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Research article

Investigations into (n, q)-symmetry in nonlinear multimappings acting on a metric space

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Abstract: In light of recent advances, this paper introduces a novel class of multimappings within the framework of metric spaces that aims to deepen the understanding of the structure and interplay of symmetric commuting tuples of mappings. A central focus of our study is the analysis of the product of two symmetric commuting tuples of mappings (c.t.m): Specifically, we consider an (m, q)-symmetric c.t.m $\mathbf{N} = (\mathcal{N}_1, \ldots, \mathcal{N}_p)$ and an (n, q)-symmetric c.t.m $\mathbf{W} = (\mathcal{W}_1, \ldots, \mathcal{W}_r)$, and establish the precise conditions under which their product $\mathbf{N} \odot \mathbf{W}$ yields an $(m+n-1, q)$-symmetric c.t.m. In addition, we investigate the behavior of (m, q)-isometric commuting tuples under composition $\mathbf{N} \cdot \mathbf{W}$, deriving new results that generalize existing theorems in the literature, most notably extending of Theorem 2.14 in Bermúdez et al., J. Operat. Theor., 72(2) (2014), 313–329 to a broader setting. The analysis of mappings in metric spaces is inherently complex, as these spaces often lack algebraic structures such as vector addition or scalar multiplication, making the study of nonlinear and multimappings particularly challenging. Our findings contribute to this area by elucidating the subtle relationships among symmetry, isometry, and commutativity in mapping tuples, thereby enabling more structured approaches to higher-dimensional problems within this framework.

Keywords: (m, q)-isometry; (n, q)-symmetry; (m, q)-isometric tuple
Mathematics Subject Classification: 54E40

1. Introduction

Within the algebra $\mathcal{B}(\mathcal{K})$ consisting of bounded linear operators on the Hilbert space $\mathcal{K}$, the structural and functional properties of m-isometric and n-symmetric operators have been studied by many authors. Two important generalizations of isometric and symmetric operators, respectively, are shown below. An operator $\mathcal{N} \in \mathcal{B}(\mathcal{K})$ is called an m-isometric [2–4] if it satisfies the identity

$$\sum_{0\le k\le m}(-1)^{m-k}\binom{m}{k}\mathcal{N}^{*k}\mathcal{N}^{k}=0, \tag{1.1}$$

which generalizes the classical notion of isometry. Similarly, n-symmetric operators extend the concept of symmetric operators through the relation

$$\sum_{0\le k\le n}(-1)^{n-k}\binom{n}{k}\mathcal{N}^{*(n-k)}\mathcal{N}^{k}=0. \tag{1.2}$$

Refer to [13, 21, 22, 24] for more details. A significant contribution to the study of metric spaces was made by T. Bermdez et al., who introduced and analyzed the notion of (m, q)-isometric mappings [9]. Consider a metric space $(\mathcal{E}, d)$, with $m \ge 1$, an integer, and $q > 0$. A map $\mathcal{N} : \mathcal{E} \to \mathcal{E}$ is defined to be an (m, q)-isometry if, for all $\psi, \omega \in \mathcal{E}$, the following condition is satisfied:

$$\sum_{0\le k\le m}(-1)^{m-k}\binom{m}{k}d\left(\mathcal{N}^{k}\psi,\mathcal{N}^{k}\omega\right)^{q}=0. \tag{1.3}$$

When $q > 0$, the case of $(1, q)$-isometry reduces to the standard notion of isometry, which means:

$$d\left(\mathcal{N}\psi,\mathcal{N}\omega\right)=d\left(\psi,\omega\right),\quad \forall\psi,\omega\in\mathcal{E}.$$

After that, the concept of (n, q)-symmetric mappings in the context of metric spaces was introduced and studied by the author Aydah in [5]. A single self-map $\mathcal{N}$ on a metric space $(\mathcal{E}, d)$ is called an (n, q)-symmetric map for some integer $n \ge 1$ and real number $q \in (0, \infty)$, if for all $\psi, \omega \in \mathcal{E}$, the following equality holds:

$$\sum_{0\le k\le n}(-1)^{n-k}\binom{n}{k}d(\mathcal{N}^{n-k}\psi,\mathcal{N}^{k}\omega)^{q}=0. \tag{1.4}$$

For $n = 1$, Eq (1.4) reduces to

$$d(\mathcal{N}\psi,\omega)=d(\psi,\mathcal{N}\omega), \tag{1.5}$$

which is called symmetric mapping. When $n = 2$, Eq (1.4) reduces to

$$d(\mathcal{N}^{2}\psi,\omega)^{q}+d(\psi,\mathcal{N}^{2}\omega)^{q}-2d(\mathcal{N}\psi,\mathcal{N}\omega)^{q}=0, \tag{1.6}$$

which is called $(2, q)$-symmetric mapping.

For $n = 3$, Eq (1.4) reduces to

$$-d(\mathcal{N}^{3}\psi,\mathcal{N}\omega)^{q}+3d(\mathcal{N}^{2}\psi,\mathcal{N}\omega)^{q}-3d(\mathcal{N}\psi,\mathcal{N}^{2}\omega)^{q}+d(\mathcal{N}\psi,\mathcal{N}^{3}\omega)^{q}=0, \tag{1.7}$$

which is called $(3, q)$-symmetric mapping.

Readers interested in further developments on mappings in metric spaces are referred to [18, 19].

Let $\mathbb{N}$ denote the set of positive integers, and define $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For a given $n \in \mathbb{N}$, define $\mathbb{N}_0^n := \mathbb{N}_0 \times \cdots \times \mathbb{N}_0$ (n-times). Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ and $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{N}_0^n$. We define:

$$|\alpha|:=\alpha_1+\cdots+\alpha_n=\sum_{k=1}^{n}\alpha_k,\quad\text{and}\quad \alpha\le\gamma\ \text{ if and only if }\ \alpha_k\le\gamma_k\ \text{ for all } k=1,\ldots,n.$$

For $\alpha \leq \gamma$, we define the multinomial coefficient as:

$$\binom{\gamma}{\alpha} := \prod_{k=1}^{n} \binom{\gamma_k}{\alpha_k}.$$

We denote a commuting tuple of mappings by c.t.m, which refers to a tuple of self-mappings $\mathbf{N} = (\mathcal{N}_1, \cdots, \mathcal{N}_p)$ that commute pairwise, i.e., for all k, j, $\mathcal{N}_k \circ \mathcal{N}_j = \mathcal{N}_j \circ \mathcal{N}_k$.

Let $\mathbf{N} = (\mathcal{N}_1, \cdots, \mathcal{N}_p)$ be a c.t.m on a metric space $\mathcal{E}$. Define

$$\mathbf{N}^{\alpha} := \mathcal{N}_1^{\alpha_1} \circ \cdots \mathcal{N}_p^{\alpha_p}, \quad \text{where } \mathcal{N}_k^{\alpha_k} = \underbrace{\mathcal{N}_k \circ \cdots \circ \mathcal{N}_k}_{\alpha_k - times} = \underbrace{\mathcal{N}_k \cdots \mathcal{N}_k}_{\alpha_k - times}.$$

Let $m \geq 1$. In [15], J. Gleason and S. Richter extended the concept of m-isometric operators by considering its application to commuting p-tuples of bounded linear operators acting on a Hilbert space. Specifically, they demonstrated that these operators satisfy a generalized form of the m-isometry condition, which is useful in understanding operator theory in infinite-dimensional spaces. Their work broadens the classical notion of isometry, providing insights into the structure and behavior of commuting operator families in operator theory. A tuple $\mathbf{N} = (\mathcal{N}_1, \cdots, \mathcal{N}_p) \in \mathcal{B}(\mathcal{K})^p$ is said to be an m-isometric tuple if

$$\sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \left(\sum_{|\alpha|=k} \frac{k!}{\alpha!} \mathbf{N}^{*\alpha} \mathbf{N}^{\alpha} \right) = 0. \tag{1.8}$$

In other words,

$$\sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \left(\sum_{|\alpha|=k} \frac{k!}{\alpha!} \|\mathbf{N}^{\alpha} \psi\|^2 \right) = 0 \text{ for all } \psi \in \mathcal{K}. \tag{1.9}$$

In [14], P. H. W. Hoffmann et al. have introduced the concept of (m, q)-isometric tuples as a generalization of isometries in the context of normed spaces. Their work extends the classical notion of isometry to families of operators or elements in normed spaces, providing a broader framework for analyzing the structure and behavior of these objects. A c.t.m $\mathbf{N} := (\mathcal{N}_1, \cdots, \mathcal{N}_p)$ with $\mathcal{N}_k : \mathcal{X} \rightarrow \mathcal{X}$ (normed space) linear is called an (m, q)-isometric tuple if for given $m \in \mathbb{N}$ and $q \in (0, \infty)$,

$$\sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \Big(\sum_{|\alpha|=k} \frac{k!}{\alpha!} \|\mathbf{N}^{\alpha} \psi\|^q \Big) = 0 \text{ for all } \psi \in \mathcal{X}. \tag{1.10}$$

