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Research article

Solving a constrained Sylvester-type system on the commutative quaternion ring

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Abstract: This paper establishes solvability conditions and general solution sets for a constrained Sylvester-type system over the commutative quaternion ring by proposing two distinct methods. As a key application, we also analyze the minimum solution of a related optimization problem when this system is solvable. Additionally, the solvability condition and the general form of the Hermitian solutions for this system over the commutative quaternion ring are established in this paper. The main results are validated through an algorithm and a numerical example.

Keywords: Sylvester-type system; commutative quaternion ring; Hermitian solution; minimum solution
Mathematics Subject Classification: 15A09, 15A03, 15A24, 15B57

1. Introduction

In 1892, Segre [1] proposed the definition of the commutative quaternion ring $\mathbb{Q}_c$, where

$$\mathbb{Q}_c = \{c = c_0 + c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k} \mid c_0,\ c_1,\ c_2,\ c_3 \in \mathbb{R} \text{ and } \mathbf{i},\ \mathbf{j},\ \mathbf{k} \notin \mathbb{R}\}.$$

Here, $\mathbf{i},\ \mathbf{j},\ \mathbf{k}$ obey the following rules:

$$\mathbf{i}^2 = \mathbf{k}^2 = -1,\ \mathbf{j}^2 = 1,\ \mathbf{ijk} = -1,\ \mathbf{ij} = \mathbf{ji} = \mathbf{k},\ \mathbf{jk} = \mathbf{kj} = \mathbf{i},\ \mathbf{ik} = \mathbf{ki} = -\mathbf{j}.$$

Commutative quaternions are extensively applied in neural networks and the fields of signal and image processing (see, e.g., [2–4]). For example, Xia et al. [2] focused on conducting a global exponential stability analysis of commutative quaternion-valued neural networks. Ding et al. [4] investigated the lower-upper (LU) decomposition in commutative quaternions and gave the application to strict image authentication.

Matrix equations are one of the research topics (see, e.g., [5–7]); they have many applications in image processing and nerve networks (see, e.g., [8–10]). Particularly, the matrix equations over the commutative quaternion ring have garnered significant interest. Kösal et al. [11] derived explicit expressions for solutions to Kalman-Yakubovich-conjugate matrix equations over the commutative quaternion ring. Subsequently, Kösal et al. [12] established the general solution to a commutative quaternion matrix equation. However, based on the information available, research on a constrained Sylvester-type system over the commutative quaternion ring is relatively limited. This paper investigates the solvability condition and general solution set for the following constrained Sylvester-type system over the commutative quaternion ring:

$$\begin{cases} A_1X = C_1,\ YB_1 = D_1, \\ A_2Z = C_2,\ ZB_2 = D_2, \\ A_3X + ZB_3 = F_1, \\ A_4ZB_4 + A_5YB_5 = F_2, \end{cases} \tag{1.1}$$

where $X,\ Y,\ Z$ are the unknowns, and the coefficient matrices are specified with compatible dimensions.

It is widely acknowledged that Hermitian matrices are of central importance in many areas, such as signal processing and control theory [13]; numerical analysis and optimization theory [14]; and nonlinear matrix equations [15]. In recent years, the study of Hermitian solutions to matrix equations over the commutative quaternion ring has attracted considerable attention from researchers. For instance, Chen et al. [16] derived the necessary and sufficient condition for solvability and the general solution expression for the following system over the commutative quaternion ring, as well as the solvability condition and the general expression for the Hermitian solutions of this system:

$$\begin{cases} A_1X_1B_1 + A_1X_2B_2 + A_2X_3B_2 = C_1, \\ E_1X_1F_1 + E_1X_2F_2 + E_2X_3F_2 = C_2, \\ G_1X_1H_1 + G_1X_2H_2 + G_2X_3H_2 = C_3. \end{cases}$$

Zhang et al. [17] established the solvability condition and general solution form for the Hermitian solutions to the following system over the commutative quaternion ring:

$$\begin{cases} A_1X = C_1, \\ YB_1 = D_1, \\ A_2Z = C_2,\ ZB_2 = D_2, \\ A_3W = C_3,\ WB_3 = D_3,\ A_4WB_4 = C_4, \\ A_5X + YB_5 + A_6ZB_6 + A_7WB_7 = C_5. \end{cases}$$

Nevertheless, currently, few researchers have investigated the Hermitian solutions problem for the system (1.1) over the commutative quaternion ring. Therefore, we investigate the solvability condition for the Hermitian solutions of the system (1.1) and present a set for these solutions over the commutative quaternion ring.

Motivated by the utility of commutative quaternions, the applications of Hermitian matrices, and the need to advance the theory of constrained Sylvester-type systems over the commutative quaternion ring, we study the solvability of the system (1.1) over the commutative quaternion ring via two approaches: the generalized inverse and the column-block matrix. Both methods yield a full set-form description of the general solution. Additionally, for the Hermitian solutions case, we employ a generalized inverse equality to obtain the solvability condition and fully characterize the solution set.

The remainder of this paper is arranged as follows: We first provide the notation used throughout the paper, and then present some basic definitions, lemmas, and relevant propositions in Section 2. Section 3 presents two distinct methods for establishing the solvability conditions and general solution sets for the system (1.1) over the commutative quaternion ring. In Section 4, we leverage our theoretical results to characterize the minimum solution of a related optimization problem. The Hermitian solutions to the system (1.1) over the commutative quaternion ring is the focus of Section 5, where we establish its solvability condition and general expression. A numerical example is included in Section 6 to validate the main results. In Section 7, we summarize this paper.

2. Preliminaries

Throughout this paper, $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively. The symbol $\mathbb{Q}_c$ refers to the commutative quaternion ring. We denote by $\mathbf{0}$ and I, the zero matrix and the identity matrix of appropriate dimensions. The set of all $m \times n$ matrices over $\mathbb{Q}_c$, $\mathbb{C}$, or $\mathbb{R}$ is denoted by $\mathbb{Q}_c^{m\times n}$, $\mathbb{C}^{m\times n}$, or $\mathbb{R}^{m\times n}$, respectively. Additionally, $\mathbb{SR}^{n\times n}$ and $\mathbb{ASR}^{n\times n}$ represent the sets of all $n \times n$ real symmetric and real skew-symmetric matrices. For any matrix A, we write A^T and A^* for its transpose and conjugate transpose, respectively. Matrix A is defined to be Hermitian if and only if $A^* = A$, and the set of such matrices is denoted by $\mathbb{HQ}_c^{n\times n}$. For $A \in \mathbb{C}^{m\times n}$, $\mathrm{Re}(A)$ and $\mathrm{Im}(A)$ represent its real and imaginary parts, while $r(A)$ denotes its rank. The Moore–Penrose inverse of A, denoted by $A^\dagger$, is the unique matrix X, satisfying the following conditions:

$$AXA = A,\ XAX = X,\ (AX)^* = AX,\ (XA)^* = XA.$$

Finally, the Kronecker product of matrices $A = (a_{ij})_{m\times n}$ and $B_{s\times t}$ is defined as $A \otimes B = (a_{ij}B)_{ms\times nt}$.

