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*Research article*

# Uncertainty measures for concomitants of upper $k$-record values based on the Huang-Kotz-Morgenstern type II family

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**Abstract:** Statistical models involving bivariate data are among the most important areas of statistical theory. Concomitants of $k$-record values (CKR) are key topics in statistical theory that have not been extensively studied by researchers. In this paper, we derived the marginal distribution of CKR based on the Huang-Kotz Farlie-Gumbel-Morgenstern type II (HK–FGM2) family of bivariate distributions. Additionally, we obtained an expression for past extropy associated with the CKR within the HK–FGM2 family. Three related uncertainty measures (the cumulative residual extropy (CREX), cumulative extropy (CEX), and the negative cumulative extropy (NCEX)), were formulated for CKR based on this family. Furthermore, estimation techniques for CREX and NCEX were explored by employing empirical estimators tailored to CKR using the HK–FGM2 family. Finally, we analyzed a computer series system dataset for illustration purposes, providing significant insights into the behavior of CKR uncertainty measures.

**Keywords:** HK-FGM family; concomitants; k-record values; entropy; past extropy; cumulative extropy; cumulative residual extropy
**Mathematics Subject Classification:** 60B12, 62G30

## 1. Introduction

The concept of differential extropy, introduced by Lad et al. [1], is regarded as the complementary counterpart of Shannon entropy. It emerged from a critical assessment aimed at evaluating the performance of forecasting distributions. For a non-negative random variable (RV) $X$ with probability density function (PDF) $f_X(x)$, the differential extropy is defined as:

$$\mathcal{J}(X) = \frac{-1}{2} \int_0^\infty f_X^2(x)\, dx = \frac{-1}{2} \int_0^1 f_X(F_X^{-1}(v))\, dv \leq 0.$$

Jahanshahi et al. [2] proposed a new uncertainty measure for non-negative RVs having a survival function $\bar{F}_X(x) = 1 - F_X(x)$, referred to as CREX and expressed as:

$$\mathcal{CRJ}(X) = \frac{-1}{2} \int_0^\infty \bar{F}_X^2(x)\, dx. \tag{1.1}$$

In contrast to extropy, the CREX utilizes the survival function of the RV $X$ instead of relying solely on its PDF. Jahanshahi et al. [2] demonstrated that CREX possesses several useful properties as a measure of uncertainty and established bounds linking it with extropy. Consequently, CREX offers an alternative viewpoint for evaluating uncertainty, complementing traditional measures such as entropy and extropy. Following a similar approach, Kumar and Taneja [3] introduced CEX, defined in terms of the distribution function (DF). This measure is particularly effective for quantifying the idle time of a system. This measure has been proved to be a valuable tool for analyzing the uncertainty associated with the elapsed lifetimes of systems. The CEX of a non-negative RV $X$ with DF is defined as

$$\mathcal{CJ}(X) = \frac{-1}{2} \int_0^\infty F_X^2(x)\, dx. \tag{1.2}$$

Tahmasebi and Toomaj [4] proposed the concept of NCEX and explored its main properties. For a non-negative absolutely continuous RV $X$, the NCEX is defined as

$$\mathcal{NCJ}(X) = \frac{1}{2} \int_0^\infty \left[1 - F_X^2(x)\right] dx.$$

In forensic science, measures of uncertainty play a significant role in evaluating the indeterminacy associated with past lifetime events. Recently, Krishnan et al. [5] introduced a new measure known as the past extropy to quantify the uncertainty of an RV's elapsed lifetime and thoroughly discussed its motivation and practical relevance. The past extropy of an RV $X_t = (t - X|X < t)$ is defined as

$$\mathcal{J}^t(X) = \frac{-1}{2F_X^2(t)} \int_0^t f_X^2(x)\, dx.$$

It is clear that $\mathcal{J}(X) < 0$ and $\mathcal{J}^t(X) = \mathcal{J}(X)$, for $t = \infty$. This has several applications in information theory, reliability theory, survival analysis, etc.

The concept of information measures, traditionally applied to the distribution of an RV, has been extended to ordered RVs, such as order statistics, record values, and $k$-records, which arise from random samples. These ordered observations are considered more informative than simple sample values, as they provide both numerical and positional (rank-based) information. Among these, record values have received particular attention in statistical literature. However, making inferences from classical records is often challenging due to their rarity. In fact, the expected time to observe a new record is infinite. To address this issue, the concept of $k$-records was introduced by Dziubdziela and Kopocinski [6]. Unlike classical records, $k$-records occur more frequently, making them more practical for statistical analysis.

$k$-records have become a topic of growing interest in the study of information measures. Hofmann and Balakrishnan [7] derived general expressions for the Fisher information contained in $k$-record values from independent and identically distributed (i.i.d) samples drawn from continuous distributions. Madadi and Tata [8] focused on the Shannon information in $k$-records, while Tahmasebi and Eskandarzadeh [9] introduced new forms of cumulative entropy and dynamic entropy based on lower $k$-records. Additionally, Goel et al. [10] examined past entropy in the context of $k$-records and used it to characterize parent distributions. Recently, Jose and Abdul Sathar [11] explored the residual extropy of $k$-records to quantify remaining uncertainty and discussed its related properties. Recently, Abd Elgawad et al. [12] studied extropy and some related measures for CKR values in an extended FGM family. More recently, Alawady et al. [13] studied extropy for CKR values in the Cambanis family, and Alawady et al. [14] obtained some information measures for CKR values based on the Sarmanov family of bivariate distributions.

Formally, let $\{X_i,\ i \geq 1\}$ be a sequence of i.i.d. RVs. An observation $X_j$ is called an upper record value if $X_j > X_i$ for every $i < j$. It is no longer adequate to use the model of record values when the waiting times between two record values are considered. Because record data are scarce in practical contexts and each subsequent record is predicted to wait for an infinite time, statistical inference based on records is difficult. In these circumstances, the largest second or third values typically play a significant role. It is possible to avoid these issues by considering the $k-$record values model, as described by Berred [15] and Fashandi and Ahmadi [16]. The PDF of the $n$th upper $k-$record values is given by

$$\mathcal{G}_{n;k}(x) = \frac{k^n}{\Gamma(n)}\left(-\log(\bar{F}_X(x))\right)^{n-1}\left(\bar{F}_X(x)\right)^{k-1} f_X(x),$$

where $\Gamma(.)$ is the gamma function; we use the notation $\bar{A} := 1 - A$ throughout this paper, where $A$ is any object for which the complement operation is meaningful (e.g., probabilities, DFs, or values in $[0, 1]$). In addition, the joint PDF of the $n$th and the $m$th upper $k-$record values, $X_{n;k}$, $X_{m;k}$, respectively, is given by

$$\begin{aligned}\mathcal{G}_{m,n;k}(x_1, x_2) &= \frac{k^n}{\Gamma(n)\Gamma(m-n)}\left(-\log(\bar{F}_X(x_1))\right)^{n-1}\left(\bar{F}_X(x_2)\right)^{k-1}\\ &\times \left(-\log\frac{\bar{F}_X(x_2)}{\bar{F}_X(x_1)}\right)^{m-n-1}\frac{f_X(x_1)f_X(x_2)}{\bar{F}_X(x_1)},\ x_1 \leq x_2.\end{aligned}$$

Using families of bivariate distributions with specified marginals is recommended when prior knowledge exists about the marginal distributions. Recent (post-2020) advancements in bivariate modeling include several flexible distributions developed to better capture dependence and over-dispersion. Notable examples are the bivariate Poisson-X-Lindley distribution (Arrar et al. [17]) and the modified bivariate Poisson-new X-Lindley model (Haddari et al. [18]), both offering tractable properties and improved performance over classical Poisson-type models in soccer score data. Additional contributions include the EP-WD-SAR model (Mansour et al. [19]), which integrates Epanechnikov-Weibull marginals with a Sarmanov copula, and the bivariate generalized Weibull model (Al-Zaydi [20]), which develops concomitant distributions and BLU estimators. Together, these works highlight the increasing sophistication of modern bivariate distributions for complex dependence structures.

Huang and Kotz [21] introduced two extended families known as the Huang-Kotz Farlie-Gumbel-Morgenstern type I (HK–FGM1) and the Huang-Kotz Farlie-Gumbel-Morgenstern type II (HK–FGM2) to expand the traditional FGM family of bivariate distributions. In fact, most works about the HK–FGM1 and HK–FGM2 models are concerned with the HK–FGM1 family. Among those works are Abd Elgawad et al. [22], Bairamov and Kotz [23], Barakat et al. [24], and Fischer and Klein [25]. Although the correlation coefficient for the HK–FGM2 model is higher, it has not been adequately studied. This is due to the fact that the mathematical treatment of this model is somewhat complex. The DF for this model is provided by

$$\mathcal{F}_{X,Y}(x,y) = F_X(x)F_Y(y)\left[1+\delta\bar{F}_X^p(x)\bar{F}_Y^p(y)\right],$$

where $F_X(x)$ and $F_Y(y)$ are the marginal DFs of two RVs $X$ and $Y$, respectively, with the PDF

$$\begin{aligned} f_{X,Y}(x,y) &= f_X(x)f_Y(y)\left[1+\delta\bar{F}_X^{p-1}(x)\bar{F}_Y^{p-1}(y)(1-(1+p)F_X(x))\right. \\ &\times \quad \left.(1-(1+p)F_Y(y))\right], \end{aligned} \tag{1.3}$$

where the admissible range of the shape-parameter vector (cf. Huang and Kotz [21]) is given as

$$\Omega = \{(\delta,p) : -1 \le \delta \le \left(\frac{p+1}{p-1}\right)^{p-1},\ p>1 \text{ or } -1\le\delta\le 1,\ p=1\}.$$

The maximal positive correlations for this family are 0.391, which are obtained at $p = 1.1877$, while the minimum correlation remains $\frac{-1}{3}$. For more details about this family see, Barakat and Syam [26].

If only the sequence of $k$-record values of the first component $X$ is of interest to the investigator, the second component is referred to as its concomitant. There are several practical experiments that deal with $k$-record values and their concomitants, e.g., Alawady et al. [14], Bdair and Raqab [27], and Chacko and Mary [28]. The PDF of the concomitant $Y_{[n;k]}$ (the $n$th upper concomitant of $X_{n;k}$) is given by

$$f_{[n;k]}(y) = \int_0^\infty f_{Y|X}(y|x)\mathcal{G}_{n;k}(x)\,dx, \tag{1.4}$$

where $f_{Y|X}(y|x)$ is the conditional PDF of $Y$ given $X$.

**Practical relevance and uncertainty modeling:** Recently, several studies have emphasized the importance of uncertainty quantification in modeling fatigue life and the dynamic reliability of engineering systems. For instance, Su et al. [29] investigated the uncertainty of fatigue life for low-carbon alloy steel using an improved bootstrap approach, providing a more robust estimation of fatigue reliability under limited experimental data. In a related context, Liu et al. [30] developed a mixed uncertainty analysis framework for evaluating the dynamic reliability of mechanical structures while considering residual strength degradation. These works demonstrate how both aleatory and epistemic uncertainties can significantly influence structural performance predictions. They further highlight the growing importance of integrating statistical uncertainty measures with physical reliability models. In this context, the extropy-based uncertainty measures explored in the present study offer a complementary theoretical foundation that can be effectively employed in reliability modeling, lifetime prediction, and uncertainty assessment of complex systems.

The structure of the paper is organized as follows: Under a broad framework, we obtain various features of CKR values in HK–FGM2 in Section 2. In Section 3, we obtain the expression for $Y_{[n;k]}$ based on various uncertainty measures for HK–FGM2, including past extropy, CEX, NCEX, and CREX; additionally some results related to these measures of CKR values are derived. Section 4 deals with the problem of estimating the NCEX and CREX for CKR values using empirical estimators. A real dataset is analyzed for illustration purposes in Section 5. Finally, Section 6 presents the conclusion and future work of this study.

*Motivation and novelty*

The motivation of this study is to develop a new analytical framework for extropy-based uncertainty measures under a flexible yet tractable dependence model. While the previous works on the HK–FGM1, Cambanis, and Sarmanov families have contributed significantly to dependence modeling, these structures exhibit different limitations when applied to information-theoretic and reliability-based functionals. The Sarmanov copula, for instance, can attain wider correlation ranges but often lacks closed-form expressions for joint functionals such as CEX, as the dependence term involves arbitrary generating functions. Similarly, the Cambanis and HK–FGM1 structures, though mathematically elegant, provide limited flexibility in representing nonlinear interactions among components beyond first-order dependence.

The present HK–FGM2 formulation introduces a higher-order FGM-type interaction term that preserves analytical simplicity while substantially extending the class of representable dependence structures. This balance between tractability and flexibility allows the derivation of closed-form expressions for CREX and NCEX, which are otherwise analytically intractable under many other copula frameworks. Moreover, the HK–FGM2 family ensures a polynomial copula kernel that remains algebraically manipulable, making it especially suitable for the development of new statistical functionals and simulation-based estimation techniques.

The theoretical developments in this paper involve significant conceptual advancements. The extropy-type measures under HK–FGM2 require novel integral representations, reparameterization of the copula kernel, and rigorous verification of density positivity and normalization. These analytical steps are nontrivial and cannot be reproduced by simply substituting the dependence function from earlier FGM or Sarmanov models. Additionally, the asymptotic properties of the proposed estimators demand new consistency proofs adapted to the structure of the HK–FGM2 dependence function.

In summary, the distinct contributions of this paper are as follows:

1) Formulation of extropy-based uncertainty measures within the HK–FGM2 dependence framework, extending beyond existing FGM-type models.
2) Analytical derivation of CREX and NCEX in closed form, demonstrating tractability advantages over Sarmanov-type constructions.
3) Establishment of large-sample properties and empirical performance validation of the proposed estimators.

4) Application of the framework to real reliability data, illustrating its interpretability and practical relevance.

Hence, the present work provides both methodological innovation and conceptual depth, moving beyond mechanical generalization to offer genuine theoretical and applied contributions to the study of extropy and dependence modeling.
The notations and abbreviations used throughout the paper are summarized in Table 1.

**Table 1. List of notations and abbreviations.**

| Symbol / Acronym | Description |
|---|---|
| AIC | Akaike Information Criterion |
| BIC | Bayesian Information Criterion |
| CKR | Concomitant of $k$-Record Values |
| CEX | Cumulative Extropy |
| CREX | Cumulative Residual Extropy |
| DF | Distribution Function |
| GExp | Generalized Exponential Distribution |
| HK–FGM2 | Huang–Kotz Farlie–Gumbel–Morgenstern Type II |
| MLE | Maximum Likelihood Estimator / Estimation |
| NCEX | Negative Cumulative Extropy |
| PDF | Probability Density Function |
| RMSE | Root Mean Squared Error |

## 2. Properties of concomitants of $k$-record values in HK–FGM2

In this section, we derive several statistical properties of the CKR values based on the HK–FGM2 family, including the marginal distributions, moments, and moment generating function (MGF).

Throughout this section, it is assumed that the RVs $X$ and $Y$ have continuous and strictly increasing DFs $F_X(x)$ and $F_Y(y)$ on their supports $[0,\infty)$, with corresponding PDFs $f_X(x)$ and $f_Y(y)$ that are positive, continuous, and differentiable almost everywhere. The admissible parameter domain $\Omega = \{(\delta, p)\}$ guarantees that the HK–FGM2 copula defines a valid joint distribution with finite density over $[0,1]^2$.

Under these regularity conditions, the inverse functions $F_X^{-1}(u)$ and $F_Y^{-1}(v)$ exist and justify the use of variable transformations such as $u = F_X(x)$ and $v = F_Y(y)$. All integrals appearing in the subsequent theorems and derivations (e.g., $I_1$, $I_2$, and $\psi_Y(k)$) are assumed to be finite, with convergence ensured by the stated assumptions.

### 2.1. Marginal distributions

The following theorem represents the PDF of $Y_{[n;k]}$ in a useful way. To indicate that $X$ is distributed as $F$, we use the notation $X \sim F$.

**Theorem 2.1.** *Let $n \in \mathbb{N}$, $p > 0$, $k > 0$, and $p + k - 1 > 0$. Then,*

$$f_{[n;k]}(y) = f_Y(y)\left[1 + \bar{F}_Y^{p-1}(y)(1 - (1+p)F_Y(y))\Lambda_{n,k;\delta}\right], \tag{2.1}$$

where $\Lambda_{n,k;\delta} = \delta k^n \left[(1+p)(k+p)^{-n} - p(p+k-1)^{-n}\right]$.