Building upon the extensions of (1.1) and (1.10) to metric spaces, Sid Ahmed et al. in [17] introduced the notion of (m, q)-isometry for c.t.m in the context of metric spaces. A tuple ($\mathbf{N} = (\mathcal{N}_1, \ldots, \mathcal{N}_p)$ of c.t.m on a metric space $\mathcal{E}$ is referred to as an (m, q)-isometric tuple if, for all $\psi, \omega \in \mathcal{E}$, the following condition is satisfied: $\mathbf{H}_m^{(q)}(\mathbf{N};\ \psi, \omega) = 0$ for all $\psi, \omega \in \mathcal{E}$ where

$$\mathbf{H}_m^{(q)}(\mathbf{N};\ \psi, \omega) := \sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \Big(\sum_{|\alpha|=k} \frac{k!}{\alpha!} d(\mathbf{N}^{\alpha} \psi, \mathbf{N}^{\alpha} \omega)^q \Big). \tag{1.11}$$

When $p = 1$, (1.11) simplifies to (1.3), which corresponds to the notion of an (m, q)-isometry for single-variable mapping, as introduced by T. Bermdez et al. in [9]. Notice that

$$\mathbf{H}_0^{(q)}(\mathbf{N};\ \psi, \omega) = d(\psi, \omega)^q$$

and for a single mapping $\mathcal{N}$,

$$\mathbf{H}_m^{(q)}(\mathcal{N};\ \psi,\omega) = f_{\mathcal{N}}(m,q;\psi,\omega) = \sum_{0\le k\le m}(-1)^{m-k}\binom{m}{k}d(\mathcal{N}^k\psi,\mathcal{N}^k\omega)^q,$$

where the symbol $f_{\mathcal{N}}$ is used in the same manner as in [9]. It was shown in [17] that if $\mathbf{N} = (\mathcal{N}_1,\ldots,\mathcal{N}_p)$ is a tuple of c.t.m on a metric space $(\mathcal{E}, d)$, the subsequent result holds:

$$\mathbf{H}_{m+1}^{(q)}(\mathbf{N};\ \omega,\psi) = \sum_{1\le k\le p}\mathbf{H}_m^{(q)}(\mathbf{N};\ \mathcal{N}_k\psi,\mathcal{N}_k\omega) - \mathbf{H}_m^{(q)}(\mathbf{N};\ \psi,\omega). \tag{1.12}$$

From Eq (1.12), it is clear that every (m, q)-isometric c.t.m is also an (n, q)-isometric c.t.m for all $n \ge m$.

The definition of n-symmetric operators was extended by M. Chō et al. [12] to include commuting p-tuples of bounded linear operators on the Hilbert space $\mathcal{K}$, along with the establishment of key spectral properties. A tuple of operators $\mathbf{N} = (\mathcal{N}_1,\ldots,\mathcal{N}_p) \in \mathcal{B}(\mathcal{K})^p$ is termed an m-symmetric commuting tuple of operators if $\mathbf{N}$ satisfies:

$$\sum_{0\le k\le n}(-1)^{n-k}(\mathcal{N}_1+\cdots+\mathcal{N}_p)^{*(n-k)}(\mathcal{N}_1+\cdots+\mathcal{N}_p)^k = 0. \tag{1.13}$$

An equivalent form of identity (1.13) was presented in [20] as follows

$$\sum_{0\le k\le n}(-1)^{n-j}\binom{n}{j}\Big(\sum_{|\alpha|=k}\frac{k!}{\alpha!}\sum_{|\gamma|=n-k}\frac{(n-k)!}{\gamma!}\mathbf{N}^{*\alpha}\mathbf{N}^{\gamma}\Big) = 0. \tag{1.14}$$

Multivariable operator theory has undergone significant development in various directions, with many authors contributing to this progress, as documented in [6–8, 12, 16]. The concept of m-isometry of Hilbert space operators in higher dimensions has been extended to the notion of (m, q)-isometry for multimappings in the context of metric spaces. In the present work, we extend the notion of n-symmetry of Hilbert space operators in higher dimensions to the concept of (n, q)-symmetry for multimappings in metric spaces. Specifically, we aim to present an extension of (n, q)-symmetric mappings in alignment with the concept of n-symmetric multimappings for multivariable Hilbert space operators.

The paper is structured as follows. In Section 2, we introduce and define a new class of multimappings and carefully analyze some of their key properties. Specifically, we derive the necessary conditions under which, if $\mathbf{N} = (\mathcal{N}_1,\cdots,\mathcal{N}_p)$ is an (m, q)-symmetric c.t.m and $\mathbf{W} = (\mathcal{W}_1,\cdots,\mathcal{W}_r)$ is an (n, q)-symmetric c.t.m, their product $\mathbf{N}\odot\mathbf{W}$ satisfies the condition of being an $(m + n - 1, q)$-symmetric c.t.m.; (Theorem 2.1). Moreover, we establish a new result concerning (m, q)-isometric commuting tuples of mappings, specifically focusing on the behavior of their product under appropriate conditions (Theorem 2.3). Our result generalizes and extends [9, Theorem 2.14], offering a broader framework for analyzing the structure and properties of such tuples in the context of metric spaces.

2. (n, q)-symmetry in nonlinear multimappings

In this section, we provide the definition and key properties of this new class of mappings. Throughout this work, $n, m \in \mathbb{N}_0$, while q denotes a positive real number.

Definition 2.1. A c.t.m $\mathbf{N} = (\mathcal{N}_1, \cdots, \mathcal{N}_p)$ where $\mathcal{N}_k : (\mathcal{E}, d) \longrightarrow (\mathcal{E}, d)$ is called (n, q)-symmetric multimappings if it satisfies

$$\sum_{0\le k\le n} (-1)^{n-k}\binom{n}{k}\Big(\sum_{|\alpha|=k} \frac{k!}{\alpha!} \sum_{|\gamma|=n-k} \frac{(n-k)!}{\gamma!} d(\mathbf{N}^\alpha\psi, \mathbf{N}^\gamma\omega)^q\Big) = 0, \tag{2.1}$$

for all $\psi, \omega \in \mathcal{E}$.

Remark 2.1. It is noteworthy to mention the following observations.

(i) When $p = 1$, Eq (2.1) reduces to (1.4).
(ii) When $p = n = 1$, Eq (2.1) reduces to $d(\mathcal{N}\psi, \omega) = d(\psi, \mathcal{N}\omega)$.
(iii) A c.t.m $\mathbf{N} = (\mathcal{N}_1, \cdots, \mathcal{N}_p)$ is a $(1, q)$-symmetric multimappings if

$$\sum_{1\le k\le p} d(\mathcal{N}_k\psi, \omega)^q - \sum_{1\le k\le p} d(\psi, \mathcal{N}_k\omega)^q = 0, \quad \forall\ \psi, \omega \in \mathcal{E}. \tag{2.2}$$

(iv) A c.t.m $\mathbf{N} = (\mathcal{N}_1, \cdots, \mathcal{N}_p)$ is a $(2, q)$-symmetric multimappings if

$$\sum_{1\le k\le p} d(\mathcal{N}_k^2\psi, \omega)^q + \sum_{1\le k\le p} d(\psi, \mathcal{N}_k^2\omega)^q + \sum_{1\le k,j\le p} d(\mathcal{N}_k\mathcal{N}_j\psi, \omega)^q$$
$$+ \sum_{1\le k,j\le p} d(\psi, \mathcal{N}_k\mathcal{N}_j\omega)^q - 2\sum_{1\le k,j\le p} d(\mathcal{N}_j\psi, \mathcal{N}_k\omega)^q = 0, \quad \forall\ \psi, \omega \in \mathcal{E}.$$

We introduce the following notation: let $\mathbf{N} = (\mathcal{N}_1, \cdots, \mathcal{N}_p)$ be a c.t.m on a metric space $\mathcal{E}$ where each $\mathcal{N}_k : \mathcal{E} \longrightarrow \mathcal{E}$ and let $\psi, \omega \in \mathcal{E}$. We define

$$Q_{(n,q)}(\mathbf{N}; \psi, \omega) := \sum_{0\le k\le n} (-1)^{n-k}\binom{n}{k}\Big(\sum_{|\alpha|=k} \frac{k!}{\alpha!} \sum_{|\gamma|=n-k} \frac{(n-k)!}{\gamma!} d(\mathbf{N}^\alpha\psi, \mathbf{N}^\gamma\omega)^q\Big). \tag{2.3}$$

Note that by definition $Q_{(0,q)}(\mathbf{N}, \psi, \omega) = d(\psi, \omega)^q$.

Consider a metric space $(\mathcal{E}, d)$. We seek two maps $\mathcal{N}_1, \mathcal{N}_2 : \mathcal{E} \to \mathcal{E}$ such that for all $\psi, \omega \in \mathcal{E}$:

1. $d(\mathcal{N}_1\psi, \omega)^q - d(\psi, \mathcal{N}_1\omega)^q \neq 0$,
2. $d(\mathcal{N}_2\psi, \omega)^q - d(\psi, \mathcal{N}_2\omega)^q \neq 0$,
3. $d(\mathcal{N}_1\psi, \omega)^q + d(\mathcal{N}_2\psi, \omega)^q = d(\psi, \mathcal{N}_1\omega)^q + d(\psi, \mathcal{N}_2\omega)^q$.

