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Research article

Reliable analytical approach to solving delay differential equations with combined proportional and constant delays

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Abstract: The investigation of delay differential equations (DDEs) spans a diverse array of practical applications. In the realm of applied sciences, DDEs typically involve either constant/pure delays or proportional delays, each posing significant analytical challenges. The task of deriving exact solutions becomes increasingly intricate when both types of delays are integrated within a single model. This paper derives an explicit unified analytical solution for a DDE with combined delays using the method of steps (MoS). A unified solution formula is presented, applicable across any sub-interval of the problem's domain. Additionally, the theoretical properties of the solution and its derivative—such as continuity and the presence of discontinuities at specific points—are meticulously examined. The proposed methodology also encompasses existing findings in the literature, with applications extending to fields including astronomy and railway electrification.

Keywords: ordinary differential equation; delay differential equation; initial value problem
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1. Introduction

Delay differential equations (DDEs) provide an effective tool for understanding population dynamic [1], physical phenomena [2,3], life science [4] and biomedical and engineering models [5,6] in which the memory effects play a crucial role. Actually, two different types of delays are known as proportional delays and constant/pure delays. For declaration, the DDE $y'(t) = \alpha y(t) + \beta y(t-\tau)$ involves $\tau > 0$ as a constant/pure delay while the pantograph delay differential equation (PDDE) $y'(t) = \alpha y(t) + \beta y(ct)$ contains c $(0 < c < 1)$ as a proportional delay [7]. The applications of the PDDE can be found in railway electrification [8–10] and electric railways [11]. The PDDE also describes the dynamics of a current collection system for an electric locomotive [12]. The PDDE has been investigated in [13–15] utilizing several techniques. Additionally, the Ambartsumian delay differential equation (ADDE) $y'(t) = -y(t) + \frac{1}{q}y\left(\frac{t}{q}\right)$, $(q > 1)$ includes $\frac{1}{q}$ as a proportional delay. In astronomy, the ADDE describes the surface brightness in the Milky Way [16]. The ADDE has been solved in classical and fractional forms in [17, 18], respectively.

In this paper, we address the incorporation of both categories of delays into a unified model. Thus, the aim of this study is to solve the following DDE:

$$\begin{aligned} &y'(t) = \alpha y(t) + \beta y(ct - \tau), \qquad 0 < c \leq 1, \ \tau \geq 0, \\ &y(t) = \phi(t) \quad \forall\, t \in [-\tau, 0), \qquad y(0) = \lambda, \end{aligned} \tag{1.1}$$

where α, β, and λ are real constants. The parameter c represents a proportional delay while τ represents a pure/constant delay, and $\phi(t)$ is the history function. If $c = 1$, the problem (1.1) becomes $y'(t) = \alpha y(t) + \beta y(t - \tau)$, which can be solved implicitly in terms of the Lambert function [19] using the method of characteristics (MoC) or numerically using nonstandard finite difference schemes [20, 21]. As $\tau \to 0$ and $\phi(t) = 0$, the DDE (1.1) transforms to the PDDE. Additionally, it reduces to the ADDE if $\alpha = -1$ and $\beta = c = 1/q$. Although real-world applications of the combined model (1.1) are currently difficult to identify, the problem itself presents a valuable mathematical challenge. Nonetheless, with ongoing advances in engineering, physical sciences, and biological sciences, such applications may well emerge in the future. The current study broadens the scope of established results for PDDE and ADDE models in the literature. Consequently, the known analytical solutions for these models arise as particular cases of our unified solution, which underscore the main strength of this work. In the absence of either the proportional delay c or the constant (pure) delay τ, each problem traditionally requires a distinct method to derive its solution. The present analysis overcomes these difficulties by providing a unified solution.

Some effective methods have been recently developed to solve various models such as the decomposition method [22, 23], the modified decomposition method [24, 25], the homotopy perturbation method [26], and the Laplace transform [27, 28]. However, the method of steps (MoS) is often the most suitable approach for solving delay models involving history functions. In particular, the ability to obtain exact solutions for certain models governed by such functions highlights the superior efficiency of the MoS compared to alternative methods [29]. Within this framework, the MoS is extended to effectively analyze the DDE (1.1), with the solution derived in an explicit form. Furthermore, the results are consistent with existing findings in the literature when the parameter c is equal to unity.

2. Analysis

The MoS operates by partitioning the given domain into a finite number of sub-intervals. Assuming $\phi(t) = 0$, the following theorem presents the solution to the current DDE over an arbitrary sub-interval. This analysis serves as an initial step toward addressing the more general case of model (1.1) with an arbitrary history function $\phi(t)$.

Theorem 1. *For $t \in I_n = [t_{n-1}, t_n)$, the solution $y_n(t)$ is*

$$y_n(t) = y_{n-1}(t_{n-1})e^{\alpha(t-t_{n-1})} + \beta e^{\alpha t}\int_{t_{n-1}}^{t} e^{-\alpha t}y_{n-1}(ct - \tau)dt, \ t \in I_n = [t_{n-1}, t_n), \ t_0 = 0, \tag{2.1}$$

where

$$t_n = \frac{\tau}{c}\sum_{k=0}^{n-1}\frac{1}{c^k} = \frac{\tau}{c}\left(\frac{c^{-n}-1}{c^{-1}-1}\right), \quad n \geq 1. \tag{2.2}$$

Proof. Let us denote I_0 to the interval $[-\tau, 0)$ and assume that the solution in this interval is denoted by $y_0(t)$ $\forall$ $t \in I_0 = [-\tau, t_0) = [-\tau, 0)$. Consider the first interval as $I_1 = [t_0, t_1) = \left[0, \frac{\tau}{c}\right)$. For $t \in I_1$, we have $-\tau \leq ct - \tau < 0$, i.e., $ct - \tau \in I_0$. Hence $y(ct-\tau) = y_0(ct-\tau)$ $\forall$ $t \in I_1 = \left[0, \frac{\tau}{c}\right)$. Accordingly, $y_1(t)$ is governed by

$$\begin{cases} y_1'(t) = \alpha y_1(t) + \beta y_0(ct-\tau), \\ y_1(0) = \lambda, \quad t \in I_1 = [t_0, t_1) = \left[0, \dfrac{\tau}{c}\right). \end{cases} \tag{2.3}$$