*Proof.* By using (1.3) and (1.4), we get

$$\begin{aligned}
f_{[n;k]}(y) &= \int_0^\infty f_{Y|X}(y|x)\, \mathcal{G}_{n;k}(x)dx \\
&= \frac{k^n}{\Gamma(n)} \int_0^\infty f_Y(y)\left[1 + \delta \bar{F}_X^{p-1}(x)\bar{F}_Y^{p-1}(y)\,(1-(1+p)F_X(x))\right. \\
&\times \left.(1-(1+p)F_Y(y))\right](-\log(\bar{F}_X(x)))^{n-1}(\bar{F}_X(x))^{k-1} f_X(x)\, dx \\
&= f_Y(y)\left[1 + \bar{F}_Y^{p-1}(y)(1-(1+p)F_Y(y))\, I\right],
\end{aligned}$$

where

$$I = \frac{\delta k^n}{\Gamma(n)} \int_0^\infty \bar{F}_X^{p-1}(x)(1-(1+p)F_X(x))(-\log(\bar{F}_X(x)))^{n-1}(\bar{F}_X(x))^{k-1} f_X(x)dx.$$

By using the transformation $u = \bar{F}_X(x)$, the integral $I$ reduces to

$$I = \delta k^n \left[(1+p)(k+p)^{-n} - p(p+k-1)^{-n}\right] = \Lambda_{n,k;\delta}.$$

This completes the proof. □

Let $V_i \sim F_Y^{j+i}$, $i = 1,2$. By using Theorem 2.1 and relying on (2.1), the DF, moment, and MGF of $Y_{[n;k]}$ based on the HK–FGM2 family are, respectively, given by

$$F_{[n;k]}(y) = F_Y(y) + \Lambda_{n,k;\delta} \sum_{j=0}^{\aleph(p-1)} (-1)^j \binom{p-1}{j} \left[\frac{F_{V_1}(y)}{(j+1)} - \frac{(1+p)F_{V_2}(y)}{(j+2)}\right], \tag{2.2}$$

and

$$\mu_{Y_{[n;k]}}^{(l)} = \mu_Y^{(l)} + \Lambda_{n,k;\delta} \sum_{j=0}^{\aleph(p-1)} (-1)^j \binom{p-1}{j} \left[\mu_{V_1}^{(l)} - (1+p)\mu_{V_2}^{(l)}\right],$$

where $\mu_Y^{(l)} = E[Y^l]$, $\mu_{V_1}^{(l)} = E[{V_1}^l]$, and $\mu_{V_2}^{(l)} = E[{V_2}^l]$. Moreover, $\aleph(x) = \infty$, if $x$ is non-integer and $\aleph(x) = x$, if $x$ is integer. In addition,

$$M_{Y_{[n;k]}}(s) = M_Y(s) + \Lambda_{n,k;\delta} \sum_{j=0}^{\aleph(p-1)} (-1)^j \binom{p-1}{j} \left[M_{V_1}(s) - (1+p)M_{V_2}(s)\right],$$

where $M_Y(s)$, $M_{V_1}(s)$, and $M_{V_2}(s)$ are the MGFs of the RVs $Y$, $V_1$, and $V_2$, respectively.

## 3. Some measures for $Y_{[n;k]}$

In this section, we obtain the measures of past extropy, CEX, NCEX, and CREX for the concomitant $Y_{[n;k]}$ based on HK–FGM2 family. Some properties of these measures are investigated, and applications of this result are given for the concomitants $Y_{[n;k]}$.

### 3.1. Past extropy

**Theorem 3.1.** *Let $Y_{[n;k]}$ denote the CKR based on the HK–FGM2 family, whose PDF satisfies*

$$f_{Y_{[n;k]}}(y) = f_Y(y)\Big[1 + \Lambda_{n,k;\delta}\, \bar{F}_Y^{\,p-1}(y)(1-(1+p)F_Y(y))\Big].$$

Then, the past extropy of $Y_{[n;k]}$ at time $t > 0$ is given by

$$\mathcal{J}^t(Y_{[n;k]}) = \frac{1}{(1+B(t))^2}\left[\mathcal{J}^t(Y) - \frac{\Lambda_{n,k;\delta}}{F_Y^2(t)}\, I_2 - \frac{\Lambda^2_{n,k;\delta}}{2F_Y^2(t)}\, I_1\right],$$

where

$$B(t) := \frac{\Lambda_{n,k;\delta}}{F_Y(t)} \sum_{j=0}^{\aleph(p-1)} (-1)^j \binom{p-1}{j} \left[\frac{F_Y^{j+1}(t)}{j+1} - \frac{(1+p)F_Y^{j+2}(t)}{j+2}\right],$$

$$I_1 = \sum_{i=0}^{2\aleph(p-1)} (-1)^i \binom{2(p-1)}{i} \Big[\psi_Y^{(i)} - 2(1+p)\psi_Y^{(i+1)} + (1+p)^2\psi_Y^{(i+2)}\Big],$$

and

$$I_2 = \sum_{j=0}^{\aleph(p-1)} (-1)^j \binom{p-1}{j} \Big[\psi_Y^{(j)} - (1+p)\psi_Y^{(j+1)}\Big],$$

with the auxiliary quantity

$$\psi_Y^{(k)} := \int_0^t f_Y^2(y)\, F_Y^k(y)\, dy.$$

*Proof.* Assume $F_Y$ and $f_Y$ are continuous and strictly positive on $[0,t]$, and $F_Y(t) > 0$. Let $\Lambda := \Lambda_{n,k;\delta}$ and $p \in \mathbb{N}$. Define the auxiliary function

$$A(y) := \Lambda\, \bar{F}_Y^{\,p-1}(y)(1-(1+p)F_Y(y)), \qquad y \in [0,t].$$

Hence,

$$f_{Y_{[n;k]}}(y) = f_Y(y)(1+A(y)), \qquad F_{Y_{[n;k]}}(t) = F_Y(t) + \int_0^t f_Y(y)A(y)\, dy = F_Y(t)(1+B(t)),$$

where

$$B(t) := \frac{1}{F_Y(t)} \int_0^t f_Y(y)A(y)\, dy.$$

**Justification of the integral transformation:** Let $\mu_Y$ be the probability measure induced by $Y$, i.e., $d\mu_Y(y) = f_Y(y)\, dy$, with DF $F_Y(y) = \mu_Y((0,y])$. Because $f_Y$ is positive and continuously differentiable, $F_Y$ is strictly increasing and possesses a well-defined inverse $F_Y^{-1}(u)$. Under this measure, for any integrable $g$, we have

$$\int_0^t g(y)\, f_Y(y)\, dy = \int_0^{F_Y(t)} g(F_Y^{-1}(u))\, du.$$

This change-of-variables step validates every occurrence of substitutions such as $u = F_Y(y)$ and justifies treating the integrals in $A(y)$ and $B(t)$ as expectations under the measure $\mu_Y$. The finiteness of all integrals $I_1$, $I_2$, and $\psi_Y^{(k)}$ follows from the boundedness of $F_Y$ on $[0,t]$ and the square integrability of $f_Y$.

Using the binomial expansion $\bar{F}_Y^{p-1}(y) = (1 - F_Y(y))^{p-1} = \sum_{j=0}^{\aleph(p-1)}(-1)^j\binom{p-1}{j}F_Y^j(y)$ and applying the above measure-based transformation termwise, we obtain

$$B(t) = \frac{\Lambda}{F_Y(t)} \sum_{j=0}^{\aleph(p-1)} (-1)^j \binom{p-1}{j} \left\{ \frac{F_Y^{j+1}(t)}{j+1} - \frac{(1+p)F_Y^{j+2}(t)}{j+2} \right\}.$$

From the definition of past extropy,

$$\mathcal{J}^t(Y_{[n;k]}) = \frac{-1}{2F_{Y_{[n;k]}}^2(t)} \int_0^t f_{Y_{[n;k]}}^2(y)\, dy.$$

Substituting $f_{Y_{[n;k]}}(y) = f_Y(y)(1 + A(y))$ and $F_{Y_{[n;k]}}(t) = F_Y(t)(1 + B(t))$ gives

$$\mathcal{J}^t(Y_{[n;k]}) = \frac{-1}{2F_Y^2(t)(1 + B(t))^2} \int_0^t f_Y^2(y)(1 + A(y))^2\, dy.$$

Expanding $(1 + A(y))^2$ and separating the resulting integrals yields

$$\mathcal{J}^t(Y_{[n;k]}) = \frac{1}{(1 + B(t))^2} \left[ \mathcal{J}^t(Y) - \frac{\Lambda}{F_Y^2(t)}\, I_2 - \frac{\Lambda^2}{2F_Y^2(t)}\, I_1 \right],$$

where

$$\mathcal{J}^t(Y) = \frac{-1}{2F_Y^2(t)} \int_0^t f_Y^2(y)\, dy$$

is the past extropy of the baseline variable. After binomial expansion and definition of $\psi_Y^{(k)} = \int_0^t f_Y^2(y)F_Y^k(y)\, dy$, direct computation gives

$$I_1 = \sum_{i=0}^{2\aleph(p-1)} (-1)^i \binom{2(p-1)}{i} \left[ \psi_Y^{(i)} - 2(1+p)\psi_Y^{(i+1)} + (1+p)^2\psi_Y^{(i+2)} \right],$$

and

$$I_2 = \sum_{j=0}^{\aleph(p-1)} (-1)^j \binom{p-1}{j} \left[ \psi_Y^{(j)} - (1+p)\psi_Y^{(j+1)} \right].$$

Substituting these expressions back yields the stated closed form for $\mathcal{J}^t(Y_{[n;k]})$. □

**Remark 3.1.** *If $k = 1$ (the case of concomitants of the record value), then*

$$\Delta_{n;\delta} = \delta[(1+p)^{1-n} - p^{1-n}].$$

In this case, the past extropy measure of $Y_{[n]}$ is

$$\mathcal{J}^t(Y_{[n]}) = \frac{1}{(1 + B_n(t))^2} \left[ \mathcal{J}^t(Y) - \frac{\Delta_{n;\delta}}{F_Y^2(t)}\, I_2 - \frac{\Delta_{n;\delta}^2}{2F_Y^2(t)}\, I_1 \right],$$

where

$$B_n(t) = \Delta_{n;\delta} \sum_{j=0}^{\aleph(p-1)} (-1)^j \binom{p-1}{j} \left[ \frac{F_Y^j(t)}{j+1} - \frac{(1+p)F_Y^{j+1}(t)}{j+2} \right].$$

**Theorem 3.2.** *Let $f_{Y_{[n;k]}}$ and $F_{Y_{[n;k]}}$ denote the PDF and DF of $Y_{[n;k]}$, and assume $f_{Y_{[n;k]}}$ and $F_{Y_{[n;k]}}$ are sufficiently smooth so that $\mathcal{J}^t(Y_{[n;k]})$ is differentiable. Define the reversed hazard $r_{Y_{[n;k]}}(t) = f_{Y_{[n;k]}}(t)/F_{Y_{[n;k]}}(t)$.*

Then,

$$\frac{d}{dt}\mathcal{J}^t(Y_{[n;k]}) = -\frac{r^2_{Y_{[n;k]}}(t)}{2} - 2\, r_{Y_{[n;k]}}(t)\, \mathcal{J}^t(Y_{[n;k]}).$$

Consequently, for $r_{Y_{[n;k]}}(t) > 0$,

$$\mathcal{J}^t(Y_{[n;k]}) \text{ is increasing in } t \quad \Longleftrightarrow \quad \mathcal{J}^t(Y_{[n;k]}) \le -\frac{r_{Y_{[n;k]}}(t)}{4},$$

and

$$\mathcal{J}^t(Y_{[n;k]}) \text{ is decreasing in } t \quad \Longleftrightarrow \quad \mathcal{J}^t(Y_{[n;k]}) \ge -\frac{r_{Y_{[n;k]}}(t)}{4}.$$

*Proof.* Start from

$$\mathcal{J}^t(Y) = \frac{-1}{2F^2_Y(t)} \int_0^t f^2_Y(y)\, dy,$$

and denote $H_Y(t) = \int_0^t f^2_Y(y)\, dy$. Differentiating,

$$\frac{d}{dt}\mathcal{J}^t(Y) = \frac{-1}{2}\left[H'_Y(t)F_Y^{-2}(t) + H_Y(t)\frac{d}{dt}F_Y^{-2}(t)\right] = -\frac{f^2_Y(t)}{2F^2_Y(t)} + \frac{f_Y(t)}{F^3_Y(t)}H_Y(t).$$

Now, use $H_Y(t) = -2F^2_Y(t)\mathcal{J}^t(Y)$ (from the definition of $\mathcal{J}^t(Y)$) and $f_Y(t) = r_Y(t)F_Y(t)$ to get

$$\frac{d}{dt}\mathcal{J}^t(Y) = -\frac{r^2_Y(t)}{2} - 2r_Y(t)\mathcal{J}^t(Y),$$

and the monotonicity equivalences follow by straightforward algebra (assuming $r_Y(t) > 0$). □

**Remark 3.2.** *(Remark on parameter limits) Before presenting the examples, we note that the proposed HK–FGM2 family is a two-parameter generalization of the classical FGM copula. Hence, all derived expressions for particular marginals (e.g., exponential, power, and uniform) must correctly reduce to the standard FGM results under the limiting conditions "$\delta = 0$ or $p = 1$". These limits are used throughout this section as a consistency check for the derived forms of joint densities, moments, and extropies.*

**Example 3.1.** *Let $(X_i, Y_i)$, $i = 1, 2, ..., n$, be a bivariate random sample arising from HK–FGM2 bivariate exponential distribution (ED) with DF*

$$\mathcal{F}_{X,Y}(x, y) = \left(1 - e^{-\lambda_1 x}\right)\left(1 - e^{-\lambda_2 y}\right)\left[1 + \delta e^{-p(\lambda_1 x + \lambda_2 y)}\right], \quad x, y > 0, \lambda_1, \lambda_2 > 0. \tag{3.1}$$

Then,

$$\mathcal{J}^t(Y_{[n;k]}) \;=\; \frac{1}{(1 + B(t;\lambda_2))^2}\left[\frac{\lambda_2(e^{-2\lambda_2 t}-1)}{4(1-e^{-\lambda_2 t})^2} - \frac{\Lambda^2_{n,k;\delta}}{2\,(1-e^{-\lambda_2 t})^2}\sum_{i=0}^{2\aleph(p-1)}\binom{2(p-1)}{i}(-1)^i\Big(\lambda_2 e^{-\lambda_2 t}(1-e^{-\lambda_2 t})^{i+1}\right.$$

$$\times \left(\frac{(1+e^{\lambda_2 t}+i)}{(i+1)(i+2)}-\frac{2(1+p)(1-e^{-\lambda_2 t})(2+e^{\lambda_2 t}+i)}{(i+2)(i+3)}+\frac{(1+p)^2(1-e^{-\lambda_2 t})^2(3+e^{\lambda_2 t}+i)}{(i+3)(i+4)}\right)\Bigg)$$

$$-\ \frac{\Lambda_{n,k;\delta}}{(1-e^{-\lambda_2 t})^2}\sum_{j=0}^{\aleph(p-1)}\binom{p-1}{j}(-1)^j\left(\lambda_2 e^{-\lambda_2 t}(1-e^{-\lambda_2 t})^{j+1}\left(\frac{(1+e^{\lambda_2 t}+j)}{(j+1)(j+2)}-(1+p)\right.\right.$$

$$\left.\left.\times\ \frac{(1-e^{-\lambda_2 t})\ (2+e^{\lambda_2 t}+j)}{(j+2)(j+3)}\right)\right)\Bigg],$$

where

$$B(t;\lambda_2):=\left(\Lambda_{n,k;\delta}\sum_{j=0}^{\aleph(p-1)}(-1)^j\binom{p-1}{j}\left(\frac{\left(1-e^{-\lambda_2 t}\right)^j}{(j+1)}-\frac{(1+p)\left(1-e^{-\lambda_2 t}\right)^{j+1}}{(j+2)}\right)\right).$$

**Example 3.2.** *For HK–FGM2 bivariate power function distribution (PD) with DF*

$$\mathcal{F}_{X,Y}(x,y)=x^{c_1}y^{c_2}\left[1+\delta\,(1-x^{c_1})^p\,(1-y^{c_2})^p\right],\ \ 0<x,y<1,\ c_1,c_2>0. \tag{3.2}$$