Example 2.1. Let $(\mathcal{E}, d)$ be a metric space where $\mathcal{E} = \mathbb{R}$ and $d(\psi, \omega) = |\psi - \omega|$, the usual absolute value distance. Define the maps $\mathcal{N}_1$ and $\mathcal{N}_2$ as follows:

$$\mathcal{N}_1(\psi) = \psi + a, \quad \mathcal{N}_2(\psi) = \psi - a,$$

where $a \neq 0$ is a fixed constant. For any $\psi, \omega \in \mathbb{R}$, we compute the distances:

$$d(\mathcal{N}_1\psi, \omega) = |\psi + a - \omega|, \quad d(\psi, \mathcal{N}_1\omega) = |\psi - (\omega + a)| = |\psi - \omega - a|,$$

$$d(\mathcal{N}_2\psi, \omega) = |\psi - a - \omega|, \quad d(\psi, \mathcal{N}_2\omega) = |\psi - (\omega - a)| = |\psi - \omega + a|.$$

Now, calculate the differences:

$$d(\mathcal{N}_1\psi,\omega)^q - d(\psi,\mathcal{N}_1\omega)^q = |\psi-\omega+a|^q - |\psi-\omega-a|^q,$$

$$d(\mathcal{N}_2\psi,\omega)^q - d(\psi,\mathcal{N}_2\omega)^q = |\psi-\omega-a|^q - |\psi-\omega+a|^q.$$

These differences are generally non−zero unless $\psi = \omega$ or $a = 0$. Finally, verify the sum condition:

$$d(\mathcal{N}_1\psi,\omega)^q + d(\mathcal{N}_2\psi,\omega)^q = |\psi-\omega+a|^q + |\psi-\omega-a|^q,$$

then

$$d(\psi,\mathcal{N}_1\omega)^q + d(\psi,\mathcal{N}_2\omega)^q = |\psi-\omega-a|^q + |\psi-\omega+a|^q.$$

Therefore, neither $\mathcal{N}_1$ nor $\mathcal{N}_2$ is a $(1,q)$-symmetric mapping for even integer q. However, the pair $\mathcal{N} = (\mathcal{N}_1,\mathcal{N}_2)$ forms a $(1,q)$-symmetric pair.

Remark 2.2. It is clear from the above example that studying (n,q)-symmetric tuples acting on a metric space is nontrivial. It is possible for a tuple $\mathcal{N} = (\mathcal{N}_1,\ldots,\mathcal{N}_p)$ to be (m,q)-symmetric even though none of its individual components $\mathcal{N}_i$ is (n,q)-symmetric.

Example 2.2. Let $\mathcal{N}_0$ be a self-mapping on a metric space $(\mathcal{E},d)$. If $\mathcal{N}_0$ is an (n,q)-symmetric single mapping, then $\mathbf{N} = (\mathcal{N}_0,\cdots,\mathcal{N}_0)$ is an (n,q)-symmetric c.t.m.

Indeed, since $\mathcal{N}_0$ is an (n,q)-symmetric single mapping, we conclude that $\mathcal{Q}_{(n,q}(\mathcal{N}_0;\psi,\omega) = 0$. Using this identity and a straightforward computation, we find:

$$\begin{aligned}
&\mathcal{Q}_{(n,q)}(\mathbf{N};\psi,\omega)\\
=&\sum_{0\le k\le n}(-1)^{n-k}\binom{n}{k}\Big(\sum_{|\alpha|=k}\frac{k!}{\alpha!}\sum_{|\gamma|=n-k}\frac{(n-k)!}{\gamma!}d(\mathbf{N}^{\alpha}\psi,\mathbf{N}^{\gamma}\omega)^q\Big)\\
=&\sum_{0\le k\le n}(-1)^{n-k}\binom{n}{k}\Big(\sum_{|\alpha|=k}\frac{k!}{\alpha!}\sum_{|\gamma|=n-k}\frac{(n-k)!}{\gamma!}d(\mathcal{N}_0^{|\alpha|}\psi,\mathcal{N}_0^{|\gamma|}\omega)^q\Big)\\
=&\sum_{0\le k\le n}(-1)^{n-k}\binom{n}{k}p^k p^{n-k}d(\mathcal{N}_0^{k}\psi,\mathcal{N}_0^{n-k}\omega)^q\\
=&\ p^n\sum_{0\le k\le n}(-1)^{n-k}\binom{n}{k}d(\mathcal{N}_0^{k}\psi,\mathcal{N}_0^{n-k}\omega)^q\\
=&\ 0.
\end{aligned}$$

Therefore $\mathbf{N}$ is an (n,q)-symmetric c.t.m.

In [5], the author proved that the following identity holds for a single mapping $\mathcal{N}:\mathcal{E}\longrightarrow\mathcal{E}$:

$$\mathcal{Q}_{(n+1,q)}(\mathcal{N};\psi,\omega) = \mathcal{Q}_{(m,q)}(\mathcal{N};\mathcal{N}\psi,\omega) - \mathcal{Q}_{(n,q)}(\mathcal{N};\psi,\mathcal{N}\omega). \tag{2.4}$$

Here, we present the corresponding identity in the case of multiple mappings.

Proposition 2.1. *Let $\mathbf{N} = (\mathcal{N}_1,\cdots,\mathcal{N}_p)$ be a c.t.m on a metric space $\mathcal{E}$. Then, for every positive integer n, every real number $q > 0$, and all $\psi,\omega\in\mathcal{E}$, the recurrence relation below holds true:*

$$\mathcal{Q}_{(n+1,q)}(\mathbf{N};\psi,\omega) = \sum_{1\le j\le p}\mathcal{Q}_{(n,q)}(\mathbf{N};\mathcal{N}_j\psi,\omega) - \sum_{1\le j\le p}\mathcal{Q}_{(n,q)}(\mathbf{N};\psi,\mathcal{N}_j\omega). \tag{2.5}$$

Proof. Taking Eq (2.3) into account, a simple calculation shows that

$$
\begin{aligned}
&\mathcal{Q}_{(n+1,q)}(\mathbf{N};\psi,\omega)\\
=\;& \sum_{0\le k\le n+1}(-1)^{n+1-k}\binom{n+1}{k}\Big(\sum_{|\alpha|=k}\frac{k!}{\alpha!}\sum_{|\gamma|=n+1-k}\frac{(n+1-k)!}{\gamma!}d(\mathbf{N}^{\alpha}\psi,\mathbf{N}^{\gamma}\omega)^{q}\Big)\\
=\;& (-1)^{n+1}\sum_{|\gamma|=n+1}\frac{(n+1)!}{\gamma!}d(\psi,\mathbf{N}^{\gamma}\omega)^{q}\\
&-\sum_{1\le k\le n}(-1)^{n-k}\Big[\binom{n}{k}+\binom{n}{k-1}\Big]\sum_{|\alpha|=k}\frac{k!}{\alpha!}\sum_{|\gamma|=n+1-k}\frac{(n+1-k)!}{\gamma!}d(\mathbf{N}^{\alpha}\psi,\mathbf{N}^{\alpha}\omega)^{q}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{|\alpha|=n+1}\frac{(n+1)!}{\alpha!}d(\mathbf{N}^{\alpha}\psi,\omega)^{q}\\
=\;& \sum_{|\alpha|=n+1}\frac{(n+1)!}{\alpha!}d(\mathbf{N}^{\alpha}\psi,\omega)^{q}-\sum_{1\le k\le n}(-1)^{n-k}\binom{n}{k-1}\Big(\sum_{|\alpha|=k}\frac{k!}{\alpha!}\sum_{|\gamma|=n+1-k}\frac{(n+1-k)!}{\gamma!}d(\mathbf{N}^{\alpha}\psi,\mathbf{N}^{\gamma}\omega)^{q}\Big)\\
&-(-1)^{n}\sum_{|\gamma|=n+1}\frac{(n+1)!}{\gamma!}d(\psi,\mathbf{N}^{\gamma}\omega)^{q}+\sum_{1\le k\le n}(-1)^{n-k}\binom{m}{k}\Big(\sum_{|\alpha|=k}\frac{k!}{\alpha!}\sum_{|\gamma|=n+1-k}\frac{(n+1-k)!}{\gamma!}d(\mathbf{N}^{\alpha}\psi,\mathbf{N}^{\gamma}\omega)^{q}\Big)\\
=\;& \sum_{1\le j\le p}\Big(\sum_{|\alpha|=n}\frac{k!}{\alpha!}d(\mathbf{N}^{\alpha}\mathcal{N}_{j}\psi,\omega)^{q}\Big)\\
&+\sum_{1\le j\le p}\Big(\sum_{0\le k\le n-1}(-1)^{n-k}\binom{n}{k}\Big(\sum_{|\alpha|=k}\sum_{|\gamma|=n-k}\frac{(n-k)!}{\gamma!}d(\mathbf{N}^{\alpha}\mathcal{N}_{j}\psi,\mathbf{N}^{\gamma}\omega)^{q}\Big)\Big)\\
&(-1)^{n}\sum_{1\le j\le p}\Big(\sum_{|\gamma|=n}\frac{k!}{\gamma!}d(\psi,\mathbf{N}^{\gamma}\mathcal{N}_{k}\omega)^{q}\Big)\\
&-\Big(\sum_{1\le j\le p}\Big(\sum_{0\le k\le n-1}(-1)^{n-k}\binom{n}{k}\Big(\sum_{|\alpha|=k}\sum_{|\gamma|=n-k}\frac{(n-k)!}{\gamma!}d(\mathbf{N}^{\alpha}\psi,\mathbf{N}^{\gamma}\mathcal{N}_{k}\omega)^{q}\Big)\Big)\\
=\;& \sum_{1\le j\le p}\Big(\sum_{0\le k\le n}\Big(\sum_{|\alpha|=k}\frac{k!}{\alpha!}\sum_{|\gamma|=n-k}\frac{(n-k)!}{\gamma!}d(\mathbf{N}^{\alpha}\mathcal{N}_{k}\psi,\mathbf{N}^{\gamma}\omega)^{q}\Big)\Big)\\
&-\sum_{1\le j\le p}\Big(\sum_{0\le k\le n}\Big(\sum_{|\alpha|=k}\frac{k!}{\alpha!}\sum_{|\gamma|=n-k}\frac{(n-k)!}{\gamma!}d(\mathbf{N}^{\alpha}\psi,\mathbf{N}\mathcal{N}_{k}^{\gamma}\omega)^{q}\Big)\\
=\;& \sum_{1\le j\le p}\mathcal{Q}_{(n,q)}(\mathbf{N};\mathcal{N}_{j}\psi,\omega)-\sum_{1\le j\le p}\mathcal{Q}_{(n,q)}(\mathbf{N};\psi,\mathcal{N}_{j}\omega).
\end{aligned}
$$