Let us define the second interval as $I_2 = [t_1, t_2) = \left[\frac{\tau}{c}, \frac{\tau}{c}\left(1+\frac{1}{c}\right)\right)$. If $t \in I_2$, then $0 \leq ct - \tau < \frac{\tau}{c}$, i.e., $ct - \tau \in I_1$. Thus $y(ct-\tau) = y_1(ct-\tau)$. Consequently, $y_2(t)$ is subjected to

$$\begin{cases} y_2'(t) = \alpha y_2(t) + \beta y_1(ct-\tau), \\ y_2(t_1) = y_1(t_1), \quad t \in I_2 = [t_1, t_2) = \left[\dfrac{\tau}{c}, \dfrac{\tau}{c}\left(1+\dfrac{1}{c}\right)\right). \end{cases} \tag{2.4}$$

The condition $y_2(t_1) = y_1(t_1)$ admits the continuity at $t = t_1$. Define

$$I_3 = [t_2, t_3) = \left[\frac{\tau}{c}\left(1+\frac{1}{c}\right), \frac{\tau}{c}\left(1+\frac{1}{c}+\frac{1}{c^2}\right)\right).$$

Therefore, $\frac{\tau}{c} \leq ct - \tau < \frac{\tau}{c}\left(1+\frac{1}{c}\right)$, i.e., $ct - \tau \in I_2$, and $y(ct-\tau) = y_2(ct-\tau)$ $\forall$ $t \in I_3$. Thus, $y_3(t)$ is governed by

$$\begin{cases} y_3'(t) = \alpha y_3(t) + \beta y_2(ct-\tau), \\ y_3(t_2) = y_2(t_2), \ t \in I_3 = [t_2, t_3) = \left[\dfrac{\tau}{c}\left(1+\dfrac{1}{c}\right), \dfrac{\tau}{c}\left(1+\dfrac{1}{c}+\dfrac{1}{c^2}\right)\right). \end{cases} \tag{2.5}$$

By induction, we have

$$\begin{cases} y_n'(t) = \alpha y_n(t) + \beta y_{n-1}(ct-\tau), \\ y_n(t_{n-1}) = y_{n-1}(t_{n-1}), \quad t \in I_n = [t_{n-1}, t_n) = \left[\dfrac{\tau}{c}\displaystyle\sum_{k=0}^{n-2}\dfrac{1}{c^k}, \dfrac{\tau}{c}\displaystyle\sum_{k=0}^{n-1}\dfrac{1}{c^k}\right), \quad n \geq 1. \end{cases} \tag{2.6}$$

Solving Eq (2.6) completes the proof. □

Lemma 1. *At* $\alpha = 0$*, the DDE* (1.1) *becomes*

$$y'(t) = \beta y(ct-\tau), \tag{2.7}$$

subject to the same initial data while $y_n(t)$ *is*

$$y_n(t) = \sum_{k=0}^{n-1}\frac{1}{k!}c^{\frac{1}{2}k(k-1)}y_{n-k-1}(t_{n-k-1})\left(\beta(t-t_{n-1})\right)^k, \ t \in I_n = [t_{n-1}, t_n), \ n \geq 1, \tag{2.8}$$

where

$$y_n(t_n) = \sum_{i=0}^{n-1}\frac{(\beta\tau)^i}{i!}c^{\frac{1}{2}i(i-1)-ni}y_{n-i-1}(t_{n-i-1}), \quad y_0(t_0) = \lambda. \tag{2.9}$$

Proof. At $\alpha = 0$, $y_n(t)$ takes the form:

$$y_n(t) = y_{n-1}(t_{n-1}) + \beta \int_{t_{n-1}}^{t} y_{n-1}(ct - \tau)dt, \quad t \in I_n = [t_{n-1}, t_n),\ t_0 = 0. \tag{2.10}$$

Applying this equation at $n = 1$ gives

$$y_1(t) = y_0(t_0) + \beta \int_{t_0}^{t} y_0(ct - \tau)dt, \quad t \in I_1 = [t_0, t_1),\ t_0 = 0. \tag{2.11}$$

If $t \in I_1$, then $ct - \tau \in I_0 = [-\tau, 0)$ and $y_0(ct - \tau) = 0\ \forall\ t \in I_1 = [t_0, t_1)$. Accordingly, $y_1(t)$ reads

$$y_1(t) = y_0(t_0), \quad t \in I_1 = [t_0, t_1). \tag{2.12}$$

At $n = 2$, we obtain

$$y_2(t) = y_1(t_1) + \beta \int_{t_1}^{t} y_1(ct - \tau)dt, \quad t \in I_2 = [t_1, t_2), \tag{2.13}$$

i.e.,

$$y_2(t) = y_1(t_1) + \beta y_0(t_0)(t - t_1), \quad t \in I_2 = [t_1, t_2). \tag{2.14}$$

Similarly, one can find that

$$y_3(t) = y_2(t_2) + \beta y_1(t_1)(t - t_2) + \frac{\beta^2 c}{2!} y_0(t_0)(t - t_2)^2,\ t \in I_3 = [t_2, t_3), \tag{2.15}$$

and

$$\begin{aligned} y_4(t) = {} & y_3(t_3) + \beta y_2(t_2)(t - t_3) + \frac{\beta^2 c}{2!} y_1(t_1)(t - t_3)^2 \\ & + \frac{\beta^3 c^3}{3!} y_0(t_0)(t - t_3)^3,\ t \in I_4 = [t_3, t_4), \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} y_5(t) = {} & y_4(t_4) + \beta y_3(t_3)(t - t_4) + \frac{\beta^2 c}{2!} y_2(t_2)(t - t_4)^2 \\ & + \frac{\beta^3 c^3}{3!} y_1(t_1)(t - t_4)^3 + \frac{\beta^4 c^6}{4!} y_0(t_0)(t - t_4)^4,\ t \in I_5 = [t_4, t_5). \end{aligned} \tag{2.17}$$

Repeating this process n times, we obtain

$$y_n(t) = \sum_{k=0}^{n-1} \frac{1}{k!} c^{\frac{1}{2}k(k-1)} y_{n-k-1}(t_{n-k-1}) \left(\beta(t - t_{n-1})\right)^k,\ t \in [t_{n-1}, t_n),\ n \geq 1. \tag{2.18}$$

