x_i\frac{\partial(\frac{1}{|x|^{n-2k}}f(x^{-1}))}{\partial x_i} + |x|^2\frac{\partial^2(\frac{1}{|x|^{n-2k}}f(x^{-1}))}{\partial x_i^2})\\ &=\frac{2n}{|x|^{n-2k}}f(x^{-1}) + 4\sum_{i=1}^{n}x_i\frac{\partial(\frac{1}{|x|^{n-2k}}f(x^{-1}))}{\partial x_i} + |x|^2\Delta(\frac{1}{|x|^{n-2k}}f(x^{-1})).\end{aligned}$$

Second, we compute $\Delta^2(\frac{1}{|x|^{n-2k-2}}f(x^{-1}))$.

We denote $\frac{2n}{|x|^{n-2k}}f(x^{-1})$, $4\sum\limits_{i=1}^{n} x_i\frac{\partial(\frac{1}{|x|^{n-2k}}f(x^{-1}))}{\partial x_i}$, $|x|^2\Delta(\frac{1}{|x|^{n-2k}}f(x^{-1}))$ by I_1, I_2, I_3, respectively. We get $\Delta I_1 = 2n\Delta(\frac{1}{|x|^{n-2k}}f(x^{-1}))$. We consider ΔI_2.

As

$$\frac{\partial(\sum\limits_{i=1}^{n} x_i\frac{\partial(\frac{1}{|x|^{n-2k}}f(x^{-1}))}{\partial x_i})}{\partial x_j} = \sum_{i=1}^{n}\delta_{ij}\frac{\partial(\frac{1}{|x|^{n-2k}}f(x^{-1}))}{\partial x_i} + \sum_{i=1}^{n} x_i\frac{\partial^2(\frac{1}{|x|^{n-2k}}f(x^{-1}))}{\partial x_i\partial x_j},$$

$$\frac{\partial(\sum\limits_{i=1}^{n}\delta_{ij}\frac{\partial(\frac{1}{|x|^{n-2k}}f(x^{-1}))}{\partial x_i})}{\partial x_j} = \sum_{i=1}^{n}\delta_{ij}\frac{\partial^2(\frac{1}{|x|^{n-2k}}f(x^{-1}))}{\partial x_i\partial x_j},$$

and

$$\frac{\partial(\sum\limits_{i=1}^{n} x_i\frac{\partial^2(\frac{1}{|x|^{n-2k}}f(x^{-1}))}{\partial x_i\partial x_j})}{\partial x_j} = \sum_{i=1}^{n}\delta_{ij}\frac{\partial^2(\frac{1}{|x|^{n-2k}}f(x^{-1}))}{\partial x_i\partial x_j} + \sum_{i=1}^{n} x_i\frac{\partial^3(\frac{1}{|x|^{n-2k}}f(x^{-1}))}{\partial x_i\partial x_j^2},$$

we get

$$\Delta I_2 = 8\Delta(\frac{1}{|x|^{n-2k}}f(x^{-1})) + 4\sum_{i=1}^{n} x_i\frac{\partial(\Delta(\frac{1}{|x|^{n-2k}}f(x^{-1})))}{\partial x_i}.$$

In addition,

$$\Delta I_3 = 2n\Delta(\frac{1}{|x|^{n-2k}}f(x^{-1})) + 4\sum_{i=1}^{n} x_i\frac{\partial(\Delta(\frac{1}{|x|^{n-2k}}f(x^{-1})))}{\partial x_i} + |x|^2\Delta^2(\frac{1}{|x|^{n-2k}}f(x^{-1})).$$

Therefore,

$$\Delta^2(\frac{1}{|x|^{n-2k-2}}f(x^{-1})) = (4n+8)\Delta(\frac{1}{|x|^{n-2k}}f(x^{-1})) + 8\sum_{i=1}^{n} x_i\frac{\partial(\Delta(\frac{1}{|x|^{n-2k}}f(x^{-1})))}{\partial x_i} + |x|^2\Delta^2(\frac{1}{|x|^{n-2k}}f(x^{-1})).$$

Third, we compute $\Delta^3(\frac{1}{|x|^{n-2k-2}}f(x^{-1}))$.