The fundamental definitions, lemmas, and propositions underlying the results of this paper are stated below. We first define the complex representation for a commutative quaternion matrix.

Definition 2.1. [18] Let $C = C_1 + C_2\mathbf{j} \in \mathbb{Q}_c^{m\times n},\ C_1,\ C_2 \in \mathbb{C}^{m\times n}$ be given. The complex representation of matrix C is defined as follows:

$$G(C) := \begin{pmatrix} C_1 & C_2 \\ C_2 & C_1 \end{pmatrix}.$$

The following statements are easily verified by applying Definition 2.1.

Proposition 2.2. *[18] Let $A,\ B \in \mathbb{Q}_c^{n\times n}$ be given. The mapping $G(\cdot)$ satisfies the following properties:*

$$\begin{aligned}
&(1)\ A = B \Leftrightarrow G(A) = G(B),\\
&(2)\ G(A + B) = G(A) + G(B),\\
&(3)\ G(AB) = G(A)G(B),\\
&(4)\ G(I_n) = I_{2n}.
\end{aligned}$$

Lemma 2.3. *[19] Let matrix $B = B_1 + B_2\mathbf{j} \in \mathbb{Q}_c^{n\times m},\ B_1,\ B_2 \in \mathbb{C}^{n\times m}$ be given. We have*

$$B_1 + B_2\mathbf{j} = B \cong \Gamma_B = \begin{pmatrix} B_1, & B_2 \end{pmatrix},$$

and the symbol $\cong$ denotes an identification.

The vectorization operator for $B = (b_{ij}) \in \mathbb{Q}_c^{n\times m}$ *is given by*

$$\mathrm{vec}(B) = (b_1,\ b_2,\ \ldots,\ b_m)^T,\ \text{where } b_j = (b_{1j},\ b_{2j},\ \ldots,\ b_{nj})\ \text{for } j = 1,2,\ldots,\ m.$$

This extends to

$$\mathrm{vec}(\widetilde{B_1}) = \begin{pmatrix}\mathrm{vec}(\mathrm{Re}(B_1))\\ \mathrm{vec}(\mathrm{Im}(B_1))\end{pmatrix},\ \mathrm{vec}(\widetilde{\mathrm{B}}) = \begin{pmatrix}\mathrm{vec}(\mathrm{Re}(B_1))\\ \mathrm{vec}(\mathrm{Im}(B_1))\\ \mathrm{vec}(\mathrm{Re}(B_2))\\ \mathrm{vec}(\mathrm{Im}(B_2))\end{pmatrix},$$

where

$$\widetilde{B_1} = \Big(\mathrm{Re}(B_1),\ \mathrm{Im}(B_1)\Big),\ \widetilde{B} = \Big(\mathrm{Re}(B_1),\ \mathrm{Im}(B_1),\ \mathrm{Re}(B_2),\ \mathrm{Im}(B_2)\Big).$$

Based on Definition 2.1 and Lemma 2.3, the following lemma can be readily derived.

Lemma 2.4. *[19] Let* $A = A_1 + A_2\mathbf{j},\ B = B_1 + B_2\mathbf{j} \in \mathbb{Q}_c^{m\times n}$ *be given, where* $A_1,\ A_2,\ B_1,\ B_2 \in \mathbb{C}^{m\times n}$. *Then*

$$\begin{aligned}&(1)\ A = B \Leftrightarrow \Gamma_A = \Gamma_B,\\ &(2)\ \Gamma_{A+B} = \Gamma_A + \Gamma_B,\\ &(3)\ \Gamma_{AB} = \Gamma_A G(B).\end{aligned}$$

Subsequently, we define the Frobenius norm as follows.

Definition 2.5. [19] Let $A = (a_{ij}) \in \mathbb{C}^{m\times n}$ be given. The Frobenius norm of A is given by

$$\|A\| = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}\|a_{ij}\|^2},\ \|a_{ij}\|^2 = (\mathrm{Re}\,a_{ij})^2 + (\mathrm{Im}\,a_{ij})^2.$$

Lemma 2.6. *[19] The Frobenius norm of a matrix* $A = A_1 + A_2\mathbf{j} \in \mathbb{Q}_c^{m\times n}$ *is*

$$\left\|\widetilde{A}\right\| = \sqrt{\|\mathrm{Re}(A_1)\|^2 + \|\mathrm{Im}(A_1)\|^2 + \|\mathrm{Re}(A_2)\|^2 + \|\mathrm{Im}(A_2)\|^2}.$$

Therefore, we have $\|\Gamma_A\| = \|\mathrm{vec}(\widetilde{A})\| = \|\widetilde{A}\|$.

Lemma 2.7. *[19] Let* $E = E_1 + E_2\mathbf{j} \in \mathbb{Q}_c^{m\times n}$, $F = F_1 + F_2\mathbf{j} \in \mathbb{Q}_c^{n\times t}$, $D = D_1 + D_2\mathbf{j} \in \mathbb{Q}_c^{t\times s}$ *be given. Then*

$$\mathrm{vec}(\Gamma_{EFD}) = \Big(G(D)^T \otimes E_1,\ G(\mathbf{j}D)^T \otimes E_2\Big) M_t \mathrm{vec}(\widetilde{F}),$$

where

$$M_t = \begin{pmatrix} I_{nt} & \mathbf{i}I_{nt} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & I_{nt} & \mathbf{i}I_{nt}\\ I_{nt} & \mathbf{i}I_{nt} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & I_{nt} & \mathbf{i}I_{nt}\end{pmatrix},$$

and I_{nt} *is the* $nt \times nt$ *identity matrix.*

Definition 2.8. [17] Consider a matrix $K = (k_{ij}) \in \mathbb{Q}_c^{n\times n}$. Let

$$k_1 = \left(k_{11},\ \sqrt{2}k_{21}, \ldots,\ \sqrt{2}k_{n1}\right),\ k_2 = \left(k_{22},\ \sqrt{2}k_{32}, \ldots,\ \sqrt{2}k_{n2}\right), \ldots,\ k_{n-1} = \left(k_{(n-1)(n-1)},\ \sqrt{2}k_{n(n-1)}\right),\ k_n = k_{nn}.$$

The vector $\mathrm{vec}_S(K)$ is represented as

$$\mathrm{vec}_S(K) = \left(k_1,\ k_2,\ \ldots,\ k_{n-1},\ k_n\right)^T \in \mathbb{Q}_c^{\frac{n(n+1)}{2}}.$$

Definition 2.9. [17] Let $L = (l_{ij}) \in \mathbb{Q}_c^{n\times n}$ be given and define

$$l_1 = \left(l_{21},\ l_{31},\ \ldots,\ l_{n1}\right),\ l_2 = \left(l_{32},\ l_{42},\ \ldots,\ l_{n2}\right),\ \ldots,\ l_{n-2} = \left(l_{(n-1)(n-2)},\ l_{n(n-2)}\right),\ l_{n-1} = l_{n(n-1)}.$$

The vector $\mathrm{vec}_A(L)$ is represented as:

$$\mathrm{vec}_A(L) = \sqrt{2}\left(l_1,\ l_2,\ \ldots,\ l_{n-2},\ l_{n-1}\right)^T \in \mathbb{Q}_c^{\frac{n(n-1)}{2}}.$$

According to the aforementioned Definitions 2.8 and 2.9, the following proposition can be established.

