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Research article

Bayesian stress-strength inference for the Moran-Downton bivariate exponential distribution under modified Type-II censoring

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Abstract: The stress-strength reliability $\delta = \Pr(X < Y)$, where (X, Y) is Moran-Downton bivariate exponential distributed, is investigated under a modified bivariate Type-II censoring scheme. Likelihood contributions for the four censoring configurations are derived via series representations of the modified Bessel function and evaluated with finite partial-sum approximations that admit geometric error bounds, enabling direct likelihood computation. Bayesian inference is developed using two Markov chain Monte Carlo strategies: (i) Metropolis-Hastings within Gibbs sampler using the likelihood based on censored data; and (ii) a data-augmentation scheme that imputes censored observations. Extensive simulations across different censoring proportions and parameter settings show a decreasing mean squared error with increasing sample size and a slightly superior correlation estimate between X and Y for the approach using a likelihood based on censored data, while both methods perform comparably for δ. Finally, the methodology is illustrated with a simulated numerical example and a real bivariate dataset, highlighting implementation details and the benefits of the proposed censoring design for estimating the model parameters and stress-strength reliability.

Keywords: dependence; incomplete data; maximum likelihood; reliability
Mathematics Subject Classification: 62F10, 62N05, 62P30

1. Introduction

Let the lifetime random variables of two components in a system be labeled by X and Y. One of the reliability characteristics defined by $\delta = \Pr(X < Y)$, known as the stress-strength model, has received considerable attention for many years and has also been applied in numerous fields, including engineering, medicine, and quality control. For biometry, X and Y may indicate the remaining lifetimes of patients under the treatments of drug A and drug B, respectively. In this case, the inference of δ would be a crucial judgment of the effectiveness of treatments from drugs A and B. For the engineering and reliability studies, δ was used to symbolize the likelihood of stress Y exceeding the component X due to exterior factors. Since [8] delivered a wonderful link for both δ and the classical Mann-Whitney statistic under the independent assumption between X and Y, the inferences of δ have gained great attention and been probed under diverse distributions and life test censoring schemes. For instance, [3, 12, 15, 22, 24–26, 31, 32, 36, 37, 41, 42, 44, 45] studied the inferences of δ for two-parameter exponential, Topp-Leone, Burr X, two- or three-parameter generalized exponential, and Weibull distributions. Additionally, the book authored by [23] delivered a brilliant collection of works on $\delta = \Pr(X < Y)$. It should be noted that these works generally treat strength and stress as stochastically independent. More recently, several studies, including [1, 10, 27, 29], have investigated stress-strength models in which strength and stress are dependent. When strength and stress are independent, the distributional parameters for each can be inferred separately, and sampling for the two components can proceed independently. Under dependence, however, the parameters must be estimated jointly from paired observations (i.e., failures recorded for matched strength-stress units), so sampling must be conducted pairwise.

Due to subject withdrawals and the increasingly long lifetimes resulting from technological advancements, obtaining complete lifetime samples in experiments has become more challenging. To address these issues, a growing literature has developed methodologies that are generally classified by stopping rules-time-based, failure-count-based, and hybrids. Let n subjects be placed on test at time $t = 0$, with ordered failure times $X_{(1)} \leq X_{(2)} \leq X_{(3)} \leq \cdots \leq X_{(n)}$. Under time-based censoring, known as Type I (time) censoring, the experiment terminates at a prespecified time, $\tau > 0$. Under failure-count-based censoring, known as Type-II censoring, the experiment terminates when a pre-determined number, $k(\leq n)$, of failures are observed (i.e., the k ordered failure time, $X_{(k)}$, is observed). There are two additional hybrid censoring schemes: the Type-I hybrid censoring scheme proposed by [13], and the Type-II hybrid censoring scheme proposed by [9]. Under Type-I hybrid censoring, the test ends at the random time $\tau^1 = \min\{X_{(k)}, \tau\}$; under Type-II hybrid censoring, the test ends at $\tau^2 = \max\{X_{(k)}, \tau\}$. All surviving items are right-censored at the stopping time for likelihood construction. The schemes above do not permit withdrawals before the stopping time, whereas in practice, items may be lost or removed for other reasons. To accommodate this, progressive censoring schemes were developed and combined with the four aforementioned basic censoring schemes. These traditional, one-dimensional censoring designs were devised to collect failure-time samples. Numerous contributions under progressive censoring include, among others, truncated censored samples [20], progressive censoring [39], progressively first-failure censoring [38], progressive Type-II censoring [30], and inference with left-censored samples [15].

Among the aforementioned commonly used censoring schemes, the progressive Type-II approach is a flexible approach that allows removing surviving subjects during the life test. More precisely, let

n items be positioned on a life test simultaneously until a pre-specified number of failures, say $m < n$, have been observed. At the jth failure time, randomly remove R_j surviving units from the test for $j = 1, 2, 3, \ldots, m$. The values R_j for $j = 1, 2, 3, \ldots, m$ are pre-decided to satisfy $n - m = \sum_{j=1}^{m} R_j$. If $R_j = 0, j = 1, 2, 3, \ldots, m$, then the censoring scheme is reduced to a Type-II censoring one. Reliability inference and analysis under progressive Type-II censoring have been extensively investigated over the years. For comprehensive reviews of progressive censoring (theory and application), readers consult the books [4–6].

The present work focuses on the inference of δ under the assumption that both strength and stress follow the Moran-Downton bivariate exponential (DBVE) distribution, originally proposed by [11, 34] in a reliability context, with parameters μ_1, μ_2, and ρ. Hereafter, $(X, Y) \sim \text{DBVE}(\mu_1, \mu_2, \rho)$ indicates that (X, Y) is DBVE distributed and has a joint probability density function (PDF),

$$f(x, y) = \frac{\mu_1 \mu_2}{1 - \rho} \exp\left\{ -\frac{\mu_1 x + \mu_2 y}{1 - \rho} \right\} I_0\left(\frac{2(\mu_1 x \mu_2 y \rho)^{1/2}}{1 - \rho} \right), \tag{1.1}$$

where $\mu_1 > 0$ and $\mu_2 > 0$ are, respectively, the marginal means of X and Y, $0 \leq \rho < 1$ is the correlation coefficient between X and Y, and $I_0(z) = \sum_{r=0}^{\infty} (z/2)^{2r}/(r!)^2$ is the modified Bessel function of the first kind of order zero. In reliability studies, [17, 28] investigated the statistical inference for ρ using incomplete samples and highlighted the usefulness of DBVE(μ_1, μ_2, ρ) for modeling the joint distribution of two dependent component lifetimes. For DBVE(μ_1, μ_2, ρ), δ involves evaluating a double integral whose integrand contains the factor $I_0(z)$, expressible as an infinite series. Since strength and stress are dependent, they should be treated as a pair during the life test. Hence, a direct extension of the traditional one-dimensional Type-II censoring scheme to collect the bivariate setting poses practical challenges, as the failure life test cannot stop once the required failures are observed for one component, but must wait until all associated component failure times have also been recorded. The present research addresses this issue by developing Bayesian inference procedures for δ via utilizing a further modified Type-II censored data to collect paired-wise censored failure times from DBVE(μ_1, μ_2, ρ).

The rest of this paper is structured as follows. Section 2 presents likelihood functions for two modified Type-II censored samples from DBVE(μ_1, μ_2, ρ). After a data augmentation approach for parameter estimation is introduced in Section 3, Section 4 addresses the Bayesian estimation of δ. Section 5 presents the performance evaluation of the proposed methods based on a Monte Carlo simulation study, and Section 6 provides a numerical example to illustrate the methodology. Concluding remarks are given in Section 7.

2. Likelihood functions

Let $(X_i, Y_i), i = 1, 2, \ldots, n$ be a size n bivariate random vectors from the DBVE(μ_1, μ_2, ρ) distribution. Section 2.1 provides more details on the DBVE(μ_1, μ_2, ρ) distribution. Then, two different modified Type-II censoring schemes and their respective likelihood functions will be presented in Sections 2.2 and 2.3.

2.1. Model description

The DBVE(μ_1, μ_2, ρ) is a special case of the bivariate gamma distribution addressed by [21] and has two marginal exponential means, $1/\mu_1$ and $1/\mu_2$, respectively. The estimation of the correlation coefficient ρ between X and Y has been studied by numerous scholars. For instance, [2] developed the moments method estimation of ρ by equating a population and sample mixed moments, as well as introduced a modified moment-method estimator. Utilizing the sample correlation coefficient, they also employed a bias-reduced estimation method. [7] derived an improved moment estimator by using bias reduction and Jackknife methods. An alternative way to derive the joint PDF of DBVE(μ_1, μ_2, ρ) is based on expanding the series $I_0(z)$, resulting in the joint PDF

$$f(x, y) = \sum_{k=0}^{\infty} \pi(k, \rho) g_{k+1}(x; \mu_1/(1-\rho)) g_{k+1}(y; \mu_2/(1-\rho)), \tag{2.1}$$

where $\pi(k, \rho) = (1-\rho)(\rho)^k, k = 0, 1, 2, \ldots$ is a geometric probability function and $g_{k+1}(x; b)$ is a gamma PDF with b and $k+1$ as rate and shape parameters, respectively. That is,

$$g_{k+1}(x; b) = \frac{b^{k+1}}{\Gamma(k+1)} x^k e^{-bx}, x > 0$$

with $\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx$. Meanwhile, let K be the random variable of $\pi(k, \rho)$. Given $K = k$, X and Y are independent random variables that have gamma distributions with $k+1$ as the common shape parameter and $\mu_1/(1-\rho)$ and $\mu_2/(1-\rho)$ as the respective rate parameters. These properties have been utilized to develop an algorithm for generating random samples from the DBVE(μ_1, μ_2, ρ), see, for example, [11]. Moreover, by utilizing the PDF of Eq (2.1), δ is obtained to be

$$\delta = \frac{(1-\rho)\mu_1}{\mu_1+\mu_2} \sum_{k=0}^{\infty} \frac{\rho^k \mu_1^k}{\Gamma(k+1)(\mu_1+\mu_2)^k} \sum_{i=0}^{k} \frac{\Gamma(i+k+1)}{\Gamma(i+1)} \frac{\mu_2^i}{(\mu_1+\mu_2)^i}. \tag{2.2}$$

2.2. Modified Type-II censoring schemes

For an observed bivariate random sample, $(x_i, y_i), i = 1, 2, \ldots, n$, from DBVE($\mu_1$, μ_2, ρ), and let $x_{1:n} \leq x_{2:n} \leq \ldots \leq x_{n:n}$ be the order statistics of X-component values $x_1, x_2, \ldots, x_n$, and $y_{[i:n]}$ be the corresponding value of the Y-component associated with $x_{i:n}$. Then, the data $D_r = \{(x_{j:n}, y_{[j:n]}), j = 1, 2, \ldots, r\}$ is an observed two-dimensional Type-II censored sample, which is a commonly used extension of the traditional one-dimensional Type-II censored sample. For example, [28] showed that the likelihood function based on the observed data D_r can be expressed as

$$\begin{aligned} L_{(r)}(\mu_1, \mu_2, \rho) \;=\; & \left\{ \left(\frac{\mu_1 \mu_2}{1-\rho} \right)^r \exp\left[-\sum_{i=1}^{r} \frac{\mu_1 x_{i:n} + \mu_2 y_{[i:n]}}{1-\rho} \right] \prod_{i=0}^{r} I_0\left(\frac{2(\mu_1 x_{i:n} \mu_2 y_{[i:n]})^{1/2}}{1-\rho} \right) \right\} \\ & \times \exp[-(n-r)\mu_1 x_{r:n}]. \end{aligned} \tag{2.3}$$

To collect the data D_r based on the aforementioned two-dimensional Type-II censoring, the life-testing experiment will not end until the first r X-samples and their concomitant Y-observations are all observed. In the case that some of $y_{[j:n]}, j = 1, 2, \ldots, r$ are larger than $x_{r:n}$, the experiment requires extra time to observe the bivariate sample D_r completely. To achieve time and cost reduction, a different modified Type-II censoring scheme is proposed as follows.