Substituting $t = t_n$, yields

$$y_n(t_n) = \sum_{k=0}^{n-1} \frac{1}{k!} c^{\frac{1}{2}k(k-1)} y_{n-k-1}(t_{n-k-1}) \left(\beta(t_n - t_{n-1})\right)^k,\ n \geq 1. \tag{2.19}$$

Using the relation $t_n - t_{n-1} = \tau / c^n$ and replacing k by i to prevent confusion, then

$$y_n(t_n) = \sum_{i=0}^{n-1} \frac{(\beta\tau)^i}{i!} c^{\frac{1}{2}i(i-1)-ni} y_{n-i-1}(t_{n-i-1}), \quad y_0(t_0) = \lambda, \tag{2.20}$$

which completes the proof. □

3. Solution

Theorem 1 is to be invested to derive the solutions $y_1(t)$, $y_2(t)$, and $y_3(t)$ in the intervals I_1, I_2, and I_3, respectively. At $n = 1$, Eq (2.1) gives

$$y_1(t) = y_0(t_0)e^{\alpha(t-t_0)} + \beta e^{\alpha t} \int_{t_0}^{t} e^{-\alpha t} y_0(ct-\tau)dt, \quad t \in [t_0, t_1). \tag{3.1}$$

Since $y_0(ct-\tau) = 0$ for $t \in [t_0, t_1) = \left[0, \frac{\tau}{c}\right)$, then

$$y_1(t) = \lambda e^{\alpha t}, \quad t \in \left[0, \frac{\tau}{c}\right). \tag{3.2}$$

At $n = 2$, Eq (2.1) becomes

$$y_2(t) = y_1(t_1)e^{\alpha(t-t_1)} + \beta e^{\alpha t} \int_{t_1}^{t} e^{-\alpha t} y_1(ct-\tau)dt, \quad t \in [t_1, t_2), \tag{3.3}$$

i.e.,

$$y_2(t) = \lambda e^{\alpha t} + \beta e^{\alpha t} \int_{\frac{\tau}{c}}^{t} e^{-\alpha t} y_1(ct-\tau)dt, \quad t \in \left[\frac{\tau}{c}, \frac{\tau}{c}\left(1 + \frac{1}{c}\right)\right). \tag{3.4}$$

Hence

$$y_2(t) = \lambda e^{\alpha t} + \lambda\beta e^{\alpha(t-\tau)} \int_{\frac{\tau}{c}}^{t} e^{\alpha(c-1)t}, \quad t \in \left[\frac{\tau}{c}, \frac{\tau}{c}\left(1 + \frac{1}{c}\right)\right), \tag{3.5}$$

which is

$$y_2(t) = \lambda e^{\alpha t} + \frac{\lambda\beta}{\alpha(c-1)} \left[e^{\alpha(ct-\tau)} - e^{\frac{\alpha}{c}(ct-\tau)}\right], \quad t \in \left[\frac{\tau}{c}, \frac{\tau}{c}\left(1 + \frac{1}{c}\right)\right), \tag{3.6}$$

or

$$y_2(t) = \lambda e^{\alpha t} + \frac{\lambda\beta}{\alpha(c-1)} \left[e^{\alpha c(t-t_1)} - e^{\alpha(t-t_1)}\right], \quad t \in [t_1, t_2)\,. \tag{3.7}$$

Using (3.7) implies

$$y_2(ct-\tau) = \lambda e^{\alpha(c-1)t - \alpha c t_1} + \frac{\lambda\beta}{\alpha(c-1)} \left[e^{\alpha(c^2-1)t - \alpha c^2 t_2} - e^{\alpha(c-1)t - \alpha c t_2}\right]. \tag{3.8}$$

Multiplying both sides of Eq (3.8) by $e^{-\alpha t}$ and then integrating from t_2 to t, we obtain

$$\int_{t_2}^{t} e^{-\alpha t} y_2(ct-\tau)dt = H_1(t) + \frac{\lambda\beta}{\alpha^2(c-1)(c^2-1)} H_2(t), \tag{3.9}$$

where

$$\begin{aligned} H_1(t) &= \frac{\lambda}{\alpha(c-1)} e^{-\alpha c t_1} \left[e^{\alpha(c-1)t} - e^{\alpha(c-1)t_2}\right], \\ H_2(t) &= e^{\alpha(c^2-1)t - \alpha c^2 t_2} - (c+1)e^{\alpha(c-1)t - \alpha c t_2} + c e^{-\alpha t_2}. \end{aligned} \tag{3.10}$$

At $n = 3$, we have

$$y_3(t) = y_2(t_2)e^{\alpha(t-t_2)} + \beta e^{\alpha t}\int_{t_2}^{t} e^{-\alpha t}y_2(ct-\tau)dt, \quad t \in [t_2, t_3). \tag{3.11}$$

Inserting (3.9) into (3.11) gives

$$y_3(t) = y_2(t_2)e^{\alpha(t-t_2)} + \beta e^{\alpha t}H_1(t) + \frac{\lambda\beta^2}{\alpha^2(c-1)(c^2-1)}e^{\alpha t}H_2(t), \quad t \in [t_2, t_3). \tag{3.12}$$

However,

$$\beta e^{\alpha t}H_1(t) = \frac{\lambda\beta}{\alpha(c-1)}\left[e^{\alpha c(t-t_1)} - e^{\alpha(t-t_1/c)}\right], \tag{3.13}$$

and

$$e^{\alpha t}H_2(t) = e^{\alpha c^2(t-t_2)} - (c+1)e^{\alpha c(t-t_2)} + ce^{\alpha(t-t_2)}. \tag{3.14}$$

Substituting (3.13) and (3.14) into (3.12) yields

$$\begin{aligned} y_3(t) &= y_2(t_2)e^{\alpha(t-t_2)} + \frac{\lambda\beta}{\alpha(c-1)}\left[e^{\alpha c(t-t_1)} - e^{\alpha(t-t_1/c)}\right] \\ &+ \frac{\lambda\beta^2}{\alpha^2(c-1)(c^2-1)}\left[e^{\alpha c^2(t-t_2)} - (c+1)e^{\alpha c(t-t_2)} + ce^{\alpha(t-t_2)}\right], \quad t \in [t_2, t_3). \end{aligned} \tag{3.15}$$

From (3.7), one can find that

$$y_2(t_2)e^{\alpha(t-t_2)} = \lambda e^{\alpha t} + \frac{\lambda\beta}{\alpha(c-1)}\left[e^{\alpha(t-t_1/c)} - e^{\alpha(t-t_1)}\right]. \tag{3.16}$$