Proposition 2.10. *[17] Let $B \in \mathbb{R}^{n\times n}$ be given. Then*
(i) *B is symmetric if and only if*

$$\mathrm{vec}(B) = K_S\,\mathrm{vec}_S(B),$$

where $K_S \in \mathbb{R}^{n^2\times\frac{n(n+1)}{2}}$ satisfies

$$K_S = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2}\varepsilon_1 & \varepsilon_2 & \ldots & \varepsilon_{n-1} & \varepsilon_n & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \varepsilon_1 & \ldots & \mathbf{0} & \mathbf{0} & \sqrt{2}\varepsilon_2 & \varepsilon_3 & \ldots & \varepsilon_n & \ldots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \varepsilon_2 & \ldots & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \varepsilon_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \ldots & \sqrt{2}\varepsilon_{n-1} & \varepsilon_n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \varepsilon_1 & \mathbf{0} & \mathbf{0} & \ldots & \varepsilon_2 & \ldots & \mathbf{0} & \varepsilon_{n-1} & \sqrt{2}\varepsilon_n \end{pmatrix}.$$

Here, ϵ_i is an n-dimensional unit column vector with a 1 *in the i-th entry and zeros elsewhere.*
(ii) *B is skew-symmetric if and only if*

$$\mathrm{vec}(B) = K_A\mathrm{vec}_A(B),$$

where the matrix $K_A \in \mathbb{R}^{n^2\times\frac{n(n-1)}{2}}$ is defined as follows:

$$K_A = \frac{1}{\sqrt{2}}\begin{pmatrix} \varepsilon_2 & \varepsilon_3 & \ldots & \varepsilon_{n-1} & \varepsilon_n & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \\ -\varepsilon_1 & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} & \varepsilon_3 & \ldots & \varepsilon_{n-1} & \varepsilon_n & \ldots & \mathbf{0} \\ \mathbf{0} & -\varepsilon_1 & \ldots & \mathbf{0} & \mathbf{0} & -\varepsilon_2 & \ldots & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \mathbf{0} & \mathbf{0} & & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \ldots & -\varepsilon_1 & \mathbf{0} & \mathbf{0} & \ldots & -\varepsilon_2 & \mathbf{0} & \ldots & \varepsilon_n \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & -\varepsilon_1 & \mathbf{0} & \ldots & \mathbf{0} & -\varepsilon_2 & \ldots & -\varepsilon_{n-1} \end{pmatrix}.$$

Here, ϵ_i is an n-dimensional unit column vector with a 1 *in the i-th entry and zeros elsewhere.*

Lemma 2.11. *[17] If* $X = X_1 + X_2\mathbf{j} \in \mathbb{HQ}_c^{n\times n}$*, where* $X_1,\ X_2 \in \mathbb{C}^{n\times n}$*, we obtain*

$$X \in \mathbb{HQ}_c^{n\times n} \Leftrightarrow \begin{cases} \mathrm{Re}(X_1)^T = \mathrm{Re}(X_1),\ \mathrm{Im}(X_1)^T = -\mathrm{Im}(X_1), \\ \mathrm{Re}(X_2)^T = -\mathrm{Re}(X_2),\ \mathrm{Im}(X_2)^T = -\mathrm{Im}(X_2). \end{cases}$$

Evidently, $\mathrm{Re}(X_1)$ *is symmetric, while* $\mathrm{Im}(X_1),\ \mathrm{Re}(X_2)$*, and* $\mathrm{Im}(X_2)$ *are skew-symmetric.*

Lemma 2.12. *[17] Let* $B = B_1 + B_2\mathbf{j} \in \mathbb{HQ}_c^{n\times n}$ *be given. Then*

$$\begin{pmatrix} \mathrm{vec}(B_1) \\ \mathrm{vec}(B_2) \end{pmatrix} = \mathbb{M} \begin{pmatrix} \mathrm{vec}_S(\mathrm{Re}(B_1)) \\ \mathrm{vec}_A(\mathrm{Im}(B_1)) \\ \mathrm{vec}_A(\mathrm{Re}(B_2)) \\ \mathrm{vec}_A(\mathrm{Im}(B_2)) \end{pmatrix},$$

where

$$\mathbb{M} = \begin{pmatrix} K_S & \mathbf{i}K_A & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & K_A & \mathbf{i}K_A \end{pmatrix}.$$

Lemma 2.13. *[20] Given matrices* $O = O_1 + O_2\mathbf{j} \in \mathbb{Q}_c^{m\times n}$*,* $P = P_1 + P_2\mathbf{j} \in \mathbb{HQ}_c^{n\times n}$*, and* $E = E_1 + E_2\mathbf{j} \in \mathbb{Q}_c^{n\times t}$*, where* $O_i \in \mathbb{C}^{m\times n}$*,* $P_i \in \mathbb{C}^{n\times n}$*, and* $E_i \in \mathbb{C}^{n\times t}$ *for* $i = 1, 2$*, we have*

$$\mathrm{vec}(\Gamma_{OPE}) = G\left[(E_1^T \otimes O_1 + E_2^T \otimes O_2) + (E_2^T \otimes O_1 + E_1^T \otimes O_2)\mathbf{j}\right]\mathbb{M} \begin{pmatrix} \mathrm{vec}_S(\mathrm{Re}(P_1)) \\ \mathrm{vec}_A(\mathrm{Im}(P_1)) \\ \mathrm{vec}_A(\mathrm{Re}(P_2)) \\ \mathrm{vec}_A(\mathrm{Im}(P_2)) \end{pmatrix},$$

where the matrix $\mathbb{M}$ *is defined as in Lemma 2.12.*

Lemma 2.14. *[21] Let* $B \in \mathbb{R}^{m\times n}$ *and* $c \in \mathbb{R}^{m\times 1}$ *be given. Then the matrix equation* $Bx = c$ *has a solution if and only if* $BB^\dagger c = c$*. When the equation is consistent, the general solution is given by*

$$x = B^\dagger c + (I_n - B^\dagger B)u,$$

where $u \in \mathbb{R}^{n\times 1}$ *is an arbitrary vector. The solution is unique if and only if* B *has full column rank, i.e.,* $r(B) = n$*, in which case it is given by* $x = B^\dagger c$*.*

3. Solutions to the system (1.1)

In this section, we aim to derive the solvability conditions and general solution sets for the quaternion matrix system (1.1) over the $\mathbb{Q}_c$ by employing two different methods. Prior to presenting the first method, we establish the following notations.

Let $A_1 = A_{11} + A_{12}\mathbf{j},\ A_2 = A_{21} + A_{22}\mathbf{j},\ C_1,\ C_2 \in \mathbb{Q}_c^{m\times n}$; $B_1,\ B_2,\ D_1,\ D_2 \in \mathbb{Q}_c^{n\times k}$; $A_3 = A_{31} + A_{32}\mathbf{j},\ B_3,\ F_1 \in \mathbb{Q}_c^{n\times n}$; $A_4 = A_{41} + A_{42}\mathbf{j},\ A_5 = A_{51} + A_{52}\mathbf{j} \in \mathbb{Q}_c^{s\times n}$; $B_4,\ B_5 \in \mathbb{Q}_c^{n\times t}$; and $F_2 \in \mathbb{Q}_c^{s\times t}$ be given.