The proposed modified Type-II censoring scheme will cease the experiment at the observed time $x_{r:n}$. In this case, $x_{j:n} \leq x_{r:n}$ and $y_{[j:n]} \leq x_{r:n}$ for $j = 1, 2, \ldots, n$ will be recorded, and $x_{j:n} > x_{r:n}$ and $y_{[j:n]} > x_{r:n}$, for $i = 1, 2, \ldots, n$ are censored and labeled by "$x^+_{r:n}$" (or simply "+" for notation convenience). Hence, the doubly-modified Type-II censored sample can be expressed as $D^* = \{(x^*_i, y^*_i, c_i), i = 1, 2, \ldots, n\}$, where $c_i = 1, 2, 3, 4$ are defined in the four cases according to the values of (x^*_i, y^*_i). These four cases and their corresponding contributions to the likelihood function are summarized as follows:

- **Case 1:** $c_i = 1$ if both variates of (x^*_i, y^*_i) are observed before or at $x_{r:n}$; i.e.,

$$(x^*_i, y^*_i) = (x_i, y_i) \quad \text{if } x^*_i \leq x_{r:n} \text{ and } y^*_i \leq x_{r:n}.$$

The likelihood of bivariate observation is (see Eq (1.1))

$$\begin{aligned} f_1(x_i, y_i) &= \frac{\mu_1\mu_2}{1-\rho} \exp\left[-\frac{\mu_1 x_i + \mu_2 y_i}{1-\rho}\right] I_0\left(\frac{2(\mu_1 x_i \mu_2 y_i)^{1/2}}{1-\rho}\right) \\ &= \sum_{k=0}^{\infty} \left\{\pi(k,\rho) g_{k+1}\left(x; \frac{\mu_1}{1-\rho}\right) g_{k+1}\left(y_i; \frac{\mu_2}{1-\rho}\right)\right\}. \end{aligned} \tag{2.4}$$

- **Case 2:** $c_i = 2$ if y_i is observed before or at the ending time $x_{r:n}$ and x_i is truncated if $x_i > x_{r:n}$; i.e.,

$$(x^*_i, y^*_i) = (+, y_i) \quad \text{if } y^*_i \leq x_{r:n} \text{ and } x^*_i > x_{r:n}.$$

The corresponding likelihood of (x^*_i, y^*_i) is

$$\begin{aligned} f_2(x_i, y_i) &= \int_{x_{r:n}}^{\infty} f(x, y_i)\, dx \\ &= \int_{x_{r:n}}^{\infty} \frac{\mu_1\mu_2}{1-\rho} \exp\left[-\frac{\mu_1 x + \mu_2 y_i}{1-\rho}\right] I_0\left(\frac{2(\mu_1 x \mu_2 y_i)^{1/2}}{1-\rho}\right) dx \\ &= \int_{x_{r:n}}^{\infty} \sum_{k=0}^{\infty} \pi(k,\rho) g_{k+1}\left(x; \frac{\mu_1}{1-\rho}\right) g_{k+1}\left(y_i; \frac{\mu_2}{1-\rho}\right) dx \\ &= \sum_{k=0}^{\infty} \left\{\pi(k,\rho) \int_{x_{r:n}}^{\infty} g_{k+1}\left(x; \frac{\mu_1}{1-\rho}\right) dx \cdot g_{k+1}\left(y_i; \frac{\mu_2}{1-\rho}\right)\right\} \\ &= \sum_{k=0}^{\infty} \left\{\pi(k,\rho)\left(1 - G_{k+1}\left(x_{r:n}; \frac{\mu_1}{1-\rho}\right)\right) g_{k+1}\left(y_i; \frac{\mu_2}{1-\rho}\right)\right\}, \end{aligned} \tag{2.5}$$

where $G_{k+1}(x; b)$ is the Gamma$(k+1, b)$ CDF.

- **Case 3:** $c_i = 3$ if x_i is observed before or at the ending time $x_{r:n}$ and y_i is truncated if $y_i > x_{r:n}$; i.e.,

$$(x^*_i, y^*_i) = (x_i, +) \quad \text{if } x_i \leq x_{r:n} \text{ and } y_i > x_{r:n}.$$

The corresponding likelihood of (x^*_i, y^*_i) is

$$f_3(x_i, y_i) = \int_{x_{r:n}}^{\infty} f(x_i, y_i)\, dy$$

$$
\begin{aligned}
&= \int_{x_{r:n}}^{\infty} \frac{\mu_1\mu_2}{1-\rho} \exp\left[-\frac{\mu_1 x_i + \mu_2 y}{1-\rho}\right] I_0\left(\frac{2(\mu_1 x_i \mu_2 y)^{1/2}}{1-\rho}\right) dy \\
&= \int_{x_{r:n}}^{\infty} \sum_{k=0}^{\infty} \pi(k,\rho) g_{k+1}\left(x_i; \frac{\mu_1}{1-\rho}\right) g_{k+1}\left(y; \frac{\mu_2}{1-\rho}\right) dy \\
&= \sum_{k=0}^{\infty} \left\{\pi(k,\rho) g_{k+1}\left(x_i; \frac{\mu_1}{1-\rho}\right)\left(1 - G_{k+1}\left(x_{r:n}; \frac{\mu_2}{1-\rho}\right)\right)\right\}. \qquad (2.6)
\end{aligned}
$$

- **Case 4:** $c_i = 4$ if $x_i > x_{r:n}$ and $y_i > x_{r:n}$; i.e.,

$$
(x_i^*, y_i^*) = (+,+) \quad \text{if } x_i^* > x_{r:n} \text{ and } y_i^* > x_{r:n}.
$$

The corresponding likelihood of (x_i^*, y_i^*) is derived as follows.

$$
\begin{aligned}
f_4(x_i, y_i) &= \int_{x_{r:n}}^{\infty} \int_{x_{r:n}}^{\infty} f(x,y)\, dy\, dx \\
&= \int_{x_{r:n}}^{\infty} \int_{x_{r:n}}^{\infty} \frac{\mu_1\mu_2}{1-\rho} \exp\left[-\frac{\mu_1 x + \mu_2 y}{1-\rho}\right] I_0\left(\frac{2(\mu_1 x \mu_2 y)^{1/2}}{1-\rho}\right) dy\, dx \\
&= \int_{x_{r:n}}^{\infty} \int_{x_{r:n}}^{\infty} \sum_{k=0}^{\infty} \pi(k,\rho) g_{k+1}\left(x; \frac{\mu_1}{1-\rho}\right) g_{k+1}\left(y; \frac{\mu_2}{1-\rho}\right) dy\, dx \\
&= \sum_{k=0}^{\infty} \left\{\pi(k,\rho) \int_{x_{r:n}}^{\infty} g_{k+1}\left(x; \frac{\mu_1}{1-\rho}\right) dx \int_{x_{r:n}}^{\infty} g_{k+1}\left(y; \frac{\mu_2}{1-\rho}\right) dy\right\} \\
&= \sum_{k=0}^{\infty} \left\{\pi(k,\rho)\left(1 - G_{k+1}\left(x_{r:n}; \frac{\mu_1}{1-\rho}\right)\right)\left(1 - G_{k+1}\left(x_{r:n}; \frac{\mu_2}{1-\rho}\right)\right)\right\}. \qquad (2.7)
\end{aligned}
$$

A flowchart in Figure 1 shows how to process a complete DBVE dataset into a modified Type-II censored sample D^*.

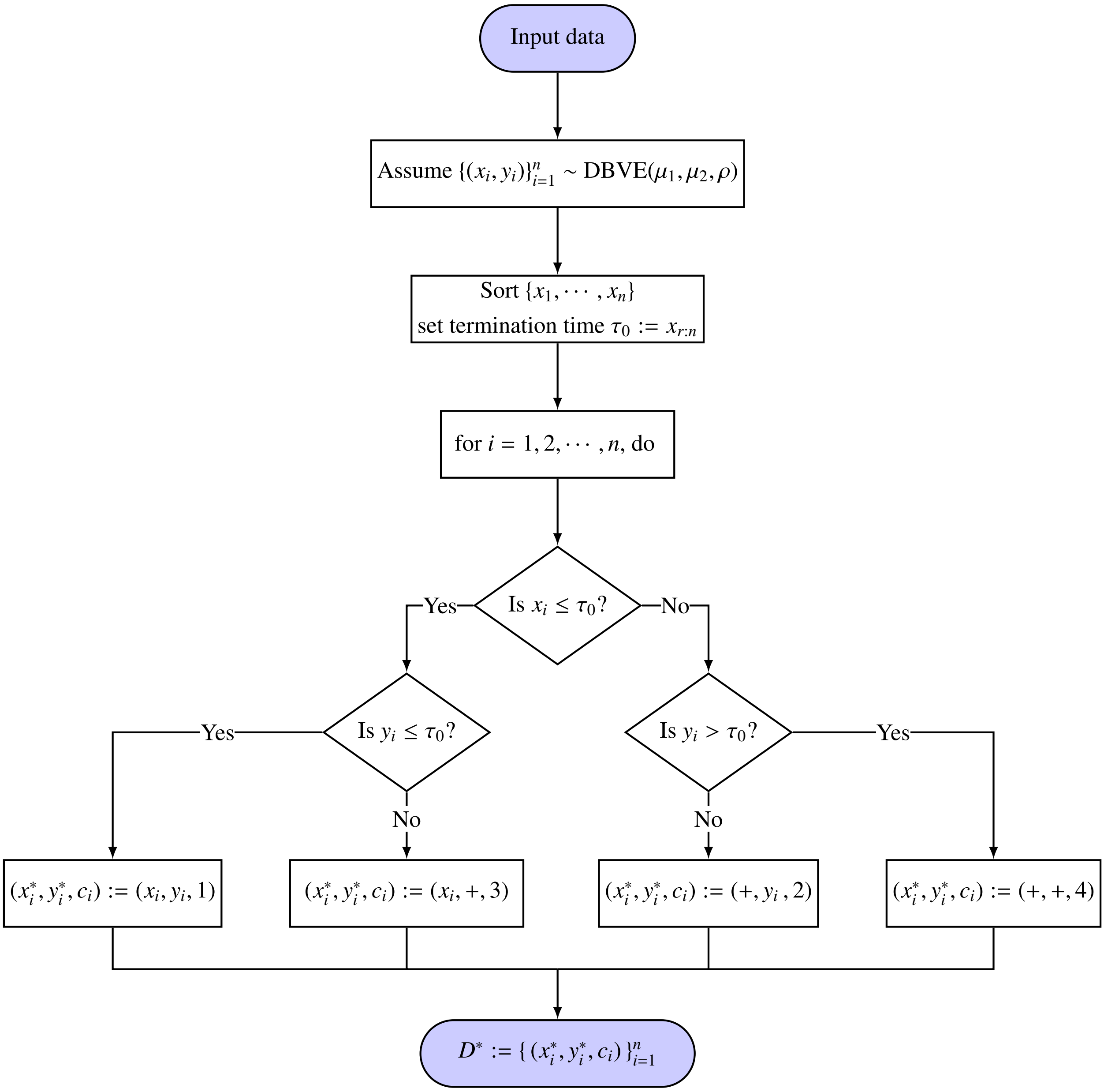

Figure 1. The data preprocessing of a modified Type-II censored sample D^*.

2.3. Likelihood function of modified Type-II censored sample

The likelihood function based on $D^* = \{(x_i^*, y_i^*, c_i), i = 1, 2, \ldots, n\}$, which is the modified Type-II censored data, can be written as the product of the likelihood of each bivariate observation (x_i^*, y_i^*) defined in Cases 1–4 above:

$$L(\mu_1, \mu_2, \rho | D^*) = \prod_{i=1}^{n} \sum_{m=1}^{4} \left\{ f_m(x_i^*, y_i^*) \, \mathrm{I}(c_i = m) \right\}, \tag{2.8}$$

where the indicator function is

$$\mathrm{I}(c_i = m) = \begin{cases} 1 & \text{if } c_i = m, \\ 0 & \text{otherwise,} \end{cases}$$

and $f_m(x_i^*, y_i^*)$, $m = 1, 2, 3, 4$, are defined in Eqs (2.4)–(2.7), respectively. Equivalently, the likelihood of D^* can be presented as

$$L(\mu_1, \mu_2, \rho | D^*) = \prod_{i \in S_1} f_1(x_i^*, y_i^*) \cdot \prod_{i \in S_2} f_2(x_i^*, y_i^*) \cdot \prod_{i \in S_3} f_3(x_i^*, y_i^*) \cdot \prod_{i \in S_4} f_4(x_i^*, y_i^*), \tag{2.9}$$

where $S_m = \{i | c_i = m, i = 1, 2, 3, \ldots, n\}$ for $m = 1, 2, 3, 4$, respectively. Note that the modified Bessel function of the first kind of order zero, $I_0(z)$, can be evaluated by using numerical algorithms provided in the statistical or mathematical software packages, for example, the function `besselI(z, nu = 0)` in the statistical language R [35]. Moreover, the likelihood functions in Eqs (2.5)–(2.7) can be approximated by their finite partial sums with L terms such that the $(L + 1)$-th term in the summation is less than ϵ, where ϵ is a very small positive quantity, due to the following theorem.