Inserting (3.16) into (3.15) and simplifying, we have

$$\begin{aligned} y_3(t) &= \lambda e^{\alpha t} + \frac{\lambda\beta}{\alpha(c-1)}\left[e^{\alpha c(t-t_1)} - e^{\alpha(t-t_1)}\right] \\ &+ \frac{\lambda\beta^2}{\alpha^2(c-1)(c^2-1)}\left[e^{\alpha c^2(t-t_2)} - (c+1)e^{\alpha c(t-t_2)} + ce^{\alpha(t-t_2)}\right], \quad t \in [t_2, t_3). \end{aligned} \tag{3.17}$$

Similarly, one can obtain $y_4(t)$ and $y_5(t)$ as

$$\begin{aligned} y_4(t) &= \lambda e^{\alpha t} + \frac{\lambda\beta}{\alpha(c-1)}\left[e^{\alpha c(t-t_1)} - e^{\alpha(t-t_1)}\right] \\ &+ \frac{\lambda\beta^2}{\alpha^2(c-1)(c^2-1)}\left[e^{\alpha c^2(t-t_2)} - (c+1)e^{\alpha c(t-t_2)} + ce^{\alpha(t-t_2)}\right] + \frac{\lambda\beta^3}{\alpha^3(c-1)(c^2-1)(c^3-1)} \\ &\times\left[e^{\alpha c^3(t-t_3)} - (c^2+c+1)e^{\alpha c^2(t-t_3)} + (c^3+c^2+c)e^{\alpha c(t-t_3)} - c^3e^{\alpha(t-t_3)}\right], \quad t \in [t_3, t_4), \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} y_5(t) &= \lambda e^{\alpha t} + \frac{\lambda\beta}{\alpha(c-1)}\left[e^{\alpha c(t-t_1)} - e^{\alpha(t-t_1)}\right] \\ &+ \frac{\lambda\beta^2}{\alpha^2(c-1)(c^2-1)}\left[e^{\alpha c^2(t-t_2)} - (c+1)e^{\alpha c(t-t_2)} + ce^{\alpha(t-t_2)}\right] + \frac{\lambda\beta^3}{\alpha^3(c-1)(c^2-1)(c^3-1)} \end{aligned}$$

$$\times\left[e^{\alpha c^3(t-t_3)}-(c^2+c+1)e^{\alpha c^2(t-t_3)}+(c^3+c^2+c)e^{\alpha c(t-t_3)}-c^3e^{\alpha(t-t_3)}\right]$$
$$+\frac{\lambda\beta^4}{\alpha^4(c-1)(c^2-1)(c^3-1)(c^4-1)}$$
$$\times\left[e^{\alpha c^4(t-t_4)}-(c+1)(c^2+1)e^{\alpha c^3(t-t_4)}+c(c^2+1)(c^2+c+1)e^{\alpha c^2(t-t_4)}\right.$$
$$\left.-c^3(c+1)(c^2+1)e^{\alpha c(t-t_4)}+c^6e^{\alpha(t-t_4)}\right],\quad t\in[t_4,t_5). \tag{3.19}$$

Following the above analysis, one can determine the solution $y_n(t)$ in any sub-interval $t \in [t_{n-1}, t_n)$, $n \geq 1$. However, a unified formula is to be deduced in the next section for $y_n(t)$ in explicit analytical form.

4. Unified formula for $y_n(t)$ & exact solution

In the previous section, it was indicated that $y_n(t)$ in any sub-interval $I_n = [t_{n-1}, t_n)$ can be calculated recurrently. However, in this section, we are able to obtain a unified formula for $y_n(t)$, $n \geq 1$, and accordingly, the exact solution of the DDE (1.1) can be established as provided by the following theorem.

Theorem 2. *For $c \neq 1$, $y_n(t)$ is given explicitly in the form:*

$$y_n(t)=\sum_{i=0}^{n-1}\sum_{m=0}^{i}\mu_{i,m}e^{\alpha c^m(t-t_i)},\quad t\in I_n,\quad n\geq 1, \tag{4.1}$$

where

$$\mu_{i,m}=\frac{\lambda(-1)^{i-m}(\beta/\alpha)^i c^{\frac{1}{2}(i-m)(i-m-1)}}{\prod_{j=1}^{i-m}(c^j-1)\prod_{r=1}^{m}(c^r-1)},\quad c\neq 1,\quad i,m\geq 0. \tag{4.2}$$

Proof. At first, let us rewrite $y_1(t)$ as

$$y_1(t)=\lambda e^{\alpha t}=\sum_{m=0}^{0}\mu_{0,m}e^{\alpha c^m(t-t_0)},\quad t\in I_1, \tag{4.3}$$

where

$$\mu_{0,0}=\lambda. \tag{4.4}$$

Also, $y_2(t)$, given by Eq (3.7), can be rewritten as

$$y_2(t)=\lambda e^{\alpha t}+\sum_{m=0}^{1}\mu_{1,m}e^{\alpha c^m(t-t_1)},\quad t\in I_2, \tag{4.5}$$

such that

$$\mu_{1,0}=-\frac{\lambda\beta}{\alpha(c-1)},\quad \mu_{1,1}=\frac{\lambda\beta}{\alpha(c-1)}. \tag{4.6}$$

From (4.3) and (4.5), we get

$$y_2(t)=\sum_{m=0}^{0}\mu_{0,m}e^{\alpha c^m(t-t_0)}+\sum_{m=0}^{1}\mu_{1,m}e^{\alpha c^m(t-t_1)},\quad t\in I_2, \tag{4.7}$$

or equivalently

$$y_2(t) = \sum_{i=0}^{1}\sum_{m=0}^{i} \mu_{i,m} e^{\alpha c^m (t-t_i)}, \quad t \in I_2. \tag{4.8}$$