We denote $(4n+8)\Delta(\frac{1}{|x|^{n-2k}}f(x^{-1}))$, $8\sum\limits_{i=1}^{n} x_i\frac{\partial(\Delta(\frac{1}{|x|^{n-2k}}f(x^{-1})))}{\partial x_i}$, $|x|^2\Delta^2(\frac{1}{|x|^{n-2k}}f(x^{-1}))$ by I_4, I_5, I_6, respectively. We have

$$\Delta I_4 = (4n+8)\Delta^2(\frac{1}{|x|^{n-2k}}f(x^{-1})),$$

$$\Delta I_5 = 16\Delta^2(\frac{1}{|x|^{n-2k}}f(x^{-1})) + 8\sum_{i=1}^{n} x_i \frac{\partial(\Delta^2(\frac{1}{|x|^{n-2k}}f(x^{-1})))}{\partial x_i},$$

$$\Delta I_6 = 2n\Delta^2(\frac{1}{|x|^{n-2k}}f(x^{-1})) + 4\sum_{i=1}^{n} x_i \frac{\partial(\Delta^2(\frac{1}{|x|^{n-2k}}f(x^{-1})))}{\partial x_i} + |x|^2\Delta^3(\frac{1}{|x|^{n-2k}}f(x^{-1})).$$

So

$$\Delta^3(\frac{1}{|x|^{n-2k-2}}f(x^{-1})) = (6n+24)\Delta^2(\frac{1}{|x|^{n-2k}}f(x^{-1})) + 12\sum_{i=1}^{n} x_i \frac{\partial(\Delta^2(\frac{1}{|x|^{n-2k}}f(x^{-1})))}{\partial x_i} + |x|^2\Delta^3(\frac{1}{|x|^{n-2k}}f(x^{-1})).$$

$$\cdots$$

Fourth, we prove Eq (3.25). Similarly, we get

$$\begin{aligned}&\Delta^k(\frac{1}{|x|^{n-2k-2}}f(x^{-1}))\\ =&(2nk+4k^2-4k)\Delta^{k-1}(\frac{1}{|x|^{n-2k}}f(x^{-1})) + 4k\sum_{i=1}^{n} x_i \frac{\partial(\Delta^{k-1}(\frac{1}{|x|^{n-2k}}f(x^{-1})))}{\partial x_i} + |x|^2\Delta^k(\frac{1}{|x|^{n-2k}}f(x^{-1})).\end{aligned}$$

By Eq (3.24), we obtain

$$\begin{aligned}&\Delta^k(\frac{1}{|x|^{n-2k-2}}f(x^{-1}))\\ =&(2nk+4k^2-4k)\Delta^{k-1}(\frac{1}{|x|^{n-2k}}f(x^{-1})) + 4k\sum_{i=1}^{n} x_i \frac{\partial(\Delta^{k-1}(\frac{1}{|x|^{n-2k}}f(x^{-1})))}{\partial x_i} + \frac{1}{|x|^{n+2k-2}}[(\Delta^k f)(x^{-1})].\end{aligned}$$

We denote $(2nk+4k^2-4k)\Delta^{k-1}(\frac{1}{|x|^{n-2k}}f(x^{-1}))$, $4k\sum_{i=1}^{n} x_i \frac{\partial(\Delta^{k-1}(\frac{1}{|x|^{n-2k}}f(x^{-1})))}{\partial x_i}$, $\frac{1}{|x|^{n+2k-2}}(\Delta^k f)(x^{-1})$ by $I_{3k-2}, I_{3k-1}, I_{3k}$, respectively.

By Eq (3.24), we have

$$\Delta I_{3k-2} = (2nk+4k^2-4k)\frac{1}{|x|^{n+2k}}[(\Delta^k f)(x^{-1})]. \tag{3.26}$$

We obtain

$$\Delta I_{3k-1} = 8k\Delta^k(\frac{1}{|x|^{n-2k}}f(x^{-1})) + 4k\sum_{i=1}^{n} x_i \frac{\partial(\Delta^k(\frac{1}{|x|^{n-2k}}f(x^{-1})))}{\partial x_i}.$$