□

Corollary 2.1. *Let* $\mathbf{N}=(\mathcal{N}_1,\cdots,\mathcal{N}_p)$ *be a c.t.m on a metric space* $(\mathcal{E},d)$*. If* $\mathbf{N}$ *is an* (n,q)*-symmetric, then it remains* (m,q)*-symmetric for every* $m\ge n$*.*

Proof. Applying the recurrence relation in Eq (2.5) we get $\mathcal{Q}_{(n+1,q)}(\mathbf{N};\psi,\omega)=0\ \forall\ \psi,\omega\in\mathcal{E}$. For the same reason, $\mathcal{Q}_{(n+2,q)}(\mathbf{N};\psi,\omega)=0$, and proceeding step by step, we obtain the desired result.

This makes the reasoning in the proof more transparent. □

Let $\mathbf{N} = (\mathcal{N}_1, \cdots, \mathcal{N}_p)$ and $\mathbf{W} = (\mathcal{W}_1, \cdots, \mathcal{W}_r)$ be a self-mappings acting on a metric space $\mathcal{E}$. We then define the product $\mathbf{N} \odot \mathbf{W}$ as the sequence of compositions

$$\mathbf{N} \odot \mathbf{W} = (\mathcal{N}_1\mathcal{W}_1, \mathcal{N}_2\mathcal{W}_1, \cdots, \mathcal{N}_p\mathcal{W}_1, \cdots, \cdots, \mathcal{N}_1\mathcal{W}_r, \mathcal{N}_2\mathcal{W}_r, \cdots, \mathcal{N}_p\mathcal{W}_r).$$

Although the following result could be formulated as a lemma, we present it as a proposition due to the technical complexity of its proof.

Proposition 2.2. *Let $\mathbf{N} = (\mathcal{N}_1, \cdots, \mathcal{N}_p)$ and $\mathbf{W} = (\mathcal{W}_1, \cdots, \mathcal{W}_r)$ be two tuples of mappings acting on a metric space $(\mathcal{E}, d)$ such that $\mathcal{N}_k\mathcal{W}_j = \mathcal{W}_j\mathcal{N}_k$ and*

$$d(\mathcal{N}_k\psi, \mathcal{W}_j\omega) = d(\psi, \omega) \text{ for } k = 1, \cdots, p, \quad j = 1, \cdots, r \text{ and } \psi, \omega \in \mathcal{E}. \tag{2.6}$$

Then

$$\mathcal{Q}_{(l,q)}(\mathbf{N} \odot \mathbf{W}; \psi, \omega) = \sum_{|\alpha|+|\gamma|=l} \binom{l}{|\alpha|} \mathcal{Q}_{(|\alpha|,q)}(\mathbf{W}; \mathbf{N}^\alpha\psi, \omega)\mathcal{Q}_{(|\gamma|,q)}(\mathbf{N}; \psi, \mathbf{W}^\gamma\omega). \tag{2.7}$$

Proof. For $l = 1$ we have

$$\begin{aligned}
& \mathcal{Q}_{(1,q)}(\mathbf{N} \odot \mathbf{W}; \psi, \omega) = \\
& \sum_{|\alpha|+|\gamma|=1} \binom{1}{|\alpha|} \mathcal{Q}_{(|\alpha|,q)}(\mathbf{W}; \mathbf{N}^\alpha\psi, \omega)\mathcal{Q}_{(|\gamma|,q)}(\mathbf{N}; \psi, \mathbf{W}^\gamma\omega) \\
= & \sum_{1\le k\le p} \mathcal{Q}_{(1,q)}(\mathbf{W}; \mathbf{N}_k\psi, \omega) + \sum_{1\le j\le r} \mathcal{Q}_{1,q)}(\mathbf{N}; \psi, \mathbf{W}_j\omega) \\
= & \sum_{1\le k\le p} \Big(\sum_{1\le j\le r} d(\mathcal{W}_j\mathcal{N}_k\psi, \omega)^q - \sum_{1\le j\le r} d(\mathcal{N}_k\psi, \mathcal{W}_j\omega)^q \Big) \\
& + \sum_{1\le j\le r} \Big(\sum_{1\le k\le p} d(\mathcal{N}_k\psi, \mathcal{W}_j\omega)^q - \sum_{1\le k\le p} d(\psi, \mathcal{N}_k\mathcal{W}_j\omega)^q \Big) \\
= & \sum_{1\le k\le p} \sum_{1\le j\le r} \Big(d(\mathcal{W}_j\mathcal{N}_k\psi, \omega)^q - d(\psi, \mathcal{N}_k\mathcal{W}_j\omega)^q \Big).
\end{aligned}$$

Based on Eq (2.5) we derive

$$\begin{aligned}
& \mathcal{Q}_{(1,q)}(\mathbf{N} \odot \mathbf{W}; \psi, \omega) \\
& \sum_{1\le k\le p} \sum_{1\le j\le r} \mathcal{Q}_{(0,q)}(\mathbf{N} \odot \mathbf{W}; \mathcal{N}_k\mathcal{W}_j\psi, \omega) - \mathcal{Q}_{(0,q)}(\mathbf{N} \odot \mathbf{W}; \psi, \mathcal{N}_k\mathcal{W}_j\omega) \\
= & \sum_{1\le k\le p} \sum_{1\le j\le r} \Big(d(\mathcal{W}_j\mathcal{N}_k\psi, \omega)^q - d(\psi, \mathcal{N}_k\mathcal{W}_j\omega)^q \Big).
\end{aligned}$$

This confirms the validity of Eq (2.7) for $l = 1$. Suppose Eq (2.7) holds for some $l \in \mathbb{N}$; we aim to show that it remains valid for $l + 1$. Using the relations given (2.5) and (2.6), it follows that