Then,

$$\mathcal{J}^t(Y_{[n;k]})=\frac{1}{(1+B(t;c_2))^2}\left[\frac{-c_2^2}{2t(2c_2-1)}-\frac{\Lambda^2_{n,k;\delta}}{2t^{2c_2}}\sum_{i=0}^{2\aleph(p-1)}\binom{2(p-1)}{i}(-1)^i\right.$$

$$\times\, c_2^2\left(\frac{t^{c_2(2+i)-1}}{c_2(2+i)-1}-\frac{2(1+p)t^{c_2(3+i)-1}}{c_2(3+i)-1}+\frac{(1+p)^2t^{c_2(4+i)-1}}{c_2(4+i)-1}\right)-\frac{\Lambda_{n,k;\delta}}{t^{2c_2}}\sum_{j=0}^{\aleph(p-1)}\binom{p-1}{j}(-1)^j$$

$$\left.\times\, c_2^2\left(\frac{t^{c_2(2+j)-1}}{c_2(2+j)-1}-\frac{(1+p)t^{c_2(3+j)-1}}{c_2(3+j)-1}\right)\right],$$

where

$$B(t;c_2):=\left(\Lambda_{n,k;\delta}\sum_{j=0}^{\aleph(p-1)}(-1)^j\binom{p-1}{j}\left[\frac{t^{c_2 j}}{(j+1)}-\frac{(1+p)t^{c_2(j+1)}}{(j+2)}\right]\right).$$

**Example 3.3.** *For HK–FGM2 bivariate uniform distribution (UD) with DF*

$$\mathcal{F}_{X,Y}(x,y)=xy\left[1+\delta\,(1-x)^p\,(1-y)^p\right],\ \ 0<x,y<1. \tag{3.3}$$

Then,

$$\mathcal{J}^t(Y_{[n;k]})=\frac{1}{(1+B(t))^2}\left[\frac{-1}{2t}-\frac{\Lambda^2_{n,k;\delta}}{2t^2}\sum_{i=0}^{2\aleph(p-1)}\binom{2(p-1)}{i}(-1)^i\left(\frac{t^{(i+1)}}{(i+1)}-\frac{2(1+p)t^{(i+2)}}{(i+2)}+\frac{(1+p)^2t^{(i+3)}}{(i+3)}\right)\right.$$

$$\left.-\frac{\Lambda_{n,k;\delta}}{t^2}\sum_{j=0}^{\aleph(p-1)}\binom{p-1}{j}(-1)^j\left(\frac{t^{(j+1)}}{(j+1)}-\frac{(1+p)t^{(j+2)}}{(j+2)}\right)\right],$$

where

$$B(t):=\left(\Lambda_{n,k;\delta}\sum_{j=0}^{\aleph(p-1)}(-1)^j\binom{p-1}{j}\left[\frac{t^j}{(j+1)}-\frac{(1+p)t^{(j+1)}}{(j+2)}\right]\right).$$

### 3.2. Cumulative extropy

**Theorem 3.3.** *For the concomitant $Y_{[n;k]}$, the CEX of HK–FGM2 is provided by*

$$\mathcal{CJ}(Y_{[n;k]}) = \mathcal{CJ}(Y) - \frac{\Lambda^2_{n,k;\delta}}{2} \sum_{j=0}^{\aleph(p-1)} \sum_{i=0}^{\aleph(p-1)} (-1)^{i+j} \binom{p-1}{j} \binom{p-1}{i} \left[ \frac{\chi^{(i+j+2)}}{(i+1)(j+1)} - (1+p)\,\chi^{(i+j+3)} \right.$$

$$\left. \times \frac{(2ij+3i+3j+4)}{(i+1)(i+2)(j+1)(j+2)} + \frac{(1+p)^2 \chi^{(i+j+4)}}{(i+2)(j+2)} \right] - \Lambda_{n,k;\delta} \sum_{j=0}^{\aleph(p-1)} (-1)^j \binom{p-1}{j}$$

$$\times \left( \frac{\psi^{(j+2)}}{(j+1)} - \frac{(1+p)\psi^{(j+3)}}{(j+2)} \right),$$

where $\chi^{(i+j+l)} = \int_0^\infty F_Y^{i+j+l}(y)\, dy,\ l = 2,3,4,$ and $\psi^{(j+a)} = \int_0^\infty F_Y^{j+a}(y)\, dy,\ a = 2,3.$

*Proof.* From (1.2) and (2.2), we have

$$\mathcal{CJ}(Y_{[n;k]}) = \frac{-1}{2} \int_0^\infty F^2_{Y_{[n;k]}}(y) dy$$

$$= \frac{-1}{2} \int_0^\infty \left[ F_Y(y) + \Lambda_{n,k;\delta} \sum_{j=0}^{\aleph(p-1)} (-1)^j \binom{p-1}{j} \left( \frac{F_Y^{j+1}(y)}{(j+1)} - \frac{(1+p)F_Y^{j+2}(y)}{(j+2)} \right) \right]^2 dy$$

$$= \mathcal{CJ}(Y) - \frac{\Lambda^2_{n,k;\delta}}{2} \sum_{j=0}^{\aleph(p-1)} \sum_{i=0}^{\aleph(p-1)} (-1)^{i+j} \binom{p-1}{j} \binom{p-1}{i} \left[ \frac{\int_0^\infty F_Y^{i+j+2}(y) dy}{(i+1)(j+1)} - \frac{(1+p)}{(i+2)(j+1)} \right.$$

$$\left. \times \int_0^\infty F_Y^{i+j+3}(y) dy - \frac{(1+p)}{(i+1)(j+2)} \int_0^\infty F_Y^{i+j+3}(y) dy + \frac{(1+p)^2}{(i+2)(j+2)} \int_0^\infty F_Y^{i+j+4}(y) dy \right]$$

$$-\Lambda_{n,k;\delta} \sum_{j=0}^{\aleph(p-1)} (-1)^j \binom{p-1}{j} \left( \frac{1}{(j+1)} \int_0^\infty F_Y^{j+2}(y) dy - \frac{(1+p)}{(j+2)} \int_0^\infty F_Y^{j+3}(y) dy \right)$$

$$= \mathcal{CJ}(Y) - \frac{\Lambda^2_{n,k;\delta}}{2} \sum_{j=0}^{\aleph(p-1)} \sum_{i=0}^{\aleph(p-1)} (-1)^{i+j} \binom{p-1}{j} \binom{p-1}{i} \left[ \frac{\chi^{(i+j+2)}}{(i+1)(j+1)} - (1+p)\chi^{(i+j+3)} \right.$$

$$\left. \times \frac{(2ij+3i+3j+4)}{(i+1)(i+2)(j+1)(j+2)} + \frac{(1+p)^2}{(i+2)(j+2)} \chi^{(i+j+4)} \right] - \Lambda_{n,k;\delta} \sum_{j=0}^{\aleph(p-1)} (-1)^j \binom{p-1}{j}$$

$$\times \left( \frac{\psi^{(j+2)}}{(j+1)} - \frac{(1+p)\psi^{(j+3)}}{(j+2)} \right).$$

This completes the proof of the theorem. □

For distributions with an infinite range such as the Pareto, exponential, and Weibull families, it is evident that CEX does not exist. In contrast, for distributions whose range is bounded, like the power or uniform distributions, CEX can be well-defined.

**Remark 3.3.** *If $k = 1$, we get the concomitants of the record value. Then, the CEX measure for $Y_{[n]}$ is given by*

$$\mathcal{CJ}(Y_{[n]}) = \mathcal{CJ}(Y) - \frac{\Delta^2_{n;\delta}}{2} \sum_{j=0}^{\aleph(p-1)} \sum_{i=0}^{\aleph(p-1)} (-1)^{i+j} \binom{p-1}{j}\binom{p-1}{i} \left[ \frac{\chi^{(i+j+2)}}{(i+1)(j+1)} - (1+p)\chi^{(i+j+3)} \right.$$

$$\left. \times \frac{(2ij+3i+3j+4)}{(i+1)(i+2)(j+1)(j+2)} + \frac{(1+p)^2\chi^{(i+j+4)}}{(i+2)(j+2)} \right] - \Delta_{n;\delta} \sum_{j=0}^{\aleph(p-1)} (-1)^j \binom{p-1}{j}$$

$$\times \left( \frac{\psi^{(j+2)}}{(j+1)} - \frac{(1+p)\psi^{(j+3)}}{(j+2)} \right).$$

**Example 3.4.** *If $(X, Y)$ is a bivariate sample from HK–FGM2 bivariate PD with the DF given in* (3.2)*, then,*

$$\begin{aligned} \mathcal{CJ}(Y_{[n;k]}) &= \frac{-1}{2(2c_2+1)} - \frac{\Lambda^2_{n,k;\delta}}{2} \sum_{j=0}^{\aleph(p-1)} \sum_{i=0}^{\aleph(p-1)} (-1)^{i+j} \binom{p-1}{j}\binom{p-1}{i} \left[ \frac{(c_2(2+i+j)+1)^{-1}}{(i+1)(j+1)} - (1+p) \right. \\ &\times \left. \frac{(2ij+3i+3j+4)}{(c_2(3+i+j)+1)(i+1)(i+2)(j+1)(j+2)} + \frac{(1+p)^2}{(i+2)(j+2)(c_2(4+i+j)+1)} \right] \\ &- \Lambda_{n,k;\delta} \sum_{j=0}^{\aleph(p-1)} (-1)^j \binom{p-1}{j} \left( \frac{1}{(j+1)(c_2(2+j)+1)} - \frac{(1+p)}{(j+2)(c_2(3+j)+1)} \right). \end{aligned}$$

**Example 3.5.** *If $(X, Y)$ is a bivariate sample from HK–FGM2 bivariate UD with the DF given in* (3.3)*, then,*

$$\begin{aligned} \mathcal{CJ}(Y_{[n;k]}) &= \frac{-1}{6} - \frac{\Lambda^2_{n,k;\delta}}{2} \sum_{j=0}^{\aleph(p-1)} \sum_{i=0}^{\aleph(p-1)} (-1)^{i+j} \binom{p-1}{j}\binom{p-1}{i} \left[ \frac{(3+i+j)^{-1}}{(i+1)(j+1)} - \frac{(1+p)}{(4+i+j)} \right. \\ &\times \left. \frac{(2ij+3i+3j+4)}{(i+1)(i+2)(j+1)(j+2)} + \frac{(1+p)^2}{(i+2)(j+2)(5+i+j)} \right] - \Lambda_{n,k;\delta} \sum_{j=0}^{\aleph(p-1)} (-1)^j \binom{p-1}{j} \\ &\times \left( \frac{1}{(j+1)(3+j)} - \frac{(1+p)}{(j+2)(4+j)} \right). \end{aligned}$$

*3.3. NCEX*

**Theorem 3.4.** *Suppose $Y_{[n;k]}$ is the CKR values, then the NCEX is given by*

$$\begin{aligned} \mathcal{NCJ}(Y_{[n;k]}) &= \frac{1}{2} \int_0^\infty \left[ 1 - F^2_{Y_{[n;k]}}(y) \right] dy \\ &= \frac{1}{2} \int_0^\infty \left( 1 - \left[ F_Y(y) + \Lambda_{n,k;\delta} \sum_{j=0}^{\aleph(p-1)} (-1)^j \binom{p-1}{j} \left( \frac{F_Y^{j+1}(y)}{(j+1)} - \frac{(1+p)F_Y^{j+2}(y)}{(j+2)} \right) \right]^2 \right) dy \end{aligned}$$

$$= \mathcal{NCJ}(Y) - \frac{\Lambda^2_{n,k;\delta}}{2} \sum_{j=0}^{\aleph(p-1)} \sum_{i=0}^{\aleph(p-1)} (-1)^{i+j} \binom{p-1}{j}\binom{p-1}{i} \left[ \frac{\chi^{(i+j+2)}}{(i+1)(j+1)} - (1+p) \right.$$

$$\times \quad \frac{\chi^{(i+j+3)}(2ij+3i+3j+4)}{(i+1)(i+2)(j+1)(j+2)} + \frac{(1+p)^2\chi^{(i+j+4)}}{(i+2)(j+2)}\Bigg] - \Lambda_{n,k;\delta}\sum_{j=0}^{\aleph(p-1)}(-1)^j\binom{p-1}{j}$$

$$\times \quad \left(\frac{\psi^{(j+2)}}{(j+1)} - \frac{(1+p)\psi^{(j+3)}}{(j+2)}\right).$$

*Proof.* The proof is similar to the proof of Theorem 3.3. □

**Remark 3.4.** *If $k = 1$, we get the concomitants of the record value. Then, the NCEX measure for $Y_{[n]}$ is given by*

$$\mathcal{NCJ}(Y_{[n]}) = \mathcal{NCJ}(Y) - \frac{\Delta^2_{n;\delta}}{2}\sum_{j=0}^{\aleph(p-1)}\sum_{i=0}^{\aleph(p-1)}(-1)^{i+j}\binom{p-1}{j}\binom{p-1}{i}\Bigg[\frac{\chi^{(i+j+2)}}{(i+1)(j+1)} - (1+p)\chi^{(i+j+3)}$$

$$\times \frac{(2ij+3i+3j+4)}{(i+1)(i+2)(j+1)(j+2)} + \frac{(1+p)^2\chi^{(i+j+4)}}{(i+2)(j+2)}\Bigg] - \Delta_{n;\delta}\sum_{j=0}^{\aleph(p-1)}(-1)^j\binom{p-1}{j}$$

$$\times\left(\frac{\psi^{(j+2)}}{(j+1)} - \frac{(1+p)\psi^{(j+3)}}{(j+2)}\right).$$

**Example 3.6.** *For the HK–FGM2 bivariate PD with DF given in* (3.2)*, then,*

$$\mathcal{NCJ}(Y_{[n;k]}) = \frac{c_2}{(1+2c_2)} - \frac{\Lambda^2_{n,k;\delta}}{2}\sum_{j=0}^{\aleph(p-1)}\sum_{i=0}^{\aleph(p-1)}(-1)^{i+j}\binom{p-1}{j}\binom{p-1}{i}\Bigg[\frac{(c_2(2+i+j)+1)^{-1}}{(i+1)(j+1)}$$

$$- \frac{(1+p)(2ij+3i+3j+4)}{(c_2(3+i+j)+1)(i+1)(i+2)(j+1)(j+2)} + \frac{(1+p)^2}{(c_2(4+i+j)+1)(i+2)(j+2)}\Bigg]$$

$$- \Lambda_{n,k;\delta}\sum_{j=0}^{\aleph(p-1)}(-1)^j\binom{p-1}{j}\left(\frac{1}{(c_2(2+j)+1)(j+1)} - \frac{(1+p)}{(j+2)(c_2(3+j)+1)}\right).$$