Then, by Eq (3.24), we have

$$\begin{aligned}\Delta I_{3k-1} =& \frac{8k}{|x|^{n+2k}}[(\Delta^k f)(x^{-1})] + 4k\sum_{i=1}^{n} x_i \frac{\partial(\frac{1}{|x|^{n+2k}}[(\Delta^k f)(x^{-1})])}{\partial x_i}\\
=&\frac{8k}{|x|^{n+2k}}[(\Delta^k f)(x^{-1})] + 4k\sum_{i=1}^{n} x_i(-(n+2k)\frac{x_i}{|x|^{n+2k+2}}[(\Delta^k f)(x^{-1})]) + 4k\sum_{i=1}^{n} x_i\frac{1}{|x|^{n+2k}}\frac{\partial[(\Delta^k f)(x^{-1})]}{\partial x_i}\\
=&\frac{8k-4k(n+2k)}{|x|^{n+2k}}[(\Delta^k f)(x^{-1})] + 4k\sum_{i=1}^{n}\frac{x_i}{|x|^{n+2k+2}}\frac{\partial[(\Delta^k f)(x^{-1})]}{\partial(x^{-1})_i},\end{aligned}$$

that is,

$$\Delta I_{3k-1} = \frac{8k-4k(n+2k)}{|x|^{n+2k}}[(\Delta^k f)(x^{-1})] + 4k\sum_{i=1}^{n}\frac{x_i}{|x|^{n+2k+2}}\frac{\partial[(\Delta^k f)(x^{-1})]}{\partial(x^{-1})_i}. \tag{3.27}$$

Furthermore, we consider I_{3k}. We have

$$\begin{aligned}\frac{\partial(\frac{1}{|x|^{n+2k-2}}[(\Delta^k f)(x^{-1})])}{\partial x_i} =&\frac{-(n+2k-2)x_i}{|x|^{n+2k}}[(\Delta^k f)(x^{-1})] + \frac{1}{|x|^{n+2k-2}}\frac{\partial[(\Delta^k f)(x^{-1})]}{\partial x_i}\\
=&\frac{-(n+2k-2)x_i}{|x|^{n+2k}}[(\Delta^k f)(x^{-1})] + \sum_{j=1}^{n}\frac{-\delta_{ij}|x|^2+2x_ix_j}{|x|^{n+2k+2}}\frac{\partial[(\Delta^k f)(x^{-1})]}{\partial(x^{-1})_j}.\end{aligned}$$

Partial derivatives are computed for the two components of the aforementioned equalities separately.

$$\begin{aligned}&\sum_{i=1}^{n}\frac{\partial(\frac{-(n+2k-2)x_i}{|x|^{n+2k}}[(\Delta^k f)(x^{-1})])}{\partial x_i}\\
=&\frac{2k(n+2k-2)}{|x|^{n+2k}}[(\Delta^k f)(x^{-1})] - (n+2k-2)\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{x_i}{|x|^{n+2k}}\frac{-\delta_{ij}|x|^2+2x_ix_j}{|x|^4}\frac{\partial[(\Delta^k f)(x^{-1})]}{\partial(x^{-1})_j}\\
=&\frac{2k(n+2k-2)}{|x|^{n+2k}}[(\Delta^k f)(x^{-1})] - (n+2k-2)\sum_{i=1}^{n}\frac{x_i}{|x|^{n+2k+2}}\frac{\partial[(\Delta^k f)(x^{-1})]}{\partial(x^{-1})_i}.\end{aligned}$$

Then

$$\frac{\partial(\sum_{j=1}^{n}\frac{-\delta_{ij}|x|^2+2x_ix_j}{|x|^{n+2k+2}}\frac{\partial[(\Delta^k f)(x^{-1})]}{\partial(x^{-1})_j})}{\partial x_i}$$
$$=\sum_{j=1}^{n}\frac{2x_j}{|x|^{n+2k+2}}\frac{\partial[(\Delta^k f)(x^{-1})]}{\partial(x^{-1})_j}+(n+2k+2)\sum_{j=1}^{n}\frac{\delta_{ij}x_i}{|x|^{n+2k+2}}\frac{\partial[(\Delta^k f)(x^{-1})]}{\partial(x^{-1})_j}$$
$$-2(n+2k+2)\sum_{j=1}^{n}\frac{x_i^2x_j}{|x|^{n+2k+4}}\frac{\partial[(\Delta^k f)(x^{-1})]}{\partial(x^{-1})_j}+\sum_{j=1}^{n}\sum_{l=1}^{n}\frac{\delta_{ij}\delta_{il}}{|x|^{n+2k+2}}\frac{\partial^2[(\Delta^k f)(x^{-1})]}{\partial(x^{-1})_j\partial(x^{-1})_l}$$
$$+\sum_{j=1}^{n}\sum_{l=1}^{n}\frac{-2\delta_{ij}x_ix_l}{|x|^{n+2k+4}}\frac{\partial^2[(\Delta^k f)(x^{-1})]}{\partial(x^{-1})_j\partial(x^{-1})_l}+\sum_{j=1}^{n}\sum_{l=1}^{n}\frac{-2\delta_{il}x_ix_j}{|x|^{n+2k+4}}\frac{\partial^2[(\Delta^k f)(x^{-1})]}{\partial(x^{-1})_j\partial(x^{-1})_l}$$
$$+4\sum_{j=1}^{n}\sum_{l=1}^{n}\frac{x_i^2x_jx_l}{|x|^{n+2k+6}}\frac{\partial^2[(\Delta^k f)(x^{-1})]}{\partial(x^{-1})_j\partial(x^{-1})_l},$$