Theorem 1. *Let* $\Sigma_2(L), \Sigma_3(L)$*, and* $\Sigma_4(L)$ *be the partial sums of the first* L *terms of* $f_2(x_i, y_i)$*,* $f_3(x_i, y_i)$*, and* $f_4(x_i, y_i)$ *in Eqs (2.5)–(2.7), respectively, and* $R_2(L)$*,* $R_3(L)$*, and* $R_4(L)$ *are their corresponding remainder terms. Then, we have*

$$\begin{aligned} R_2(L) &= \sum_{k=L+1}^{\infty} \left\{ \pi(k, \rho) \left(1 - G_{k+1}\left(x_{r:n}; \frac{\mu_1}{1-\rho} \right) \right) g_{k+1}\left(y_i; \frac{\mu_2}{1-\rho} \right) \right\} = o\left(f_2(x_i, y_i) \right), \\ R_3(L) &= \sum_{k=L+1}^{\infty} \left\{ \pi(k, \rho) g_{k+1}\left(x_i; \frac{\mu_2}{1-\rho} \right) \left(1 - G_{k+1}\left(x_{r:n}; \frac{\mu_2}{1-\rho} \right) \right) \right\} = o\left(f_3(x_i, y_i) \right), \\ R_4(L) &= \sum_{k=L+1}^{\infty} \left\{ \pi(k, \rho) \left(1 - G_{k+1}\left(x_{r:n}; \frac{\mu_1}{1-\rho} \right) \right) \left(1 - G_{k+1}\left(x_{r:n}; \frac{\mu_2}{1-\rho} \right) \right) \right\} = o\left(f_4(x_i, y_i) \right). \end{aligned}$$

Proof. Given $x > 0$, $\mu > 0$, and $0 < \rho < 1$, both $g_{k+1}\left(x; \frac{\mu}{1-\rho}\right)$ and $\left(1 - G_{k+1}\left(x; \frac{\mu}{1-\rho}\right)\right)$ are decreasing functions of k, for $k \geq L_0$, where, L_0, a large positive integer. Therefore, for any given $L > L_0$,

$$\begin{aligned} 0 \leq \frac{R_2(L)}{f_2(x_i, y_i)} &\leq \frac{\sum_{k=L+1}^{\infty} \left\{ \pi(k, \rho) \left(1 - G_{k+1}\left(x_{r:n}; \frac{\mu_1}{1-\rho} \right) \right) g_{k+1}\left(y_i; \frac{\mu_2}{1-\rho} \right) \right\}}{\sum_{k=1}^{L} \left\{ \pi(k, \rho) \left(1 - G_{k+1}\left(x_{r:n}; \frac{\mu_1}{1-\rho} \right) \right) g_{k+1}\left(y_i; \frac{\mu_2}{1-\rho} \right) \right\}} \\ &\leq \frac{\sum_{k=L+1}^{\infty} \pi(k, \rho) \left(1 - G_{L+1}(x_{r:n}; \frac{\mu_1}{1-\rho}) \right) g_{L+1}(y_i; \frac{\mu_2}{1-\rho})}{\sum_{k=1}^{L} \pi(k, \rho) \left(1 - G_{L+1}(x_{r:n}; \frac{\mu_1}{1-\rho}) \right) g_{L+1}(y_i; \frac{\mu_2}{1-\rho})} \\ &= \frac{\sum_{k=L+1}^{\infty} (1-\rho)\rho^{k-1}}{\sum_{k=1}^{L} (1-\rho)\rho^{k-1}} \\ &= \frac{\rho^L}{1 - \rho^{L+1}}. \end{aligned}$$

Therefore, we have

$$\lim_{L \to \infty} \frac{R_2(L)}{f_2(x_i, y_i)} = 0.$$

Similarly, we have

$$\lim_{L\to\infty}\frac{R_3(L)}{f_3(x_i,y_i)}=0 \text{ and } \lim_{L\to\infty}\frac{R_4(L)}{f_4(x_i,y_i)}=0.$$

□

As $L\to\infty$ with fixed $0<\rho<1$, the remainders $R_2(L)$, $R_3(L)$, and $R_4(L)$ decay geometrically to zero. Hence, for large L, the partial sums $\Sigma_2(L)$, $\Sigma_3(L)$, and $\Sigma_4(L)$ provide sufficiently accurate approximations to $f_2(x_i,y_i)$, $f_3(x_i,y_i)$, and $f_4(x_i,y_i)$. In our numerical studies, $L=50$ was adequate for the likelihood calculations. Hence, the maximum likelihood estimates of the model parameters μ_1, μ_2, and ρ can be obtained by direct maximization of the likelihood function in Eq (2.8) or (2.9) based on the observed data D^*.

3. Data augmentation of censored data

For parameter estimation under the modified Type-II censoring scheme, we can view the censored data as latent variables, and the corresponding pseudo-observations greater than $X_{r:n}$ can be imputed by simulation. Then, the pseudo-complete data $\hat{D}=\{(\hat{x}_i,\hat{y}_i), i=1,2,\ldots,n\}$ can be used in both the likelihood and Bayesian inferential approaches.

Specifically, given the modified Type-II censored sample $D^*=\{(x_i^*,y_i^*,c_i), i=1,2,\ldots,n\}$, the pseudo complete data $\hat{D}=\{(\hat{x}_i,\hat{y}_i), i=1,2,\ldots,n\}$ will be generated as follows.

(1) If $c_i=1$: $x_i^*\le x_{r:n}$ and $y_i^*\le x_{r:n}$. $(\hat{x}_i,\hat{y}_i)=(x_i^*,y_i^*)$; no data augmentation is necessary.
(2) If $c_i=2$: $x_i^*>x_{r:n}$ and $y_i^*\le x_{r:n}$. The latent variate $\hat{x}_i$ is simulated by utilizing Algorithm 1, established below.
(3) If $c_i=3$: $x_i^*\le x_{r:n}$ and $y_i^*>x_{r:n}$. The latent variate $\hat{y}_i$ is simulated by utilizing Algorithm 2, established below.
(4) If $c_i=4$: $x_i^*>x_{r:n}$ and $y_i^*>x_{r:n}$. The latent variates $(\hat{x}_i,\ \hat{y}_i)$ are simulated in Algorithm 3, established below.

Utilizing the joint PDF, $f(x,y)$, in Eq (2.1), the conditional distribution required for the rejection Algorithm 1, can be derived from the ratio of partial sums. The result is presented in the following theorem.

Theorem 2. Let $(X,Y)\sim f(x,y)$, and denote $H_2: X>x_{r:n}$ and $Y=\hat{y}_i\le x_{r:n}$. The conditional PDF of X given H_2 is given by

$$f_{H_2}(x)=\frac{\sum_{k=0}^{\infty}\pi(k,\rho)g_{k+1}(x;\frac{\mu_1}{1-\rho})g_{k+1}(\hat{y}_i;\frac{\mu_2}{1-\rho})}{\sum_{k=0}^{\infty}\pi(k,\rho)\left(1-G_{k+1}(x;\frac{\mu_1}{1-\rho})\right)g_{k+1}(\hat{y}_i;\frac{\mu_2}{1-\rho})},\ x>x_{r:n}. \tag{3.1}$$

Proof. Given $0\le\hat{y}_i\le x_{r:n}$, let $x>x_{r:n}$

$$F_{H_2}(x)=\Pr(X\le x|X>x_{r:n},Y=\hat{y}_i)=\frac{\Pr(X\le x,Y=\hat{y}_i)}{\Pr(x_{r:n}<X,Y=\hat{y}_i)}$$
$$\overset{(2.1)}{=}\frac{\int_{x_{r:n}}^{x}\sum_{k=0}^{\infty}\pi(k,\rho)g_{k+1}(s;\frac{\mu_1}{1-\rho})g_{k+1}(\hat{y}_i;\frac{\mu_2}{1-\rho})ds}{\int_{x_{r:n}}^{\infty}\sum_{k=0}^{\infty}\pi(k,\rho)g_{k+1}(s,\frac{\mu_1}{1-\rho})g_{k+1}(\hat{y}_i,\frac{\mu_2}{1-\rho})ds}$$

$$= \frac{\sum_{k=0}^{\infty}\pi(k,\rho)\left(G_{k+1}(x;\frac{\mu_1}{1-\rho}) - G_{k+1}(x_{r:n};\frac{\mu_1}{1-\rho})\right)g_{k+1}(\hat{y}_i;\frac{\mu_2}{1-\rho})}{\sum_{k=0}^{\infty}\pi(k,\rho)\left(1 - G_{k+1}(x_{r:n};\frac{\mu_1}{1-\rho})\right)g_{k+1}(\hat{y}_i;\frac{\mu_2}{1-\rho})}$$

$$f_{H_2}(x) = \frac{d}{dx}\frac{\sum_{k=0}^{\infty}\pi(k,\rho)\left(G_{k+1}(x;\frac{\mu_1}{1-\rho}) - G_{k+1}(x_{r:n};\frac{\mu_1}{1-\rho})\right)g_{k+1}(\hat{y}_i;\frac{\mu_2}{1-\rho})}{\sum_{k=0}^{\infty}\pi(k,\rho)\left(1 - G_{k+1}(x_{r:n};\frac{\mu_1}{1-\rho})\right)g_{k+1}(\hat{y}_i;\frac{\mu_2}{1-\rho})}$$

$$= \frac{\sum_{k=0}^{\infty}\pi(k,\rho)g_{k+1}(x;\frac{\mu_1}{1-\rho})\,g_{k+1}(\hat{y}_i;\frac{\mu_2}{1-\rho})}{\sum_{k=0}^{\infty}\pi(k,\rho)\left(1 - G_{k+1}(x_{r:n};\frac{\mu_1}{1-\rho})\right)g_{k+1}(\hat{y}_i;\frac{\mu_2}{1-\rho})}.$$

□

Similarly, the random variate of $\hat{x}_i$ can be simulated from a conditional distribution. Since the marginal distribution of X is an exponential distribution, we consider the truncated exponential distribution as the candidate distribution.

Algorithm 1 Generate $X = \hat{x}_i$ given $H_2 : X > x_{r:n}$ and $Y = \hat{y}_i \leq x_{r:n}$

Let $f_V(x;\mu_1)$, $x > x_{r:n}$, denote a truncated exponential distribution with PDF $f_V(x;\mu_1) = \mu_1 e^{-\mu_1(x-x_{r:n})}$, $x > x_{r:n}$, and $M = \sup_{x\leq x_{r:n}} \frac{f_{H_2}(x)}{f_V(x)}$.

1: Propose $V \sim f_V(x;\mu_1)$.
2: Generate $U \sim \text{Uniform}(0,1)$.
3: If $U \leq \frac{f_{H_2}(V)}{Mf_V(V;\mu_1)}$, then 'stop'; otherwise, go back to Step 2.
4: Set $\hat{x}_i \leftarrow V$.

Here, we provide the justification of Algorithm 1: Let $x > x_{r:n}$. Then we have

$$\begin{aligned} F_V(x) &= \Pr(V \leq x \mid \text{stop at Step 3}) = \Pr\left(V \leq x \mid U \leq \frac{f_{H_2}(V)}{Mf_V(V)}\right) \\ &= \frac{\Pr(V \leq x, U \leq \frac{f_{H_2}(V)}{f_V(V)})}{\Pr(U \leq \frac{f_{H_2}(V)}{Mf_V(V)}, V \leq x_{r:n})} = \frac{\int_0^{\frac{f_{H_2}(v)}{Mf_V(v)}}\int_{x_{r:n}}^{x} f_V(v)dvdu}{\int_0^{\frac{f_{H_2}(v)}{Mf_V(v)}}\int_{x_{r:n}}^{\infty} f_V(v)dvdu} \\ &= \frac{\int_{x_{r:n}}^{x}\frac{1}{M}\frac{f_{H_2}(v)}{f_V(v)}f_V(v)f_V(v)dv}{\int_{x_{r:n}}^{\infty}\frac{1}{M}\frac{f_{H_2}(v)}{f_V(v)}f_V(v)dv} = \int_{x_{r:n}}^{x} f_{H_2}(v)dv. \end{aligned} \tag{3.2}$$

Hence, we can obtain

$$f_V(x) = \frac{d}{dx}\int_{x_{r:n}}^{x} f_{H_2}(v)dv = f_{H_2}(x). \tag{3.3}$$

The conditional PDF of $Y > x_{r:n}$ given $X \leq x_{r:n}, Y > x_{r:n}$ can be obtained based on the following theorem.

Theorem 3. Denote $H_3 : X = \hat{x}_i \leq x_{r:n}$ and $Y > x_{r:n}$. The conditional PDF of X given H_3 is given by

$$f_{H_3}(y) = \frac{\sum_{k=0}^{\infty}\pi(k,\rho)g_{k+1}\left(\hat{x}_i;\frac{\mu_1}{1-\rho}\right)g_{k+1}\left(y;\frac{\mu_2}{1-\rho}\right)}{\sum_{k=0}^{\infty}\pi(k,\rho)g_{k+1}\left(\hat{x}_i;\frac{\mu_1}{1-\rho}\right)\left(1 - G_{k+1}\left(x_{r:n};\frac{\mu_2}{1-\rho}\right)\right)},\ y > x_{r:n}. \tag{3.4}$$

The random variate $\hat{y}_i$ can be simulated from a similar distribution that has the same support as the conditional PDF $f_{H_3}(x)$.

Algorithm 2 Generate $\hat{Y} = \hat{y}_i$ given $H_3 : X = \hat{x}_i \leq x_{r:n}, Y > x_{r:n}$:

Let $f_W(y;\mu_2)$, denote a truncated exponential distribution $f_W(y;\mu_2) = \mu_2 e^{-\mu_2(y-x_{r:n})}$, $xy > x_{r:n}$, and $M = \max\limits_{y>x_{r:n}} \dfrac{f_{H_3}(y)}{f_W(y)}$.

1: Propose $W \sim f_W(y;\mu_2)$.
2: Generate $U \sim \text{Uniform}(0,1)$.
3: If $U \leq \dfrac{f_{H_3}(W)}{M f_W(W;\,\mu_2)}$, then 'stop'; otherwise, go back to Step 2.
4: Set $\hat{y}_i \leftarrow W$.

By the same argument, the joint PDF of (X, Y) given $X > x_{r:n}$ and $Y > x_{r:n}$ can be obtained in the following theorem.

Theorem 4. Denote $H_4 : X > x_{r:n}, Y > x_{r:n}$. The conditional PDF of X and Y given H_4 is

$$f_{H_4}(x,y) = \frac{\sum_{k=0}^{\infty} \pi(k,\rho) g_{k+1}(x; \frac{\mu_1}{1-\rho}) g_{k+1}(y; \frac{\mu_2}{1-\rho})}{\sum_{k=0}^{\infty} \pi(k,\rho) \left(1 - G_{k+1}(x_{r:n}; \frac{\mu_1}{1-\rho})\right)\left(1 - G_{k+1}(x_{r:n}; \frac{\mu_2}{1-\rho})\right)}, \ x > x_{r:n}, y > x_{r:n}. \tag{3.5}$$

The random variates $\hat{x}_i, \hat{y}_i$ can be simulated from a similar distribution that has the same support as the conditional PDF $f_{H_3}(x)$.