Similarly, $y_3(t)$, $y_4(t)$, and $y_5(t)$ can be written as

$$\begin{aligned} y_3(t) &= \sum_{i=0}^{2}\sum_{m=0}^{i} \mu_{i,m} e^{\alpha c^m (t-t_i)}, \quad t \in I_3, \\ y_4(t) &= \sum_{i=0}^{3}\sum_{m=0}^{i} \mu_{i,m} e^{\alpha c^m (t-t_i)}, \quad t \in I_4, \\ y_5(t) &= \sum_{i=0}^{4}\sum_{m=0}^{i} \mu_{i,m} e^{\alpha c^m (t-t_i)}, \quad t \in I_5, \end{aligned} \tag{4.9}$$

with the coefficients:

$$\begin{cases} \mu_{2,0} = \dfrac{\lambda\beta^2 c}{\alpha^2(c-1)(c^2-1)}, \\ \mu_{2,1} = -\dfrac{\lambda\beta^2(c+1)}{\alpha^2(c-1)(c^2-1)} = -\dfrac{\lambda\beta^2}{\alpha^2(c-1)^2}, \\ \mu_{2,2} = \dfrac{\lambda\beta^2}{\alpha^2(c-1)(c^2-1)}, \\ \mu_{3,0} = -\dfrac{\lambda\beta^3 c^3}{\alpha^3(c-1)(c^2-1)(c^3-1)}, \\ \mu_{3,1} = \dfrac{\lambda\beta^3 c^3(c^3+c^2+c)}{\alpha^3(c-1)(c^2-1)(c^3-1)} = \dfrac{\lambda\beta^3 c^4}{\alpha^3(c-1)^2(c^2-1)}, \\ \mu_{3,2} = -\dfrac{\lambda\beta^3 c^3(c^2+c+1)}{\alpha^3(c-1)(c^2-1)(c^3-1)} = -\dfrac{\lambda\beta^3 c^3}{\alpha^3(c-1)^2(c^2-1)}, \\ \mu_{3,3} = \dfrac{\lambda\beta^3}{\alpha^3(c-1)(c^2-1)(c^3-1)}, \\ \mu_{4,0} = \dfrac{\lambda\beta^4 c^6}{\alpha^4(c-1)(c^2-1)(c^3-1)(c^4-1)}, \\ \mu_{4,1} = -\dfrac{\lambda\beta^4 c^3}{\alpha^4(c-1)^2(c^2-1)(c^3-1)}, \\ \mu_{4,2} = \dfrac{\lambda\beta^4 c}{\alpha^4(c-1)^2(c^2-1)^2}, \\ \mu_{4,3} = -\dfrac{\lambda\beta^4}{\alpha^4(c-1)^2(c^2-1)(c^3-1)}, \\ \mu_{4,4} = \dfrac{\lambda\beta^4}{\alpha^4(c-1)(c^2-1)(c^3-1)(c^4-1)}. \end{cases}$$

The above calculations reveal that

$$\mu_{i,0} = \frac{\lambda(-1)^i(\beta/\alpha)^i c^{\frac{1}{2}i(i-1)}}{\prod_{k=1}^{i}(c^k-1)}, \quad \mu_{i,i} = \frac{\lambda(\beta/\alpha)^i}{\prod_{k=1}^{i}(c^k-1)}, \quad i \geq 0, \tag{4.10}$$

while

$$\mu_{i,m} = \frac{\lambda(-1)^{i-m}(\beta/\alpha)^i c^{\frac{1}{2}(i-m)(i-m-1)}}{\prod_{j=1}^{i-m}(c^j-1)\prod_{r=1}^{m}(c^r-1)}, \quad i,m \geq 0. \tag{4.11}$$

Repeating the process (4.9) n-times leads to

$$y_n(t) = \sum_{i=0}^{n-1}\sum_{m=0}^{i}\mu_{i,m}e^{\alpha c^m(t-t_i)}, \quad t \in I_n, \quad n \geq 1, \tag{4.12}$$

where $\mu_{i,m}$ is already obtained in Eq (4.11), which completes the proof. □

Remark 1. *At $m = 0$, Eq (4.11) agrees with the first formula in (4.10). Additionally, for $i = m$, Eq (4.11) reduces to the second formula in (4.10). In addition, for $i = m = 0$, we have from (4.11) that $\mu_{0,0} = \lambda$, which agrees with (4.4). Thus, Eq (4.11) unifies the coefficients $\mu_{i,m}$ $\forall$ $i,m \geq 0$. Moreover, the general formula (4.1) along with the coefficients (4.11) have been explicitly verified for $n = 1,2,3$. For $n > 3$, verification can be readily performed using computational software due to the increasing complexity of the expressions.*

Lemma 2. *The solution $y_n(t)$ can be expressed as*

$$y_n(t) = \lambda\sum_{i=0}^{n-1}(\beta/\alpha)^i\sum_{m=0}^{i}\frac{(-1)^m c^{\frac{1}{2}(i-m)(i-m-1)}e^{\alpha c^m(t-t_i)}}{(c:c)_{i-m}(c:c)_m}, \quad c \neq 1, \quad t \in I_n, \tag{4.13}$$

where $(c:c)_m$ denote the Pochhammer symbol

$$(c:c)_m = \prod_{r=0}^{m-1}\left(1-c^{r+1}\right) = \prod_{r=1}^{m}(1-c^r). \tag{4.14}$$

Proof. In view of the quantum calculus [30], we have the product

$$(a:b)_m = \prod_{r=0}^{m-1}(1-ab^r). \tag{4.15}$$

For $a = b = c$, we have

$$(c:c)_m = \prod_{r=0}^{m-1}\left(1-c^{r+1}\right) = \prod_{r=1}^{m}(1-c^r). \tag{4.16}$$

Now, one can rewrite the product in the denominator of $\mu_{i,m}$ as

$$\prod_{j=1}^{i-m}(c^j-1)\prod_{r=1}^{m}(c^r-1) = (-1)^{i-m}\prod_{j=1}^{i-m}(1-c^j)\times(-1)^m\prod_{r=1}^{m}(1-c^r) = (-1)^n(c:c)_{i-m}(c:c)_m. \tag{4.17}$$

Inserting this result in Eqs (4.1) and (4.2) completes the proof. □

5. Characteristics of the solution

Here, we focus on proving some characteristics regarding the obtained solution and its derivative. To achieve this target, let us introduce the following basic relationship:

$$y_{n+1}(t) = y_n(t) + \sum_{m=0}^{n} \mu_{n,m} e^{\alpha c^m (t-t_n)}, \quad n \geq 1, \tag{5.1}$$

between the solutions $y_n(t)$ and $y_{n+1}(t)$ in the intervals I_n and I_{n+1}, respectively. Differentiating (5.1) once with respect to t, then

$$y'_{n+1}(t) = y'_n(t) + \alpha \sum_{m=0}^{n} c^m \mu_{n,m} e^{\alpha c^m (t-t_n)}, \quad n \geq 1. \tag{5.2}$$

At $t = t_n$, the difference between the right derivative $y'_{n+1}(t_n) = y'_+(t_n)$ and the left derivative $y'_n(t_n) = y'_-(t_n)$ reads

$$y'_+(t_n) - y'_-(t_n) = \alpha \sum_{m=0}^{n} c^m \mu_{n,m}, \quad n \geq 1. \tag{5.3}$$

This relation is essential for proving the next theorem.