and thus,

$$\sum_{i=1}^{n}\frac{\partial(\sum_{j=1}^{n}\frac{-\delta_{ij}|x|^2+2x_ix_j}{|x|^{n+2k+2}}\frac{\partial[(\Delta^k f)(x^{-1})]}{\partial(x^{-1})_j})}{\partial x_i}=(n-2k-2)\sum_{i=1}^{n}\frac{x_i}{|x|^{n+2k+2}}\frac{\partial[(\Delta^k f)(x^{-1})]}{\partial(x^{-1})_i}+\frac{1}{|x|^{n+2k+2}}[(\Delta^{k+1}f)(x^{-1})].$$

Hence

$$\Delta I_{3k}=2k(n+2k-2)\frac{1}{|x|^{n+2k}}[(\Delta^k f)(x^{-1})]-4k\sum_{i=1}^{n}\frac{x_i}{|x|^{n+2k+2}}\frac{\partial[(\Delta^k f)(x^{-1})]}{\partial(x^{-1})_i}+\frac{1}{|x|^{n+2k+2}}[(\Delta^{k+1}f)(x^{-1})]. \quad (3.28)$$

By Eqs (3.26)–(3.28), Eq (3.25) holds. By mathematical induction, we draw the conclusion. □

Theorem 3.2. *If $f\in C^k(\Omega, Cl_{0,n})$, then we have*

$$D^k(\frac{(\overline{x})^k}{|x|^n}f(x^{-1}))=\frac{(\overline{x})^k}{|x|^{n+2k}}[(D^k f)(x^{-1})], \quad (3.29)$$

where $k\in\mathbf{N}^*$, and $k<n$.

Proof. Case 1: When $k=2j$, where $j\in\mathbf{N}^*$, by Theorem 3.1, we draw the conclusion.

Case 2: When $k=2j-1$, where $j\in\mathbf{N}^*$, we need to prove

$$D^{2j-1}(\frac{(\overline{x})^{2j-1}}{|x|^n}f(x^{-1}))=\frac{(\overline{x})^{2j-1}}{|x|^{n+4j-2}}[(D^{2j-1}f)(x^{-1})]. \quad (3.30)$$

Step 1: For $j=1$, by Eq (3.3), we have

$$D(\frac{\overline{x}}{|x|^n}f(x^{-1}))=\frac{\overline{x}}{|x|^{n+2}}[(Df)(x^{-1})].$$

Step 2: Assume that Eq (3.30) holds. First, we have

$$D(\frac{\overline{x}}{|x|^{n-2j}}f(x^{-1})) = \frac{2j}{|x|^{n-2j}}f(x^{-1}) + \frac{\overline{x}}{|x|^{n-2j+2}}[(Df)(x^{-1})]. \tag{3.31}$$

Second, by Theorem 3.1, we get

$$D^{2j}(\frac{2j}{|x|^{n-2j}}f(x^{-1})) = (-1)^j \cdot 2j \cdot \frac{1}{|x|^{n+2j}}[(\Delta^j f)(x^{-1}))] = 2j\frac{1}{|x|^{n+2j}}[(D^{2j}f)(x^{-1}))]. \tag{3.32}$$