Algorithm 3 Generate $X = \hat{x}_i, Y = \hat{y}_i$ given $H_4 : X > x_{r:n}, Y > x_{r:n}$

Let $f_V(v;\mu_1)$ and $f_W(y;\mu_2)$ denote two independent truncated exponential distributions greater than $x_{r:n}$ and $M = \max\limits_{x,y \leq x_{r:n}} \dfrac{f_{H_4}(x,y)}{f_V(x) f_W(y)}$.

1: Propose $V \sim f_W(x;\mu_1)$ and $W \sim f_W(y;\mu_2)$.
2: Generate $U \sim \text{Uniform}(0,1)$.
3: If $U \leq \dfrac{f_{H_4}(V,W)}{M f_V(V;\mu_1) f_W(W;\mu_2)}$, then 'stop'; otherwise, go back to Step 2.
4: $\hat{x}_i \leftarrow V$ and $\hat{y}_i \leftarrow W$.

4. Bayesian inference

For the Bayesian inference, μ_1, μ_2, and ρ are assumed to be stochastic independent, and μ_l, $l = 1, 2$, have gamma priors with PDF

$$\pi_l(\mu_l;\alpha_l,\lambda_l) = \frac{1}{\Gamma(\alpha_l)\lambda_l^{\alpha_l}} \mu_l^{\alpha_l - 1} \exp\left(-\frac{\mu_l}{\lambda_l}\right), \quad \mu_l > 0, \tag{4.1}$$

with $\alpha_l > 0$ and $\lambda_l > 0$ as shape and scale hyperparameters, respectively, and ρ has beta prior with PDF

$$\pi_3(\rho;\beta_1,\beta_2) = \frac{\Gamma(\beta_1 + \beta_2)}{\Gamma(\beta_1)\Gamma(\beta_2)} \rho^{\beta_1 - 1}(1-\rho)^{\beta_2 - 1}, \quad 0 < \rho \leq 1, \tag{4.2}$$

with $\beta_1 > 0$ and $\beta_2 > 0$ be the hyperparameters. Note that the gamma distribution is a natural prior for exponential-type likelihoods because it is conjugate to them, yielding gamma posteriors with simple closed-form updates and hyperparameters that admit an intuitive interpretation as prior exposure time and event counts. The aforementioned priors were also considered by [19, 28] for the Bayesian inference of the DBVE(μ_1, μ_2, ρ) distribution. To specify the hyperparameters (α_l, λ_l) in the gamma prior for μ_l, recall that the mean and variance are $E(\mu_l) = \alpha_l/\lambda_l$ and $Var(\mu_l) = \alpha_l/\lambda_l^2$, respectively. A diffuse (essentially non-informative) prior can be obtained by choosing a very large variance while keeping the mean at a convenient reference value. Following a common practical choice used in Bayesian software packages for posterior computation (see, for example, [40]), we take a small value, such as $\alpha_l = \lambda_l = 0.001$, which yields a standardized prior mean of 1 together with a very large prior variance.

Based on these prior distributions, given a modified Type-II censored sample, D^*, the joint posterior PDF of μ_1, μ_2, and ρ can be derived via Eqs (2.8), (4.1), and (4.2) as

$$\Pi(\mu_1,\mu_2,\rho|D) \;\propto\; \left\{\prod_{i=1}^{n}\sum_{k=1}^{4}\{f_k(x_i^*,y_i^*)\,\mathrm{I}(c_i=k)\}\right\}\pi_1(\mu_1;\alpha_1,\lambda_1)\pi_2(\mu_2;\alpha_2,\lambda_2)\pi_3(\rho;\beta_1,\beta_2). \quad (4.3)$$

Due to the complexity of the joint posterior, closed-form expressions for the posterior distributions of μ_1, μ_2, and ρ are unavailable, and high-dimensional numerical integration is also impractical. We therefore employ the Markov-chain Monte Carlo (MCMC) approach to approximate the posterior densities. We implement MCMC via two approaches. The first approach uses the likelihood function based on the censored sample, as described in Eq (2.8). The second approach uses a data augmentation technique, as detailed in Section 3.

4.1. MCMC based on likelihood of censored data

The first strategy utilizes the likelihood of proposed modified Type-II censored data. In this case, the Metropolis-Hastings (M-H) within Gibbs sampler [14, 16, 33] will be utilized to draw posterior samples for μ_1, μ_2, and ρ. That is, the Markov chain $\{\theta^{(\ell)}, \ell = 1, 2, \ldots\}$ of a given parameter, θ, will be created via utilizing the M-H algorithm. Given θ, let $q(\theta^{(*)}|\theta)$ be a proposed conditional transition PDF for getting $\theta^{(*)}$. Hence, the current state value, $\theta^{(\ell-1)}$, of θ, a candidate for the next state, $\theta^{(*)}$, can be simulated by $q(\theta^{(*)}|\theta^{(\ell-1)})$, with probability of $\min\left\{1, \dfrac{\Pi(\theta^{(*)}|D)q(\theta^{(\ell)}|\theta^{(*)})}{\Pi(\theta^{(\ell)}|D)q(\theta^{(*)}|\theta^{(\ell)})}\right\}$; otherwise, the next state $\theta^{(\ell)} = \theta^{(\ell-1)}$. Here, let $q_1(\mu_1^{(b)}|\mu_1^{(a)})$, $q_2(\mu_2^{(b)}|\mu_2^{(a)})$, and $q_3(\rho^{(b)}|\rho^{(a)})$ be the transition probabilities from $\mu_1^{(a)}$ to $\mu_1^{(b)}$, from $\mu_2^{(a)}$ to $\mu_2^{(b)}$, and from $\rho^{(a)}$ to $\rho^{(b)}$, respectively, and $U(0, 1)$ be the continuous uniform distribution over the interval $(0, 1)$. Specifically, the sequence $\{\mu_1^{(\ell)}, \mu_2^{(\ell)}, \rho^{(\ell)}\}$ forms a Markov chain generated by the iterative steps given below:

(1) Generate $\mu_1^{(*)}$ from $q_1(\mu_1^{(*)}|\mu_1^{(\ell)})$ and u_1 from $U(0, 1)$ independently, and then obtain

$$\mu_1^{(\ell+1)} = \begin{cases} \mu_1^{(\ell)}, & \text{if } u_1 > \min\left\{1, \dfrac{\Pi_1(\mu_1^{(*)}|D,\mu_2^{(\ell)},\rho^{(\ell)})q_1(\mu_1^{(\ell)}|\mu_1^{(*)})}{\Pi_1(\mu_1^{(\ell)}|D,\mu_2^{(\ell)},\rho^{(\ell)})q_1(\mu_1^{(*)}|\mu_1^{(\ell)})}\right\}, \\ \mu_1^{(*)}, & \text{otherwise}. \end{cases}$$

where the conditional posterior density function $\Pi_1(\mu_1|D,\mu_2,\rho) \propto \Pi(\mu_1,\mu_2,\rho|D)\pi_1(\mu_1;\alpha_1,\lambda_1)$.

(2) Generate $\mu_2^{(*)}$ from $q_2(\mu_2^{(*)}|\mu_2^{(\ell)})$ and u_2 from $U(0,1)$ independently, and then obtain

$$\mu_2^{(\ell+1)} = \begin{cases} \mu_2^{(\ell)}, & \text{if } u_2 > \min\left\{1, \dfrac{\Pi_2(\mu_2^{(*)}|D, \mu_1^{(\ell+1),}\rho^{(\ell)})q_2(\mu_2^{(\ell)}|\mu_2^{(*)})}{\Pi_2(\mu_2^{(\ell)}|D, \mu_1^{(\ell+1),}\rho^{(\ell)})q_2(\mu_2^{(*)}|\mu_2^{(\ell)})}\right\}, \\ \mu_2^{(*)}, & \text{otherwise.} \end{cases}$$

where the conditional posterior density function $\Pi_1(\mu_2|D,\mu_1,\rho) \propto \Pi(\mu_1,\mu_2,\rho|D)\pi_2(\mu_2;\alpha_2,\lambda_2)$.

(3) Generate $\rho^{(*)}$ from $q_3(\rho^{(*)}|\rho^{(\ell)})$ and u_3 from $U(0,1)$ independently, and then obtain

$$\rho^{(\ell+1)} = \begin{cases} \rho^{(\ell)}, & \text{if } u_3 > \min\left\{1, \dfrac{\Pi_3(\rho^{(*)}|D, \mu_1^{(\ell+1)}, \mu_2^{(\ell)})q_3(\rho^{(\ell)}|\rho^{(*)})}{\Pi_3(\rho^{(\ell)}|D, \mu_1^{(\ell+1),}\mu_2^{(\ell+1)})q_3(\rho^{(*)}|\rho^{(\ell)})}\right\}, \\ \rho^{(*)}, & \text{otherwise.} \end{cases}$$

where the conditional posterior density function

$$\Pi_3(\rho|D,\mu_1,\mu_2) \propto \Pi(\mu_1,\mu_2,\rho|D)\pi_3(\rho;\beta_1,\beta_2). \tag{4.4}$$

The iterative scheme is run for N iterations, starting from the initial values $\mu_1^{(0)}$, $\mu_2^{(0)}$, and $\rho^{(0)}$. After a prespecified burn-in period N_b ($< N$), we retain the sequences $\left\{\mu_1^{(\ell)}; \ell = N_b+1, N_b+2, \ldots, N\right\}$, $\left\{\mu_2^{(\ell)}; \ell = N_b+1, \ldots, N\right\}$, and $\left\{\rho^{(\ell)}; j = N_b+1, \ldots, N\right\}$. These sequences are used to compute the Bayes point estimates for parameters μ_1, μ_2, and ρ under various loss functions. For example:

- Under squared-error loss, the Bayes estimate is the posterior mean, i.e., the mean of the sequence after burn-in;
- Under absolute-error loss, the Bayes estimate is the posterior median, i.e., the median of the sequence after burn-in.

4.2. *MCMC with data augmentation*

This strategy utilizes the data augmentation presented in Section 3 to obtain a pseudo-complete sample for the Bayesian inference. Let $\pi_i(\mu_i;\alpha_i,\lambda_i)$, $i = 1,2$ and $\pi_3(\rho,\beta_1,\beta_2)$ be the prior distributions in Eqs (4.1) and (4.2). The joint posterior PDF of μ_1, μ_2 and ρ, based on a pseudo-complete sample $\hat{D}$, can be expressed by

$$\begin{aligned} L(\mu_1,\mu_2,\rho|\hat{D}) &\propto L(\mu_1,\mu_2,\rho)\cdot\ \pi_1(\mu_1;\alpha_1,\lambda_1)\,\pi_2(\mu_2;\alpha_2,\lambda_2)\,\pi_3(\rho;\beta_1,\beta_2) \\ &\propto \prod_{i=1}^{n}\left[\frac{\mu_1\mu_2}{1-\rho}\exp\left\{-\frac{\mu_1\hat{x}_i+\mu_2\hat{y}_i}{1-\rho}\right\} I_0\left\{\frac{2(\mu_1\hat{x}_i\mu_2\hat{y}_i\rho)^{1/2}}{1-\rho}\right\}\right] \\ &\quad \cdot\pi_1(\mu_1;\alpha_1,\lambda_1)\,\pi_2(\mu_2;\alpha_2,\lambda_2)\,\pi_3(\rho;\beta_1,\beta_2). \end{aligned} \tag{4.5}$$

By integrating the PDF in Eq (4.5) with respect to μ_2, and ρ, the marginal posterior PDF of μ_1, which involves the data $\{\hat{x}_i,\hat{y}_i\}_{i=1}^n$ and the product of the likelihood in Eq (1.1), can be derived. Alternatively, it is simpler and more effective to use the fact that μ_1 is gamma distributed, given the pseudo-complete data $\hat{D}$, in view of [19]. With the gamma conjugate prior π_1, the marginal posterior PDF of μ_1 can be presented as

$$L_1(\mu_1|\hat{D}) \propto \left\{\prod_{i=1}^{n}\mu_1\exp(-\mu_1\hat{x}_i)\right\}\mu_1^{\alpha_1-1}\exp(-\lambda_1\mu_1)$$

$$\propto \mu_1^{n+\alpha_1-1} \exp\left[-\mu_1\left(\sum_{i=1}^{n} \hat{x}_i + \lambda_1\right)\right]. \tag{4.6}$$

Similarly, the marginal posterior PDF of μ_2 is

$$L_2(\mu_2|\hat{D}) \propto \left\{\prod_{i=1}^{n} \mu_2 \exp(-\mu_2 \hat{y}_i)\right\} \mu_2^{\alpha_2-1} \exp(-\lambda_2\mu_2)$$
$$\propto \mu_2^{n+\alpha_2-1} \exp\left[-\mu_2\left(\sum_{i=1}^{n} \hat{y}_i + \lambda_2\right)\right], \tag{4.7}$$

Observe that the pseudo-complete data $\hat{D} = \{D^*, \tilde{D}\}$, where D^* is the collection of the observations and $\tilde{D}$ is the augmented data imputed from the previous state values of the model parameters (μ_1, μ_2, ρ). That is, $\tilde{D}$ contains prior information about the model parameters. Updating all model parameters given the pseudo-complete data may cause a non-identifiability problem. It is therefore recommended to update the marginal posterior PDF of ρ given μ_1, μ_2, and the observations D by Π_3 in Eq (4.4).