Assign

$$
L=\begin{pmatrix} G(I)^T\otimes A_{11} & G(\mathbf{j}I)^T\otimes A_{12}\\ \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\\ G(I)^T\otimes A_{31} & G(\mathbf{j}I)^T\otimes A_{32}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}M_n,\ J=\begin{pmatrix} \mathbf{0} & \mathbf{0}\\ G(B_1)^T\otimes I & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\\ G(B_5)^T\otimes A_{51} & G(\mathbf{j}B_5)^T\otimes A_{52}\end{pmatrix}M_n,
$$
$$
H=\begin{pmatrix} \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\\ G(I)^T\otimes A_{21} & G(\mathbf{j}I)^T\otimes A_{22}\\ G(B_2)^T\otimes I & \mathbf{0}\\ G(B_3)^T\otimes I & \mathbf{0}\\ G(B_4)^T\otimes A_{41} & G(\mathbf{j}B_4)^T\otimes A_{42}\end{pmatrix}M_n,\ K=\begin{pmatrix}\mathrm{vec}(\Gamma_{C_1})\\ \mathrm{vec}(\Gamma_{D_1})\\ \mathrm{vec}(\Gamma_{C_2})\\ \mathrm{vec}(\Gamma_{D_2})\\ \mathrm{vec}(\Gamma_{F_1})\\ \mathrm{vec}(\Gamma_{F_2})\end{pmatrix}, \tag{3.1}
$$

$$
K_1=\begin{pmatrix}\mathrm{vec}(\mathrm{Re}\,\Gamma_{C_1})\\ \mathrm{vec}(\mathrm{Re}\,\Gamma_{D_1})\\ \mathrm{vec}(\mathrm{Re}\,\Gamma_{C_2})\\ \mathrm{vec}(\mathrm{Re}\,\Gamma_{D_2})\\ \mathrm{vec}(\mathrm{Re}\,\Gamma_{F_1})\\ \mathrm{vec}(\mathrm{Re}\,\Gamma_{F_2})\end{pmatrix},\ K_2=\begin{pmatrix}\mathrm{vec}(\mathrm{Im}\,\Gamma_{C_1})\\ \mathrm{vec}(\mathrm{Im}\,\Gamma_{D_1})\\ \mathrm{vec}(\mathrm{Im}\,\Gamma_{C_2})\\ \mathrm{vec}(\mathrm{Im}\,\Gamma_{D_2})\\ \mathrm{vec}(\mathrm{Im}\,\Gamma_{F_1})\\ \mathrm{vec}(\mathrm{Im}\,\Gamma_{F_2})\end{pmatrix},\ \eta=\begin{pmatrix}K_1\\ K_2\end{pmatrix}, \tag{3.2}
$$

$$
\begin{aligned} &L_1=\mathrm{Re}\,L,\ L_2=\mathrm{Im}\,L,\ J_1=\mathrm{Re}\,J,\ J_2=\mathrm{Im}\,J,\ H_1=\mathrm{Re}\,H,\ H_2=\mathrm{Im}\,H,\\ &W_1=\begin{pmatrix}L_1, & J_1, & H_1\end{pmatrix},\ W_2=\begin{pmatrix}L_2, & J_2, & H_2\end{pmatrix}.\end{aligned} \tag{3.3}
$$

Using the notations above, we present the first method for deriving the solvability condition and general solution set for the system (1.1).

Theorem 3.1. *Given matrices $A_1,\ A_2,\ C_1,\ C_2\in\mathbb{Q}_c^{m\times n}$; $B_1,\ B_2,\ D_1,\ D_2\in\mathbb{Q}_c^{n\times k}$; $A_3,\ B_3,\ F_1\in\mathbb{Q}_c^{n\times n}$; $A_4,\ A_5\in\mathbb{Q}_c^{s\times n}$; $B_4,\ B_5\in\mathbb{Q}_c^{n\times t}$; and $F_2\in\mathbb{Q}_c^{s\times t}$. The symbols η, W_1, and W_2 are defined in* (3.2) *and* (3.3). *The system* (1.1) *is solvable if and only if the following condition holds:*

$$
\begin{pmatrix}W_1\\ W_2\end{pmatrix}\begin{pmatrix}W_1\\ W_2\end{pmatrix}^{\dagger}\eta=\eta. \tag{3.4}
$$

The general solution set of the system (1.1) *can be written as*

$$
\Theta_1=\left\{(X,Y,Z)\ \middle|\ \begin{pmatrix}\mathrm{vec}(\widetilde{X})\\ \mathrm{vec}(\widetilde{Y})\\ \mathrm{vec}(\widetilde{Z})\end{pmatrix}=\begin{pmatrix}W_1\\ W_2\end{pmatrix}^{\dagger}\eta+\left(I_{12n^2}-\begin{pmatrix}W_1\\ W_2\end{pmatrix}^{\dagger}\begin{pmatrix}W_1\\ W_2\end{pmatrix}\right)u\right\}, \tag{3.5}
$$

where u is an arbitrary vector with compatible dimension. Moreover, if (3.4) *holds, then the system of matrix equations* (1.1) *has a unique solution if and only if*

$$
r\begin{pmatrix}W_1\\ W_2\end{pmatrix}=12n^2, \tag{3.6}
$$

and the solution set consisting of the unique solution is

$$\Theta_2 = \left\{ (X, Y, Z) \left| \begin{pmatrix} \mathrm{vec}(\widetilde{X}) \\ \mathrm{vec}(\widetilde{Y}) \\ \mathrm{vec}(\widetilde{Z}) \end{pmatrix} = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}^{\dagger} \eta \right. \right\}. \tag{3.7}$$

Proof.

$$(1.1) \xLeftrightarrow{Lemma\ 2.4} \begin{cases} \Gamma_{A_1X} = \Gamma_{C_1},\ \Gamma_{YB_1} = \Gamma_{D_1}, \\ \Gamma_{A_2Z} = \Gamma_{C_2},\ \Gamma_{YB_2} = \Gamma_{D_2}, \\ \Gamma_{A_3X} + \Gamma_{ZB_3} = \Gamma_{F_1}, \\ \Gamma_{A_4ZB_4} + \Gamma_{A_5YB_5} = \Gamma_{F_2}, \end{cases}$$

$$\xLeftrightarrow[and\ (3.1)]{Lemma\ 2.7} L\mathrm{vec}(\widetilde{X}) + J\mathrm{vec}(\widetilde{Y}) + H\mathrm{vec}(\widetilde{Z}) = K,$$