By Eq (3.30), we have

$$D^{2j-1}(\frac{\overline{x}}{|x|^{n-2j+2}}[(Df)(x^{-1})]) = (-1)^{j+1}\frac{(\overline{x})^{2j-1}}{|x|^{n+4j-2}}[(D^{2j}f)(x^{-1})] = \frac{\overline{x}}{|x|^{n+2j}}[(D^{2j}f)(x^{-1})]. \tag{3.33}$$

Then

$$\begin{aligned}
&D(\frac{\overline{x}}{|x|^{n+2j}}[(D^{2j}f)(x^{-1})]) \\
=& -\sum_{i=1}^{n} e_i \frac{e_i|x|^{n+2j} - (n+2j)|x|^{n+2j-2}x_i x}{|x|^{2n+4j}}[(D^{2j}f)(x^{-1})] + \sum_{i=1}^{n} \frac{xe_i xx}{|x|^{n+2j+4}} \frac{\partial[(D^{2j}f)(x^{-1})]}{\partial(x^{-1})_i} \\
=& -\frac{2j}{|x|^{n+2j}}[(D^{2j}f)(x^{-1})] + \frac{\overline{x}}{|x|^{n+2j+2}}[(D^{2j+1}f)(x^{-1})].
\end{aligned} \tag{3.34}$$

By Eqs (3.31)–(3.34), we have

$$D^{2j+1}(\frac{(\overline{x})^{2j+1}}{|x|^n}f(x^{-1})) = \frac{(\overline{x})^{2j+1}}{|x|^{n+4j+2}}[(D^{2j+1}f)(x^{-1})].$$

Hence, Eq (3.30) holds.

By mathematical induction, we finish the proof. □

Remark 3.1. *When k is an even number, Theorem 3.2 reduces to Theorem 3.1. When $k = 2, 3, 4$, Theorem 3.2 is identical to Properties 3.1, 3.3, 3.5, respectively.*

Lemma 3.1. *[5, 9] If $f \in C^1(\Omega, Cl_{0,n})$, then we have*

$$\begin{cases} D[f(T_a(x))] = (Df)(T_a(x)), \text{ where } T_a(x) = x + a, a \in \mathbf{R}^n, \\ D[f(T_a(x))] = \lambda(Df)(T_a(x)), \text{ where } T_a(x) = \lambda x, \lambda \in \mathbf{R}, \\ D[af(T_a(x))] = \overline{a}(Df)(T_a(x)), \text{ where } T_a(x) = axa, a \in \mathbf{R}^n, |a| = 1. \end{cases}$$

By Lemma 3.1, we can prove the following property.

Property 3.6. *If $f \in C^k(\Omega, Cl_{0,n})$, then the following equalities hold:*

$$\begin{cases} D^k[f(T_a(x))] = (D^k f)(T_a(x)), \text{ where } T_a(x) = x + a, a \in \mathbf{R}^n, \\ D^k[f(T_a(x))] = \lambda^k(D^k f)(T_a(x)), \text{ where } T_a(x) = \lambda x, \lambda \in \mathbf{R}, \\ D^k[af(T_a(x))] = (-1)^k a(D^k f)(T_a(x)), \text{ where } T_a(x) = axa, a \in \mathbf{R}^n, |a| = 1. \end{cases}$$

Lemma 3.2. *[9] If $f \in C^1(\Omega, Cl_{0,n})$, then we have*

$$\begin{cases} [f(T_a(x))]D = (fD)(T_a(x)), & \text{where } T_a(x) = x + a, a \in \mathbf{R}^n, \\ [f(T_a(x))]D = \lambda(fD)(T_a(x)), & \text{where } T_a(x) = \lambda x, \lambda \in \mathbf{R}, \\ [f(T_a(x))a]D = (fD)(T_a(x))\overline{a}, & \text{where } T_a(x) = axa, a \in \mathbf{R}^n, |a| = 1. \end{cases}$$

Property 3.7. *If $f \in C^k(\Omega, Cl_{0,n})$, then the following equalities hold:*

$$\begin{cases} [f(T_a(x))]D^k = (fD^k)(T_a(x)), & \text{where } T_a(x) = x + a, a \in \mathbf{R}^n, \\ [f(T_a(x))]D^k = \lambda^k(fD^k)(T_a(x)), & \text{where } T_a(x) = \lambda x, \lambda \in \mathbf{R}, \\ [f(T_a(x))a]D^k = (-1)^k(fD^k)(T_a(x))a, & \text{where } T_a(x) = axa, a \in \mathbf{R}^n, |a| = 1. \end{cases}$$

Theorem 3.3. *Let $T_a : B(0,1) \to B(0,1)$ be the Möbius transformation, $T_a(x) = (x+a)(1+\overline{a}x)^{-1}$, where $x = \sum_{i=1}^{n} x_i e_i$. If f is a left 2-monogenic function in $B(0,1)$, then the function F defined by*

$$F(x) = \frac{1}{|1+\overline{a}x|^{n-2}} \frac{a}{|a|}(f \circ T_a(x)) \tag{3.35}$$

is also left 2-monogenic in $B(0,1)$.