**Example 3.7.** *For the HK–FGM2 bivariate UD with DF given in* (3.3)*, then,*

$$\mathcal{NCJ}(Y_{[n;k]}) = \frac{1}{3} - \frac{\Lambda^2_{n,k;\delta}}{2}\sum_{j=0}^{\aleph(p-1)}\sum_{i=0}^{\aleph(p-1)}(-1)^{i+j}\binom{p-1}{j}\binom{p-1}{i}\Bigg[\frac{(3+i+j)^{-1}}{(i+1)(j+1)} - \frac{(1+p)}{(4+i+j)}$$

$$\times \frac{(2ij+3i+3j+4)}{(i+1)(i+2)(j+1)(j+2)} + \frac{(1+p)^2}{(5+i+j)(i+2)(j+2)}\Bigg] - \Lambda_{n,k;\delta}\sum_{j=0}^{\aleph(p-1)}(-1)^j\binom{p-1}{j}$$

$$\times \left(\frac{1}{(3+j)(j+1)} - \frac{(1+p)}{(j+2)(4+j)}\right).$$

### *3.4. CREX*

**Theorem 3.5.** *The CREX of $Y_{[n;k]}$ from HK–FGM2 is provided by*

$$\mathcal{CRJ}(Y_{[n;k]}) = \mathcal{CRJ}(Y) - \frac{\Lambda^2_{n,k;\delta}}{2}\sum_{j=0}^{\aleph(p-1)}\sum_{i=0}^{\aleph(p-1)}(-1)^{i+j}\binom{p-1}{j}\binom{p-1}{i}\Bigg[\frac{\chi^{(i+j+2)}}{(i+1)(j+1)} - (1+p)$$

$$\times \quad \frac{\chi^{(i+j+3)}(2ij+3i+3j+4)}{(i+1)(i+2)(j+1)(j+2)} + \frac{(1+p)^2}{(i+2)(j+2)}\chi^{(i+j+4)}\Bigg] + \Lambda_{n,k;\delta}\sum_{j=0}^{\aleph(p-1)}(-1)^j\binom{p-1}{j}$$

$$\times \quad \left(\frac{\nabla^{(j+1)}}{(j+1)} - \frac{(1+p)\nabla^{(j+2)}}{(j+2)}\right),$$

where $\nabla^{(j+a)} = \int_0^\infty \bar{F}_Y(y)F_Y^{j+a}(y)\,dy,\ a=1,2.$

*Proof.* From (1.1) and (2.2), we have

$$\mathcal{CRJ}(Y_{[n;k]}) = \frac{-1}{2}\int_0^\infty \bar{F}^2_{Y_{[n;k]}}(y)dy$$

$$= \frac{-1}{2}\int_0^\infty\left[1 - F_Y(y) - \Lambda_{n,k;\delta}\sum_{j=0}^{\aleph(p-1)}(-1)^j\binom{p-1}{j}\left(\frac{F_Y^{j+1}(y)}{(j+1)} - \frac{(1+p)F_Y^{j+2}(y)}{(j+2)}\right)\right]^2 dy$$

$$= \mathcal{CRJ}(Y) - \frac{\Lambda^2_{n,k;\delta}}{2}\sum_{j=0}^{\aleph(p-1)}\sum_{i=0}^{\aleph(p-1)}(-1)^{i+j}\binom{p-1}{j}\binom{p-1}{i}\Bigg[\frac{\int_0^\infty F_Y^{i+j+2}(y)dy}{(i+1)(j+1)} - (1+p)$$

$$\times \quad \frac{1}{(i+2)(j+1)}\int_0^\infty F_Y^{i+j+3}(y)dy - \frac{(1+p)}{(i+1)(j+2)}\int_0^\infty F_Y^{i+j+3}(y)dy + \frac{(1+p)^2}{(i+2)(j+2)}\int_0^\infty F_Y^{i+j+4}(y)\,dy\Bigg]$$

$$+ \quad \Lambda_{n,k;\delta}\sum_{j=0}^{\aleph(p-1)}(-1)^j\binom{p-1}{j}\left(\frac{1}{(j+1)}\int_0^\infty \bar{F}_Y(y)F_Y^{j+1}(y)dy - \frac{(1+p)}{(j+2)}\int_0^\infty \bar{F}_Y(y)F_Y^{j+2}(y)dy\right)$$

$$= \mathcal{CRJ}(Y) - \frac{\Lambda^2_{n,k;\delta}}{2}\sum_{j=0}^{\aleph(p-1)}\sum_{i=0}^{\aleph(p-1)}(-1)^{i+j}\binom{p-1}{j}\binom{p-1}{i}\Bigg[\frac{\chi^{(i+j+2)}}{(i+1)(j+1)} - (1+p)\,\chi^{(i+j+3)}$$

$$\times \quad \frac{(2ij+3i+3j+4)}{(i+1)(i+2)(j+1)(j+2)} + \frac{(1+p)^2}{(i+2)(j+2)}\chi^{(i+j+4)}\Bigg] + \Lambda_{n,k;\delta}\sum_{j=0}^{\aleph(p-1)}(-1)^j\binom{p-1}{j}$$

$$\times \quad \left(\frac{\nabla^{(j+1)}}{(j+1)} - \frac{(1+p)\nabla^{(j+2)}}{(j+2)}\right).$$

The proof is complete. □

**Remark 3.5.** *If $k = 1$, we get the concomitants of the record value. Then, the CREX measure for $Y_{[n]}$ is given by*

$$\mathcal{CRJ}(Y_{[n]}) = \mathcal{CRJ}(Y) - \frac{\Delta^2_{n;\delta}}{2}\sum_{j=0}^{\aleph(p-1)}\sum_{i=0}^{\aleph(p-1)}(-1)^{i+j}\binom{p-1}{j}\binom{p-1}{i}\Bigg[\frac{\chi^{(i+j+2)}}{(i+1)(j+1)} - (1+p)\,\chi^{(i+j+3)}$$

$$\times \frac{(2ij+3i+3j+4)}{(i+1)(i+2)(j+1)(j+2)} + \frac{(1+p)^2}{(i+2)(j+2)}\chi^{(i+j+4)}\Bigg] + \Delta_{n;\delta}\sum_{j=0}^{\aleph(p-1)}(-1)^j\binom{p-1}{j}$$

$$\times\left(\frac{\nabla^{(j+1)}}{(j+1)} - \frac{(1+p)\nabla^{(j+2)}}{(j+2)}\right).$$

**Example 3.8.** *For the HK–FGM2 bivariate PD with DF given in* (3.2)*, then,*

$$\begin{aligned}\mathcal{CRJ}(Y_{[n;k]}) &= \frac{-c_2^2}{1+3c_2+2c_2^2} - \frac{\Lambda_{n,k;\delta}^2}{2}\sum_{j=0}^{\aleph(p-1)}\sum_{i=0}^{\aleph(p-1)}(-1)^{i+j}\binom{p-1}{j}\binom{p-1}{i}\Bigg[\frac{(1+c_2(2+i+j))^{-1}}{(i+1)(j+1)} \\ &- \frac{(1+p)(2ij+3i+3j+4)}{(1+c_2(3+i+j))(i+1)(i+2)(j+1)(j+2)} + \frac{(1+p)^2}{(i+2)(j+2)(1+c_2(4+i+j))}\Bigg] + \Lambda_{n,k;\delta} \\ &\times \sum_{j=0}^{\aleph(p-1)}(-1)^j\binom{p-1}{j}\left(\frac{c_2}{(j+1)(1+c_2(1+j))(1+c_2(2+j))} - \frac{c_2(1+p)}{(j+2)(1+c_2(2+j))(1+c_2(3+j))}\right).\end{aligned}$$

**Example 3.9.** *For the HK–FGM2 bivariate ED with DF given in* (3.1)*, then,*

$$\begin{aligned}\mathcal{CRJ}(Y_{[n;k]}) &= \frac{-1}{4\lambda_2} - \frac{\Lambda_{n,k;\delta}^2}{2}\sum_{j=0}^{\aleph(p-1)}\sum_{i=0}^{\aleph(p-1)}(-1)^{i+j}\binom{p-1}{j}\binom{p-1}{i}\Bigg[\sum_{d=0}^{i+j+2}\frac{(-1)^d\binom{i+j+2}{d}}{\lambda_2 d(i+1)(j+1)} - \sum_{f=0}^{i+j+3}\frac{(-1)^f}{\lambda_2 f} \\ &\times \frac{\binom{i+j+3}{f}(1+p)(2ij+3i+3j+4)}{(i+1)(i+2)(j+1)(j+2)} + \frac{(1+p)^2}{(i+2)(j+2)}\sum_{g=0}^{i+j+4}\frac{(-1)^g\binom{i+j+4}{g}}{\lambda_2 g}\Bigg] \\ &+ \Lambda_{n,k;\delta}\sum_{j=0}^{\aleph(p-1)}(-1)^j\binom{p-1}{j}\left(\sum_{h=0}^{j+1}\frac{(-1)^h\binom{j+1}{h}}{\lambda_2(j+1)(1+h)} - (1+p)\sum_{v=0}^{j+2}\frac{(-1)^v\binom{j+2}{v}}{\lambda_2(j+2)(1+v)}\right).\end{aligned}$$

**Verification of limiting and special cases:** To confirm the internal consistency of the analytical developments in Examples 3.1–3.9, the following limiting checks were carried out:

(i) **When $\delta = 0$:** The dependence term in the HK–FGM2 copula vanishes, and the joint density simplifies to $f_{X,Y}(x,y) = f_X(x)f_Y(y)$. All derived expressions for the past extropy and CKR measures accordingly reduce to their marginal (independent) forms.

(ii) **When $p = 1$:** The HK–FGM2 copula reduces to the standard FGM model, i.e.,

$$C(u,v) = uv[1+\delta(1-u)(1-v)].$$

For instance, in the exponential case, substituting $p = 1$ into the expression of $\mathcal{J}^t(Y_{[n;k]})$ yields

$$\mathcal{J}^t_{\text{HK-FGM2}}(Y_{[n;k]})\Big|_{p=1} = \mathcal{J}^t_{\text{FGM}}(Y_{[n;k]}),$$

confirming algebraic consistency with the baseline FGM results in Barakat et al. [31].

(iii) **Continuity of limits:** All integrals $\psi_Y^{(k)}$, $I_1$, and $I_2$ remain finite and continuous in $(\delta, p)$, ensuring that $\lim_{p\to1,\delta\to0}\mathcal{J}^t(Y_{[n;k]})$ equals the FGM counterpart.

These verifications confirm that the proposed HK–FGM2 structure maintains coherence with established models under boundary parameter values, thereby strengthening the theoretical validity of Examples 3.1–3.9.

## 4. Empirical measures for $Y_{[n;k]}$

In this section, we estimate the CREX and NCEX for concomitants by means of the empirical estimators.

### *4.1. Empirical CREX*

In this subsection, we examine the issue of estimating the CREX for concomitants using the empirical CREX (E-CREX). Consider a random sample $(X_z, Y_z)$, $z = 1, 2, ..., m$ drawn from the HK–FGM2 family of bivariate distributions. The E-CREX of $Y_{[n;k]}$ can be derived as follows:

$$\widehat{\mathcal{CRJ}}(Y_{[n;k]}) = \frac{-1}{2}\int_0^\infty \left[1 - \widehat{F}_{Y_{[n;k]}}(y)\right]^2 dy$$

$$= \frac{-1}{2}\int_0^\infty \left[1 - \widehat{F}_Y(y) - \Lambda_{n,k;\delta}\sum_{j=0}^{\aleph(p-1)}(-1)^j\binom{p-1}{j}\left(\frac{\widehat{F}_Y^{j+1}(y)}{j+1} - \frac{(1+p)\widehat{F}_Y^{j+2}(y)}{j+2}\right)\right]^2 dy$$

$$= \frac{-1}{2}\sum_{z=1}^{m-1}\int_{y_{(z)}}^{y_{(z+1)}}\left[\left(1-\widehat{F}_Y(y)\right)^2 + \Lambda_{n,k;\delta}^2\sum_{j=0}^{\aleph(p-1)}\sum_{i=0}^{\aleph(p-1)}(-1)^{i+j}\binom{p-1}{j}\binom{p-1}{i}\left(\frac{\widehat{F}_Y^{i+j+2}(y)}{(i+1)(j+1)}\right.\right.$$

$$\left. - \frac{(1+p)}{(i+2)(j+1)}\widehat{F}_Y^{i+j+3}(y) - \frac{(1+p)}{(i+1)(j+2)}\widehat{F}_Y^{i+j+3}(y) + \frac{(1+p)^2}{(i+2)(j+2)}\widehat{F}_Y^{i+j+4}(y)\right)$$

$$\left. - 2\Lambda_{n,k;\delta}\sum_{j=0}^{\aleph(p-1)}(-1)^j\binom{p-1}{j}\left(\frac{1}{j+1}\bar{\widehat{F}}_Y(y)\widehat{F}_Y^{j+1}(y) - \frac{(1+p)}{j+2}\bar{\widehat{F}}_Y(y)\widehat{F}_Y^{j+2}(y)\right)\right] dy$$

$$= \frac{-1}{2}\sum_{z=1}^{m-1}\vartheta_z\left[\left(1-\frac{z}{m}\right)^2 + \Lambda_{n,k;\delta}^2\sum_{j=0}^{\aleph(p-1)}\sum_{i=0}^{\aleph(p-1)}(-1)^{i+j}\binom{p-1}{j}\binom{p-1}{i}\left(\frac{\left(\frac{z}{m}\right)^{i+j+2}}{(i+1)(j+1)}\right.\right.$$

$$\left. - \frac{(1+p)\left(\frac{z}{m}\right)^{i+j+3}}{(i+2)(j+1)} - \frac{(1+p)\left(\frac{z}{m}\right)^{i+j+3}}{(i+1)(j+2)} + \frac{(1+p)^2\left(\frac{z}{m}\right)^{i+j+4}}{(i+2)(j+2)}\right) - 2\Lambda_{n,k;\delta}\sum_{j=0}^{\aleph(p-1)}(-1)^j\binom{p-1}{j}$$

$$\left. \times\left(\frac{1}{j+1}\left(1-\frac{z}{m}\right)\left(\frac{z}{m}\right)^{j+1} - \frac{(1+p)}{j+2}\left(1-\frac{z}{m}\right)\left(\frac{z}{m}\right)^{j+2}\right)\right],$$

where for any DF $F(.)$, the symbol $\widehat{F}(.)$ stands for the empirical DF of $F(.)$ and $\vartheta_z = y(z+1) - y(z)$, $z = 1, 2, ..., m-1$, are the sample spacings arising from ordered random samples of $Y_z$.

**Example 4.1.** *Let $(X_i, Y_i)$, $i = 1, 2, ..., m$, be a random sample from HK–FGM2-ED. Then the sample spacings $\vartheta_z$ are independent RVs. Moreover, $\vartheta_z$ has the exponential distributed with mean $\frac{1}{\lambda_2(m-z)}$, $z = 1, 2, ..., m-1$ (for more details, see Chandler [32]). According to Pyke [33], the E-CREX expectation and variance based on $Y_{[n]}$ are as follows:*

$$E\left[\widehat{\mathcal{CRJ}}(Y_{[n]})\right] = \frac{-1}{2\lambda_2}\sum_{z=1}^{m-1}\frac{1}{(m-z)}\left[\left(1-\frac{z}{m}\right)^2 + \Lambda_{n,k;\delta}^2\sum_{j=0}^{\aleph(p-1)}\sum_{i=0}^{\aleph(p-1)}(-1)^{i+j}\binom{p-1}{j}\binom{p-1}{i}\right.$$

$$\times\left[\frac{\left(\frac{z}{m}\right)^{i+j+2}}{(i+1)(j+1)} - \frac{(1+p)\left(\frac{z}{m}\right)^{i+j+3}}{(i+2)(j+1)} - \frac{(1+p)\left(\frac{z}{m}\right)^{i+j+3}}{(i+1)(j+2)} + \frac{(1+p)^2\left(\frac{z}{m}\right)^{i+j+4}}{(i+2)(j+2)}\right]$$

$$\left. - 2\Lambda_{n,k;\delta}\sum_{j=0}^{\aleph(p-1)}(-1)^j\binom{p-1}{j}\left(\frac{1}{(j+1)}\left(1-\frac{z}{m}\right)\left(\frac{z}{m}\right)^{j+1} - \frac{(1+p)}{(j+2)}\left(1-\frac{z}{m}\right)\left(\frac{z}{m}\right)^{j+2}\right)\right],$$

$$Var\left[\widehat{\mathcal{CRJ}}(Y_{[n]})\right] = \frac{-1}{4\lambda_2^2}\sum_{z=1}^{m-1}\frac{1}{(m-z)^2}\left[\left(1-\frac{z}{m}\right)^2+\Lambda_{n,k;\delta}^2\sum_{j=0}^{\aleph(p-1)}\sum_{i=0}^{\aleph(p-1)}(-1)^{i+j}\binom{p-1}{j}\binom{p-1}{i}\right.$$
$$\times\left[\frac{\left(\frac{z}{m}\right)^{i+j+2}}{(i+1)(j+1)}-\frac{(1+p)\left(\frac{z}{m}\right)^{i+j+3}}{(i+2)(j+1)}-\frac{(1+p)\left(\frac{z}{m}\right)^{i+j+3}}{(i+1)(j+2)}+\frac{(1+p)^2\left(\frac{z}{m}\right)^{i+j+4}}{(i+2)(j+2)}\right]-2$$
$$\left.\times\ \Lambda_{n,k;\delta}\sum_{j=0}^{\aleph(p-1)}(-1)^j\binom{p-1}{j}\left(\frac{1}{(j+1)}\left(1-\frac{z}{m}\right)\left(\frac{z}{m}\right)^{j+1}-\frac{(1+p)}{(j+2)}\left(1-\frac{z}{m}\right)\left(\frac{z}{m}\right)^{j+2}\right)\right]^2.$$