$$\Leftrightarrow (\mathrm{Re}\ L + \mathbf{i}\mathrm{Im}\ L)\mathrm{vec}(\widetilde{X}) + (\mathrm{Re}\ J + \mathbf{i}\mathrm{Im}\ J)\mathrm{vec}(\widetilde{Y}) + (\mathrm{Re}\ H + \mathbf{i}\mathrm{Im}\ H)\mathrm{vec}(\widetilde{Z}) = \mathrm{Re}\ K + \mathbf{i}\mathrm{Im}\ K,$$

$$\Leftrightarrow \begin{pmatrix} \mathrm{Re}\ L & \mathrm{Re}\ J & \mathrm{Re}\ H \\ \mathrm{Im}\ L & \mathrm{Im}\ J & \mathrm{Im}\ H \end{pmatrix} \begin{pmatrix} \mathrm{vec}(\widetilde{X}) \\ \mathrm{vec}(\widetilde{Y}) \\ \mathrm{vec}(\widetilde{Z}) \end{pmatrix} = \begin{pmatrix} \mathrm{Re}\ K \\ \mathrm{Im}\ K \end{pmatrix},$$

$$\xLeftrightarrow{(3.2)\ and\ (3.3)} \begin{pmatrix} L_1 & J_1 & H_1 \\ L_2 & J_2 & H_2 \end{pmatrix} \begin{pmatrix} \mathrm{vec}(\widetilde{X}) \\ \mathrm{vec}(\widetilde{Y}) \\ \mathrm{vec}(\widetilde{Z}) \end{pmatrix} = \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = \eta,$$

$$\xLeftrightarrow{(3.3)} \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \begin{pmatrix} \mathrm{vec}(\widetilde{X}) \\ \mathrm{vec}(\widetilde{Y}) \\ \mathrm{vec}(\widetilde{Z}) \end{pmatrix} = \eta.$$

By Lemma 2.14, the necessary and sufficient condition for the solvability of the system (1.1) is the validity of (3.4). Under this condition, the general solution is given by the set Θ_1. Moreover, if (3.4) holds, then the solution is unique if and only if (3.6) holds, in which case the solution set becomes the singleton set Θ_2. □

Theorem 3.1 provides a method to characterize the solvability condition and general solution of the system (1.1). To obtain an alternative form, we now consider the generalized inverse of a column-block matrix. For this purpose, we define the following notations:

$$\begin{aligned}
&p = 4m \times n + 4k \times n + 2n^2 + 2s \times t, \\
&S = (I_{12n^2} - W_1^{\dagger} W_1) W_2^T, \\
&N = (I_p + (I_p - S^{\dagger} S) W_2 W_1^{\dagger} W_1^{\dagger T} W_2^T (I_p - S^{\dagger} S))^{-1}, \\
&\Omega = S^{\dagger} + (I_p - S^{\dagger} S) N W_2 W_1^{\dagger} W_1^{\dagger T} (I_{12n^2} - W_2^T S^{\dagger}), \\
&\alpha_1 = I_p - W_1 W_1^{\dagger} + W_1^{\dagger T} W_2^T N (I_p - S^{\dagger} S) W_2 W_1^{\dagger}, \\
&\alpha_2 = -W_1^{\dagger T} W_2^T (I_p - S^{\dagger} S) N, \\
&\alpha_3 = (I_p - S^{\dagger} S) N.
\end{aligned}$$

According to the findings of Magnus [22], we have

$$\begin{pmatrix} W_1 \\ W_2 \end{pmatrix}^{\dagger} = \left(W_1^{\dagger} - \Omega^T W_2 W_1^{\dagger},\ \Omega^T \right),\ \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}^{\dagger} \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = W_1^{\dagger} W_1 + S S^{\dagger}, \tag{3.8}$$

$$I_{2p} - \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}^{\dagger} = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_2^T & \alpha_3 \end{pmatrix}. \tag{3.9}$$

Theorem 3.2. *The system of matrix equations* (1.1) *is solvable if and only if*

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_2^T & \alpha_3 \end{pmatrix} \eta = \boldsymbol{0}. \tag{3.10}$$

Under this condition, the general solution to the system (1.1) *is given by the set*

$$\Theta_3 = \left\{ (X, Y, Z) \left| \begin{pmatrix} \mathrm{vec}(\widetilde{X}) \\ \mathrm{vec}(\widetilde{Y}) \\ \mathrm{vec}(\widetilde{Z}) \end{pmatrix} = \left(W_1^{\dagger} - \Omega^T W_2 W_1^{\dagger},\ \Omega^T \right) \eta + (I_{12n^2} - W_1^{\dagger} W_1 - S S^{\dagger}) u \right. \right\}, \tag{3.11}$$

where u is an arbitrary vector with compatible dimension. Furthermore, if (3.10) *holds, then the system* (1.1) *has a unique solution if and only if* (3.6) *holds, in which case the solution set is*

$$\Theta_4 = \left\{ (X, Y, Z) \left| \begin{pmatrix} \mathrm{vec}(\widetilde{X}) \\ \mathrm{vec}(\widetilde{Y}) \\ \mathrm{vec}(\widetilde{Z}) \end{pmatrix} = \left(W_1^{\dagger} - \Omega^T W_2 W_1^{\dagger},\ \Omega^T \right) \eta \right. \right\}. \tag{3.12}$$

Proof. From Theorem 3.1, we combine the solvability condition (3.4) with (3.9) to derive the alternative solvability condition (3.10), and we use the general solution (3.5) with (3.8) to obtain the alternative general solution form (3.11). The solution is unique and is given by (3.12) if and only if conditions (3.10) and (3.6) hold. □

4. A computational method for optimization task

Based on Theorem 3.2, we present the unique minimum-norm solution to the following problem:

$$\min_{(X,Y,Z)\in\Theta_3} \left(\|\Gamma_X\|^2 + \|\Gamma_Y\|^2 + \|\Gamma_Z\|^2 \right). \tag{4.1}$$

Theorem 4.1. *Under the condition of Theorem 3.2, the optimization problem* (4.1) *admits a unique minimizer (X_m, Y_m, Z_m), and this minimum solution satisfies*

$$\begin{pmatrix} \mathrm{vec}(\widetilde{X_m}) \\ \mathrm{vec}(\widetilde{Y_m}) \\ \mathrm{vec}(\widetilde{Z_m}) \end{pmatrix} = \left(W_1^{\dagger} - \Omega^T W_2 W_1^{\dagger},\ \Omega^T \right) \eta. \tag{4.2}$$

Proof. The solution set Θ_3 in (3.11) is a nonempty closed convex set. Hence, according to Definition 2.5 and Lemma 2.6, we obtain the minimization equivalence

$$\begin{aligned}
&\min_{(X,Y,Z)\in\Theta_3}\left(\|\Gamma_X\|^2+\|\Gamma_Y\|^2+\|\Gamma_Z\|^2\right)\\
&=\min_{(X,Y,Z)\in\Theta_3}\left(\|\widetilde{X}\|^2+\|\widetilde{Y}\|^2+\|\widetilde{Z}\|^2\right)\\
&=\min_{(X,Y,Z)\in\Theta_3}\left(\|\mathrm{vec}(\widetilde{X})\|^2+\|\mathrm{vec}(\widetilde{Y})\|^2+\|\mathrm{vec}(\widetilde{Z})\|^2\right)\\
&=\min_{(X,Y,Z)\in\Theta_3}\left\|\begin{pmatrix}\mathrm{vec}(\widetilde{X})\\ \mathrm{vec}(\widetilde{Y})\\ \mathrm{vec}(\widetilde{Z})\end{pmatrix}\right\|^2.
\end{aligned}\tag{4.3}$$

Given that $(X,Y,Z)\in\Theta_3$, it follows from (4.3) that (4.2) holds. □

5. Hermitian solution to the system (1.1) over $\mathbb{Q}_c$

In this section, we establish the necessary and sufficient condition for the existence of a Hermitian solution to the system (1.1) over $\mathbb{Q}_c$ and derive its general expression.