Proof. Let $y = T_a(x) = (x+a)(1+\overline{a}x)^{-1} = (x+a)\dfrac{1+\overline{x}a}{|1+\overline{a}x|^2}$. Then $y = \dfrac{a}{|a|^2} + \dfrac{1-|a|^2}{|a|}\dfrac{a}{|a|}\left(\dfrac{-a}{|a|} - |a|x\right)^{-1}\dfrac{a}{|a|}$, which maps $B(0,1)$ onto $B(0,1)$. Notice that $T_a = T_a^6 \circ T_a^5 \circ T_a^4 \circ T_a^3 \circ T_a^2 \circ T_a^1$, where

$$T_a^6(x) = x + \frac{a}{|a|^2}; \qquad T_a^5(x) = \frac{1-|a|^2}{|a|}x; \qquad T_a^4(x) = \frac{a}{|a|}x\frac{a}{|a|};$$
$$T_a^3(x) = x^{-1}; \qquad T_a^2(x) = x - \frac{a}{|a|}; \qquad T_a^1(x) = -|a|x.$$

As f is left 2-monogenic in $B(0,1)$ and by Property 3.6, $f \circ T_a^6$ is left 2-monogenic in $T_a^5 \circ T_a^4 \circ T_a^3 \circ T_a^2 \circ T_a^1(B(0,1))$. By Property 3.6, $f \circ T_a^6 \circ T_a^5$ is left 2-monogenic in $T_a^4 \circ T_a^3 \circ T_a^2 \circ T_a^1(B(0,1))$. Similarly, by Property 3.6, $\dfrac{a}{|a|}f \circ T_a^6 \circ T_a^5 \circ T_a^4$ is left 2-monogenic in $T_a^3 \circ T_a^2 \circ T_a^1(B(0,1))$. Applying Property 3.1, the function

$$g(x) = \frac{1}{|x|^{n-2}}\left(\frac{a}{|a|}f \circ T_a^6 \circ T_a^5 \circ T_a^4 \circ T_a^3\right)(x)$$

is left 2-monogenic in $T_a^2 \circ T_a^1(B(0,1))$. Then, by Property 3.6, the function

$$g \circ T_a^2(x) = \frac{1}{|x - \frac{a}{|a|}|^{n-2}}\left(\frac{a}{|a|}f \circ T_a^6 \circ T_a^5 \circ T_a^4 \circ T_a^3 \circ T_a^2\right)(x)$$

is left 2-monogenic in $T_a^1(B(0,1))$. Consequently, by Property 3.6, the function

$$g \circ T_a^2 \circ T_a^1(x) = \frac{1}{|-|a|x - \frac{a}{|a|}|^{n-2}} \frac{a}{|a|} f \circ T_a(x)$$

is left 2-monogenic in $B(0,1)$. As

$$\left||a|x+\frac{a}{|a|}\right|=\left|\frac{a(\overline{a}x+1)}{|a|}\right|=|1+\overline{a}x|,$$

we draw the conclusion. □

After similar calculations, we can achieve the following theorem.

Theorem 3.4. *Let $T_a : B(0,1) \to B(0,1)$ be the Möbius transformation, $T_a(x) = (x+a)(1+\overline{a}x)^{-1}$, where $x = \sum_{i=1}^{n} x_i e_i$. If f is a left 3-monogenic function in $B(0,1)$, then the function F defined by*

$$F(x) = \frac{(1+x\overline{a})}{|1+\overline{a}x|^{n-2}}(f \circ T_a(x)) \tag{3.36}$$

is also left 3-monogenic in $B(0,1)$.