Theorem 3. *For $c \neq 1$, $y'(t)$ is discontinuous at $t_0 = 0$ and $t_1 = \tau/c$ provided that λ, α, and β are nonzeros. Moreover, $y'(t)$ is continuous $\forall\ t = t_n$, $n \geq 2$ for $\lambda, \alpha, \beta \in \mathbb{R}$.*

Proof. From the initial conditions, the left derivative at $t = t_0 = 0$ is $y'_-(0) = 0$. The right derivative is determined from the solution in $I_1 = [t_0, t_1) = [0, \tau/c)$, i.e., $y'_+(0) = \alpha\lambda$. By this, $y'_-(0) \neq y'_+(0)$, so $y'(t)$ is discontinuous at $t = t_0 = 0$, where $\alpha\lambda \neq 0$. At $n = 1$, Eq (5.3) yields $y'_+(t_1) - y'_-(t_1) = \alpha(\mu_{1,0} + c\mu_{1,1}) = \lambda\beta \neq 0$, which implies discontinuity at $t = t_1$. For $n = 2$, we have

$$y'_+(t_2) - y'_-(t_2) = \alpha\left(\mu_{2,0} + c\mu_{2,1} + c^2\mu_{2,2}\right) = \frac{\lambda\beta^2}{\alpha(c-1)(c^2-1)}\left[c - c(c+1) + c^2\right] = 0.$$

Hence, $y'(t)$ is continuous at $t = t_2$. Similarly, we have at $n = 3$ that

$$y'_+(t_3) - y'_-(t_3) = \alpha\left(\mu_{3,0} + c\mu_{3,1} + c^2\mu_{3,2} + c^3\mu_{3,3}\right) = 0,$$

where the values of $\mu_{3,0}$, $\mu_{3,1}$, $\mu_{3,2}$, and $\mu_{3,3}$ are implemented. Thus, $y'(t)$ is continuous at $t = t_3$. It is noticed from above that the coefficients $\mu_{n,m}$ satisfy the relation $\sum_{m=0}^{n} c^m \mu_{n,m} = 0\ \forall\ n \geq 2$, which needs a proof by induction. This relation directly leads to $y'_+(t_n) = y'_-(t_n)$, i.e., $y'(t)$ is continuous at $t = t_n$ $\forall\ n \geq 2$.

Instead, one can use Eq (2.6) to accomplish this point as follows: From Eq (2.6), one can write

$$\begin{cases} y'_-(t_n) = y'_n(t_n) = \alpha y_n(t_n) + \beta y_{n-1}(ct_n - \tau), \\ y'_+(t_n) = y'_{n+1}(t_n) = \alpha y_{n+1}(t_n) + \beta y_n(ct_n - \tau), \quad n \geq 1. \end{cases} \tag{5.4}$$

Since $ct_n - \tau = t_{n-1}$ (from Eq (2.2)) and $y_{n+1}(t_n) = y_n(t_n)$ (from the continuity condition at $t = t_n$), then

$$y'_+(t_n) - y'_-(t_n) = \beta\left(y_n(t_{n-1}) - y_{n-1}(t_{n-1})\right). \tag{5.5}$$

For $n = 1$, we have

$$y'_+(t_1) - y'_-(t_1) = \beta\left(y_n(t_0) - y_0(t_0)\right) = \beta\lambda \neq 0,\ t_0 = 0, \tag{5.6}$$

which confirms the discontinuity of $y'(t)$ at $t = t_1$ without resorting to the coefficients $\mu_{n,m}$ as made above. Since $y(t)$ is continuous at $t = t_{n-1}$ $\forall\ n \geq 2$, then Eq (5.5) ensures that $y'_+(t_n) = y'_-(t_n)$ $\forall\ n \geq 2$. This finalizes the proof. □

6. Solutions of the PDDE and the ADDE as special cases

This section recovers some existing results in the literature of some models such as the PDDE [7] and the ADDE [31] as special cases of our solution. This step may be essential in the context of validating the correctness of the present calculations/results.

6.1. Solution of the PDDE

As $\tau \to 0$, the present model reduces to the PDDE [7]:

$$y'(t) = \alpha y(t) + \beta y\left(ct\right), \quad y(t) = 0, \quad y(0) = \lambda, \quad 0 < c \leq 1, \quad t \geq 0. \tag{6.1}$$

It is noted in this case that as $\tau \to 0$, then $t_n = \frac{\tau}{c}\sum_{k=0}^{n-1}\frac{1}{c^k} \to 0\ \forall\ n \geq 1$. Thus, all the intervals $I_n = [t_{n-1}, t_n)$ violate and collapse to the initial point $t = 0$. Hence, the delay model (1.1) transforms to the PDDE in the domain $[0, \infty)$, and accordingly the solution of the PDDE can be recovered from Eq (4.13) as $n \to \infty$ and $\tau \to 0$

$$y(t) = \lim_{n\to\infty,\tau\to 0} y_n(t) = \lambda\sum_{i=0}^{\infty}(\beta/\alpha)^i\sum_{m=0}^{i}\frac{(-1)^m c^{\frac{1}{2}(i-m)(i-m-1)}e^{\alpha c^m t}}{(c:c)_{i-m}(c:c)_m}, \tag{6.2}$$

which is in full agreement with the solution of the PDDE obtained in [9].