$$\begin{aligned}
& \mathcal{Q}_{(l+1,q)}(\mathbf{N} \odot \mathbf{W}; \psi, \omega) = \\
& \sum_{\substack{1 \le k \le p \\ 1 \le j \le r}} \mathcal{Q}_{(l,q)}(\mathbf{N} \odot \mathbf{W}; \mathcal{N}_k\mathcal{W}_j\psi, \omega)
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{\substack{1 \le k \le p \\ 1 \le j \le r}} Q_{(l,q)}(\mathbf{N} \odot \mathbf{W}; \psi, \mathcal{N}_k \mathcal{W}_j \omega) \\
= &\sum_{|\alpha|+|\gamma|=l} \binom{l}{|\alpha|} \sum_{\substack{1 \le k \le p \\ 1 \le j \le r}} \Big(Q_{(|\alpha|,q)}(\mathbf{W}, \mathbf{N}^{\alpha} \mathcal{N}_k \mathcal{W}_j \psi, \omega) Q_{(|\gamma|,q)}(\mathbf{N}, \psi, \mathbf{W}^{\gamma} \omega) \Big) \\
&- \sum_{|\alpha|+|\gamma|=l} \binom{l}{|\alpha|} \sum_{\substack{1 \le k \le p \\ 1 \le j \le r}} \Big(Q_{(|\alpha|,q)}(\mathbf{W}; \mathbf{N}^{\alpha} \psi, \omega) Q_{(|\gamma|,q)}(\mathbf{N}; \psi, \mathbf{W}^{\gamma} \mathcal{N}_k \mathcal{W}_j \omega) \Big) \\
= &\sum_{|\alpha|+|\gamma|=l} \binom{l}{|\alpha|} \sum_{1 \le k \le p} \Big(Q_{|\alpha|+1,q}(\mathbf{W}, \mathbf{N}^{\alpha} \mathcal{N}_k \psi, \omega) + \sum_{1 \le j \le r} Q_{(|\alpha|,q)}(\mathbf{W}; \mathbf{N}^{\alpha} \mathcal{N}_k \psi, \mathcal{W}_j \omega) \Big) \times \\
&Q_{(|\gamma|,q)}(\mathbf{N}; \psi, \mathbf{W}^{\gamma} \omega) \\
- &\sum_{|\alpha|+|\gamma|=l} \binom{l}{|\alpha|} \sum_{1 \le j \le r} \Big(- Q_{(|\gamma|+1,q)}(\mathbf{N}, \psi, \mathbf{W}^{\gamma} \mathcal{W}_j \omega) + \sum_{1 \le k \le p} Q_{(|\gamma|,q)}(\mathbf{N}; \mathcal{N}_k \psi, \mathbf{W}^{\gamma} \mathcal{W}_j \omega) \Big) \times \\
&Q_{(|\alpha|,q)}(\mathbf{W}, \mathbf{N}^{\alpha} \psi, \omega) \\
= &\sum_{|\alpha|+|\gamma|=l} \binom{l}{|\alpha|} \sum_{1 \le k \le p} \Big(Q_{|\alpha|+1,q}(\mathbf{W}; \mathbf{N}^{\alpha} \mathcal{N}_k \psi, \omega) \Big) Q_{(|\gamma|,q)}(\mathbf{N}, \psi, \mathbf{W}^{\gamma} \omega) \\
&+ \sum_{|\alpha|+|\gamma|=l} \binom{l}{|\alpha|} \sum_{1 \le j \le r} \Big(Q_{(|\gamma|+1,q)}(\mathbf{N}; \psi, \mathbf{W}^{\gamma} \mathcal{W}_j \omega) \Big) Q_{(|\alpha|,q)}(\mathbf{W}, \mathbf{N}^{\alpha} \psi, \omega).
\end{aligned}
$$

Depending on the identity involved, $\binom{l+1}{|\alpha|} = \binom{l}{|\alpha|} + \binom{l}{|\alpha|-1}$ for $|\alpha| \ge 1$, we can write

$$
\begin{aligned}
&Q_{(l+1,q)}(\mathbf{N} \odot \mathbf{W}; \psi, \omega) \\
= &\sum_{|\alpha|+|\gamma|=l+1} \binom{l+1}{|\alpha|} Q_{(|\alpha|,q)}(\mathbf{W}, \mathbf{N}^{\alpha} \psi, \omega) Q_{(|\gamma|,q)}(\mathbf{N}; \psi, \mathbf{W}^{\gamma} \omega) \\
= &\sum_{|\alpha|+|\gamma|=l+1} \left(\binom{l}{|\alpha|} + \binom{l}{|\alpha|-1} \right) Q_{(|\alpha|,q)}(\mathbf{W}; \mathbf{N}^{\alpha} \psi, \omega) Q_{(|\gamma|,q)}(\mathbf{N}, \psi, \mathbf{W}^{\gamma} \omega) \\
= &\sum_{|\alpha|+|\gamma|=l+1} \binom{l}{|\alpha|} Q_{(|\alpha|,q)}(\mathbf{W}; \mathbf{N}^{\alpha} \psi, \omega) Q_{(|\gamma|,q)}(\mathbf{N}, \psi, \mathbf{W}^{\gamma} \omega) \\
&+ \sum_{|\alpha|+|\gamma|=l+1} \binom{l}{|\alpha|-1} Q_{(|\alpha|,q)}(\mathbf{W}; \mathbf{N}^{\alpha} \psi, \omega) Q_{(|\gamma|,q)}(\mathbf{N}; \psi, \mathbf{W}^{\gamma} \omega) \\
= &\sum_{|\alpha|+|\gamma|-1=1} \binom{l}{|\alpha|} Q_{(|\alpha|,q)}(\mathbf{W}; \mathbf{N}^{\alpha} \psi, \omega) Q_{(|\gamma|,q)}(\mathbf{N}, \psi, \mathbf{W}^{\gamma} \omega) \\
&+ \sum_{|\alpha|-1+|\gamma|=l} \binom{l}{|\alpha|-1} Q_{(|\alpha|,q)}(\mathbf{W}; \mathbf{N}^{\alpha} \psi, \omega) Q_{(|\gamma|,q)}(\mathbf{N}, \psi, \mathbf{W}^{\gamma} \omega)
\end{aligned}
$$

$$= \sum_{|\alpha|+|\gamma'|=1} \binom{l}{|\alpha|} \sum_{1\le j\le r} Q_{(|\alpha|,q)}(\mathbf{W};\mathbf{N}^{\alpha}\psi,\omega) Q_{(|\gamma'|+1,q)}(\mathbf{N};\psi,\mathbf{W}^{\gamma'}\mathcal{W}_j\omega)$$
$$+ \sum_{|\alpha'|+|\gamma|=l} \binom{l}{|\alpha'|} \sum_{1\le k\le p} Q_{(|\alpha'|+1,q)}(\mathbf{W};\mathbf{N}^{\alpha'}\mathcal{N}_k\psi,\omega) Q_{(|\gamma|,q)}(\mathbf{N};\psi,\mathbf{W}^{\gamma}\omega).$$

From the preceding arguments, we conclude that (2.7) holds. □

Example 2.3. Consider the upper half-plane

$$\mathbb{H} = \{\psi = x + iy \in \mathbb{C} : y > 0\}.$$

The hyperbolic distance between two points $z, w \in \mathbb{H}$ is defined by

$$d_{\mathbb{H}}(\psi,\omega) = \operatorname{arccosh}\left(1 + \frac{|\psi-\omega|^2}{2\, Im(\psi)\, Im(\omega)}\right).$$

Consider the Möbius maps as follows

$$\mathcal{N}(\psi) = \frac{a\psi + b}{c\psi - a}, \quad a,b,c \in \mathbb{R}, \quad a^2 + bc = -1$$

and

$$\mathcal{W}(\psi) = \frac{a\psi - b}{c\psi + a}, \quad a,b,c \in \mathbb{R}, \quad a^2 + bc = -1.$$

We have

$$Im(\mathcal{N}(\psi)) = \frac{Im(\psi)}{|c\psi - a|^2}; \qquad Im(\mathcal{W}(\omega)) = \frac{Im(\omega)}{|c\omega + a|^2}$$

and

$$|\mathcal{N}(\psi)) - \mathcal{W}(\omega)|^2 = \frac{|\psi-\omega|^2}{|c\psi - a|^2|c\omega + a|^2}.$$

Taking the general expansion from above and collecting factors, we get

$$\begin{aligned} d_{\mathbb{H}}(\mathcal{N}(\psi),\mathcal{W}(\omega)) &= arccosh\left(1 + \frac{|\mathcal{N}(\psi) - \mathcal{W}(\omega)|^2}{2Im(\mathcal{N}(\psi))Im(\mathcal{W}(\omega))}\right) \\ &= \operatorname{arccosh}\left(1 + \frac{|\psi-\omega|^2}{|c\psi + a|^2|c\omega - a|^2} \times \frac{|c\psi - a|^2|c\omega + a|^2}{2Im(\psi)Im(\omega)}\right) \\ &= \operatorname{arccosh}\left(1 + \frac{|\psi-\omega|^2}{2\, Im(\psi)\, Im(\omega)}\right) \\ &= d_{\mathbb{H}}(\psi,\omega). \end{aligned}$$

Therefore, for the tuples

$$N_0 = (\mathcal{N},\ldots,\mathcal{N}), \qquad W_0 = (\mathcal{W},\ldots,\mathcal{W}),$$

we have, for every pair of indices k, j,

$$d_{\mathbb{H}}(N_k(\psi),\ W_j(\omega)) = d_{\mathbb{H}}(\psi,\ \omega),$$

and hence condition (2.20) is satisfied.

In Hilbert space operator theory, the product $\mathbf{N}\odot\mathbf{W}$ has been studied in various ways, notably in [16, Theorem 4] and [1, Theorem 3.11] for (m, C)-isometric tuples and related classes. An analogous result for (n, A)-symmetric tuples can be found in [20, Theorem 3.2]. In the following theorem, we extend the investigation to the metric space setting by presenting a property of the product $\mathbf{N}\odot\mathbf{W}$, where $\mathbf{N}$ and $\mathbf{W}$ are mappings defined on a metric space. This provides a generalized perspective that complements the known results in Hilbert spaces.