Theorem 3.5. *Let $T_a : B(0,1) \to B(0,1)$ be the Möbius transformation, $T_a(x) = (x+a)(1+\overline{a}x)^{-1}$, where $x = \sum_{i=1}^{n} x_i e_i$. If f is a left 4-monogenic function in $B(0,1)$, then the function F defined by*

$$F(x) = \frac{1}{|1+\overline{a}x|^{n-4}}\frac{a}{|a|}(f \circ T_a(x)) \tag{3.37}$$

is also left 4-monogenic in $B(0,1)$.

Theorem 3.6. *Let $T_a : B(0,1) \to B(0,1)$ be the Möbius transformation, $T_a(x) = (x+a)(1+\overline{a}x)^{-1}$, where $x = \sum_{i=1}^{n} x_i e_i$. If f is a left k-monogenic function in $B(0,1)$, then the function F defined by*

$$F(x) = \frac{(\overline{-|a|x-\frac{a}{|a|}})^k}{|1+\overline{a}x|^{n}}\frac{a}{|a|}(f \circ T_a(x)) \tag{3.38}$$

is also left k-monogenic in $B(0,1)$.

Remark 3.2. *To clarify the relationships among Theorems 3.3–3.6, the function F in these theorems is denoted as F_i with $i = 3.3, 3.4, 3.5, 3.6$. When $k=2$, $F_{3.6}(x) = -F_{3.3}(x)$; when $k=3$, $F_{3.6}(x) = F_{3.4}(x)$; and when $k=4$, $F_{3.6}(x) = F_{3.5}(x)$.*

4. Applications for Möbius transformations

Example 4.1. *Let $f(x) = x \in C^2(\overline{B(0,\ 1)}, Cl_{0,3})$, where $B(0,1) \in \mathbf{R}^3$. Then $f(0) = 0$, $D^2 f(x) = 0$, and $|f(x)| = |x| \le 1$ for $\forall x \in B(0,\ 1)$. $T_a(x) = (x+a)(1+\overline{a}x)^{-1}$, where $x \in B(0,\ 1)$. Taking $a = \frac{1}{2}e_1$, we find that $f(T_a(x))$ is not left monogenic. But $\frac{a}{|1+\overline{a}x||a|}(f \circ T_a(x))$ is left monogenic.*

Lemma 4.1. *[5] If $f \in C^2(\overline{B(0,1)}, Cl_{0,n})$, $\Delta f = 0$ in $B(0,1)$, $f(0) = 0$, and for any $x \in B(0,1)$, we have $|f(x)| \le 1$, then for any $x \in B(0,1)$, we have*

$$|f(x)| \le \frac{1}{\sqrt[n]{2}-1}|x|.$$

Theorem 4.1. *(Schwarz-Pick-type lemma for harmonic functions) If* $f \in C^2(\overline{B(0,1)}, Cl_{0,n})$*,* $\Delta f = 0$ *in* $B(0,1)$*,* $f(a) = 0$ *for some* $a \in B(0,1)$*, and for any* $x \in B(0,1)$*, we have* $|f(x)| \leq 1$*, then for any* $x \in B(0,1)$*, we have*

$$|f(x)| \leq \frac{(1+|a|)^{n-2}}{\sqrt[n]{2}-1} \frac{|x-a|}{|1-\overline{a}x|^{n-1}}.$$

Proof. Let $y = T_a(x) = (x-a)(1-\overline{a}x)^{-1}$. Then $x = T_a^{-1}(y) = (y+a)(1+\overline{a}y)^{-1} = (y+a)\dfrac{1+\overline{y}a}{|1+\overline{a}y|^2}$.

Let

$$F(y) = (1-|a|)^{n-2} \frac{1}{|1+\overline{a}y|^{n-2}} \frac{a}{|a|} (f \circ T_a^{-1}(y)).$$

By Theorem 3.3, F is harmonic in $B(0,1)$ with respect to y. In addition, $F(0) = 0$, $|F(y)| \leq 1$, and according to Lemma 4.1, we draw the conclusion. □

Remark 4.1. *Theorem 4.1 is the same as the result in [10]. However, [10] is derived using conclusions from several complex variables, whereas we provide a direct calculation of its four types of basic forms within the framework of Clifford analysis.*

Based on the relationship between harmonic functions and inframonogenic functions, we can obtain a version of the Schwarz-Pick-type lemma for inframonogenic functions as follows.