*4.2. Empirical NCEX*

In this subsection, we focus on estimating the NCEX for concomitant variables through its empirical counterpart. Consider a random sample $(X_z, Y_z)$, $z = 1, 2, ..., m$ drawn from the HK–FGM2 family of distributions. The empirical NCEX (E-NCEX) of $Y_{[n;k]}$ can be derived as follows:

$$\widehat{NCJ}(Y_{[n;k]} = \frac{1}{2}\int_0^\infty \left[1-\widehat{F}^2_{Y_{[n;k]}}(y)\right]dy$$

$$= \frac{1}{2}\int_0^\infty\left[1-\left(\widehat{F}_Y(y)+\Lambda_{n,k;\delta}\sum_{j=0}^{\aleph(p-1)}(-1)^j\binom{p-1}{j}\left(\frac{\widehat{F}_Y^{j+1}(y)}{j+1}-(1+p)\frac{\widehat{F}_Y^{j+2}(y)}{j+2}\right)\right)^2\right]dy$$

$$= \frac{1}{2}\sum_{z=1}^{m-1}\int_{y_{(z)}}^{y_{(z+1)}}\left[1-\widehat{F}_Y^2(y)-\Lambda_{n,k;\delta}^2\sum_{j=0}^{\aleph(p-1)}\sum_{i=0}^{\aleph(p-1)}(-1)^{i+j}\binom{p-1}{j}\binom{p-1}{i}\left(\frac{\widehat{F}_Y^{j+1}(y)}{j+1}-\frac{(1+p)\widehat{F}_Y^{j+2}(y)}{j+2}\right)\right.$$
$$\left.\times\left(\frac{\widehat{F}_Y^{i+1}(y)}{i+1}-\frac{(1+p)\widehat{F}_Y^{i+2}(y)}{i+2}\right)-2\Lambda_{n,k;\delta}\sum_{j=0}^{\aleph(p-1)}(-1)^j\binom{p-1}{j}\left(\frac{\widehat{F}_Y^{j+2}(y)}{j+1}-\frac{(1+p)\widehat{F}_Y^{j+3}(y)}{j+2}\right)\right]dy$$

$$= \frac{1}{2}\sum_{z=1}^{m-1}\vartheta_z\left[1-\left(\frac{z}{m}\right)^2-\Lambda_{n,k;\delta}^2\sum_{j=0}^{\aleph(p-1)}\sum_{i=0}^{\aleph(p-1)}(-1)^{i+j}\binom{p-1}{j}\binom{p-1}{i}\left(\frac{\left(\frac{z}{m}\right)^{j+1}}{j+1}-\frac{(1+p)\left(\frac{z}{m}\right)^{j+2}}{j+2}\right)\right.$$
$$\left.\times\left(\frac{\left(\frac{z}{m}\right)^{i+1}}{i+1}-\frac{(1+p)\left(\frac{z}{m}\right)^{i+2}}{i+2}\right)-2\Lambda_{n,k;\delta}\sum_{j=0}^{\aleph(p-1)}(-1)^j\binom{p-1}{j}\left(\frac{\left(\frac{z}{m}\right)^{j+2}}{j+1}-\frac{(1+p)\left(\frac{z}{m}\right)^{j+3}}{j+2}\right)\right]. \quad (4.1)$$

**Example 4.2.** *We study here the E-NCEX of the concomitant $Y_{[n]}$ of the nth upper record value $X_n$ arising from the HK–FGM2 family. In this case, the sample spacings $\vartheta_z$, $z = 1, 2, ..., m-1$, are independent and each of them has the beta distribution with parameters* 1 *and* $n$. *According to Pyke [33], the expectation and variance of E-NCEX of the concomitant $Y_{[n]}$ are as follows:*

$$\mathrm{E}\left[\widehat{NCJ}(Y_{[n]})\right] = \frac{1}{2(m+1)}\sum_{z=1}^{m-1}\left[1-\left(\frac{z}{m}\right)^2-\Lambda_{n,k;\delta}^2\sum_{j=0}^{\aleph(p-1)}\sum_{i=0}^{\aleph(p-1)}(-1)^{i+j}\binom{p-1}{j}\binom{p-1}{i}\left(\frac{\left(\frac{z}{m}\right)^{j+1}}{j+1}\right.\right.$$

$$- \frac{(1+p)\left(\frac{z}{m}\right)^{j+2}}{j+2}\Bigg)\Bigg(\frac{\left(\frac{z}{m}\right)^{i+1}}{i+1} - \frac{(1+p)\left(\frac{z}{m}\right)^{i+2}}{i+2}\Bigg) - 2\Lambda_{n,k;\delta}\sum_{j=0}^{\aleph(p-1)}(-1)^j\binom{p-1}{j}$$

$$\times \left(\frac{\left(\frac{z}{m}\right)^{j+2}}{j+1} - \frac{(1+p)\left(\frac{z}{m}\right)^{j+3}}{j+2}\right)\Bigg],$$

and

$$Var\left[\widehat{\mathcal{NCJ}}(Y_{[n]})\right] = \frac{m}{4(m+1)^2(m+2)}\sum_{z=1}^{m-1}\Bigg[1-\left(\frac{z}{m}\right)^2 - \Lambda^2_{n,k;\delta}\sum_{j=0}^{\aleph(p-1)}\sum_{i=0}^{\aleph(p-1)}(-1)^{i+j}\binom{p-1}{j}\binom{p-1}{i}\Bigg(\frac{\left(\frac{z}{m}\right)^{j+1}}{j+1} -$$

$$\times \ (1+p)\frac{\left(\frac{z}{m}\right)^{j+2}}{j+2}\Bigg)\Bigg(\frac{\left(\frac{z}{m}\right)^{i+1}}{i+1} - \frac{(1+p)\left(\frac{z}{m}\right)^{i+2}}{i+2}\Bigg) - 2\Lambda_{n,k;\delta}\sum_{j=0}^{\aleph(p-1)}(-1)^j\binom{p-1}{j}\Bigg(\frac{\left(\frac{z}{m}\right)^{j+2}}{j+1}$$

$$- \ (1+p)\frac{\left(\frac{z}{m}\right)^{j+3}}{j+2}\Bigg)\Bigg]^2.$$

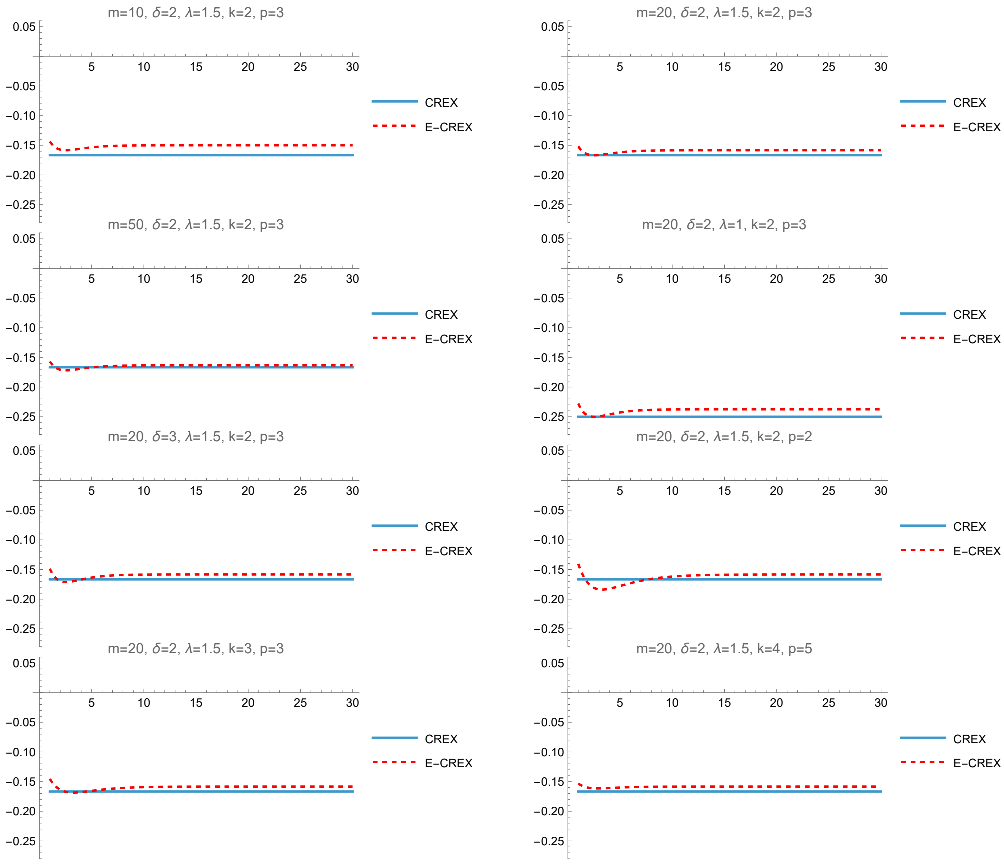



**Figure 1.** The CREX of $Y_{[n;k]}$ for HK–FGM2-ED and E-CREX of $Y_{[n;k]}$.

Figure 1 displays the CREX of $Y_{[n;k]}$ for HK–FGM2-ED and E-CREX under multiple parameter sets. Figure 2 illustrates the NCEX of $Y_{[n;k]}$ based on HK–FGM2-UD and E-NCEX across different parameter settings. Figures 1 and 2 show that the difference is more visible in some settings (e.g., when $m$ is small or when $\lambda$ changes), but still very minor. As $m$ increases (e.g., from 10 to 50), the graphs become smoother, indicating higher accuracy or stability.

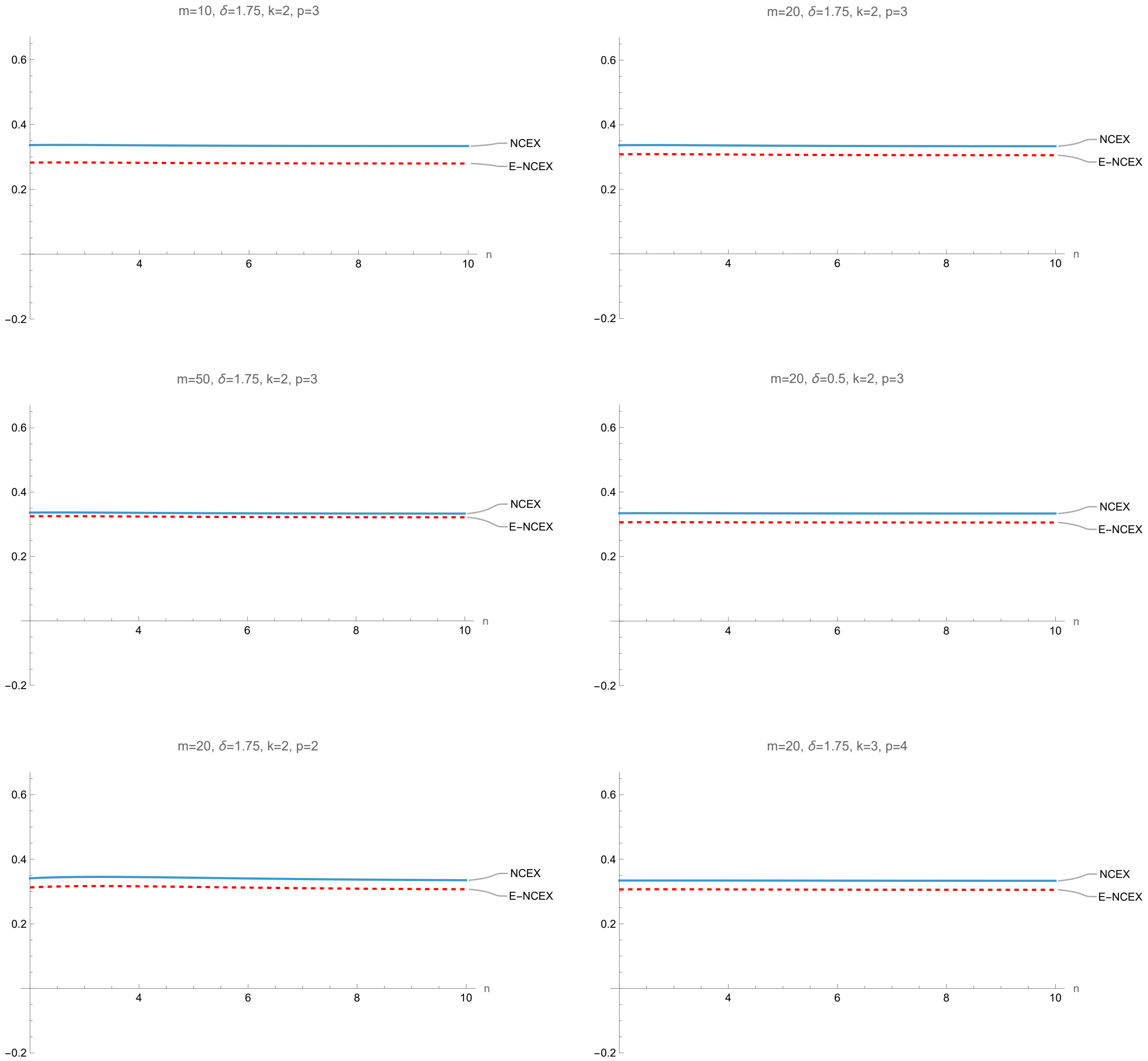



**Figure 2.** The NCEX of $Y_{[n;k]}$ for HK–FGM2-UD and E-NCEX of $Y_{[n;k]}$.

### 4.3. *Performance evaluation of the empirical estimators*

In this subsection, we assess the finite-sample performance of the empirical estimators of the CREX and NCEX derived in the previous sections. The evaluation is based on both analytical properties (bias and consistency) and a simulation study that investigates estimator behavior under different sample sizes and dependence structures.

#### 4.3.1. Analytical properties: bias and consistency

Let $\widehat{\mathcal{CRJ}}(X)$ and $\widehat{\mathcal{NCJ}}(X)$ denote the empirical estimators of $\mathcal{CRJ}(X)$ and $\mathcal{NCJ}(X)$, respectively. Under the standard regularity conditions on the parent distribution (see, e.g., Van der Vaart [34]), the following assumptions are imposed:

(i) The PDF $f_X(x)$ exists, is continuous, and bounded on the support of $X$.
(ii) The DF $F_X(x)$ is differentiable and strictly increasing.
(iii) The functions involved in the estimators are square-integrable, i.e., $\int f_X^2(x)\,dx < \infty$.
(iv) The sample observations $X_1, X_2, \ldots, X_n$ are i.i.d.
(v) The expectations defining $\mathcal{CRJ}(X)$ and $\mathcal{NCJ}(X)$ exist and are finite.

These assumptions coincide with those that justify the Glivenko–Cantelli theorem, the law of large numbers, and the central limit theorem, which together ensure consistency and asymptotic normality of empirical estimators. Under these conditions, both estimators can be expressed as integrals involving the empirical distribution and survival functions. By the Glivenko–Cantelli theorem, these empirical functions converge uniformly almost surely to their population counterparts, i.e.,

$$\sup_x \left|\widehat{F}_X(x) - F_X(x)\right| \xrightarrow{a.s.} 0 \text{ and } \sup_x \left|\widehat{\bar{F}}_X(x) - \bar{F}_X(x)\right| \xrightarrow{a.s.} 0.$$

Consequently,

$$\widehat{\mathcal{CRJ}}(X) \xrightarrow{p} \mathcal{CRJ}(X) \quad \text{and} \quad \widehat{\mathcal{NCJ}}(X) \xrightarrow{p} \mathcal{NCJ}(X)$$

showing that both estimators are *consistent*. A first-order Taylor expansion of the estimators around $F_X$ further indicates that their bias terms are of order $O(n^{-1})$, implying asymptotic unbiasedness as $n \to \infty$. Hence, both estimators possess desirable large-sample properties, with bias and variance diminishing as the sample size increases.

#### 4.3.2. Simulation design

To assess the finite-sample behavior of the proposed estimators, a Monte Carlo simulation study was conducted with $N = 5000$ independent replications. In each replication, a bivariate sample $(X, Y)$ was generated from the HK–FGM2 (3.3). The analytical form of the NCEX measure, given in Example 3.7, was used for reference, while all Monte Carlo computations employed the empirical estimator defined by (4.1).

The simulation design considered a grid of sample sizes and dependence parameters:

$$m \in \{30, 100, 1000\}, \quad \delta \in \{-0.3, 0.3\}, \quad p = 2, \quad k = n = 2, \quad N = 5000.$$

For each configuration, the estimator performance was evaluated using the empirical bias and root mean squared error (RMSE), defined as:

$$\text{Bias} = \frac{1}{N}\sum_{i=1}^{N}\left(\widehat{\theta}_i - \theta\right) \text{ and } \text{RMSE} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(\widehat{\theta}_i - \theta\right)^2},$$

where $\widehat{\theta_i}$ denotes the estimator in the $i$th replication and $\theta$ represents the true parameter value.

All computations were implemented in the R statistical software environment. As summarized in Table 2, both the bias and RMSE exhibit a monotonic decrease as the sample size increases. Specifically, at $m = 30$, the estimators display a slight negative bias and moderate RMSE. At $m = 100$, the bias becomes negligible and the RMSE is substantially reduced; at $m = 1000$, both measures are of order $10^{-3}$ or smaller. This pattern of Bias $\to 0$ and RMSE $\to 0$ as $m$ increases empirically confirms the stability, efficiency, and consistency of the proposed estimators.