For convenience, we define the following notations: $A_1=A_{11}+A_{12}\mathbf{j},\ A_2=A_{21}+A_{22}\mathbf{j},\ C_1,C_2\in\mathbb{Q}_c^{m\times n}$; $B_1=B_{11}+B_{12}\mathbf{j},\ B_2=B_{21}+B_{22}\mathbf{j},\ D_1,D_2\in\mathbb{Q}_c^{n\times k}$; $A_3=A_{31}+A_{32}\mathbf{j},\ B_3=B_{31}+B_{32}\mathbf{j},\ F_1\in\mathbb{Q}_c^{n\times n}$; $A_4=A_{41}+A_{42}\mathbf{j},\ A_5=A_{51}+A_{52}\mathbf{j}\in\mathbb{Q}_c^{s\times n}$; $B_4=B_{41}+B_{42}\mathbf{j},\ B_5=B_{51}+B_{52}\mathbf{j}\in\mathbb{Q}_c^{n\times t}$; $F_2\in\mathbb{Q}_c^{s\times t}$; and $X=X_1+X_2\mathbf{j},\ Y=Y_1+Y_2\mathbf{j},\ Z=Z_1+Z_2\mathbf{j}\in\mathbb{HQ}_c^{n\times n}$. Set

$$\begin{aligned}
K_1&=\begin{pmatrix}G[(I\otimes A_{11})+(I\otimes A_{12})\mathbf{j}]\\ \mathbf{0}\\ \mathbf{0}\\ \mathbf{0}\\ G[(I\otimes A_{31})+(I\otimes A_{32})\mathbf{j}]\\ \mathbf{0}\end{pmatrix}\mathbb{M},\ W=\begin{pmatrix}K_S&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&K_A&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&K_A&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&K_A\end{pmatrix},\\
K_2&=\begin{pmatrix}\mathbf{0}\\ G[(B_{11}^T\otimes I)+(B_{12}^T\otimes I)\mathbf{j}]\\ \mathbf{0}\\ \mathbf{0}\\ \mathbf{0}\\ G[(B_{51}^T\otimes A_{51}+B_{52}^T\otimes A_{52})+(B_{52}^T\otimes A_{51}+B_{51}^T\otimes A_{52})\mathbf{j}]\end{pmatrix}\mathbb{M},\\
K_3&=\begin{pmatrix}\mathbf{0}\\ \mathbf{0}\\ G[(I\otimes A_{21})+(I\otimes A_{22})\mathbf{j}]\\ G[(B_{21}^T\otimes I)+(B_{22}^T\otimes I)\mathbf{j}]\\ G[(B_{31}^T\otimes I)+(B_{32}^T\otimes I)\mathbf{j}]\\ G[(B_{41}^T\otimes A_{41}+B_{42}^T\otimes A_{42})+(B_{42}^T\otimes A_{41}+B_{41}^T\otimes A_{42})\mathbf{j}]\end{pmatrix}\mathbb{M},
\end{aligned}\tag{5.1}$$

$$K_{11} = \text{Re } K_1,\ K_{12} = \text{Im } K_1,\ K_{21} = \text{Re } K_2,$$
$$K_{22} = \text{Im } K_2,\ K_{31} = \text{Re } K_3,\ K_{32} = \text{Im } K_3,$$
$$L_1 = (K_{11}, K_{21}, K_{31}),\ L_2 = (K_{12}, K_{22}, K_{32}),\ O = \begin{pmatrix} L_1 \\ L_2 \end{pmatrix},$$
$$V = \begin{pmatrix} \text{vec}(\Gamma_{C_1}) \\ \text{vec}(\Gamma_{D_1}) \\ \text{vec}(\Gamma_{C_2}) \\ \text{vec}(\Gamma_{D_2}) \\ \text{vec}(\Gamma_{F_1}) \\ \text{vec}(\Gamma_{F_2}) \end{pmatrix},\ V_1 = \begin{pmatrix} \text{vec}(\text{Re}(\Gamma_{C_1})) \\ \text{vec}(\text{Re}(\Gamma_{D_1})) \\ \text{vec}(\text{Re}(\Gamma_{C_2})) \\ \text{vec}(\text{Re}(\Gamma_{D_2})) \\ \text{vec}(\text{Re}(\Gamma_{F_1})) \\ \text{vec}(\text{Re}(\Gamma_{F_2})) \end{pmatrix},\ V_2 = \begin{pmatrix} \text{vec}(\text{Im}(\Gamma_{C_1})) \\ \text{vec}(\text{Im}(\Gamma_{D_1})) \\ \text{vec}(\text{Im}(\Gamma_{C_2})) \\ \text{vec}(\text{Im}(\Gamma_{D_2})) \\ \text{vec}(\text{Im}(\Gamma_{F_1})) \\ \text{vec}(\text{Im}(\Gamma_{F_2})) \end{pmatrix}, \tag{5.2}$$
$$\text{vec}(\check{X}) = \begin{pmatrix} \text{vec}_S(\text{Re}(X_1)) \\ \text{vec}_A(\text{Im}(X_1)) \\ \text{vec}_A(\text{Re}(X_2)) \\ \text{vec}_A(\text{Im}(X_2)) \end{pmatrix},\ \text{vec}(\check{Y}) = \begin{pmatrix} \text{vec}_S(\text{Re}(Y_1)) \\ \text{vec}_A(\text{Im}(Y_1)) \\ \text{vec}_A(\text{Re}(Y_2)) \\ \text{vec}_A(\text{Im}(Y_2)) \end{pmatrix},$$
$$\text{vec}(\check{Z}) = \begin{pmatrix} \text{vec}_S(\text{Re}(Z_1)) \\ \text{vec}_A(\text{Im}(Z_1)) \\ \text{vec}_A(\text{Re}(Z_2)) \\ \text{vec}_A(\text{Im}(Z_2)) \end{pmatrix},\ M = \begin{pmatrix} W & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & W & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & W \end{pmatrix},\ \xi = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}.$$