Theorem 2.1. *Let* $\mathbf{N} = (\mathcal{N}_1, \cdots, \mathcal{N}_p)$ *and* $\mathbf{W} = (\mathcal{W}_1, \cdots, \mathcal{W}_r)$ *be two tuples of mappings acting on a metric space* $(\mathcal{E}, d)$ *such that* $\mathbf{N}$ *is an* (m, q)*-symmetric c.t.m and* $\mathbf{W}$ *is an* (n, q)*-symmetric c.t.m. If* $\mathcal{N}_k\mathcal{W}_j = \mathcal{W}_j\mathcal{N}_k$ *and* $d(\mathcal{N}_k\psi, \mathcal{W}_j\omega) = d(\psi, \omega)$ *for* $k = 1, \cdots, p,\quad j = 1, \cdots, r$ *and* $\forall\ \psi, \omega \in \mathcal{E}$*, then* $\mathbf{N}\odot\mathbf{W}$ *is an* $(m + n - 1, q)$*-symmetric* (pr)*-c.t.m.*

Proof. Our goal is to show that $Q_{(m+n-1,q)}(\mathbf{N}\odot\mathbf{W}; \psi, \omega)$ for all $\psi, \omega \in \mathcal{E}$. For this purpose, we consider the identity in (2.7):

$$Q_{(m+n-1,q)}(\mathbf{N}\odot\mathbf{W}; \psi, \omega) = \sum_{|\alpha|+|\gamma|=m+n-1} \binom{m+n-1}{|\alpha|} Q_{(|\alpha|,q)}(\mathbf{W}; \mathbf{N}^{\alpha}\psi, \omega) Q_{(|\gamma|,q)}(\mathbf{N}; \psi, \mathbf{W}^{\gamma}\omega).$$

Given that $\mathbf{N}$ is an (m, q)-symmetric c.t.m and $\mathbf{W}$ is an (n, q)-symmetric c.t.m, we deduce that for $|\gamma| \geq n$ then $Q_{(|\gamma|,q)}(\mathbf{W}; \psi, \omega) = 0$, and for $|\gamma| \leq n - 1$ we have $|\alpha| = m + n - 1 - |\gamma| \geq m$ and hence $Q_{(|\alpha|,q)}(\mathbf{N}; \psi, \omega) = 0$. Accordingly, we find that $Q_{(m+n-1,q)}(\mathbf{N}\odot\mathbf{W}; \psi, \omega) = 0$. The required consequence is now established. □

Let $\mathbf{N} = (\mathcal{N}_1, \cdots, \mathcal{N}_p)$ and $\mathbf{W} = (\mathcal{W}_1, \cdots, \mathcal{W}_p)$ be **c.t.m** on a metric space $(\mathcal{E}, d)$. We set $\mathbf{N}\cdot\mathbf{W} = (\mathcal{N}_1\mathcal{W}_1, \cdots, \mathcal{N}_p\mathcal{W}_p)$.

Although the following result could be formulated as a lemma, we present it as a theorem due to the technical complexity of its proof.

Proposition 2.3. *Let* $\mathbf{N} = (\mathcal{N}_1, \cdots, \mathcal{N}_p)$*, and* $\mathbf{W} = (\mathcal{W}_1, \cdots, \mathcal{W}_p)$ *be* **c.t.m** *on a metric space* $(\mathcal{E}, d)$ *satisfying the conditions.* $\mathcal{N}_j\mathcal{W}_k = \mathcal{W}_k\mathcal{N}_j$, $\mathcal{N}_k^2\mathcal{W}_k = \mathcal{N}_k^2, \mathcal{W}_k^2\mathcal{N}_k = \mathcal{W}_k^2$*, and* $\mathcal{W}_j\mathcal{W}_k\mathcal{N}_k = \mathcal{W}_j$ *if* $k \neq j$ *for all* $j, k = 1, \cdots, p$*. Then the equality below is satisfied for* $\psi \neq \omega$*:*

$$\mathcal{H}_l^{(q)}(\mathbf{N}\cdot\mathbf{W}; \psi, \omega) = d(\psi, \omega)^{-pq} \sum_{0\leq k\leq l} \sum_{|\gamma|=k} \binom{l}{k}\binom{k}{\gamma} \mathcal{H}_{l-k}^{(q)}(\mathbf{N}, \psi, \omega) \prod_{1\leq i\leq p} \mathcal{H}_{\gamma_i}^{(q)}(\mathcal{W}_i, \mathbf{N}^{\gamma}\psi, \mathbf{N}^{\gamma}\omega). \tag{2.8}$$

Proof. We will prove (2.8) by mathematical induction. For $l = 1$, we have

$$\begin{aligned}
&\mathcal{H}_1^{(q)}(\mathbf{N}\cdot\mathbf{W}; \psi, \omega)\\
= \ & d(\psi, \omega)^{-pq} \sum_{0\leq k\leq 1} \sum_{|\gamma|=k} \binom{1}{k}\binom{k}{\gamma} \mathcal{H}_{1-k}^{(q)}(\mathbf{N}; \psi, \omega) \prod_{1\leq i\leq p} \mathcal{H}_{\gamma_i}^{(q)}(\mathcal{W}_i, \mathbf{N}^{\gamma}\psi, \mathbf{N}^{\gamma}\omega)\\
= \ & d(\psi, \omega)^{-pq}\Big(\mathcal{H}_1^{(q)}(\mathbf{N}, \psi, \omega)d(\psi, \omega)^{pq}\\
&+ \sum_{1\leq j\leq p} d(\psi, \omega)^q \mathcal{H}_1^q(\mathcal{W}_j; \mathcal{N}_j\psi, \mathcal{N}_j\omega) d(\psi, \omega)^{(p-1)q}\Big)\\
= \ & \mathcal{H}_1^{(q)}(\mathbf{N}; \psi, \omega) + \sum_{1\leq j\leq p} \mathcal{H}_1^{(q)}(\mathcal{W}_j; \mathcal{N}_j\psi, \mathcal{N}_j\omega))
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{1\le j\le p} d(\mathcal{N}_j\psi, \mathcal{N}_j\omega)^q - d(\psi,\omega)^q + \sum_{1\le j\le p}\Big(d(\mathcal{W}_j\mathcal{N}_j\psi, \mathcal{W}_j\mathcal{N}_j\omega)^q - d(\mathcal{N}_j\psi, \mathcal{N}_j\omega)^q\Big)\\
&= \sum_{1\le j\le p} d(\mathcal{W}_j\mathcal{N}_j\psi, \mathcal{W}_j\mathcal{N}_j\omega)^q - d(\psi,\omega)^q.
\end{aligned}
$$

Based on (1.12), it can be concluded that

$$
\mathcal{H}_1^{(q)}(\mathbf{N}\cdot\mathbf{W};\psi,\omega) = .\sum_{1\le j\le p} d(\mathcal{W}_j\mathcal{N}_j\psi, \mathcal{W}_j\mathcal{N}_j\omega)^q - d(\psi,\omega)^q.
$$

So, (2.8) is true for $l = 1$. We assume it is true for $l = n$ and prove it for $l = n + 1$. By Proposition 2.1, $\mathcal{N}_j\mathcal{W}_k = \mathcal{W}_k\mathcal{N}$ for all $j, k = 1, \cdots, p$. Using the induction hypothesis, it follows that