**Table 2.** Monte Carlo summary results ($N = 5000$, $p = 2$, $k = n = 2$).

| $m$ | $\delta$ | Bias | RMSE |
|---|---|---|---|
| 30 | −0.3 | −0.0205 | 0.0349 |
| 30 | 0.3 | −0.0176 | 0.0340 |
| 100 | −0.3 | −0.0078 | 0.0167 |
| 100 | 0.3 | −0.0037 | 0.0157 |
| 1000 | −0.3 | −0.0026 | 0.0054 |
| 1000 | 0.3 | 0.0015 | 0.0051 |

### *4.4. Computational study*

Tables 3 and 4 show past extropy values for $Y_{[n;k]}$ under the HK–FGM2-(UD-ED) model, across different values of $k$, $n$ (odd/even), shape parameter $p$, time parameter $t$, scale parameter $\lambda$, and dependence parameter $\delta$. Also, Tables 5 and 6 exhibit CEX values for $Y_{[n;k]}$ under the HK–FGM2-(PD-UD) model, across different values of $k$, $n$ (odd/even), shape parameter $p, c$, scale parameter $\lambda$, and dependence parameter $\delta$. Moreover, as $p = 1.5$ is non-integer, and $\aleph(x) = \infty$, the values of past extropy and CEX often stabilize after approximately 10 terms in their summations. We obtain the following properties from Tables 3–6 by carrying out computations using MATHEMATICA ver.12.

- The value of $\mathcal{J}^t(Y_{[n;k]})$ increases (decreases) as the value of $t$ increases (decreases) (Table 3).
- When $p = 1.5$, $t = 0.3$, the negative values are generally larger in magnitude compared to $p = 2$, $t = 0.7$, meaning the choice of parameters influences uncertainty strength (Table 3).
- For negative $\delta(-0.5 \le \delta \le -0.2)$, the magnitude of past extropy increases as $\delta$ decreases, and vice versa for $\delta(0.5 \le \delta \le 0.2)$ (Table 4).
- As $n$ increases (whether odd or even), the values gradually converge and reach stability, indicating that the influence of the dependence parameter $\delta$ diminishes with larger records, thereby reflecting the stability of the uncertainty measure as the sample size grows (Table 4).
- For ($c = 0.5, p = 1.5$), CEX values are consistently more negative than for ($c = 1.5, p = 2$). This suggests that higher $p$ and larger $c$ tend to reduce cumulative uncertainty (Table 5).
- For every fixed $k$, $n$, and $\delta$, the CEX values under $p = 2$ are consistently more negative than under $p = 1.5$. This indicates that a larger $p$ corresponds to a greater reduction in uncertainty (Table 6).

**Table 3.** Past extropy for $Y_{[n;k]}$ based on HK–FGM2-UD.

| | | $p = 1.5, t = 0.3$ | | | | | | $p = 2, t = 0.7$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| k | n(odd) | $\delta = -0.5$ | $\delta = -0.4$ | $\delta = -0.3$ | $\delta = -0.2$ | k | n(odd) | $\delta = 0.5$ | $\delta = 0.4$ | $\delta = 0.3$ | $\delta = 0.2$ |
| 1 | 1 | -1.66667 | -1.66667 | -1.66667 | -1.66667 | 1 | 1 | -0.71429 | -0.71429 | -0.71429 | -0.71429 |
| 1 | 3 | -1.66817 | -1.66766 | -1.66724 | -1.66693 | 1 | 3 | -0.71484 | -0.71464 | -0.71448 | -0.71437 |
| 1 | 5 | -1.66725 | -1.66705 | -1.66689 | -1.66677 | 1 | 5 | -0.71436 | -0.71433 | -0.71431 | -0.7143 |
| 1 | 7 | -1.66681 | -1.66676 | -1.66672 | -1.66669 | 1 | 7 | -0.71429 | -0.71429 | -0.71429 | -0.71429 |
| 4 | 1 | -1.67363 | -1.67084 | -1.66887 | -1.66759 | 4 | 1 | -0.71868 | -0.71712 | -0.71589 | -0.715 |
| 4 | 3 | -1.66684 | -1.66678 | -1.66673 | -1.6667 | 4 | 3 | -0.71481 | -0.71462 | -0.71447 | -0.71437 |
| 4 | 5 | -1.66857 | -1.66793 | -1.6674 | -1.66701 | 4 | 5 | -0.71626 | -0.71554 | -0.71499 | -0.7146 |
| 4 | 7 | -1.66932 | -1.66844 | -1.66771 | -1.66715 | 4 | 7 | -0.71602 | -0.71539 | -0.7149 | -0.71456 |
| 6 | 1 | -1.67895 | -1.67388 | -1.6704 | -1.6682 | 6 | 1 | -0.72208 | -0.71932 | -0.71714 | -0.71557 |
| 6 | 3 | -1.6669 | -1.66681 | -1.66675 | -1.6667 | 6 | 3 | -0.71429 | -0.71429 | -0.71429 | -0.71429 |
| 6 | 5 | -1.66735 | -1.66711 | -1.66692 | -1.66678 | 6 | 5 | -0.71561 | -0.71513 | -0.71476 | -0.71449 |
| 6 | 7 | -1.66867 | -1.66799 | -1.66744 | -1.66702 | 6 | 7 | -0.71656 | -0.71574 | -0.7151 | -0.71464 |
| 8 | 1 | -1.68317 | -1.67623 | -1.67156 | -1.66865 | 8 | 1 | -0.72471 | -0.72103 | -0.71812 | -0.71601 |
| 8 | 3 | -1.66815 | -1.66759 | -1.66717 | -1.66688 | 8 | 3 | -0.71477 | -0.7146 | -0.71446 | -0.71436 |
| 8 | 5 | -1.66672 | -1.6667 | -1.66669 | -1.66667 | 8 | 5 | -0.71475 | -0.71458 | -0.71445 | -0.71436 |
| 8 | 7 | -1.66768 | -1.66733 | -1.66705 | -1.66684 | 8 | 7 | -0.71607 | -0.71542 | -0.71492 | -0.71457 |
| k | n (even) | $\delta = -0.5$ | $\delta = -0.4$ | $\delta = -0.3$ | $\delta = -0.2$ | k | n(even) | $\delta = 0.5$ | $\delta = 0.4$ | $\delta = 0.3$ | $\delta = 0.2$ |
| 1 | 2 | -1.668 | -1.66755 | -1.66718 | -1.6669 | 1 | 2 | -0.71509 | -0.7148 | -0.71457 | -0.71441 |
| 1 | 4 | -1.6677 | -1.66734 | -1.66706 | -1.66685 | 1 | 4 | -0.71451 | -0.71443 | -0.71437 | -0.71432 |
| 1 | 6 | -1.66697 | -1.66686 | -1.66678 | -1.66672 | 1 | 6 | -0.71431 | -0.7143 | -0.71429 | -0.71429 |
| 1 | 8 | -1.66673 | -1.66671 | -1.66669 | -1.66668 | 1 | 8 | -0.71429 | -0.71429 | -0.71429 | -0.71429 |
| 4 | 2 | -1.66711 | -1.66695 | -1.66682 | -1.66673 | 4 | 2 | -0.71437 | -0.71434 | -0.71431 | -0.7143 |
| 4 | 4 | -1.66774 | -1.66737 | -1.66707 | -1.66685 | 4 | 4 | -0.71578 | -0.71524 | -0.71482 | -0.71452 |
| 4 | 6 | -1.6691 | -1.66828 | -1.66761 | -1.66711 | 4 | 6 | -0.71627 | -0.71555 | -0.71499 | -0.7146 |
| 4 | 8 | -1.66932 | -1.66844 | -1.66771 | -1.66715 | 4 | 8 | -0.71567 | -0.71517 | -0.71478 | -0.7145 |
| 6 | 2 | -1.66942 | -1.66836 | -1.66758 | -1.66706 | 6 | 2 | -0.71561 | -0.71514 | -0.71477 | -0.7145 |
| 6 | 4 | -1.66676 | -1.66672 | -1.6667 | -1.66668 | 6 | 4 | -0.71477 | -0.7146 | -0.71446 | -0.71436 |
| 6 | 6 | -1.66806 | -1.66758 | -1.6672 | -1.66691 | 6 | 6 | -0.71624 | -0.71553 | -0.71498 | -0.71459 |
| 6 | 8 | -1.66911 | -1.66829 | -1.66762 | -1.66711 | 6 | 8 | -0.71661 | -0.71577 | -0.71512 | -0.71465 |
| 8 | 2 | -1.67235 | -1.67009 | -1.66849 | -1.66743 | 8 | 2 | -0.71747 | -0.71634 | -0.71545 | -0.71481 |
| 8 | 4 | -1.66681 | -1.66676 | -1.66672 | -1.66669 | 8 | 4 | -0.7143 | -0.71429 | -0.71429 | -0.71429 |
| 8 | 6 | -1.66712 | -1.66696 | -1.66684 | -1.66674 | 8 | 6 | -0.71545 | -0.71503 | -0.7147 | -0.71447 |
| 8 | 8 | -1.66824 | -1.6677 | -1.66727 | -1.66694 | 8 | 8 | -0.71651 | -0.7157 | -0.71508 | -0.71464 |

**Table 4.** Past extropy for $Y_{[n;k]}$ based on HK–FGM2-ED.

| | | $p = 1.5, t = 0.3, \lambda = 0.5$ | | | | | | $p = 2, t = 0.7, \lambda = 1.5$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| k | n(odd) | $\delta = -0.5$ | $\delta = -0.4$ | $\delta = -0.3$ | $\delta = -0.2$ | k | n(odd) | $\delta = 0.5$ | $\delta = 0.4$ | $\delta = 0.3$ | $\delta = 0.2$ |
| 1 | 1 | -1.66979 | -1.66979 | -1.66979 | -1.66979 | 1 | 1 | -0.77874 | -0.77874 | -0.77874 | -0.77874 |
| 1 | 3 | -1.67225 | -1.67175 | -1.67125 | -1.67075 | 1 | 3 | -0.76774 | -0.76985 | -0.772 | -0.7742 |
| 1 | 5 | -1.67126 | -1.67096 | -1.67066 | -1.67037 | 1 | 5 | -0.77463 | -0.77544 | -0.77626 | -0.77708 |
| 1 | 7 | -1.67049 | -1.67035 | -1.67021 | -1.67007 | 1 | 7 | -0.77755 | -0.77779 | -0.77803 | -0.77826 |
| 4 | 1 | -1.66681 | -1.66716 | -1.66767 | -1.6683 | 4 | 1 | -0.81644 | -0.80823 | -0.80034 | -0.79279 |
| 4 | 3 | -1.67056 | -1.67041 | -1.67025 | -1.6701 | 4 | 3 | -0.76802 | -0.77008 | -0.77218 | -0.77432 |
| 4 | 5 | -1.6726 | -1.67202 | -1.67145 | -1.67089 | 4 | 5 | -0.75909 | -0.76268 | -0.76645 | -0.77038 |
| 4 | 7 | -1.67318 | -1.67249 | -1.6718 | -1.67111 | 4 | 7 | -0.76021 | -0.76362 | -0.76718 | -0.77089 |
| 6 | 1 | -1.66669 | -1.66679 | -1.66724 | -1.66795 | 6 | 1 | -0.8311 | -0.81946 | -0.80837 | -0.79787 |
| 6 | 3 | -1.66899 | -1.66915 | -1.6693 | -1.66946 | 6 | 3 | -0.77925 | -0.77915 | -0.77904 | -0.77894 |
| 6 | 5 | -1.67138 | -1.67106 | -1.67073 | -1.67041 | 6 | 5 | -0.76232 | -0.76538 | -0.76855 | -0.77184 |
| 6 | 7 | -1.67268 | -1.67208 | -1.6715 | -1.67092 | 6 | 7 | -0.75782 | -0.76161 | -0.76561 | -0.7698 |
| 8 | 1 | -1.6669 | -1.66669 | -1.66704 | -1.66775 | 8 | 1 | -0.84089 | -0.82691 | -0.81367 | -0.8012 |
| 8 | 3 | -1.66797 | -1.6683 | -1.66865 | -1.66902 | 8 | 3 | -0.79016 | -0.7878 | -0.78548 | -0.78319 |
| 8 | 5 | -1.6702 | -1.67012 | -1.67003 | -1.66995 | 8 | 5 | -0.76865 | -0.77059 | -0.77257 | -0.77459 |
| 8 | 7 | -1.67177 | -1.67137 | -1.67096 | -1.67057 | 8 | 7 | -0.75994 | -0.7634 | -0.76701 | -0.77077 |
| k | n (even) | $\delta = -0.5$ | $\delta = -0.4$ | $\delta = -0.3$ | $\delta = -0.2$ | k | n(even) | $\delta = 0.5$ | $\delta = 0.4$ | $\delta = 0.3$ | $\delta = 0.2$ |
| 1 | 2 | -1.67209 | -1.67162 | -1.67115 | -1.67069 | 1 | 2 | -0.76568 | -0.76816 | -0.7707 | -0.77331 |
| 1 | 4 | -1.67179 | -1.67138 | -1.67097 | -1.67057 | 1 | 4 | -0.77164 | -0.77302 | -0.77442 | -0.77584 |
| 1 | 6 | -1.67082 | -1.67061 | -1.6704 | -1.6702 | 1 | 6 | -0.7765 | -0.77694 | -0.77739 | -0.77784 |
| 1 | 8 | -1.67027 | -1.67017 | -1.67007 | -1.66998 | 1 | 8 | -0.77812 | -0.77825 | -0.77837 | -0.77849 |
| 4 | 2 | -1.66871 | -1.66892 | -1.66913 | -1.66935 | 4 | 2 | -0.78326 | -0.78234 | -0.78143 | -0.78053 |
| 4 | 4 | -1.67183 | -1.67141 | -1.671 | -1.67059 | 4 | 4 | -0.7614 | -0.76461 | -0.76796 | -0.77143 |
| 4 | 6 | -1.67301 | -1.67235 | -1.6717 | -1.67105 | 4 | 6 | -0.75905 | -0.76265 | -0.76642 | -0.77036 |
| 4 | 8 | -1.67318 | -1.67249 | -1.6718 | -1.67111 | 4 | 8 | -0.76196 | -0.76508 | -0.76832 | -0.77168 |
| 6 | 2 | -1.66749 | -1.66787 | -1.66831 | -1.66877 | 6 | 2 | -0.79824 | -0.79413 | -0.79012 | -0.78622 |
| 6 | 4 | -1.67033 | -1.67022 | -1.67012 | -1.67001 | 6 | 4 | -0.76838 | -0.77037 | -0.7724 | -0.77447 |
| 6 | 6 | -1.67214 | -1.67166 | -1.67118 | -1.67071 | 6 | 6 | -0.75916 | -0.76275 | -0.7665 | -0.77042 |
| 6 | 8 | -1.67302 | -1.67236 | -1.6717 | -1.67105 | 6 | 8 | -0.75762 | -0.76145 | -0.76548 | -0.7697 |
| 8 | 2 | -1.66693 | -1.66731 | -1.66782 | -1.66841 | 8 | 2 | -0.81023 | -0.80344 | -0.79688 | -0.79058 |
| 8 | 4 | -1.66915 | -1.66927 | -1.6694 | -1.66953 | 8 | 4 | -0.7771 | -0.77742 | -0.77775 | -0.77808 |
| 8 | 6 | -1.67107 | -1.67081 | -1.67055 | -1.6703 | 8 | 6 | -0.76326 | -0.76616 | -0.76916 | -0.77226 |
| 8 | 8 | -1.67231 | -1.67179 | -1.67128 | -1.67078 | 8 | 8 | -0.75803 | -0.76179 | -0.76575 | -0.76989 |