Theorem 5.1. *Given matrices $A_1, A_2, C_1, C_2 \in \mathbb{Q}_c^{m\times n}$; $B_1, B_2, D_1, D_2 \in \mathbb{Q}_c^{n\times k}$; $A_3, B_3, F_1 \in \mathbb{Q}_c^{n\times n}$; $A_4, A_5 \in \mathbb{Q}_c^{s\times n}$; $B_4, B_5 \in \mathbb{Q}_c^{n\times t}$; and $F_2 \in \mathbb{Q}_c^{s\times t}$, let the symbols M, O, and ξ be defined in* (5.2). *The system of matrix equations* (1.1) *has a Hermitian solution if and only if*

$$OO^\dagger \xi = \xi. \tag{5.3}$$

The set of Hermitian solutions to the system (1.1) *is*

$$\Phi_1 = \left\{ (X, Y, Z) \middle| \begin{pmatrix} \text{vec}(\widetilde{X}) \\ \text{vec}(\widetilde{Y}) \\ \text{vec}(\widetilde{Z}) \end{pmatrix} = MO^\dagger \xi + M(I_{6n^2-3n} - O^\dagger O)u \right\}, \tag{5.4}$$

where u is an arbitrary column vector of appropriate dimension. Furthermore, if

$$r(O) = 6n^2 - 3n, \tag{5.5}$$

then the system (1.1) *admits a unique Hermitian solution, and the solution set is given by*

$$\Phi_2 = \left\{ (X, Y, Z) \middle| \begin{pmatrix} \text{vec}(\widetilde{X}) \\ \text{vec}(\widetilde{Y}) \\ \text{vec}(\widetilde{Z}) \end{pmatrix} = MO^\dagger \xi \right\}. \tag{5.6}$$

Proof.

$$(1.1) \overset{Lemma\ 2.4}{\Longleftrightarrow} \begin{cases} \Gamma_{A_1X} = \Gamma_{C_1},\ \Gamma_{YB_1} = \Gamma_{D_1}, \\ \Gamma_{A_2Z} = \Gamma_{C_2},\ \Gamma_{YB_2} = \Gamma_{D_2}, \\ \Gamma_{A_3X} + \Gamma_{ZB_3} = \Gamma_{F_1}, \\ \Gamma_{A_4ZB_4} + \Gamma_{A_5YB_5} = \Gamma_{F_2}, \end{cases}$$

$$\Leftrightarrow \begin{cases} \mathrm{vec}(\Gamma_{A_1X}) = \mathrm{vec}(\Gamma_{C_1}),\ \mathrm{vec}(\Gamma_{YB_1}) = \mathrm{vec}(\Gamma_{D_1}), \\ \mathrm{vec}(\Gamma_{A_2Z}) = \mathrm{vec}(\Gamma_{C_2}),\ \mathrm{vec}(\Gamma_{YB_2}) = \mathrm{vec}(\Gamma_{D_2}), \\ \mathrm{vec}(\Gamma_{A_3X}) + \mathrm{vec}(\Gamma_{ZB_3}) = \mathrm{vec}(\Gamma_{F_1}), \\ \mathrm{vec}(\Gamma_{A_4ZB_4}) + \mathrm{vec}(\Gamma_{A_5YB_5}) = \mathrm{vec}(\Gamma_{F_2}), \end{cases}$$

$$\xLeftrightarrow[Lemma\ 2.13]{(5.1)\ and\ (5.2)} K_1 \begin{pmatrix} \mathrm{vec}_S(\mathrm{Re}(X_1)) \\ \mathrm{vec}_A(\mathrm{Im}(X_1)) \\ \mathrm{vec}_A(\mathrm{Re}(X_2)) \\ \mathrm{vec}_A(\mathrm{Im}(X_2)) \end{pmatrix} + K_2 \begin{pmatrix} \mathrm{vec}_S(\mathrm{Re}(Y_1)) \\ \mathrm{vec}_A(\mathrm{Im}(Y_1)) \\ \mathrm{vec}_A(\mathrm{Re}(Y_2)) \\ \mathrm{vec}_A(\mathrm{Im}(Y_2)) \end{pmatrix} + K_3 \begin{pmatrix} \mathrm{vec}_S(\mathrm{Re}(Z_1)) \\ \mathrm{vec}_A(\mathrm{Im}(Z_1)) \\ \mathrm{vec}_A(\mathrm{Re}(Z_2)) \\ \mathrm{vec}_A(\mathrm{Im}(Z_2)) \end{pmatrix} = V,$$

$$\Leftrightarrow (\mathrm{Re}(K_1) + \mathbf{i}\mathrm{Im}(K_1)) \begin{pmatrix} \mathrm{vec}_S(\mathrm{Re}(X_1)) \\ \mathrm{vec}_A(\mathrm{Im}(X_1)) \\ \mathrm{vec}_A(\mathrm{Re}(X_2)) \\ \mathrm{vec}_A(\mathrm{Im}(X_2)) \end{pmatrix} + (\mathrm{Re}(K_2) + \mathbf{i}\mathrm{Im}(K_2)) \begin{pmatrix} \mathrm{vec}_S(\mathrm{Re}(Y_1)) \\ \mathrm{vec}_A(\mathrm{Im}(Y_1)) \\ \mathrm{vec}_A(\mathrm{Re}(Y_2)) \\ \mathrm{vec}_A(\mathrm{Im}(Y_2)) \end{pmatrix}$$

$$+ (\mathrm{Re}(K_3) + \mathbf{i}\mathrm{Im}(K_3)) \begin{pmatrix} \mathrm{vec}_S(\mathrm{Re}(Z_1)) \\ \mathrm{vec}_A(\mathrm{Im}(Z_1)) \\ \mathrm{vec}_A(\mathrm{Re}(Z_2)) \\ \mathrm{vec}_A(\mathrm{Im}(Z_2)) \end{pmatrix} = \mathrm{Re}(V) + \mathbf{i}\mathrm{Im}(V),$$

$$\xLeftrightarrow{(5.2)} \begin{pmatrix} K_{11} & K_{21} & K_{31} \\ K_{12} & K_{22} & K_{32} \end{pmatrix} \begin{pmatrix} \mathrm{vec}(\check{X}) \\ \mathrm{vec}(\check{Y}) \\ \mathrm{vec}(\check{Z}) \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix},$$

$$\xLeftrightarrow{(5.2)} \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} \begin{pmatrix} \mathrm{vec}(\check{X}) \\ \mathrm{vec}(\check{Y}) \\ \mathrm{vec}(\check{Z}) \end{pmatrix} = \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \xLeftrightarrow{(5.2)} O \begin{pmatrix} \mathrm{vec}(\check{X}) \\ \mathrm{vec}(\check{Y}) \\ \mathrm{vec}(\check{Z}) \end{pmatrix} = \xi.$$

Lemma 2.14 establishes that a solution to the system (1.1) exists if and only if (5.3) holds. Under this condition, the general solution is

$$\begin{pmatrix} \mathrm{vec}(\check{X}) \\ \mathrm{vec}(\check{Y}) \\ \mathrm{vec}(\check{Z}) \end{pmatrix} = O^{\dagger}\xi + (I_{6n^2-3n} - O^{\dagger}O)u,$$

where u is an arbitrary column vector of appropriate dimension. Moreover, if (5.5) holds, then the system (1.1) has a unique solution

$$\begin{pmatrix} \mathrm{vec}(\check{X}) \\ \mathrm{vec}(\check{Y}) \\ \mathrm{vec}(\check{Z}) \end{pmatrix} = O^{\dagger}\xi.$$