The Markov chain $\{\mu_1^{(b)}, \mu_2^{(b)}, \rho^{(b)}\}_{b=1}^{N}$, where N is a large number of periods, can be generated through the iterative processes addressed below.

Step 0. (Data Augmentation) The pseudo-complete data, $\hat{D}$, can be obtained by imputing the latent variable X or Y from D^* through Algorithms 1–3 established in Section 3.

Step 1. Set $b = 0$, and start with initial values, $\mu_1^{(0)}$, $\mu_2^{(0)}$, and $\rho^{(0)}$.

Step 2. By utilizing Eq (4.6), draw the $(b + 1)$th state value of μ_1 from gamma distribution with two parameters $n + \alpha_1$ and $\sum_{i=1}^{n} \hat{x}_i + \lambda_1$; i.e.,

$$\mu_1^{(b+1)} \sim \text{Gamma}\left(n + \alpha_1, \sum_{i=1}^{n} \hat{x}_i + \lambda_1\right).$$

Step 3. By utilizing (4.7), draw the $(b + 1)$th state value of μ_2 from gamma distribution with two parameters $n + \alpha_2$ and $\sum_{i=1}^{n} \hat{y}_i + \lambda_2$; i.e.,

$$\mu_2^{(b+1)} \sim \text{Gamma}\left(n + \alpha_1, \sum_{i=1}^{n} \hat{y}_i + \lambda_2\right).$$

Step 4. Generate $\rho^{(b+1)}$ from $q_3(\rho^{(*)}|\rho^{(b)})$ and u_3 from $U(0, 1)$ independently, and set

$$\rho^{(b+1)} = \begin{cases} \rho^{(b)} & \text{if } u_3 > \min\left\{1, \dfrac{\Pi_3(\rho^{(*)}|\hat{D}, \mu_1^{(b+1)}, \mu_2^{(b+1)})\, q_3(\rho^{(b)}|\rho^{(*)})}{\Pi_3(\rho^{(b)}|\hat{D}, \mu_1^{(b+1),} \mu_2^{(b+1)})\, q_3(\rho^{(*)}|\rho^{(b)})}\right\}, \\ \rho^{(*)} & \text{otherwise.} \end{cases}$$

Step 5. The $(b+1)$th state value of the latent variable δ can be thought of the realization of δ in Eq (2.2) by plugging the parameters $\mu_1 = \mu_1^{(b+1)}$, $\mu_2 = \mu_2^{(b+1)}$, and $\rho = \rho^{(b+1)}$. Specifically, given $\mu_1^{(b+1)}$, $\mu_2^{(b+1)}$, and $\rho^{(b+1)}$,

$$\delta^{(b+1)} = \frac{(1 - \rho^{(b+1)})\mu_1^{(b+1)}}{\mu_1^{(b+1)} + \mu_2^{(b+1)}} \sum_{k=0}^{\infty} \frac{(\rho^{(b+1)})^k (\mu_1^{(b+1)})^k}{\Gamma(k + 1)(\mu_1^{(b+1)} + \mu_2^{(b+1)})^k} \sum_{i=0}^{k} \frac{\Gamma(i + k + 1)}{\Gamma(i + 1)} \frac{(\mu_2^{(b+1)})^i}{(\mu_1^{(b+1)} + \mu_2^{(b+1)})^i}. \tag{4.8}$$

Step 6. Let $b = b + 1$. Move to Step 2, if $b < N$; otherwise, stop.

4.3. Bayes estimates

Let $\{\mu_1^{(j)}\}_{j=1}^N, \{\mu_1^{(j)}\}_{j=1}^N, \{\rho^{(j)}\}_{j=1}^N$, and $\{\delta^{(j)}\}_{j=1}^N$ be the realizations of Markov chain of μ_1, μ_2, ρ, and δ by either the MCMC with M-H algorithm in Section 4.1, or the data augmentation method in Section 4.2, respectively. If the loss function is the squared-error loss function, then the Bayes estimates of μ_1, μ_2, ρ, and δ are the means of empirical distribution from $\{\mu_1^{(j)}\}_{j=1}^N, \{\mu_1^{(j)}\}_{j=1}^N, \{\rho^{(j)}\}_{j=1}^N$, and $\{\delta^{(j)}\}_{j=1}^N$ after burn-in period N_b, i.e., the Bayes estimates of μ_1, μ_2, and ρ under the squared loss function are, respectively,

$$\bar{\mu}_1 = \frac{1}{N-N_b}\sum_{b=N_b+1}^{N}\mu_1^{(b)},\ \bar{\mu}_2 = \frac{1}{N-N_b}\sum_{b=N_b+1}^{N}\mu_2^{(b)},\ \bar{\rho} = \frac{1}{N-N_b}\sum_{b=N_b+1}^{N}\rho^{(b)},\ \bar{\delta} = \frac{1}{N-N_b}\sum_{j=N_b+1}^{N}\delta^{(j)}. \quad (4.9)$$

If the loss function is the absolute-error loss function, then the Bayes estimates are the medians of the empirical distribution of $\{\mu_1^{(j)}\}_{j=1}^N, \{\mu_1^{(j)}\}_{j=1}^N, \{\rho^{(j)}\}_{j=1}^N$, and $\{\delta^{(j)}\}_{j=1}^N$ after burn-in period N_b. Specifically, let $K = N - N_b$, $\mu_1^{([1])} \le \mu_1^{([2])} \le \ldots \le \mu_1^{([K])}$ be the ordered values of $\{\mu_1^{(j)}\}_{j=N_b+1}^N$, $\mu_2^{([1])} \le \mu_2^{([2])} \le \ldots \le \mu_2^{([K])}$ be the ordered values of $\{\mu_2^{(j)}\}_{j=N_b+1}^N$, $\rho^{([1])} \le \rho^{([2])} \le \ldots \le \rho^{([K])}$ be the ordered values of $\{\rho^{(j)}\}_{j=N_b+1}^N$, and $\delta^{([1])} \le \delta^{([2])} \le \ldots \le \delta^{([K])}$ be ordered values of $\{\delta^{(j)}\}_{j=N_b+1}^N$. Then the Bayes estimates of μ_1, μ_2, and ρ under the absolute-error loss function are, respectively,

$$\begin{cases}\hat{\mu}_1 = \frac{\mu_1^{([d])}+\mu_1^{([d+1])}}{2},\ \hat{\mu}_2 = \frac{\mu_2^{([d])}+\mu_2^{([d+1])}}{2},\ \hat{\rho} = \frac{\rho^{([d])}+\rho^{([d+1])}}{2},\ \hat{\delta} = \frac{\delta^{([d])}+\delta^{([d+1])}}{2} \text{ with } d = \frac{K}{2}, & \text{if } K \text{ is even,}\\ \hat{\mu}_1 = \mu_1^{([d])},\ \hat{\mu}_2 = \mu_2^{([d])},\ \hat{\rho} = \rho^{([d])},\ \hat{\delta} = \delta^{([d])} \text{ with } d = \frac{K+1}{2}, & \text{if } K \text{ is odd.}\end{cases}$$

5. Monte Carlo simulation study

In this section, a Monte Carlo simulation study is used to investigate the performance of proposed Bayesian estimation procedures for μ_1, μ_2, ρ, and δ based on D^* from DBVE(μ_1, μ_2, ρ). The modified Type-II censored samples D^* are generated from DBVE(μ_1, μ_2, ρ) with the following four parameter settings of (μ_1, μ_2, ρ):

S1. $\mu_1 = 0.5, \mu_2 = 0.5, \rho = 0.5$ ($\delta = 0.5000$),
S2. $\mu_1 = 1.0, \mu_2 = 1.2, \rho = 0.7$ ($\delta = 0.4178$),
S3. $\mu_1 = 1.0, \mu_2 = 0.8, \rho = 0.3$, ($\delta = 0.5662$),
S4. $\mu_1 = 2.0, \mu_2 = 1.2, \rho = 0.6$, ($\delta = 0.6889$),

and sample size settings (n, r) = (60, 30), (60,45), (100,50), (100,75), (200,100), (200,150), (500,250), and (500, 375).

We apply the Bayesian estimation procedures using MCMC with the exact likelihood function based on the censored sample in Section 4.1 (denoted as EL) and MCMC with data augmentation in Section 4.2 (denoted as DA). We consider the prior distributions

$$\pi_1(\mu_1) \sim \text{Gamma}\,(0.001, 0.001), \pi_2(\mu_2) \sim \text{Gamma}\,(0.001, 0.001), \pi_3(\rho) \sim \text{Beta}\,(0.01, 0.01). \quad (5.1)$$

Note that, in the above settings, the supremum value for the DA approach to apply algorithms 1–3 is $M = 3.0$. The mean squared errors (MSE) and bias of the Bayes estimates μ_1, μ_2, ρ, and the stress-strength reliability δ by the EL and DA approaches are computed based on 1000 simulations and they are presented in Tables 1–4 for parameter settings S1–S4 and censoring proportion r/n = 75%, 60%, or 50%.

Table 1. Simulated MSEs and bias (in parentheses) of the Bayes estimators of μ_1, μ_2, ρ, and δ under the modified Type-II censoring scheme with parameter setting S1 and censoring proportion $r/n = 75\%$.

$(\mu_1, \mu_2, \rho, \delta) = (0.5, 0.5, 0.5, 0.5)$, $r/n = 75\%$									
n	r	μ_1		μ_2		ρ		δ	
EL approach									
60	45	0.006384	(0.02731)	0.005661	(0.01486)	0.012334	(0.00363)	0.003208	(0.00379)
100	75	0.003864	(0.01744)	0.003456	(0.01269)	0.008604	(-0.03034)	0.001831	(0.00298)
200	150	0.001784	(0.00907)	0.001587	(0.00375)	0.004032	(-0.01591)	0.000998	(0.00354)
500	375	0.000671	(0.00338)	0.000699	(0.00352)	0.001624	(-0.00592)	0.000386	(-0.00011)
DA approach									
60	45	0.006980	(0.02119)	0.006229	(0.01485)	0.012204	(-0.03893)	0.003124	(0.01048)
100	75	0.003352	(0.00934)	0.003051	(0.00382)	0.008358	(-0.02716)	0.001729	(0.00558)
200	150	0.001800	(0.00728)	0.001719	(0.00337)	0.004090	(-0.01347)	0.000838	(0.00363)
500	375	0.000734	(0.00546)	0.000763	(0.00529)	0.001590	(-0.00709)	0.000321	(0.00051)

Table 2. Simulated MSEs and bias (in parentheses) of the Bayes estimators of μ_1, μ_2, ρ, and δ under the modified Type-II censoring scheme with parameter setting S1 and censoring proportion $r/n = 50\%$.

$(\mu_1, \mu_2, \rho, \delta) = (0.5, 0.5, 0.5, 0.5)$, $r/n = 50\%$									
n	r	μ_1		μ_2		ρ		δ	
EL approach									
60	30	0.012199	(0.04093)	0.008763	(0.01865)	0.018902	(-0.05395)	0.005203	(0.01317)
100	50	0.005726	(0.01796)	0.005333	(0.01551)	0.013356	(-0.04197)	0.003293	(0.00121)
200	100	0.002825	(0.01204)	0.002532	(0.00839)	0.007041	(-0.02293)	0.001544	(0.00224)
500	250	0.000955	(0.00546)	0.000993	(0.00473)	0.002355	(-0.00682)	0.000623	(0.00055)
DA approach									
60	30	0.010438	(0.02572)	0.009079	(0.00476)	0.017639	(-0.05334)	0.005489	(0.01769)
100	50	0.005973	(0.01995)	0.005441	(0.00392)	0.013380	(-0.03916)	0.003642	(0.01252)
200	100	0.002560	(0.00755)	0.002429	(-0.00116)	0.006397	(-0.02245)	0.001548	(0.00698)
500	250	0.001040	(0.00390)	0.001016	(0.00052)	0.002574	(-0.00758)	0.000614	(0.00293)

Table 3. Simulated MSEs and bias (in parentheses) of the Bayes estimators of μ_1, μ_2, ρ, and δ under the modified Type-II censoring scheme with parameter setting S2 and censoring proportion $r/n = 75\%$.

$(\mu_1, \mu_2, \rho, \delta) = (1.0, 1.2, 0.7, 0.4178)$, $r/n = 75\%$									
n	r	μ_1		μ_2		ρ		δ	
EL approach									
60	45	0.029143	(0.06289)	0.034766	(0.06553)	0.005672	(-0.03690)	0.003007	(0.00810)
100	75	0.016844	(0.04046)	0.021387	(0.03633)	0.003757	(-0.02313)	0.001959	(0.00660)
200	150	0.007233	(0.01671)	0.010532	(0.02079)	0.001692	(-0.01214)	0.000935	(0.00113)
500	375	0.002698	(0.00731)	0.003471	(0.00737)	0.000587	(-0.00484)	0.000379	(0.00099)
DA approach									
60	45	0.030288	(0.04638)	0.033920	(0.02035)	0.006824	(-0.04487)	0.002977	(0.02041)
100	75	0.015235	(0.03631)	0.019006	(0.02088)	0.003964	(-0.03047)	0.001831	(0.01375)
200	150	0.007154	(0.03045)	0.008667	(0.00834)	0.001451	(-0.01733)	0.000954	(0.01321)
500	375	0.003149	(0.02082)	0.003499	(0.00565)	0.000615	(-0.01049)	0.000351	(0.00859)

Table 4. Simulated MSEs and bias (in parentheses) of the Bayes estimators of μ_1, μ_2, ρ, and δ under the modified Type-II censoring scheme with parameter setting S3 and censoring proportion $r/n = 75\%$.