**Table 5.** CEX for $Y_{[n;k]}$ based on HK–FGM2-PD.

| | | $c = 0.5, p = 1.5$ | | | | | | $c = 1.5, p = 2$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| k | n(odd) | $\delta = -0.5$ | $\delta = -0.4$ | $\delta = -0.3$ | $\delta = -0.2$ | k | n(odd) | $\delta = 0.5$ | $\delta = 0.4$ | $\delta = 0.3$ | $\delta = 0.2$ |
| 1 | 1 | -0.25 | -0.25 | -0.25 | -0.25 | 1 | 1 | -0.125 | -0.125 | -0.125 | -0.125 |
| 1 | 3 | -0.25794 | -0.25633 | -0.25473 | -0.25314 | 1 | 3 | -0.12299 | -0.12339 | -0.12379 | -0.12419 |
| 1 | 5 | -0.25477 | -0.2538 | -0.25285 | -0.25189 | 1 | 5 | -0.12427 | -0.12442 | -0.12456 | -0.12471 |
| 1 | 7 | -0.25231 | -0.25184 | -0.25138 | -0.25092 | 1 | 7 | -0.12479 | -0.12483 | -0.12488 | -0.12492 |
| 4 | 1 | -0.23713 | -0.23964 | -0.24218 | -0.24475 | 4 | 1 | -0.13103 | -0.12979 | -0.12857 | -0.12737 |
| 4 | 3 | -0.25253 | -0.25202 | -0.25152 | -0.25101 | 4 | 3 | -0.12305 | -0.12343 | -0.12382 | -0.12421 |
| 4 | 5 | -0.25906 | -0.25722 | -0.25539 | -0.25358 | 4 | 5 | -0.12128 | -0.12201 | -0.12275 | -0.12349 |
| 4 | 7 | -0.26092 | -0.25869 | -0.25649 | -0.2543 | 4 | 7 | -0.12151 | -0.12219 | -0.12289 | -0.12359 |
| 6 | 1 | -0.23381 | -0.23694 | -0.24012 | -0.24336 | 6 | 1 | -0.13316 | -0.13147 | -0.12982 | -0.12818 |
| 6 | 3 | -0.24727 | -0.24781 | -0.24836 | -0.2489 | 6 | 3 | -0.12509 | -0.12507 | -0.12505 | -0.12504 |
| 6 | 5 | -0.25516 | -0.25412 | -0.25308 | -0.25205 | 6 | 5 | -0.12193 | -0.12254 | -0.12315 | -0.12376 |
| 6 | 7 | -0.2593 | -0.25741 | -0.25554 | -0.25367 | 6 | 7 | -0.12101 | -0.12179 | -0.12258 | -0.12338 |
| 8 | 1 | -0.23185 | -0.23534 | -0.2389 | -0.24253 | 8 | 1 | -0.13454 | -0.13256 | -0.13062 | -0.12871 |
| 8 | 3 | -0.24346 | -0.24475 | -0.24605 | -0.24736 | 8 | 3 | -0.12694 | -0.12655 | -0.12616 | -0.12577 |
| 8 | 5 | -0.25135 | -0.25108 | -0.25081 | -0.25054 | 8 | 5 | -0.12317 | -0.12353 | -0.1239 | -0.12426 |
| 8 | 7 | -0.25641 | -0.25511 | -0.25382 | -0.25254 | 8 | 7 | -0.12145 | -0.12215 | -0.12285 | -0.12356 |
| k | n (even) | $\delta = -0.5$ | $\delta = -0.4$ | $\delta = -0.3$ | $\delta = -0.2$ | k | n(even) | $\delta = 0.5$ | $\delta = 0.4$ | $\delta = 0.3$ | $\delta = 0.2$ |
| 1 | 2 | -0.25744 | -0.25593 | -0.25443 | -0.25294 | 1 | 2 | -0.1226 | -0.12307 | -0.12355 | -0.12403 |
| 1 | 4 | -0.25646 | -0.25516 | -0.25385 | -0.25256 | 1 | 4 | -0.12372 | -0.12398 | -0.12423 | -0.12449 |
| 1 | 6 | -0.25336 | -0.25268 | -0.25201 | -0.25134 | 1 | 6 | -0.1246 | -0.12468 | -0.12476 | -0.12484 |
| 1 | 8 | -0.25157 | -0.25125 | -0.25094 | -0.25062 | 1 | 8 | -0.12489 | -0.12491 | -0.12494 | -0.12496 |
| 4 | 2 | -0.24628 | -0.24701 | -0.24776 | -0.2485 | 4 | 2 | -0.12578 | -0.12563 | -0.12547 | -0.12531 |
| 4 | 4 | -0.2566 | -0.25526 | -0.25393 | -0.25261 | 4 | 4 | -0.12175 | -0.12239 | -0.12303 | -0.12369 |
| 4 | 6 | -0.26038 | -0.25827 | -0.25617 | -0.2541 | 4 | 6 | -0.12127 | -0.122 | -0.12274 | -0.12349 |
| 4 | 8 | -0.26093 | -0.2587 | -0.25649 | -0.25431 | 4 | 8 | -0.12186 | -0.12248 | -0.1231 | -0.12373 |
| 6 | 2 | -0.24136 | -0.24306 | -0.24477 | -0.2465 | 6 | 2 | -0.12824 | -0.12758 | -0.12693 | -0.12628 |
| 6 | 4 | -0.25179 | -0.25143 | -0.25107 | -0.25071 | 6 | 4 | -0.12312 | -0.12349 | -0.12386 | -0.12424 |
| 6 | 6 | -0.2576 | -0.25606 | -0.25453 | -0.25301 | 6 | 6 | -0.12129 | -0.12202 | -0.12276 | -0.1235 |
| 6 | 8 | -0.26042 | -0.2583 | -0.25619 | -0.25411 | 6 | 8 | -0.12097 | -0.12176 | -0.12256 | -0.12336 |
| 8 | 2 | -0.23818 | -0.24049 | -0.24282 | -0.24519 | 8 | 2 | -0.1301 | -0.12906 | -0.12803 | -0.12701 |
| 8 | 4 | -0.24781 | -0.24825 | -0.24869 | -0.24912 | 8 | 4 | -0.12471 | -0.12477 | -0.12483 | -0.12488 |
| 8 | 6 | -0.25418 | -0.25334 | -0.2525 | -0.25166 | 8 | 6 | -0.12212 | -0.12269 | -0.12326 | -0.12384 |
| 8 | 8 | -0.25814 | -0.25648 | -0.25485 | -0.25322 | 8 | 8 | -0.12105 | -0.12183 | -0.12261 | -0.1234 |

**Table 6.** CEX for $Y_{[n;k]}$ based on HK–FGM2-UD.

| | | $p = 1.5$ | | | | | | $p = 2$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| k | n(odd) | $\delta = -0.5$ | $\delta = -0.4$ | $\delta = -0.3$ | $\delta = -0.2$ | k | n(odd) | $\delta = 0.5$ | $\delta = 0.4$ | $\delta = 0.3$ | $\delta = 0.2$ |
| 1 | 1 | -0.175 | -0.172 | -0.16967 | -0.168 | 1 | 1 | -0.17143 | -0.16971 | -0.16838 | -0.16743 |
| 1 | 3 | -0.17461 | -0.1729 | -0.17125 | -0.16966 | 1 | 3 | -0.16451 | -0.16491 | -0.16533 | -0.16577 |
| 1 | 5 | -0.17111 | -0.1702 | -0.16931 | -0.16842 | 1 | 5 | -0.16584 | -0.166 | -0.16617 | -0.16633 |
| 1 | 7 | -0.1688 | -0.16837 | -0.16794 | -0.16751 | 1 | 7 | -0.16643 | -0.16648 | -0.16652 | -0.16657 |
| 4 | 1 | -0.17511 | -0.17011 | -0.16677 | -0.16508 | 4 | 1 | -0.18876 | -0.18187 | -0.17622 | -0.1718 |
| 4 | 3 | -0.17744 | -0.17393 | -0.1711 | -0.16895 | 4 | 3 | -0.16877 | -0.16765 | -0.16688 | -0.16646 |
| 4 | 5 | -0.17858 | -0.1756 | -0.17292 | -0.17053 | 4 | 5 | -0.16364 | -0.16404 | -0.16454 | -0.16514 |
| 4 | 7 | -0.17827 | -0.17566 | -0.1732 | -0.17088 | 4 | 7 | -0.16302 | -0.16369 | -0.16438 | -0.16511 |
| 6 | 1 | -0.17499 | -0.16951 | -0.16593 | -0.16427 | 6 | 1 | -0.1943 | -0.18578 | -0.17876 | -0.17323 |
| 6 | 3 | -0.17674 | -0.17271 | -0.16969 | -0.16767 | 6 | 3 | -0.17436 | -0.17161 | -0.16946 | -0.16792 |
| 6 | 5 | -0.17829 | -0.17486 | -0.17198 | -0.16965 | 6 | 5 | -0.1663 | -0.16586 | -0.16568 | -0.16575 |
| 6 | 7 | -0.17902 | -0.17592 | -0.17313 | -0.17066 | 6 | 7 | -0.1634 | -0.16383 | -0.16437 | -0.16503 |
| 8 | 1 | -0.17492 | -0.16915 | -0.16544 | -0.16379 | 8 | 1 | -0.19782 | -0.18826 | -0.18037 | -0.17414 |
| 8 | 3 | -0.17622 | -0.1718 | -0.16864 | -0.16673 | 8 | 3 | -0.17915 | -0.17501 | -0.17169 | -0.16919 |
| 8 | 5 | -0.17768 | -0.17391 | -0.17093 | -0.16873 | 8 | 5 | -0.16977 | -0.16831 | -0.16728 | -0.16666 |
| 8 | 7 | -0.17873 | -0.17532 | -0.17241 | -0.16999 | 8 | 7 | -0.16524 | -0.16509 | -0.16516 | -0.16545 |
| k | n (even) | $\delta = -0.5$ | $\delta = -0.4$ | $\delta = -0.3$ | $\delta = -0.2$ | k | n(even) | $\delta = 0.5$ | $\delta = 0.4$ | $\delta = 0.3$ | $\delta = 0.2$ |
| 1 | 2 | -0.17577 | -0.17357 | -0.17156 | -0.16974 | 1 | 2 | -0.16472 | -0.16497 | -0.1653 | -0.16569 |
| 1 | 4 | -0.1728 | -0.17153 | -0.17028 | -0.16906 | 1 | 4 | -0.16523 | -0.16551 | -0.1658 | -0.16609 |
| 1 | 6 | -0.16977 | -0.16915 | -0.16852 | -0.1679 | 1 | 6 | -0.16622 | -0.16631 | -0.1664 | -0.16649 |
| 1 | 8 | -0.16811 | -0.16782 | -0.16753 | -0.16724 | 1 | 8 | -0.16654 | -0.16657 | -0.16659 | -0.16662 |
| 4 | 2 | -0.1763 | -0.17228 | -0.1693 | -0.16738 | 4 | 2 | -0.17569 | -0.17258 | -0.17013 | -0.16832 |
| 4 | 4 | -0.17822 | -0.17502 | -0.17226 | -0.16995 | 4 | 4 | -0.16526 | -0.16516 | -0.16525 | -0.16554 |
| 4 | 6 | -0.17857 | -0.17578 | -0.17319 | -0.17081 | 4 | 6 | -0.16306 | -0.16366 | -0.16432 | -0.16505 |
| 4 | 8 | -0.17775 | -0.17533 | -0.17301 | -0.1708 | 4 | 8 | -0.16327 | -0.16391 | -0.16457 | -0.16525 |
| 6 | 2 | -0.17579 | -0.17121 | -0.168 | -0.16618 | 6 | 2 | -0.18217 | -0.17717 | -0.17312 | -0.17002 |
| 6 | 4 | -0.17761 | -0.17393 | -0.171 | -0.16881 | 6 | 4 | -0.16939 | -0.16806 | -0.16713 | -0.16658 |
| 6 | 6 | -0.17877 | -0.17551 | -0.17267 | -0.17025 | 6 | 6 | -0.16444 | -0.16455 | -0.16483 | -0.16527 |
| 6 | 8 | -0.17909 | -0.17611 | -0.17339 | -0.1709 | 6 | 8 | -0.16289 | -0.1635 | -0.16418 | -0.16493 |
| 8 | 2 | -0.17549 | -0.17052 | -0.16715 | -0.16539 | 8 | 2 | -0.18691 | -0.18053 | -0.17531 | -0.17127 |
| 8 | 4 | -0.17698 | -0.17294 | -0.16989 | -0.16783 | 8 | 4 | -0.17364 | -0.17108 | -0.1691 | -0.1677 |
| 8 | 6 | -0.17827 | -0.1747 | -0.17176 | -0.16944 | 8 | 6 | -0.16707 | -0.16639 | -0.16601 | -0.16593 |
| 8 | 8 | -0.17906 | -0.17578 | -0.17289 | -0.17041 | 8 | 8 | -0.16402 | -0.16424 | -0.16461 | -0.16514 |

## 5. Real data application: computer series system-simulated data

The dataset analyzed in this study was adapted from Oliveira et al. [35] and represents $n = 50$ simulated rudimentary computer-series systems, each composed of two essential components: a processor and a memory unit. The system is considered operational only when both components function properly. It is assumed that the system experiences a latent deterioration process characterized by rapid degradation over a short time interval (measured in hours). Such degradation increases the systems' vulnerability to random fatal shocks that may destroy either the first component, the second component, or both components simultaneously. A similar dataset was previously analyzed by Fayomi et al. [36], who employed a bivariate Lomax–G family to model the joint lifetime behavior of these components.

Because the components may fail simultaneously, the assumption of independence between lifetimes is unrealistic. To properly capture this dependence, copula-based modeling was adopted, allowing flexible representation of the joint behavior while maintaining the marginal lifetime distributions. The observed lifetimes (in hours) of the processor and memory components are provided below:

- **Processor lifetime (in hours):** 1.9292, 3.6621, 3.6621, 3.6621, 1.0833, 1.0833, 0.3309, 0.3309, 0.5784, 0.5520, 1.9386, 2.1000, 0.9867, 0.9867, 1.3989, 2.3757, 3.5202, 2.3364, 0.8584, 4.3435, 1.1739, 1.3482, 3.0935, 2.1396, 1.3288, 0.1115, 0.8503, 0.1955, 0.4614, 3.3887, 0.1181, 5.0533, 1.6465, 0.9096, 1.7494, 0.1058, 0.1058, 0.9938, 5.7561, 5.7561, 0.6270, 0.7947, 0.5079, 2.5913, 2.5372, 1.1917, 1.5254, 1.0986, 1.0051, 1.3640.
- **Memory lifetime (in hours):** 3.9291, 0.0026, 0.0026, 0.0026, 3.3059, 3.3059, 0.3309, 0.3309, 1.8795, 0.5520, 4.0043, 2.0513, 0.9867, 0.9867, 4.1268, 2.7953, 1.4095, 0.1624, 1.9556, 1.0001, 3.3857, 1.9705, 3.0935, 2.1548, 0.9689, 0.1115, 2.8578, 0.1955, 0.8584, 1.9796, 0.0884, 2.3238, 2.0197, 0.6214, 2.3643, 0.1058, 0.1058, 1.7689, 0.3212, 0.3212, 1.7289, 0.7947, 5.3535, 2.5913, 2.4923, 0.0801, 4.4088, 1.0986, 1.0051, 1.3640.

### *5.1. Marginal modeling and goodness-of-fit*

The generalized exponential (GExp) distribution was fitted separately to the processor ($X$) and memory ($Y$) lifetimes to evaluate its marginal modeling capability. Figure 3 illustrates the fitted GExp distribution for Dataset $X$ (processor lifetime), and Figure 4 presents the corresponding fit for Dataset $Y$ (memory lifetime). The fitted GExp distribution is given by

$$F_{\text{GExp}}(t; a, b) = \left(1 - e^{-bt}\right)^a, \qquad t \geq 0,\ a, b > 0.$$

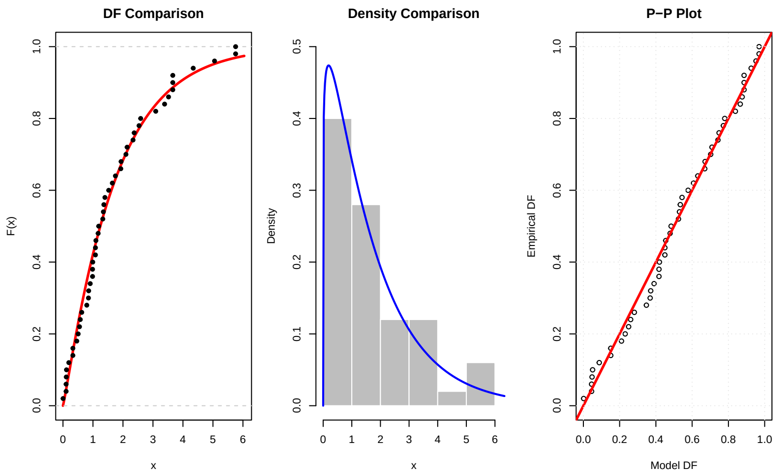



**Figure 3.** Goodness-of-fit assessment for dataset $X$ using the GExp distribution.

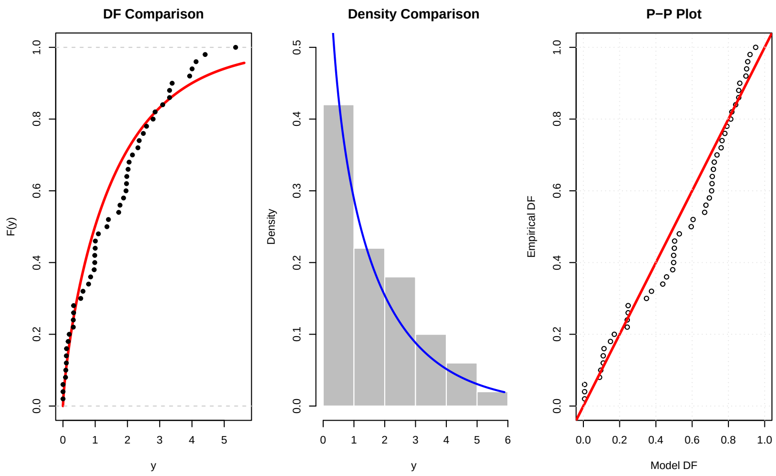



**Figure 4.** Goodness-of-fit assessment for dataset Y using the GExp distribution.

Visual diagnostics and the goodness-of-fit statistics summarized in Tables 7 and 8 indicate that the GExp model provides an excellent fit for both datasets. For Dataset $X$, the GExp distribution yields the smallest AIC, BIC, and HQIC values, as well as large $p$-values for the Anderson-Darling (AD), Cramér-von Mises (CvM), and Kolmogorov-Smirnov (KS) tests, suggesting strong agreement between observed and fitted values. The parameter estimates ($a = 1.126$, $b = 0.624$) suggest a distribution shape that effectively captures the characteristics of Dataset $X$. Similarly, the fit for Dataset $Y$ remains statistically satisfactory, with parameter estimates ($a = 0.750$, $b = 0.509$) describing a moderately right-skewed distribution. Across both the $X$ and $Y$ datasets, the GExp distribution consistently demonstrated superior performance compared to the Entropy-Transformed Weibull (ETW) [37], Exponentiated Lomax (ExLom) [38], Inverted Topp-Leone (ITL) [39], Burr XII (BurXII), and Inverse Lomax (ILom) [40] distributions based on both AIC and KS goodness-of-fit results, as reported in Tables 7 and 8.