6.2. Solution of the ADDE

The ADDE [31] is given as

$$y'(t) = -y(t) + \frac{1}{q}y\left(\frac{t}{q}\right), \quad y(0) = \lambda, t \geq 0. \tag{6.3}$$

Comparing Eq (6.3) with our model gives $\tau = 0$, $\alpha = -1$, and $\beta = c = \frac{1}{q}$ $q > 1$, where $1/q$ is a proportional delay parameter. Substituting these values into Eqs (3.7) and (3.17), respectively, gives

$$y_2(t) = \lambda e^{-t} + \frac{\lambda}{(q-1)}\left[e^{-t/q} - e^{-t}\right], \tag{6.4}$$

and

$$y_3(t) = \lambda e^{-t} + \frac{\lambda}{(q-1)}\left[e^{-t/q} - e^{-t}\right] + \frac{\lambda}{(q-1)(q^2-1)}\left[qe^{-t/q^2} - (q+1)e^{-t/q} + e^{-t}\right], \tag{6.5}$$

which agree with the obtained results for the two-term and the three-term solutions of the ADDE using the Adomian decomposition method (ADM) [31].

As an interesting result, one can obtain a new closed-form solution for the ADDE through substituting $\alpha = -1$ and $\beta = c = \frac{1}{q}$ into the solution (6.2) of the PDDE, thus

$$y(t) = \lambda \sum_{i=0}^{\infty} (-q)^{-i} \sum_{m=0}^{i} \frac{(-1)^m q^{-\frac{1}{2}(i-m)(i-m-1)} e^{-t/q^m}}{(1/q : 1/q)_{i-m} (1/q : 1/q)_m}. \tag{6.6}$$

It may be important to note that the solution obtained in Eq (6.6) has not been reported in the literature for the ADDE, which reflects the effectiveness of the present analysis.

6.3. Reduced delay model: $c = 1$

At $c = 1$, we have the reduced DDE

$$y'(t) = \alpha y(t) + \beta y(t - \tau), \quad y(t) = 0 \ \forall \ t \in [-\tau, 0), \ y(0) = \lambda, \ \tau \geq 0. \tag{6.7}$$

In this case, Eq (2.2) implies $t_n = n\tau$, and the solution $y_n(t)$ in the interval $I_n = [(n-1)\tau, n\tau)$ can be determined by calculating the limit as $c \to 1$ of the corresponding $y_n(t)$ in Section 3. For declaration, the solution $y_2(t)$ in the interval $I_2 = [\tau, 2\tau)$ is evaluated from Eq (3.7) as

$$y_2(t) = \lambda e^{\alpha t} + \lim_{c \to 1} \frac{\lambda\beta}{\alpha(c-1)} \left[e^{\alpha c(t-\tau)} - e^{\alpha(t-\tau)} \right], \quad t \in [\tau, 2\tau), \tag{6.8}$$

i.e.,

$$y_2(t) = \lambda e^{\alpha t} + \beta\lambda(t - \tau) e^{\alpha(t-\tau)}, \quad t \in [\tau, 2\tau). \tag{6.9}$$

By similar analysis, one can find that

$$y_3(t) = \lambda e^{\alpha t} + \beta\lambda(t - \tau) e^{\alpha(t-\tau)} + \frac{1}{2}\beta^2\lambda(t - 2\tau)^2 e^{\alpha(t-2\tau)}, \quad t \in [2\tau, 3\tau), \tag{6.10}$$

and

$$y_4(t) = \lambda e^{\alpha t} + \beta\lambda(t - \tau) e^{\alpha(t-\tau)} + \frac{1}{2!}\beta^2\lambda(t - 2\tau)^2 e^{\alpha(t-2\tau)} + \frac{1}{3!}\beta^3\lambda(t - 3\tau)^3 e^{\alpha(t-3\tau)}, \quad t \in [3\tau, 4\tau). \tag{6.11}$$

It is obvious from above that the general solution $y_n(t)$ in the interval $I_n = [(n-1)\tau, n\tau)$ reads

$$y_n(t) = \lambda \sum_{k=0}^{n-1} \frac{\beta^k}{k!} (t - k\tau)^k e^{\alpha(t-k\tau)}, \quad t \in [(n-1)\tau, n\tau), \quad n \geq 1, \tag{6.12}$$

which agrees with the results obtained in [32].

7. Numerical examples

This section examines the behavior of the solution and its derivative through graphical representations. The forthcoming plots demonstrate that the previously established theoretical properties of the solution and its derivative hold within a finite set of intervals. For illustration, the first five intervals of the problem's domain are considered. Nevertheless, the solution and its derivative can be plotted over any finite number of intervals, as permitted by the analytical solution derived in the preceding sections.

In order to extract numerical results, the parameters α, β, and τ are assigned specific values while λ is fixed at 1. Figures 1 and 2 show the behavior of $y(t)$ in the first five intervals at $\alpha = -1, \beta = 1, c = \frac{1}{2}$, and $\tau = 1$ (Figure 1) and at $\alpha = -1, \beta = -1, c = \frac{3}{4}$, and $\tau = 3$ (Figure 2). In these figures, the black dots indicate the junctions between the five intervals under consideration, where the continuity of the solution $y(t)$ is clearly visible. However, it should be noted that $y(t)$ is discontinuous at $t = 0$ as a consequence of the specified initial conditions.

With respect to the continuity of the derivative $y'(t)$, Figures 3 and 4 illustrate its behavior using the same parameter values employed in generating Figures 1 and 2. These figures corroborate our theoretical findings regarding the discontinuity of the derivative $y'(t)$ at $t = t_1 = \frac{\tau}{c}$ while also confirming that $y'(t)$ remains continuous at $t = t_n$, $n \geq 2$.

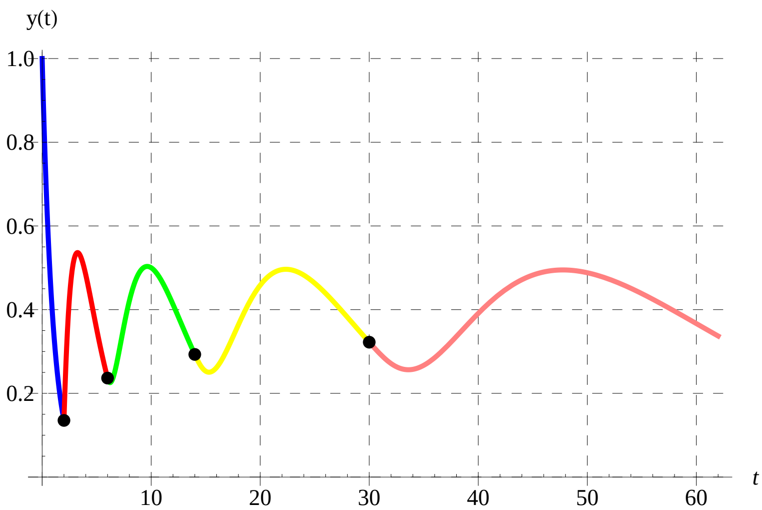

Figure 1. Behavior of the solution $y(t)$ in the first five intervals at $\lambda = 1, \alpha = -1, \beta = 1, c = \frac{1}{2}$, and $\tau = 1$.