$(\mu_1, \mu_2, \rho, \delta) = (1.0, 0.8, 0.3, 0.5662)$, $r/n = 75\%$									
n	r	μ_1		μ_2		ρ		δ	
EL approach									
60	45	0.028624	(0.04155)	0.016273	(0.02959)	0.010715	(-0.00430)	0.003103	(0.00072)
100	75	0.014591	(0.01549)	0.010377	(0.00935)	0.007808	(0.00411)	0.001854	(0.00062)
200	150	0.007269	(0.01748)	0.005182	(0.01127)	0.006037	(-0.01881)	0.000942	(0.00029)
500	375	0.002749	(0.00520)	0.001898	(0.00456)	0.002781	(-0.00483)	0.000370	(-0.00026)
DA approach									
60	45	0.022803	(0.02401)	0.016535	(0.02001)	0.012820	(0.00343)	0.002950	(0.00318)
100	75	0.014082	(0.02189)	0.009968	(0.00550)	0.009351	(-0.01769)	0.001788	(0.00559)
200	150	0.006406	(0.00721)	0.004526	(0.00178)	0.005960	(-0.01037)	0.000855	(0.00196)
500	375	0.002599	(-0.00215)	0.001777	(0.00098)	0.002551	(-0.00475)	0.000340	(-0.00083)

From the simulation results, we observe that across all parameter settings, the MSEs decrease as n increases, indicating that both the EL and DA approaches are consistent, i.e., larger samples yield more precise estimates. It can be seen that the MSE and bias are inversely proportional to the sample size. Overall, DA is slightly more accurate because it fully updates μ_1 and μ_2 marginally via Gibbs, and ρ is updated without information from the augmentation data. In contrast, EL updates (μ_1, μ_2, ρ) jointly at each MCMC iteration and requires more iterations for the MCMC processes to converge.

On the other hand, the censoring proportion clearly affects the performance of the estimation procedures. With $r/n = 0.75$ (Table 1), the stopping time of the experiment $\tau = x_{r:n}$ is stochastically higher, so more observations are recorded before or at τ on average, leading to smaller MSEs, roughly 66%, and smaller absolute values of the biases, about 60% $\sim$ 80% for μ_1, μ_2, and ρ, than with $r/n = 0.50$ (Table 2) under the same parameter setting. This pattern also carries over to the reliability δ, in which the MSEs are comparable across Tables 1, 3, and 4 (all at $r/n = 0.75$), but increase notably when $r/n = 0.50$.

The values of rate parameters μ_1 and μ_2 also play a role in the performance of Bayesian estimation procedures. When rate parameters halve (e.g., from 1.0 to 0.5), the MSE for μ_1 and μ_2 increases by about a factor of four, consistent with the squared-rate relationship $(1.0^2/0.5^2 = 4)$ (see Tables 3 and 4). When $\mu_2 = 1.2 > \mu_1 = 1.0$, the MSE for μ_2 is only modestly larger (about 20%) because the marginal mean of Y is smaller ($1/1.2 \approx 0.833$ times), so a greater proportion of Y values (exceeding 75%) are observed, slightly improving the precision of the estimation procedures. Estimates of ρ are most precise in Table 1, where μ_1 and μ_2 are estimated most accurately, and this improved accuracy propagates to ρ.

For visual comparison, the MSEs (in log base 10 scale) for the simulation settings in Tables 1–5 are shown as line plots in Figure 2, and the corresponding biases are displayed as bar charts in Figure 3.

Table 5. Simulated MSEs and bias (in parentheses) of the Bayes estimators of μ_1, μ_2, ρ, and δ under the modified Type-II censoring scheme with parameter setting S4 and censoring proportion $r/n = 60\%$.

EL approach		$(\mu_1, \mu_2, \rho, \delta) = (2.0, 1.2, 0.6, 0.6889)$, $r/n = 60\%$							
n	r	μ_1		μ_2		ρ		δ	
60	45	0.158527	(0.14267)	0.068219	(0.06856)	0.018486	(-0.06644)	0.004057	(-0.00691)
100	75	0.075883	(0.07851)	0.036306	(0.04376)	0.009213	(-0.03705)	0.002383	(-0.00509)
200	150	0.036369	(0.03863)	0.017014	(0.01441)	0.005028	(-0.01881)	0.001243	(-0.00101)
500	375	0.015768	(0.01967)	0.007807	(0.01047)	0.001788	(-0.00916)	0.000614	(-0.00026)
DA approach									
60	45	0.122806	(0.07266)	0.056482	(0.02999)	0.018930	(-0.06603)	0.003690	(-0.00572)
100	75	0.076322	(0.05450)	0.031674	(0.03421)	0.010818	(-0.04654)	0.002293	(-0.00748)
200	150	0.032638	(-0.00488)	0.017652	(0.02640)	0.004168	(-0.02003)	0.001300	(-0.00119)
500	375	0.013872	(0.00815)	0.006972	(0.03363)	0.001711	(-0.00882)	0.000596	(-0.00908)

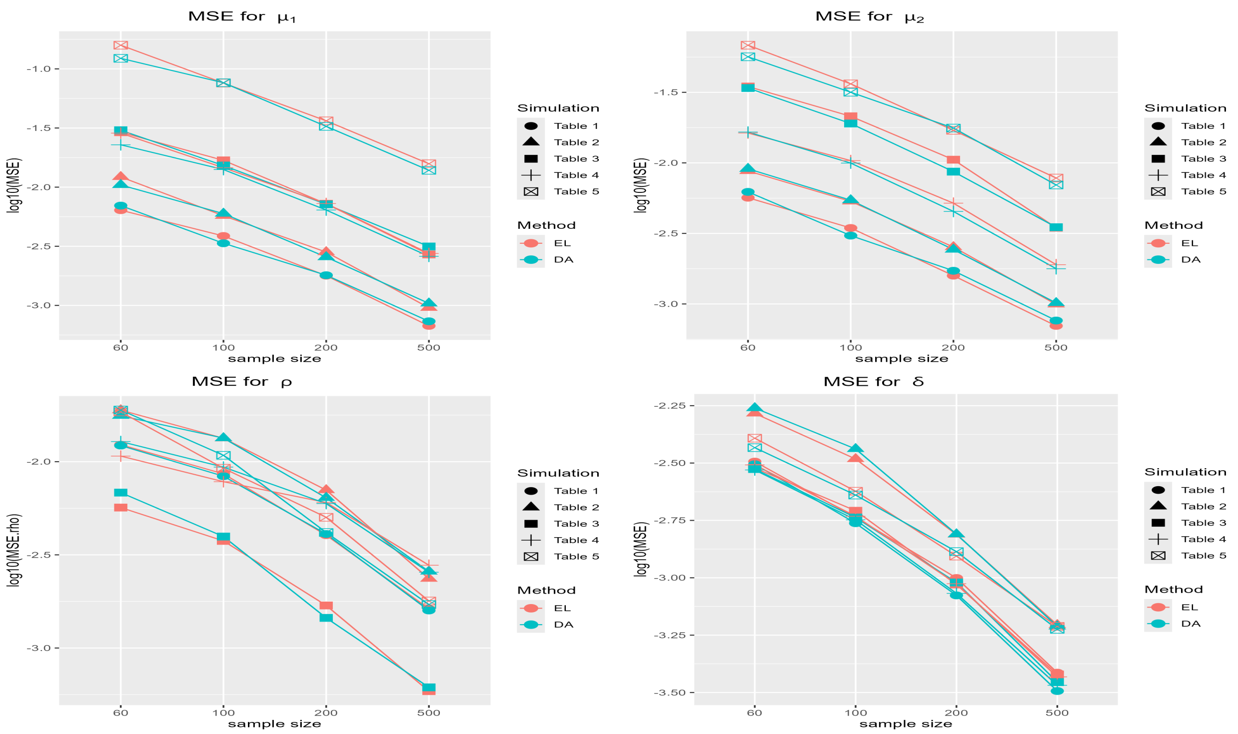

Figure 2. Line charts of $\log_{10}$(MSE) of the model parameters for the simulation settings in Tables 1–5.

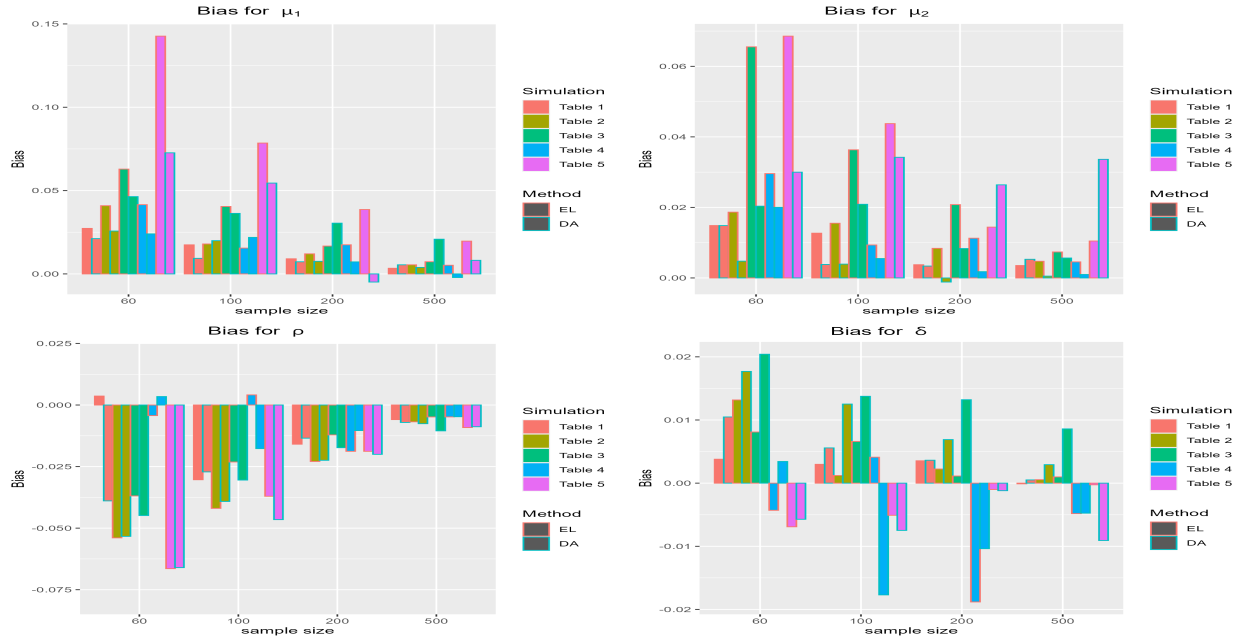

Figure 3. Bar charts of the biases of the model parameters for the simulation settings in Tables 1–5.

6. Numerical examples

In this section, the Bayesian estimation procedures introduced in this paper are illustrated using a simulated sample and a real dataset subject to modified Type-II bivariate censoring.

6.1. Simulated dataset

In this subsection, we illustrate the proposed methodology with a simulated dataset. A sample, D_1, of size $n = 60$ is simulated from BDVE$\,(\mu_1 = 0.5,\ \mu_2 = 0.5,\ \rho = 0.5)$. Under the modified Type-II bivariate censoring scheme, set $r = 45$, or $1 - r/n = 25\%$, as the censored rate in the simulation study. Hence, the termination time, $X_{45:60}$, is then the 45th ordered value among all X sample observations. In this example, an x or y value greater than $X_{45,60} = 2.54$ in each data pair (x, y) will be recorded as “+”. The resulting censored dataset, D_1^*, is a modified Type-II bivariate censored sample and is reported in Table 6.

To visualize the data $D_1^* = (x_i^*, y_i^*, c_i)$ and illustrate the idea of data augmentation, the data augmentation method described in Section 3 is applied to obtain a pseudo-complete dataset of size $n = 60$, $\hat{D}_1 = (\hat{x}_i, \hat{y}_i), i = 1, 2, \ldots, 60$. Figure 4 displays the scatter plot of the simulated data $\hat{D}_1$ in Table 6 along with univariate histograms.

Table 6. The simulated modified Type-II censored data. The + sign indicates that the values of x or y is greater than $\tau = x_{45:60} = 2.54$.