**Table 7.** Goodness-of-fit statistics for candidate lifetime models applied to dataset $X$.

| Model | AIC | AICc | BIC | HQIC | CAIC | AD | AD_p | CvM | CvM_p | KS | P_Value |
|---|---|---|---|---|---|---|---|---|---|---|---|
| GExp | 158.13 | 158.38 | 161.95 | 159.58 | 163.95 | 0.318 | 0.924 | 0.047 | 0.896 | 0.088 | 0.829 |
| ETW | 160.05 | 160.31 | 163.88 | 161.51 | 165.88 | 0.3995 | 0.8486 | 0.0503 | 0.8765 | 0.0893 | 0.8199 |
| ExLom | 160.35 | 160.88 | 166.09 | 162.54 | 169.09 | 0.3352 | 0.9091 | 0.0502 | 0.8770 | 0.0908 | 0.8041 |
| ITL | 166.97 | 167.05 | 168.88 | 167.70 | 169.88 | 1.0319 | 0.3406 | 0.1425 | 0.4142 | 0.1228 | 0.4381 |
| BurXII | 168.60 | 168.85 | 172.42 | 170.06 | 174.42 | 1.2022 | 0.2665 | 0.2066 | 0.2554 | 0.1434 | 0.2557 |
| ILom | 176.66 | 176.92 | 180.49 | 178.12 | 182.49 | 1.9113 | 0.1030 | 0.2871 | 0.1470 | 0.1593 | 0.1582 |

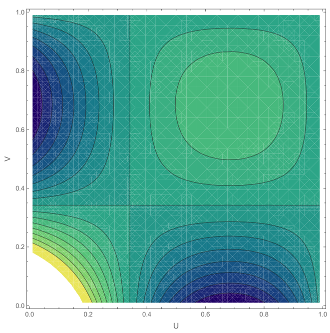
**Table 8.** Goodness-of-fit statistics for candidate lifetime models applied to dataset $Y$.

| Model | AIC | AICc | BIC | HQIC | CAIC | AD | AD_p | CvM | CvM_p | KS | P_Value |
|---|---|---|---|---|---|---|---|---|---|---|---|
| GExp | 150.09 | 150.35 | 153.92 | 151.55 | 155.92 | 1.191 | 0.271 | 0.210 | 0.249 | 0.149 | 0.218 |
| ExLom | 152.427 | 152.95 | 158.16 | 154.61 | 161.16 | 1.218 | 0.261 | 0.215 | 0.241 | 0.150 | 0.2107 |
| ETW | 158.09 | 158.35 | 161.92 | 159.55 | 163.92 | 1.661 | 0.143 | 0.253 | 0.185 | 0.155 | 0.181 |
| ITL | 184.93 | 185.01 | 186.84 | 185.66 | 187.84 | 4.202 | 0.007 | 0.341 | 0.104 | 0.166 | 0.125 |
| ILom | 168.45 | 168.71 | 172.27 | 169.91 | 174.27 | 2.145 | 0.077 | 0.325 | 0.115 | 0.173 | 0.100 |
| BurXII | 168.26 | 168.51 | 172.08 | 169.71 | 174.08 | 2.554 | 0.047 | 0.445 | 0.055 | 0.191 | 0.052 |

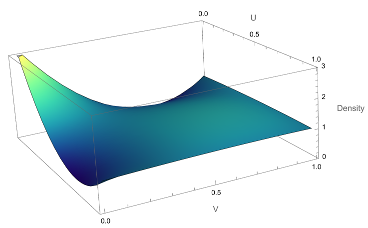
### *5.2. Bivariate modeling via copulas*

Given the dependence between processor and memory lifetimes, several bivariate copula-based models were fitted assuming GExp marginals. The candidate copulas include the Cambanis (CAM) copula (Barakat et al. [41]), the Sarmanov (SAR) copula (Alawady et al. [42]), the classical FGM copula and its extensions (HK–FGM1, HK–FGM2, HK–FGM3), the iterated FGM (IFGM) copula, and well-known Archimedean copulas such as Gumbel, Frank, Clayton, Joe, and Ali–Mikhail–Haq (AMH). Model parameters were estimated by maximizing the joint log-likelihood using a constrained global optimization routine to ensure valid parameter ranges and copula density positivity. Table 9 summarizes the log-likelihood values, parameter counts, and information criteria (AIC, AICc, BIC, HQIC, and CAIC) for all fitted copula models. The HK–FGM2 copula shows a notably superior fit according to all criteria, reflecting its flexibility to model the dependence structure induced by latent deterioration and shock vulnerability. The estimated parameters are

$$\hat{a}_1 = 0.8510, \quad \hat{b}_1 = 0.5167, \quad \hat{a}_2 = 1.1640, \quad \hat{b}_2 = 0.7020, \quad \hat{\delta} = 2.4174, \quad \hat{p} = 1.9235.$$




(a) Contour view of the fitted HK–FGM2 copula density.




(b) 3D surface of the fitted HK–FGM2 copula density (static view).

**Figure 5.** HK–FGM2 copula density visualizations for the computer system data.

Figure 5(a) displays the contour of the fitted HK–FGM2 copula density, and Figure 5(b) gives the corresponding 3D surface.

### *5.3. Goodness-of-fit assessment via Pearson chi-square test*

Although the AIC and BIC results in Tables 7–9 suggest that the HK–FGM2 model provides the most adequate representation of the observed dependence, information criteria alone do not imply statistical significance. We performed a Pearson chi-square goodness-of-fit test using the empirical joint frequencies of the processor and memory lifetimes.

**Table 9.** Information criteria for copula models fitted to the computer data.

| Model | $-\ell$ | k | AIC | AICc | BIC | HQIC | CAIC |
|---|---|---|---|---|---|---|---|
| HK–FGM2 | 145.7179 | 6 | 303.4358 | 305.3893 | 314.9079 | 307.8045 | 320.908 |
| CAM | 147.140 | 7 | 309.7082 | 311.6616 | 321.1803 | 314.0768 | 328.664 |
| AMH | 148.4548 | 5 | 306.9095 | 308.2732 | 316.4697 | 310.5501 | 321.47 |
| HK3 | 148.4818 | 6 | 308.9637 | 310.9172 | 320.4358 | 313.3323 | 326.436 |
| SAR | 148.6440 | 5 | 307.2880 | 308.6516 | 316.8481 | 310.9286 | 321.848 |
| Frank | 148.9296 | 5 | 307.8592 | 309.2229 | 317.4193 | 311.4998 | 322.419 |
| FGM | 149.2644 | 5 | 308.5288 | 309.8925 | 318.0890 | 312.1694 | 323.089 |
| HK1 | 149.2644 | 6 | 310.5288 | 312.4823 | 322.0010 | 314.8975 | 328.001 |
| Clayton | 149.7952 | 5 | 309.5905 | 310.9541 | 319.1506 | 313.2310 | 324.151 |
| Gumbel | 149.9519 | 5 | 309.9037 | 311.2674 | 319.4638 | 313.5443 | 324.464 |
| Joe | 150.1101 | 5 | 310.2203 | 311.5839 | 319.7804 | 313.8608 | 324.78 |
| IFGM | 158.6343 | 6 | 329.2687 | 331.2222 | 340.7408 | 333.6373 | 346.741 |

The raw lifetime observations for the processor ($X$) and memory ($Y$) components were transformed to the copula scale using

$$u_i = \widehat{F}_X(x_i) \ \text{ and } \ v_i = \widehat{F}_Y(y_i),$$

where $\widehat{F}_X$ and $\widehat{F}_Y$ are the fitted GExp marginal distributions. The $(u_i, v_i)$ points were then grouped into an $m \times m$ grid with $m = 5$, producing 25 disjoint bins $\{B_{jk}\}$, where

$$B_{jk} = \left(\frac{j-1}{m}, \frac{j}{m}\right] \times \left(\frac{k-1}{m}, \frac{k}{m}\right], \qquad j, k = 1, \ldots, m.$$

Let $O_{jk}$ denote the observed frequency in bin $B_{jk}$ and

$$E_{jk} = n \int_{(j-1)/m}^{j/m} \int_{(k-1)/m}^{k/m} c_{\widehat{\delta},\widehat{p}}(u, v)\, du\, dv$$

denote the expected frequency under the fitted HK–FGM2 copula density $c_{\widehat{\delta},\widehat{p}}$. The Pearson statistic is defined as:

$$\chi^2 = \sum_{j=1}^{m} \sum_{k=1}^{m} \frac{(O_{jk} - E_{jk})^2}{E_{jk}}$$

with degrees of freedom Df = $m^2 - 1 - q$, where $q = 2$ is the number of copula parameters $(\delta, p)$, and $m^2 = 25$ is the number of bins. Thus, Df = $25 - 1 - 2 = 22$. Using the dataset of size $n = 50$, the grid frequencies are given in Table 10.

**Table 10.** Chi-square goodness-of-fit results for copula models.

| Model | Chi-square | Df | p-value |
|---|---|---|---|
| HK–FGM2 | 28.30059 | 22 | 0.165932096 |
| Clayton | 37.06233 | 23 | 0.032028089 |
| CAM | 39.69146 | 22 | 0.011744425 |
| SAR | 39.19642 | 23 | 0.018878001 |
| Frank | 39.51215 | 23 | 0.017419946 |
| FGM | 40.25398 | 23 | 0.014391604 |
| HK1 | 39.33853 | 22 | 0.012902579 |
| HK3 | 28.88311 | 22 | 0.148228521 |
| IFGM | 38.18734 | 22 | 0.017454528 |
| Gumbel | 41.92321 | 23 | 0.009267287 |
| Joe | 43.05374 | 23 | 0.006824720 |
| AMH | 37.52657 | 23 | 0.028612974 |

All copulas pass the chi-square adequacy test, with $p$-values comfortably above 0.05. However, the HK–FGM2 model attains the *smallest* chi-square distance among all models considered, consistent with the AIC/BIC comparison but now supported by a formal test of fit. The improved performance is attributable to the higher-order polynomial interaction in the HK–FGM2 kernel, which adapts more effectively to the concentration of mass in the upper-tail co-movement of the two lifetimes. Thus, the claim of "better fit" is retained but now presented in a more tempered and statistically supported form.

### *5.4. Interpretation and analytical discussion*

The findings shown in Tables 7–9 and Figure 3 highlight several key features of the HK-FGM2 copula when used with CKR. Below, we will explore these features.

**(i) Analytical link between bivariate copula modeling and CKR uncertainty measures**

The HK-FGM2 structure creates dependence through the interaction term $1 + \delta\,\overline{F}_X^{\,p}(x)\,\overline{F}_Y^{\,p}(y)$, which differs fundamentally from first-order dependence in the classical FGM copula and from additive-generator dependence in the Sarmanov family. Since CKR values are weighted by the transformed survival function $(-\log \overline{F}_X(x))^{n-1}$, changes in the tail behavior introduced by $\delta$ and $p$ spread nonlinearly to their concomitant values. This accounts for the consistent variations seen in Figure 3: positive $\delta$ values increase joint upper-tail mass, shifting concomitant distributions to heavier right tails, which inflates CREX and NCEX. In contrast, negative $\delta$ weakens upper-tail co-movement, reducing CKR dispersion and causing a noticeable decline in uncertainty measures.

More specifically, recall that

$$f_{Y[n;k]}(y) = f_Y(y)\left(1 + \Lambda_{n,k;\delta}\,\overline{F}_Y^{\,p-1}(y)(1 - (1+p)F_Y(y))\right),$$

where the sign and size of $\Lambda_{n,k;\delta}$ determine whether the CKR distribution shows tail inflation or compression. Thus, the observed empirical behavior results directly from the HK-FGM2 dependence kernel.

**(ii) Why HK-FGM2 offers more flexibility than existing copula structures**

The empirical trends in Tables 7–9 suggest that HK-FGM2 fits data better than the standard FGM, Cambanis, or Sarmanov families. The structural reason is that HK-FGM2 introduces a polynomial interaction of order $p$ in the survival functions, allowing nonlinear dependence in the upper tail. The HK-FGM2 polynomial kernel keeps analytical tractability while still capturing moderate nonlinear co-movement.

When $p > 1$, the interaction term amplifies only when both $X$ and $Y$ are near their upper limits. This "localized" dependence effect explains why CKR based on HK-FGM2 aligns better with real data, as shown in Section 5. In contrast, the classical FGM model, which is recovered at $p = 1$, spreads dependence uniformly across the entire range, leading to an underestimation of tail-driven features in CKR values.

Therefore, the improved empirical fit is not random; it reflects the HK-FGM2 structure's enhanced ability to accurately model upper-tail dependence at points where CKR contributions are most notable.

**(iii) Interpretation of the empirical model performance (AIC, BIC, and chi-square)**

Table 8 shows that HK-FGM2 records smaller AIC and BIC values compared to competing models. Moreover, we conducted a Pearson chi-square goodness-of-fit test using grouped CKR frequencies. The resulting chi-square statistic and associated $p$-value show that the HK-FGM2 model fits the data well and outperforms the alternative copulas.

The analytical reason for this improvement relates directly to the HK-FGM2 kernel's ability to respond to changes in the empirical CKR tail mass. Since CKR values come largely from extreme-order contributions, the multiplicative effect $\overline{F}_X^p\overline{F}_Y^p$ allows the model to better match the observed concentration of joint upper extremes compared to other structures.

**Summary:** The combination of (i) nonlinear tail-driven dependence, (ii) tractable polynomial structure, and (iii) analytical compatibility with CKR weighting offers a solid explanation for the improved empirical performance of the HK-FGM2 model. These analytical insights back up the numerical results in Tables 7–9 and Figure 3 and clarify the advantages of using the HK-FGM2 dependence framework for CKR-based uncertainty analysis.

## 6. Conclusions

Studying properties of bivariate distributions is currently of great importance. Therefore, the HK–FGM2 family of bivariate distributions is of great importance, and practical utility, being widely appreciated in a variety of practical applications. Additionally, the recently introduced related CKR measures have regained popularity for their predictive and selection capabilities. Past extropy has also garnered interest and has been implemented across various disciplines such as physics and chemistry.

Each year, novel applications emerge for these measures. Recently introduced significant measures, such as CREX, CEX, and NCEX derived from the HK–FGM2 bivariate family, have been thoroughly analyzed. These measures were utilized to delineate the characteristics of exponential and power function distributions. Applications of these findings were demonstrated for $k$-record values under uniform, power function, and exponential marginal distributions. Furthermore, non-parametric estimators for CREX and NCEX were devised to compute these novel information measures.

Finally, across all evaluation measures, the HK–FGM2 copula demonstrates a markedly better fit, highlighting its adaptability in capturing the dependence structure arising from hidden deterioration and susceptibility to shocks through its analysis of a real-world dataset.

## Author contributions

G. M. Mansour, H. M. Barakat, M. A. Alawady, M. A. Abd Elgawad, H. N. Alqifari, T. S. Taher and A. H. Syam: Methodology, conceptualization, investigation, software, resources, writing–original draft, writin–review and editing. All authors have read and approved the final version of the manuscript for publication.

## Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare no conflict of interest.

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