Theorem 4.2. *(Schwarz-Pick-type lemma for inframonogenic functions) If* $f \in C^2(\overline{B(0,1)}, Cl_{0,n})$ *is inframonogenic in* $B(0,1)$*,* $f(a) = 0$ *for some* $a \in B(0,1)$*,* $|f(x)| \leq 1$ *for any* $x \in B(0,1)$*, then for any* $x \in B(0,1)$*, we have*

$$|f(x)| \leq (1 + \frac{1}{2}k + \frac{1}{2}|(fD)(a)||a|) \frac{(1+|a|)^{n-2}}{\sqrt[n]{2}-1} \frac{|x-a|}{|1-\overline{a}x|^{n-1}} + \frac{1}{2}k|x| + \frac{1}{2}|(fD)(a)||a|,$$

where $k = \sup\limits_{x \in \overline{B(0,1)}} |(fD)(x)|$.

Proof. Let $g(x) = (f(x) + \frac{1}{2}[(fD)(x)]x - \frac{1}{2}[(fD)(a)]a)(1 + \frac{1}{2}k + \frac{1}{2}|(fD)(a)||a|)^{-1}$, where f is inframonogenic in $B(0,1)$, $k = \sup\limits_{x \in \overline{B(0,1)}} |(fD)(x)|$. Then g is harmonic in $B(0,1)$. The detailed proof can be found in [11]. In addition, $g(a) = 0$, $|g(x)| \leq 1$ holds for any $x \in B(0,1)$.

By Theorem 4.1, we have

$$|g(x)| \leq \frac{(1+|a|)^{n-2}}{\sqrt[n]{2}-1} \frac{|x-a|}{|1-\overline{a}x|^{n-1}},$$

that is,

$$|(f(x) + \frac{1}{2}[(fD)(x)]x - \frac{1}{2}[(fD)(a)]a)(1 + \frac{1}{2}k + \frac{1}{2}|(fD)(a)||a|)^{-1}| \leq \frac{(1+|a|)^{n-2}}{\sqrt[n]{2}-1} \frac{|x-a|}{|1-\overline{a}x|^{n-1}}.$$

From the properties of inequalities, we conclude

$$|f(x)| - \frac{1}{2}|[(fD)(x)]x| - \frac{1}{2}|(fD)(a)||a| \leq (1 + \frac{1}{2}k + \frac{1}{2}|(fD)(a)||a|) \frac{(1+|a|)^{n-2}}{\sqrt[n]{2}-1} \frac{|x-a|}{|1-\overline{a}x|^{n-1}}.$$

Furthermore, we have

$$|f(x)|\leq (1+\frac{1}{2}k+\frac{1}{2}|(fD)(a)||a|)\frac{(1+|a|)^{n-2}}{\sqrt[n]{2}-1}\frac{|x-a|}{|1-\overline{a}x|^{n-1}}+\frac{1}{2}|[(fD)(x)]x|+\frac{1}{2}|(fD)(a)||a|$$

$$\leq(1+\frac{1}{2}k+\frac{1}{2}|(fD)(a)||a|)\frac{(1+|a|)^{n-2}}{\sqrt[n]{2}-1}\frac{|x-a|}{|1-\overline{a}x|^{n-1}}+\frac{1}{2}k|x|+\frac{1}{2}|(fD)(a)||a|,$$

where $k=\sup_{x\in\overline{B(0,1)}}|(fD)(x)|$. □

5. Conclusions

In this paper, we have discussed that the composite function of a k-monogenic function and a Möbius transformation when $k = 2, 3, 4$ and in the general case, which extends the functions associated with Lemma 1.1, we proved the Schwarz-Pick-type lemma for harmonic functions through calculations in Clifford analysis rather than methods from several complex variables, and we obtained the Schwarz-Pick-type lemma for inframonogenic functions. The function classes corresponding to Lemma 1.2 were generalized.

In future studies, we will focus on the Schwarz-Pick-type lemma for k-monogenic functions, explore its applications, and examine relevant fuzzy algorithms.

Author contributions

Xiaotong Liang: Conceptualization, writing-original draft, methodology, writing-review and editing; Chunxue Duan: writing-review and language editing; Zihan Su: writing-review and language editing; Yonghong Xie: Conceptualization, methodology, writing-review, supervision, language editing, and funding acquisition. All of the authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors state that there are no conflicts of interest in this paper.

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