(0.5540, 0.2258)	(+, 0.2528)	(0.4687, 0.3408)	(2.3425, +)	(2.3468, 1.7150)
(+, +)	(1.9392, 0.8435)	(2.4696, +)	(0.3773, 2.4315)	(0.3289, 2.3639)
(0.7449, 0.6773)	(1.2685, 0.8404)	(+, +)	(0.4708, 1.742)	(0.0935, 1.0565)
(1.8856, 0.0830)	(0.8414, 0.4172)	(+, 1.5263)	(+, 0.9109)	(1.7430, 0.6049)
(1.4001, 1.9605)	(1.8765, 1.2550)	(+, +)	(0.4962, 0.0817)	(0.1582, 0.8124)
(1.5456, 0.2479)	(2.0673, +)	(0.2763, 0.2299)	(1.9434, 1.2940)	(0.1826, 0.0543)
(+, +)	(0.7692, 0.4001)	(2.5750, 0.7569)	(0.0622, 0.4237)	(1.3726, 1.0168)
(+, +)	(+, +)	(+, 2.3924)	(1.7944, +)	(0.4744, 0.4195)
(1.7599, 1.7672)	(0.7200, +)	(+, 1.2740)	(0.1246, 0.1502)	(0.5214, 0.0701)
(+, 1.5965)	(0.3068, 0.5997)	(1.2189, 0.8119)	(0.7418, +)	(+, +)
(+, 1.9426)	(0.8942, 1.1803)	(0.9054, 0.4693)	(0.6728, 0.9177)	(1.3966, 1.7966)

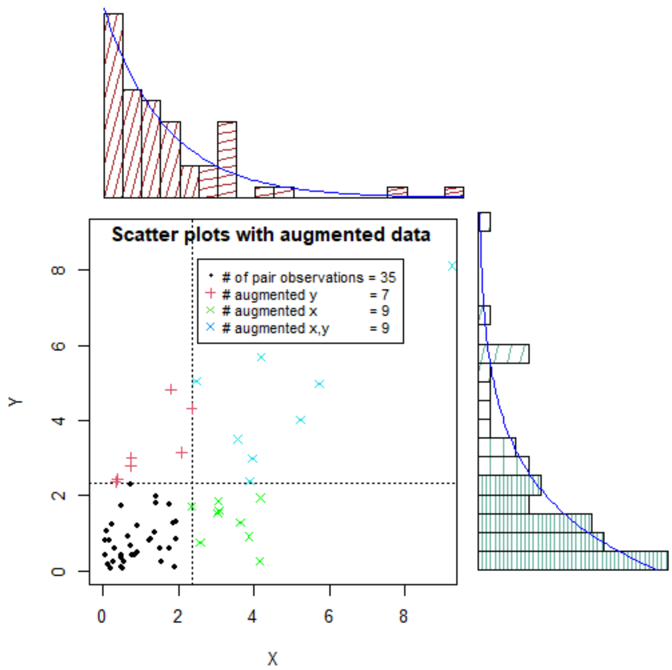

Figure 4. Scatter plot and univariate histograms for the simulated modified Type-II censored sample in Table 6.

With the priors in Eq (5.1), Bayes estimates are taken as posterior means after discarding a burn-in of 3,000 iterations. For each parameter, 95% credible intervals are obtained from the 2.5th and 97.5th percentiles of the retained MCMC draws of 30,000 MCMC iterations. Figure 5 shows trace plots from two independent chains for each method, the EL approach (EL-1 and EL-2), and the data-augmentation approach (DA-1 and DA-2), for μ_1, μ_2, ρ, and the stress-strength reliability δ with Gelman-Rubin statistics for μ_1, μ_2, ρ, and δ are 1.00, 100, 1.00, 1.00 between EL-1 and EL-2 chains and 1.00, 100, 1.00, 1.00 between DA-1 and DA-2 chains, respectively.

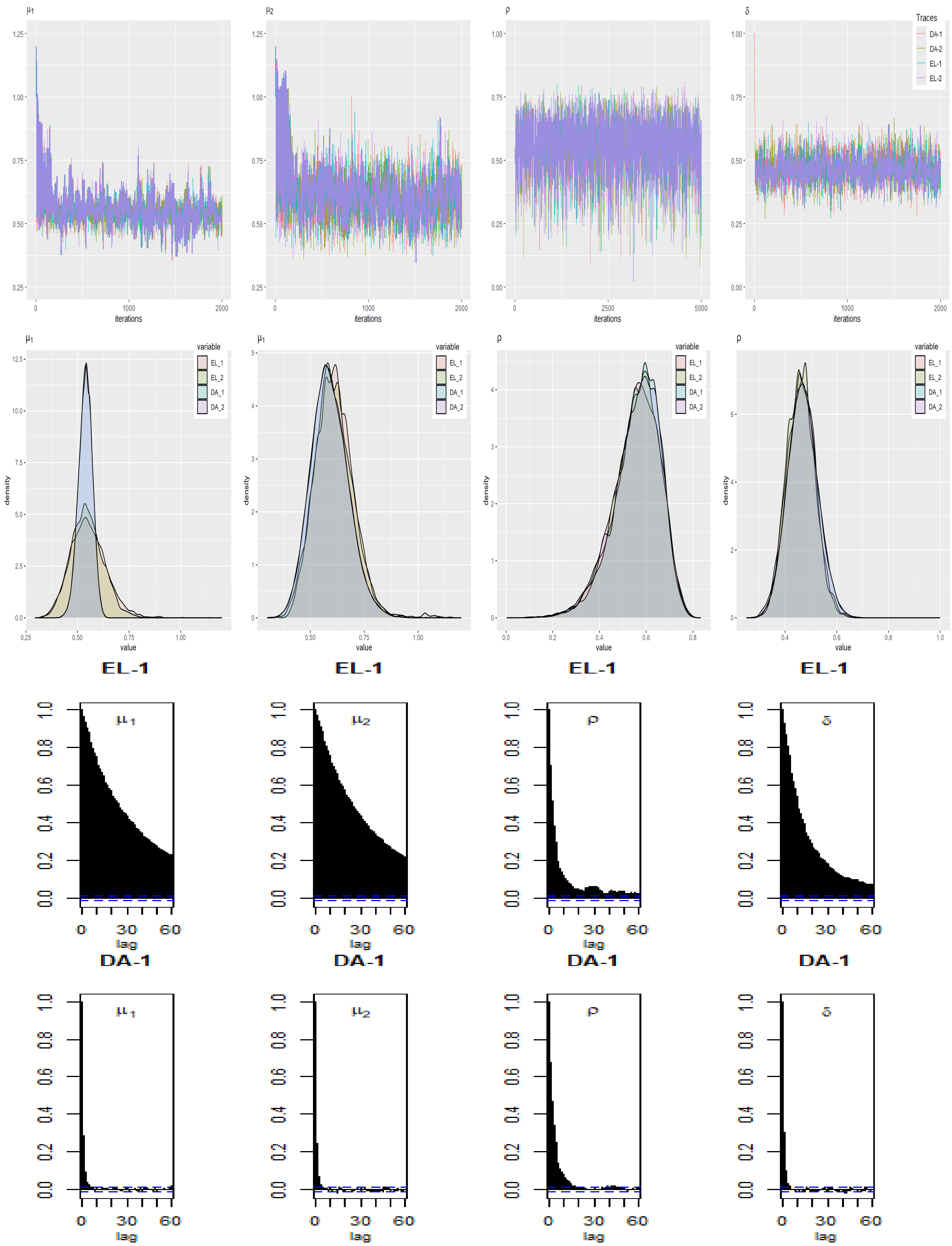

Figure 5. Trace plots, density plots, and ACF plots of two different MCMC chains for model parameters μ_1, μ_2, ρ, and stress-strength reliability δ based on the EL and DA approaches, denoted by EL-1 and DA-1, respectively, for the dataset in Table 6.

For the data in Table 6, the Bayes estimates and their corresponding 95% credible intervals for the model parameters and the stress-strength reliability δ based on the two MCMC chains are presented in Table 7. Figure 5 presents the trace plots and the ACF plots for the MCMC chains with thinning interval 1. From Figure 5, we observe that MCMC chains are stationary, stable, and centered around their sample means after burn-in. It indicates that the resulting Bayes estimates for each parameter are close between the two MCMC chains. To assess the sensitivity of the estimates of the priors, additional weak priors are also applied. Specifically, $\pi_1(\mu_1) \sim \text{Gamma}(0.001, 0.001)$, $\pi_2(\mu_2) \sim \text{Gamma}(0.001, 0.001)$, and $\pi_3(\rho) \sim \text{Uniform}(0, 1)$ are considered. The Bayes estimates are close to the results in Table 7, which indicates that our approaches are robust to the choice of weak priors.

Table 7. Bayes estimates and their corresponding 95% credible intervals for the model parameters and the stress-strength reliability δ based on the two MCMC chains using the EL and DA approaches.

	EL approach			
	Based on EL-1		Based on EL-2	
Parameter	Estimate	95% credible interval	Estimate	95% credible interval
μ_1	0.54901	(0.4098, 0.7089)	0.55077	(0.4057, 0.7211)
μ_2	0.60908	(0.4593, 0.7878)	0.61131	(0.4497, 0.8049)
ρ	0.55531	(0.3297, 0.7227)	0.55583	(0.3253, 0.7220)
δ	0.46085	(0.3605, 0.5681)	0.46097	(0.3594, 0.5677)
	DA approach			
	Based on DA-1		Based onDA-2	
Parameter	Estimate	95% credible interval	Estimate	95% credible interval
μ_1	0.53849	(0.4710, 0.5996)	0.53879	(0.4712, 0.5996)
μ_2	0.59373	(0.5231, 0.6548)	0.59375	(0.5232, 0.6557)
ρ	0.56372	(0.3414, 0.7226)	0.56627	(0.3409, 0.7232)
δ	0.46254	(0.4101, 0.5145)	0.46260	(0.4096, 0.5153)

In addition, Bayesian inference allows us to assess whether the censored data D_1^* provide evidence of independence between X and Y. When $\rho = 0$, the DBVE model reduces to two independent exponential distributions. Thus, testing independence is equivalent to testing the hypotheses $H_0 : \rho = 0$ versus $H_a : \rho > 0$. A Bayesian approach is to examine the 95% credible intervals for ρ. For the MCMC chains EL-1 and EL-2, the intervals are (0.35, 0.71) and (0.35, 0.73), and for the MCMC chains DA-1 and DA-2, the intervals are (0.43, 0.72) and (0.44, 0.68). Since zero does not fall within any of these intervals, the null hypothesis of independence is rejected, providing evidence of positive correlation between X and Y at the 5% significance level.

Regarding computational cost, both EL and DA are computationally efficient. For dataset D_1^* with sample size 60, generating 30,000 MCMC iterations took 27.67 seconds for the EL approach and 36.00 seconds for the DA approach on a desktop computer with an Intel i7-6700 CPU, and standard MCMC diagnostics (e.g., trace plots and effective sample sizes) indicated satisfactory convergence for both methods.

6.2. Real dataset

In this subsection, we illustrate the proposed methodology with a real bivariate dataset. [18] analyzed data from the Diabetic Retinopathy Study (DRS), which is available in the `survival` package in R [43] as `retinopathy`. In this study, 197 patients with high-risk diabetic retinopathy were randomized to receive laser treatment in one eye, with the other eye receiving no treatment, and the survival time for each eye was recorded as the time to visual acuity dropping below 5/200. Let X and Y denote the survival times of the patient who had left-eye or right-eye laser treatment, respectively. Here, the value $\delta = \Pr(X < Y)$ is the probability that, if we randomly select one patient whose left eye received laser treatment and another whose right eye received laser treatment, the time to severe vision loss in the treated eye is shorter for the left-eye-treated patient than for the right-eye-treated patient. In other words, it measures how often left-eye-treated patients experience vision loss earlier than right-eye-treated patients, as defined in this study.

Based on the Kolmogorov-Smirnov tests for exponentiality of the samples X and Y with p-values 0.0711 and 0.1940, respectively, we do not reject the null hypothesis that the samples X and Y can be modeled by exponential distributions marginally. To fit the modified Type-II censored data using the DBVE model, pairs with the same x and y values are removed. Moreover, the observations in data pairs are censored if they are greater than the threshold defined as the 75th percentile of the x-sample (i.e., $x_{76:102} = 49.97$). The DRS modified Type-II censored data used in the data analysis are summarized in Table 8.

Table 8. The DRS modified Type-II censored data. The + sign indicates values of x or y greater than $x_{76:102} = 49.97$.

(42.5, 31.3)	(0.3, 38.77)	(+, +)	(+, 10.8)	(+, 13.83)	(48.53, 46.43)
(44.4, 7.9)	(30.83, 38.57)	(+, 14.1)	(6.9, 20.17)	(41.4, +)	(+, 0.6)
(10.27, 1.63)	(13.83, 5.67)	(+, 29.97)	(+, 26.37)	(1.33, 5.77)	(35.53, 5.9)
(21.9, 25.63)	(14.8, 33.9)	(6.2, 1.73)	(22, 46.9)	(22, 30.2)	(25.8, 13.87)
(48.3, 5.73)	(1.9, +)	(2.67, 46.73)	(18.73, 13.83)	(32.03, 4.27)	(13.9, +)
(1.77, 43.03)	(21.57, 18.43)	(6.53, 18.7)	(+, 22.23)	(+, 14)	(5.33, 10.7)
(+, +)	(5.83, +)	(+, 2.17)	(48.43, 14.3)	(9.6, 13.33)	(7.6, 14.27)
(1.8, 34.57)	(4.3, +)	(12.2, 4.1)	(38.07, 12.73)	(+, 9.4)	(9.9, 21.57)
(40.03, 26.23)	(41.6, 18.03)	(7.07, +)	(1.5, 45.73)	(27.6, +)	(38.47, 1.63)
(25.3, +)	(46.2, +)	(9.87, 1.7)	(10.33, 0.83)	(+, 6.13)	(25.93, 43.67)
(38.77, 19.4)	(21.97, 38.07)	(+, 26.2)	(18.03, +)	(1.57, 13.83)	(46.5, 13.37)
(1.97, 11.07)	(42.47, 22.2)	(46.5, 6.1)	(11.3, 2.1)	(17.73, 42.3)	(4.97, 12.93)
(26.47, +)	(9.87, 24.43)	(30.4, 13.97)	(+, 38.57)	(+, 48.87)	(20.07, 8.83)
(+, 22.13)	(6.3, +)	(+, 18.93)	(19, 13.8)	(5.43, 13.57)	(26.17, +)
(24.73, +)	(+, 10.97)	(21.1, +)	(+, 43.7)	(+, 14.37)	(14.37, 1.5)
(+, 38.4)	(+, 2.83)	(49.93, 6.57)	(34.37, 42.27)	(+, +)	(+, 6.57)
(38.87, +)	(46.63, 42.43)	(+, 13.1)	(42.33, +)	(49.97, 2.9)	(45.9, 1.43)

Figure 6 displays the scatter plot of the DRS Type-II censored data in Table 8. In Figure 6, 59 data pairs of Case 1 are fully observed and plotted on the bottom left corner; 23 data pairs of Case 2 with x-values censored and y-values observed are plotted on the bottom right corner; 17 data pairs of Case 3 with x-values observed and y-values censored are plotted on the top left corner; and 3 data pairs of Case 4 with both x and y values censored are plotted on the top right corner.