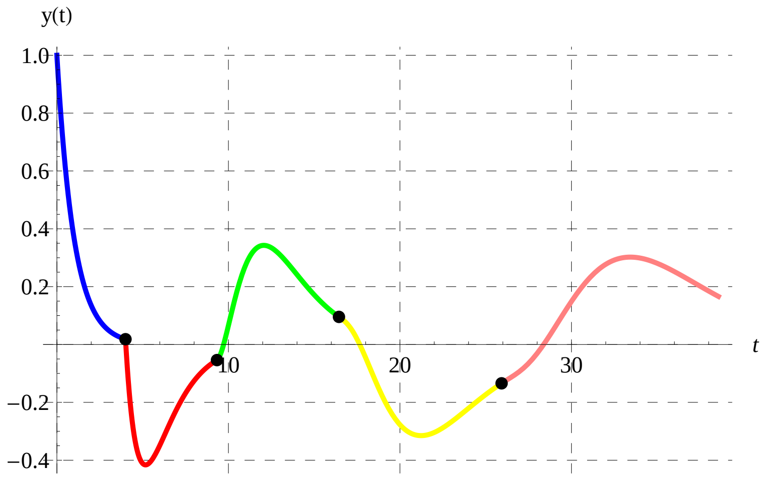

Figure 2. Behavior of the solution $y(t)$ in the first five intervals at $\lambda = 1, \alpha = -1, \beta = -1, c = \frac{3}{4}$, and $\tau = 3$.

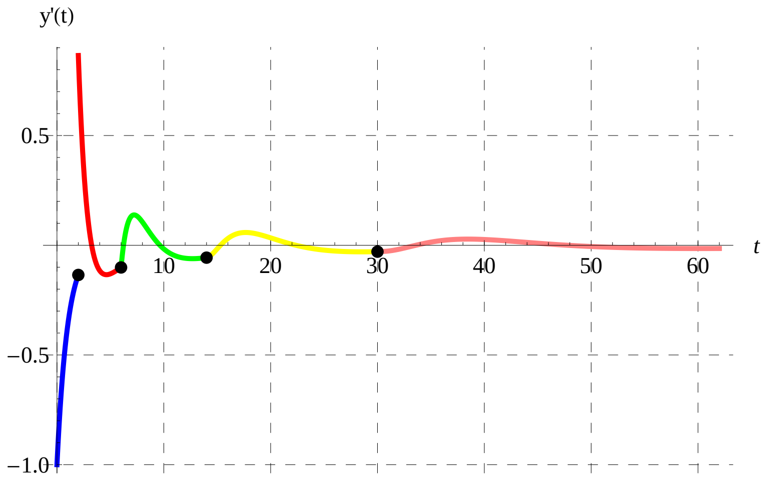

Figure 3. Behavior of the derivative $y'(t)$ in the first five intervals at $\lambda = 1$, $\alpha = -1$, $\beta = 1$, $c = \frac{1}{2}$, and $\tau = 1$.

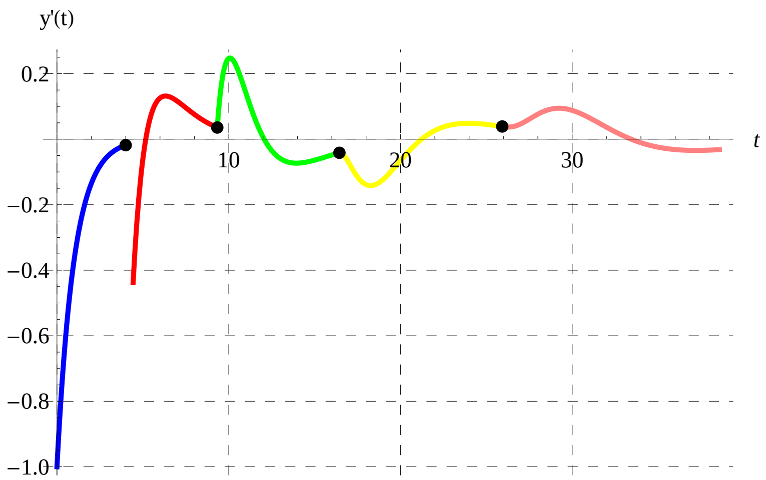

Figure 4. Behavior of the derivative $y'(t)$ in the first five intervals at $\lambda = 1$, $\alpha = -1$, $\beta = -1$, $c = \frac{3}{4}$, and $\tau = 3$.

8. Conclusions

An effective approach was introduced in this paper to solve the DDE $y'(t) = \alpha y(t) + \beta y(ct - \tau)$, $0 < c \leq 1$, $\tau > 0$. The exact solution was obtained under prescribed initial conditions, which assumed discontinuity at the initial point. As an advantage of the current work over the previous studies, the exact solution was constructed explicitly and analytically via the aid of the MoS. Furthermore, a general formula was established, which expressed the solution in any finite sub-interval of the domain of the problem. It was shown that the problem reduces to some existing delay models in the literature as the number of the sub-intervals tends to infinity, at particular choices of the involved coefficients. Also, a theoretical analysis was presented to detect some properties about the solution and its derivative. Finally, the solutions of some basic models in the literature such as the PDDE and the ADDE were calculated as special cases. Accordingly, the present results generalized the previous ones in the relevant literature [32] in which the parameter c equals unity. This proves the efficiency of the proposed approach. Moreover, the suggested analysis may be extended to include other forms of DDEs in fractional calculus and fractional delay integro-differential systems such as those addressed in [33, 34].

Author contributions

Essam R. El-Zahar: Conceptualization, methodology, validation, formal analysis, investigation, writing-review and editing, visualization; Abdelhalim Ebaid: Conceptualization, methodology, validation, formal analysis, investigation, writing-review and editing; Laila F. Seddek: Methodology, validation, formal analysis, investigation, writing-original draft preparation; Mona D. Aljoufi: Conceptualization, methodology, software, validation, formal analysis, investigation, data curation, writing-review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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