$$
\begin{aligned}
&\mathcal{H}_{n+1}^{(q)}(\mathbf{N}\cdot\mathbf{W};\psi,\omega)\\
=&\sum_{1\le j\le p}\mathcal{H}_n^{(q)}(\mathbf{N}\cdot\mathbf{W};\mathcal{W}_j\mathcal{N}_j\psi,\mathcal{W}_j\mathcal{N}_j\omega) - \mathcal{H}_n^{(q)}(\mathbf{N}\cdot\mathbf{W};\psi,\omega)\\
=&\sum_{1\le j\le p}\Big(\sum_{0\le k\le n}\sum_{|\gamma|=k}\binom{n}{k}\binom{k}{\gamma}\mathcal{H}_{n-k}^{(q)}(\mathbf{N};\mathcal{W}_j\mathcal{N}_j\psi,\mathcal{W}_j\mathcal{N}_j\omega)\prod_{1\le i\le p}\mathcal{H}_{\gamma_i}^{(q)}(\mathcal{W}_i;\mathbf{N}^\gamma\mathcal{W}_j\mathcal{N}_j\psi,\mathbf{N}^\gamma\mathcal{W}_j\mathcal{N}_j\omega)\Big)\\
&-\sum_{0\le k\le n}\sum_{|\gamma|=k}\binom{n}{k}\binom{k}{\gamma}\mathcal{H}_{n-k}^{(q)}(\mathbf{N};\psi,\omega)\prod_{1\le i\le d}\mathcal{H}_{\gamma_i}^{(q)}(\mathcal{W}_i;\mathbf{N}^\gamma\psi,\mathbf{N}^\gamma\omega)\\
=&\Big(\sum_{0\le k\le n}\sum_{|\gamma|=k}\binom{n}{k}\binom{k}{\gamma}\sum_{1\le j\le p}\mathcal{H}_{n-k}^{(q)}(\mathbf{N};\mathcal{W}_j\mathcal{N}_j\psi,\mathcal{W}_j\mathcal{N}_j\omega)\prod_{1\le i\le p}\mathcal{H}_{\gamma_i}^{(q)}(\mathcal{W}_i;\mathbf{N}^\gamma\mathcal{W}_j\mathcal{N}_j\psi,\mathbf{N}^\gamma\mathcal{W}_j\mathcal{N}_j\omega)\Big)\\
&-\sum_{0\le k\le n}\sum_{|\gamma|=k}\binom{n}{k}\binom{k}{\gamma}\mathcal{H}_{n-k}^{(q)}(\mathbf{N};\psi,\omega)\prod_{1\le i\le d}\mathcal{H}_{\gamma_i}^{(q)}(\mathcal{W}_i;\mathbf{N}^\gamma\psi,\mathbf{N}^\gamma\omega)\\
=&\sum_{0\le k\le n}\sum_{|\gamma|=k}\binom{n}{k}\binom{k}{\gamma}\Big(\sum_{1\le j\le d}\mathcal{H}_{n-k}^{(q)}(\mathbf{N};\mathcal{N}_j\psi,\mathcal{N}_j\omega)\\
&\mathcal{H}_{\gamma_1}^{(q)}(\mathcal{W}_1;\mathbf{N}^\gamma\psi,\mathbf{N}^\gamma\omega)\cdots\underbrace{\mathcal{H}_{\gamma_j}^{(q)}(\mathcal{W}_j;\mathbf{N}^\gamma\mathcal{W}_j\psi,\mathbf{N}^\gamma\mathcal{W}_j\omega)}\cdots\\
&\mathcal{H}_{\gamma_p}^{(q)}(\mathcal{W}_p;\mathbf{N}^\gamma\psi,\mathbf{N}^\gamma\omega)\Big)\\
&-\sum_{0\le k\le n}\sum_{|\gamma|=k}\binom{n}{k}\binom{k}{\gamma}\mathcal{H}_{n-k}^{(q)}(\mathbf{N};\psi,\omega)\prod_{1\le i\le d}\mathcal{H}_{\gamma_i}^{(q)}(\mathcal{W}_i;\mathbf{N}^\gamma\psi,\mathbf{N}^\gamma\omega)\\
=&\sum_{0\le k\le n}\sum_{|\gamma|=k}\binom{n}{k}\binom{k}{\gamma}\Big(\sum_{1\le j\le d}\mathcal{H}_{n-k}^{(q)}(\mathbf{N};\mathcal{N}_j\psi,\mathcal{N}_j\omega)\\
&\mathcal{H}_{\gamma_1}^{(q)}(\mathcal{W};\mathbf{N}^\gamma\psi,\mathbf{N}^\gamma\omega)\cdots\Big(\underbrace{\mathcal{H}_{\gamma_j+1}^{(q)}(\mathcal{W}_j;\mathbf{N}^\gamma\psi,\mathbf{N}^\gamma\omega)+\mathcal{H}_{\gamma_j}^{(q)}(\mathcal{W}_j;\mathbf{N}^\gamma\psi,\mathbf{N}^\gamma\omega)}\Big)\cdots\\
&\mathcal{H}_{\gamma_p}^{(q)}(\mathcal{W}_p;\mathbf{N}^\gamma\psi,\mathbf{N}^\gamma\omega)\Big)\\
&-\sum_{0\le k\le n}\sum_{|\gamma|=k}\binom{n}{k}\binom{k}{\gamma}\mathcal{H}_{n-k}^{(q)}(\mathbf{N};\psi,\omega)\prod_{1\le i\le d}\mathcal{H}_{\gamma_i}^{(q)}(\mathcal{W}_i;\mathbf{N}^\gamma\psi,\mathbf{N}^\gamma\omega)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{0\le k\le n}\sum_{|\gamma|=k}\binom{n}{k}\binom{k}{\gamma}\sum_{1\le j\le p}\mathcal{H}^{(q)}_{n-k}(\mathbf{N};\mathcal{N}_j\psi,\mathcal{N}_j\omega)\prod_{1\le i\le p}\mathcal{H}_{\gamma_i}(\mathcal{W}_i;\mathbf{N}^\gamma\psi,\mathbf{N}^\gamma\omega)\\
&\quad+\sum_{0\le k\le n}\sum_{|\gamma|=k}\binom{n}{k}\binom{k}{\gamma}\sum_{1\le j\le p}\mathcal{H}^{(q)}_{n-k}(\mathbf{N};\mathcal{N}_j\psi,\mathcal{N}_j\omega)\mathcal{H}^{(q)}_{\gamma_1}(\mathcal{W}_1;\mathbf{N}^\gamma\psi,\mathbf{N}^\gamma\omega)\\
&\quad\cdots\mathcal{H}^{(q)}_{\gamma_j+1}(\mathcal{W}_j;\mathbf{N}^\gamma\psi,\mathbf{N}^\gamma\omega)\cdots\mathcal{H}^{(q)}_{\gamma_p}(\mathcal{W}^{(q)}_p;\mathbf{N}^\gamma\psi,\mathbf{N}^\gamma\omega)\\
&\quad-\sum_{0\le k\le n}\sum_{|\gamma|=k}\binom{n}{k}\binom{k}{\gamma}\mathcal{H}^{(q)}_{n-k}(\mathbf{N};\psi,\omega)\prod_{1\le i\le d}\mathcal{H}^{(q)}_{\gamma_i}(\mathcal{W}_i;\mathbf{N}^\gamma\psi,\mathbf{N}^\gamma\omega)\\
&=\\
&= \sum_{0\le k\le n}\sum_{|\gamma|=k}\binom{n}{k}\binom{k}{\gamma}\Big(\mathcal{H}^{(q)}_{n-k+1}(\mathbf{N};\psi,\omega)+\mathcal{H}^{(q)}_{n-k}(\mathbf{N};\psi,\omega)\Big)\prod_{1\le i\le p}\mathcal{H}_{\gamma_i}(\mathcal{W}_i;\mathbf{N}^\gamma\psi,\mathbf{N}^\gamma\omega)\\
&\quad+\sum_{0\le k\le n}\sum_{|\gamma|=k}\binom{n}{k}\binom{k}{\gamma}\sum_{1\le j\le p}\mathcal{H}^{(q)}_{n-k}(\mathbf{N};\psi,\omega)\mathcal{H}^{(q)}_{\gamma_1}(\mathcal{W}_1;\mathbf{N}^\gamma\mathcal{N}_j\psi,\mathbf{N}^\gamma\mathcal{N}_j\omega)\\
&\quad\cdots\mathcal{H}^{(q)}_{\gamma_j+1}(\mathcal{W}_j;\mathbf{N}^\gamma\mathcal{N}_j\psi,\mathbf{N}^\gamma\mathcal{N}_j\omega)\cdots\mathcal{H}^{(q)}_{\gamma_p}(\mathcal{W}^{(q)}_p;\mathbf{N}^\gamma\psi,\mathbf{N}^\gamma\omega)\\
&\quad-\sum_{0\le k\le n}\sum_{|\gamma|=k}\binom{n}{k}\binom{k}{\gamma}\mathcal{H}^{(q)}_{n-k}(\mathbf{N};\psi,\omega)\prod_{1\le i\le d}\mathcal{H}^{(q)}_{\gamma_i}(\mathcal{W}_i;\mathbf{N}^\gamma\psi,\mathbf{N}^\gamma\omega)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{0\le k\le n}\sum_{|\gamma|=k}\binom{n}{k}\binom{k}{\gamma}\Big(\mathcal{H}^{(q)}_{n-k+1}(\mathbf{N};\psi,\omega)\Big)\prod_{1\le i\le p}\mathcal{H}_{\gamma_i}(\mathcal{W}_i;\mathbf{N}^\gamma\psi,\mathbf{N}^\gamma\omega)\\
&\quad+\sum_{0\le k\le n}\sum_{|\gamma|=k}\binom{n}{k}\binom{k}{\gamma}\Big(\mathcal{H}^{(q)}_{n-k}(\mathbf{N};\mathcal{W}_j\psi,\mathcal{W}_j\omega)\Big)\prod_{1\le i\le p}\mathcal{H}_{\gamma_i}(\mathcal{W}_i;\mathbf{N}^\gamma\psi,\mathbf{N}^\gamma\mathcal{N}_j\omega)\\
&\quad+\sum_{0\le k\le n}\sum_{|\gamma|=k}\binom{n}{k}\binom{k}{\gamma}\sum_{1\le j\le p}\mathcal{H}^{(q)}_{n-k}(\mathbf{N};\mathcal{W}_j\mathcal{N}_j\psi,\mathcal{W}_j\mathcal{N}_j\omega)\mathcal{H}^{(q)}_{\gamma_1}(\mathcal{W}_1;\mathbf{N}^\gamma\mathcal{N}_j\psi,\mathbf{N}^\gamma\mathcal{N}_j\omega)\\
&\quad\cdots\mathcal{H}^{(q)}_{\gamma_j+1}(\mathcal{W}_j;\mathbf{N}^\gamma\mathcal{N}_j\psi,\mathbf{N}^\gamma\mathcal{N}_j\omega)\cdots\mathcal{H}^{(q)}_{\gamma_p}(\mathcal{W}^{(q)}_p;\mathbf{N}^\gamma\mathcal{N}_j\psi,\mathbf{N}^\gamma\mathcal{N}_j\omega)\\
&\quad-\sum_{0\le k\le n}\sum_{|\gamma|=k}\binom{n}{k}\binom{k}{\gamma}\mathcal{H}^{(q)}_{n-k}(\mathbf{N};\psi,\omega)\prod_{1\le i\le d}\mathcal{H}^{(q)}_{\gamma_i}(\mathcal{W}_i;\mathbf{N}^\gamma\psi,\mathbf{N}^\gamma\omega)\\
&= \sum_{0\le k\le n}\sum_{|\gamma|=k}\binom{n}{k}\binom{k}{\gamma}\mathcal{H}^{(q)}_{n+1-k}(\mathbf{N};\psi,\omega)\prod_{1\le i\le d}\mathcal{H}_{\gamma_i}(\mathcal{W}_i;\mathbf{N}^\gamma\psi,\mathbf{N}^\gamma\omega)\\
&\quad+\sum_{0\le k\le n}\sum_{|\gamma|=k+1}\binom{n}{k}\binom{k+1}{\gamma}\mathcal{H}^{(q)}_{n-k}(\mathbf{N};\psi,\omega)\prod_{1\le i\le d}\mathcal{H}_{\gamma_i}(\mathcal{W}_i;\mathbf{N}^\gamma\psi,\mathbf{N}^\gamma\omega).
\end{aligned}
$$