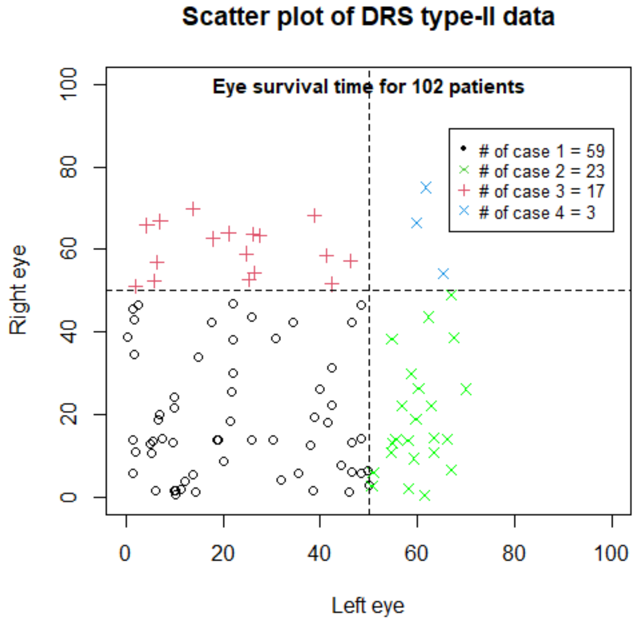

Figure 6. Scatter plot of the DRS Type-II censored data in Table 8.

Based on the Type-II censored data in Table 8, using the priors in Eq (5.1), both the EL and DA approaches are applied to simulate MCMC samplers of model parameters μ_1, μ_2, ρ, and the strength stress parameter δ with Gelman-Rubin statistics for μ_1, μ_2, , ρ , and δ are 1.01, 1.01, 1.00, 1.00 between D-EL-1 and D-EL-2 chains and all 1.00, 1.00, 1.00, 1.00 between D-DA-1 and D-DA-2 chains, respectively. According to their MCMC trace plots, density plots, and ACF plots in Figure 7, these MCMC chains with thinning interval 1 appear stationary and are not strongly correlated. With a total of 30,000 MCMC iterations and 3000 burn-in, the Bayes estimates and the corresponding 95% credible sets for the model parameter and the strength stress parameter, along with the EL and DA approaches based on a single MCMC chain, are summarized in Table 9. The Bayes estimates based on the EL and DA approaches are $\mu_1 = 0.0254$, $\mu_2 = 0.0322$, $\rho = 0.065$, and $\delta = 0.44$.

Table 9. DRS Type-II data, Bayes estimates, and their corresponding 95% credible intervals for the model parameters and the stress-strength reliability δ based on one MCMC chain using the EL and DA approaches.

	EL approach		DA approach	
Parameter	Estimate	95% credible interval	Estimate	95% credible interval
μ_1	0.02534	(0.01995, 0.03118)	0.02536	(0.02274, 0.02771)
μ_2	0.03213	(0.02570, 0.03930)	0.03220	(0.02567, 0.03953)
ρ	0.06576	(0.00197, 0.20260)	0.06541	(0.01513, 0.16096)
δ	0.43895	(0.36603, 0.51523)	0.44035	(0.38182, 0.50243)

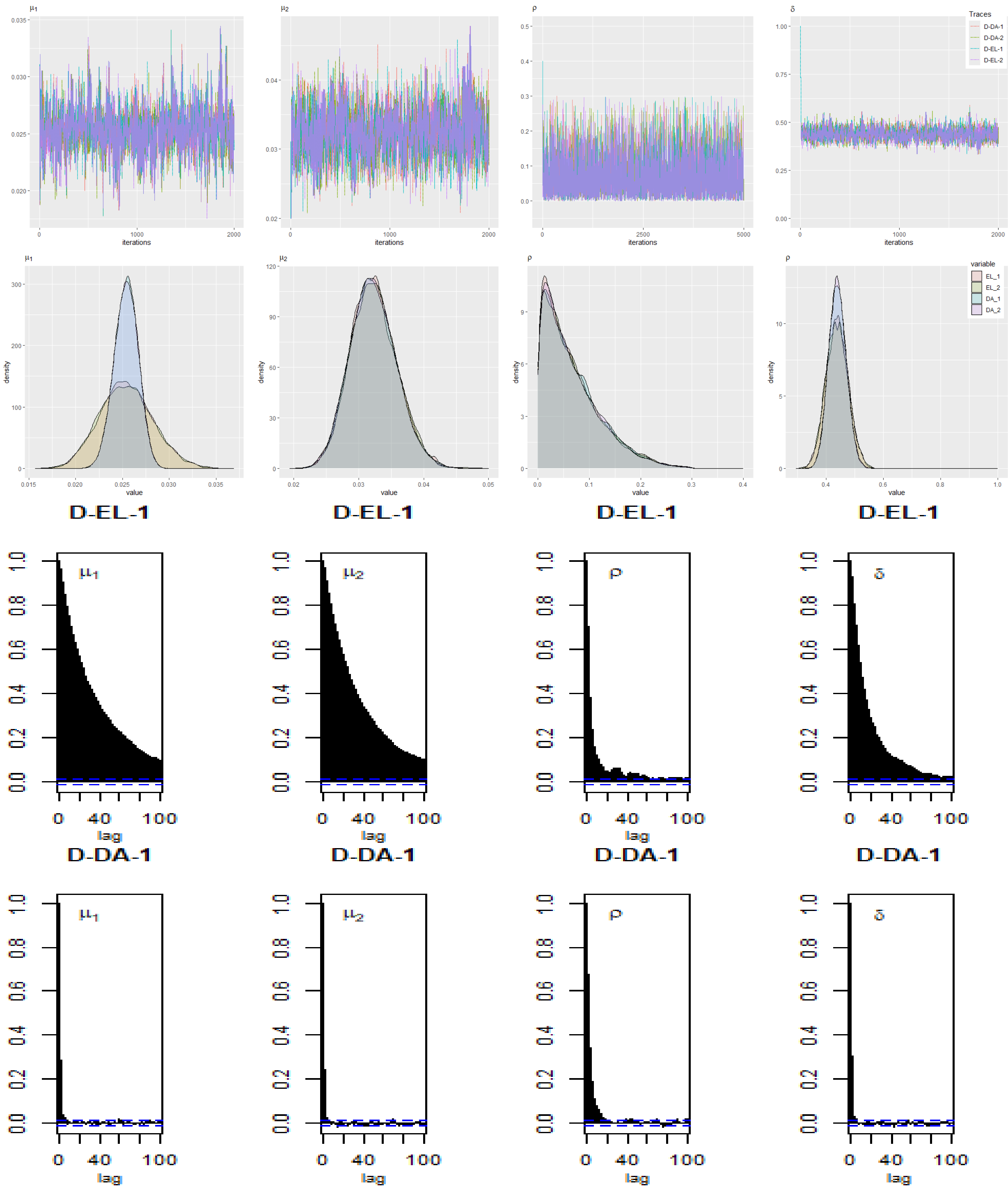

Figure 7. Trace plots, posterior density plots, and ACF plots of MCMC chains by the EL and DA approaches, denoted by D-EL-1 and D-DA-1, for model parameters μ_1, μ_2, ρ, and δ, respectively, for the DRS Type-II censored data in Table 8.

To compare the DBVE model with a simpler model based on independent exponential distributions for X and Y, suppose the ordered sample $x_{i:102}$ and the corresponding concomitants $y_{[i:102]}$, $i = 1, 2, \ldots, 102$, arise independently from $\text{Exp}(\mu_1)$ and $\text{Exp}(\mu_2)$, respectively. Under this assumption, the MLEs of μ_1, μ_2, and δ are

$$\begin{aligned}
\tilde{\mu}_1 &= \frac{76}{\sum_{i=1}^{76} x_{i:102} + (102-76)x_{76:102}} = 0.01378, \\
\tilde{\mu}_2 &= \frac{82}{\sum_{i=1}^{82} y_{[i:102]} + (102-82)x_{76:102}} = 0.03209, \\
\tilde{\delta} &= \frac{\tilde{\mu}_1}{\tilde{\mu}_1 + \tilde{\mu}_2} = 0.30043,
\end{aligned}$$

where 82 concomitants $y_{[i:102]}$ are less than or equal to the threshold $x_{76:102} = 49.97$. These MLEs are close to the corresponding Bayes estimates obtained under the DBEV model, which is consistent with the finding that the dependence between X and Y is weak.

To evaluate the performance of the Bayes estimates obtained by the EL and DA approaches, we computed the sample means of the corresponding posterior estimates based on 1000 MCMC draws. As summarized in Table 10, the sample standard deviations of the EL-based Bayes estimates are generally larger than those from the DA approach, indicating that the DA method provides more precise inference. From Table 9, both 95% credible intervals for δ include 0.5, so the hypothesis $\delta = 0.5$ is not rejected. Nevertheless, the DA-based Bayes estimate of δ is 0.44995 with a small standard deviation of 0.00089, suggesting a slightly lower but precisely estimated value.

Table 10. Sample means and sample standard deviations of Bayes estimates of model parameters and the stress-strength reliability, denoted by $\hat{\mu}_1$, $\hat{\mu}_2$, $\hat{\rho}$, and $\hat{\delta}$, respectively, based on the 1000 MCMC chains using the EL and DA approaches.

	$\hat{\mu}_1$		$\hat{\mu}_2$		$\hat{\rho}$		$\hat{\delta}$	
	mean	sd	mean	sd	mean	sd	mean	sd
EL	0.02539	0.00012	0.03230	0.00017	0.07208	0.03202	0.43828	0.00152
DA	0.02540	0.00008	0.03223	0.00008	0.07128	0.03123	0.43995	0.00089

7. Conclusions

In this paper, we investigated the Bayesian estimation of model parameters and the stress-strength reliability $\delta = \Pr(X < Y)$ for the Moran-Downton bivariate exponential distribution under a modified Type-II censoring scheme. Two Bayesian strategies were developed: one that directly uses the likelihood of the censored data within a Metropolis-Hastings sampling framework, and another that employs data augmentation to impute censored observations within a Gibbs sampler. Through extensive simulation studies, we demonstrated that both approaches yield consistent estimates, with performance improving as the sample size increases. The MCMC approach that directly uses the likelihood of the censored data generally provides slightly more accurate estimates of the correlation parameter ρ, while the data-augmentation method offers additional computational convenience.

The proposed modified censoring scheme was shown to improve efficiency compared with conventional censoring schemes, particularly when censoring rates are high. A numerical example illustrated the practical implementation of the methods and highlighted the utility of the modified censoring approach for real data analysis.

Overall, this study provides feasible Bayesian estimation methods for handling dependent lifetime data subject to censoring, offering both theoretical development and practical tools for reliability analysis. Future research may extend these approaches to more general bivariate or multivariate lifetime models, such as multivariate normal or copula-based models, and explore adaptive censoring designs that further balance cost and information efficiency, such as progressive first-failure censoring schemes.

In addition, several methodological extensions are worth pursuing. First, hierarchical prior structures for $(\mu_1, \mu_2, \rho, \delta)$ could be developed to borrow strength across multiple experiments, product lines, or test conditions, thereby accommodating between-system heterogeneity and incorporating expert reliability judgments. Second, the present framework can be embedded into regression-type stress-strength models in which the parameters depend on observed covariates through generalized linear or accelerated life structures. Finally, the modified Type-II censoring design considered here can be generalized to progressive censoring schemes with removals at intermediate failure times, providing greater flexibility for controlling test duration and cost.

Author contributions

Conceptualization: H. K. T. Ng, L. Wang, Y. L. Lio and Y. J. Lin; Methodology: H. K. T. Ng and Y. J. Lin; Software: Y. J. Lin; Validation: H. K. T. Ng, Y. L. Lio, T. R. Tsai and Y. J. Lin; Investigation: Y. L. Lio, T. R. Tsai and Y. J. Lin; Resources: Y. L. Lio, T. R. Tsai and Y. J. Lin; Data curation: Y. L. Lio, T. R. Tsai and Y. J. Lin; Writing-original draft preparation: Y. L. Lio, T. R. Tsai and Y. J. Lin; Writing-review and editing: H. K. T. Ng, Y L. Lio, T. R. Tsai and Y. J. Lin; Visualization: Y. J. Lin; Project administration: Y. L. Lio and T. R. Tsai.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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