Alternatively,

$$
\begin{aligned}
&\sum_{0\le k\le n+1}\sum_{|\gamma|=k}\binom{n+1}{k}\binom{k}{\gamma}\mathcal{H}^{(q)}_{n+1-k}(\mathbf{N};\psi,\omega)\prod_{1\le i\le p}\mathcal{H}^{(q)}_{\gamma_i}(\mathcal{W}_i,\mathbf{N}^\gamma\psi,\mathbf{N}^\gamma\omega)\\
&= \mathcal{H}^{(q)}_{n+1}(\mathbf{N};\psi,\omega)d(\psi,\omega)^q+\sum_{1\le k\le n}\sum_{|\gamma|=k}\binom{n+1}{k}\binom{k}{\gamma}\mathcal{H}^{(q)}_{n+1-k}(\mathbf{N};\psi,\omega)\prod_{1\le i\le p}\mathcal{H}^{(q)}_{\gamma_i}(\mathcal{W}_i,\mathbf{N}^\gamma\psi,\mathbf{N}^\gamma\omega))
\end{aligned}
$$

$$+\sum_{|\gamma|=n+1}\binom{n+1}{\gamma}\mathcal{H}_0^{(q)}(\mathbf{N};\psi,\omega)\prod_{1\le i\le p}\mathcal{H}_{\gamma_i}^{(q)}(W_i;\mathbf{N}^\gamma\psi,\mathbf{N}^\gamma\omega)$$

$$=\mathcal{H}_{n+1}^{(q)}(\mathbf{N};\psi,\omega)d(\psi,\omega)^q+\sum_{1\le k\le n}\sum_{|\gamma|=k}\left(\binom{n}{k}+\binom{n}{k-1}\right)\binom{k}{\gamma}\mathcal{H}_{n+1-k}^{(q)}(\mathbf{N};\psi,\omega)\prod_{1\le i\le p}\mathcal{H}_{\gamma_i}^{(q)}(\mathcal{W}_i,\mathbf{N}^\gamma\psi,\mathbf{N}^\gamma\omega))$$

$$+\sum_{|\gamma|=n+1}\binom{n+1}{\gamma}\mathcal{H}_0^{(q)}(\mathbf{N};\psi,\omega)\prod_{1\le i\le p}\mathcal{H}_{\gamma_i}^{(q)}(W_i;\mathbf{N}^\gamma\psi,\mathbf{N}^\gamma\omega)$$

$$=\sum_{0\le k\le n}\sum_{|\gamma|=k}\binom{n}{k}\binom{k}{\gamma}\mathcal{H}_{n+1-k}^{(q)}(\mathbf{N};\psi,\omega)\prod_{1\le i\le p}\mathcal{H}_{\gamma_i}^{(q)}(\mathcal{W}_i,\mathbf{N}^\gamma\psi,\mathbf{N}^\gamma\omega)$$

$$+\sum_{0\le k\le n}\sum_{|\gamma|=k+1}\binom{n}{k}\binom{k+1}{\gamma}\mathcal{H}_{n+1-k}^{(q)}(\mathbf{N};\psi,\omega)\prod_{1\le i\le p}\mathcal{H}_{\gamma_i}^{(q)}(\mathcal{W}_i,\mathbf{N}^\gamma\psi,\mathbf{N}^\gamma\omega).$$

Accordingly, identity (2.8) is established. □

Theorem 2.2. *[9, Theorem 2.14] Let $\mathcal{N}$ and $\mathcal{W}$ be two commuting mappings on a metric space $(\mathcal{E},D)$, i.e., $\mathcal{N}\mathcal{W}=\mathcal{W}\mathcal{N}$. If $\mathcal{N}$ is an (m,q)-isometry and $\mathcal{W}$ is an (n,q)-isometry, then the composition $\mathcal{N}\mathcal{W}$ is an $(m+n-1,q)$-isometry.*

For the special case where $\mathcal{E}$ is a Hilbert space, a version in [10, Theorem 2.2].

In the following theorem, we discuss the product of (m,q)-isometric tuples. To the best of our knowledge, this result has not been previously established in the literature, and thus represents a novel contribution of our work.

Theorem 2.3. *Let $\mathbf{N}=(\mathcal{N}_1,\cdots,\mathcal{N}_p)$ and $\mathbf{W}=(\mathcal{W}_1,\cdots,\mathcal{W}_p)$ be two* **c.t.m** *on a metric space $(\mathcal{E},d)$ satisfying the conditions. $\mathcal{N}_j\mathcal{W}_k=\mathcal{W}_k\mathcal{N}_j$, $\mathcal{N}_k^2\mathcal{W}_k=\mathcal{N}_k^2$, $\mathcal{W}_k^2\mathcal{N}_k=\mathcal{W}_k^2$, and $\mathcal{W}_j\mathcal{W}_k\mathcal{N}_k=\mathcal{W}_j$ if $k\neq j$ for all $j,k=1,\cdots,p$. If $\mathbf{N}$ is an (m,q)-isometry and each $\mathcal{W}_k$ is an (n_k,q)-isometry for $k=1,\cdots,p$, then $\mathbf{N}\cdot\mathbf{W}$ is an $(m+\sum_{1\le k\le p}n_k-p,q)$-isometric c.t.m.*

Proof. Set $n=\sum_{1\le k\le p}n_j$ and $l=m+n-p$. By Proposition 2.3, we have for $\psi\neq\omega$

$$\mathcal{H}_l^{(q)}(\mathbf{N}\cdot\mathbf{W};\psi,\omega)=d(\psi,\omega)^{-pq}\sum_{0\le k\le l}\sum_{|\gamma|=k}\binom{l}{k}\binom{k}{\gamma}\mathcal{H}_{l-k}^{(q)}(\mathbf{N};\psi,\omega)\prod_{1\le i\le p}\mathcal{H}_{\gamma_i}^{(q)}(\mathcal{W}_i;\mathbf{N}^\gamma\psi,\mathbf{N}^\gamma\omega).$$

We are required to verify that $\mathcal{H}_l^{(q)}(\mathbf{N}\cdot\mathbf{W};\psi,\omega)=0$ which is obvious for $\omega=\psi$. For $\omega\neq\psi$, the methodology employed for this task is summarized below:

When $0\le k\le n-p$, we have $l-k\ge m$, and thus $\mathcal{H}_{l-k}^{(q)}(\mathbf{N};\psi,\omega)=0$ (by (1.12). When $k>n-p$, then one of the γ_j satisfies $\gamma_j\ge n_j$ since $|\gamma|=k$. In this case, $\mathcal{H}_{\gamma_i}(\mathcal{W}_i;\psi,\omega)=0$. Accordingly $\mathcal{H}_l^{(q)}(\mathbf{N}\cdot\mathbf{W};\psi,\omega)=0$. We conclude that $\mathbf{N}\cdot\mathbf{W}$ is an $(m+n-1,q)$-isometric **c.t.m.**. □

Remark 2.3. In Hilbert space operator theory, the product $\mathbf{N}\cdot\mathbf{W}$ has been studied in various ways, notably in [16, Theorem 5] and [1, Theorem 3.13] for (m,C)-isometric tuples and related classes.

3. Conclusions

In light of recent advances, this paper introduces a new class of multimappings within the framework of metric spaces, designed to enhance the understanding of the structure and interplay of symmetric commuting tuples of mappings. A central contribution of our work is the analysis of the product of two symmetric c.t.m: Specifically, an (m, q)-symmetric c.t.m $N = (N_1, \ldots, N_p)$ and an (n, q)-symmetric c.t.m $W = (W_1, \ldots, W_r)$. We establish precise conditions under which their product $N\,W$ forms an $(m + n - 1, q)$-symmetric c.t.m, thereby clarifying the behavior of symmetry under such compositions. Furthermore, we investigate the structural properties of (m, q)-isometric commuting tuples under composition $N \circ W$, obtaining new results that significantly generalize existing theorems in the literature-most notably extending [9, Theorem 2.14] to a broader and more flexible setting. These developments demonstrate the robustness of the proposed framework and its compatibility with various nonlinear classes of mappings.

Overall, our findings pave the way for more structured approaches to higher-dimensional and nonlinear problems in metric spaces, offering new tools and perspectives for future research in this rapidly developing field.

Author contributions

All authors contributed equally to the conception, execution, and writing of this manuscript. All authors read and approved the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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