Next, we proceed to find the Hermitian solutions to the system (1.1). According to Lemma 2.3, we obtain

$$\widetilde{X} = (\mathrm{Re}(X_1),\ \mathrm{Im}(X_1),\ \mathrm{Re}(X_2),\ \mathrm{Im}(X_2)),$$
$$\widetilde{Y} = (\mathrm{Re}(Y_1),\ \mathrm{Im}(Y_1),\ \mathrm{Re}(Y_2),\ \mathrm{Im}(Y_2)),$$
$$\widetilde{Z} = (\mathrm{Re}(Z_1),\ \mathrm{Im}(Z_1),\ \mathrm{Re}(Z_2),\ \mathrm{Im}(Z_2)),$$

and given that $X = X_1 + X_2\mathbf{j},\ Y = Y_1 + Y_2\mathbf{j},\ Z = Z_1 + Z_2\mathbf{j} \in \mathbb{HQ}_c^{n\times n}$, it follows from Proposition 2.10

and Lemma 2.11 that

$$\mathrm{vec}(\widetilde{X}) = \begin{pmatrix} \mathrm{vec}(\mathrm{Re}(X_1)) \\ \mathrm{vec}(\mathrm{Im}(X_1)) \\ \mathrm{vec}(\mathrm{Re}(X_2)) \\ \mathrm{vec}(\mathrm{Im}(X_2)) \end{pmatrix} = \begin{pmatrix} K_S & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & K_A & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & K_A & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & K_A \end{pmatrix} \begin{pmatrix} \mathrm{vec}_S(\mathrm{Re}(X_1)) \\ \mathrm{vec}_A(\mathrm{Im}(X_1)) \\ \mathrm{vec}_A(\mathrm{Re}(X_2)) \\ \mathrm{vec}_A(\mathrm{Im}(X_2)) \end{pmatrix} = W\mathrm{vec}(\check{X}).$$

In a similar manner, we can derive

$$\mathrm{vec}(\widetilde{Y}) = W\mathrm{vec}(\check{Y}),\ \mathrm{vec}(\widetilde{Z}) = W\mathrm{vec}(\check{Z}).$$

Therefore,

$$\begin{pmatrix} \mathrm{vec}(\widetilde{X}) \\ \mathrm{vec}(\widetilde{Y}) \\ \mathrm{vec}(\widetilde{Z}) \end{pmatrix} = \begin{pmatrix} W & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & W & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & W \end{pmatrix} \begin{pmatrix} \mathrm{vec}(\check{X}) \\ \mathrm{vec}(\check{Y}) \\ \mathrm{vec}(\check{Z}) \end{pmatrix} = M \begin{pmatrix} \mathrm{vec}(\check{X}) \\ \mathrm{vec}(\check{Y}) \\ \mathrm{vec}(\check{Z}) \end{pmatrix} = MO^{\dagger}\xi + M(I_{6n^2-3n} - O^{\dagger}O)u.$$

Moreover, when (5.5) holds,

$$\begin{pmatrix} \mathrm{vec}(\widetilde{X}) \\ \mathrm{vec}(\widetilde{Y}) \\ \mathrm{vec}(\widetilde{Z}) \end{pmatrix} = MO^{\dagger}\xi.$$

To sum up, the system (1.1) admits a Hermitian solution if and only if (5.3) holds. Furthermore, the solution is unique when condition (5.5) is satisfied.

The above proof shows that when condition (5.3) holds, the Hermitian solutions set to the system (1.1) is (5.4). Moreover, if (5.5) is satisfied, then the unique Hermitian solution is given by (5.6). □

6. Numerical exemplification

The main results of this paper are validated in this section through a concrete numerical example.

6.1. Algorithm

Algorithm 1

1: **Input**: $A_i,\ B_i\ (i = \overline{1,5}),\ C_j,\ D_j,\ F_j\ (j = \overline{1,2})$.
2: Compute the values of $W_1,\ W_2,\ S,\ N,\ \Omega,\ \alpha_1,\ \alpha_2,\ \alpha_3$ and η.
3: If (3.6) and (3.10) are satisfied, then the unique solution $(X, Y, Z) \in \Theta_4$ is calculated using (3.12).
4: If (3.10) holds, then $(X, Y, Z) \in \Theta_3$ is obtained through (3.11). Otherwise, terminate.
5: **Output** the matrix: (X, Y, Z).

If the system (1.1) is solvable, we obtain sufficiently small values for

$$\gamma_1 = \left\| \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}^{\dagger} \eta - \eta \right\|,\ \gamma_2 = \left\| \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_2^T & \alpha_3 \end{bmatrix} \eta \right\|,$$
$$\gamma_3 = \left\| I_{2p} - \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}^{\dagger} - \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_2^T & \alpha_3 \end{bmatrix} \right\|.$$

6.2. Example

Consider the following commutative quaternion matrices for the system (1.1):

$$
\begin{aligned}
&A_1 = \begin{pmatrix} \mathbf{k} & 0.5\mathbf{j} - \mathbf{k} \\ 0.5\mathbf{k} & 1 + 0.5\mathbf{i} \end{pmatrix},\ A_2 = \begin{pmatrix} 2 - 0.5\mathbf{i} + 0.5\mathbf{j} + \mathbf{k} & 2\mathbf{j} \\ \mathbf{i} + 0.5\mathbf{j} & 1 + \mathbf{i} + \mathbf{j} \end{pmatrix},\\
&B_2 = \begin{pmatrix} 0.5 + 3\mathbf{i} + 0.5\mathbf{j} & -2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \\ 1 + 0.5\mathbf{i} + \mathbf{k} & 0.5\mathbf{i} \end{pmatrix},\ A_5 = \begin{pmatrix} 0 & 1 + \mathbf{j} + 0.5\mathbf{k} \\ 0 & 1 + 2\mathbf{j} \end{pmatrix},\\
&C_1 = \begin{pmatrix} -0.5 + 2\mathbf{i} + 0.5\mathbf{j} + 1.5\mathbf{k} & -0.75 + 0.5\mathbf{i} - \mathbf{j} \\ 0.5 + 2\mathbf{i} - 1.5\mathbf{j} + 0.5\mathbf{k} & -0.5 + 0.5\mathbf{i} + 0.25\mathbf{k} \end{pmatrix},\ B_3 = \begin{pmatrix} \mathbf{j} & \mathbf{k} \\ 0 & \mathbf{i} \end{pmatrix},\\
&A_3 = \begin{pmatrix} 1 + 0.5\mathbf{k} & \mathbf{j} \\ 0 & 1 + \mathbf{j} \end{pmatrix},\ A_4 = \begin{pmatrix} 1 & \mathbf{j} \\ \mathbf{i} & \mathbf{k} \end{pmatrix},\ B_4 = \begin{pmatrix} \mathbf{i} + 0.5\mathbf{k} & -\mathbf{k} \\ 0.5 + \mathbf{i} & 0 \end{pmatrix},\\
&C_2 = \begin{pmatrix} 1.5 - 2\mathbf{i} + 2\mathbf{j} + 2.5\mathbf{k} & 0.125 + 2.5\mathbf{i} - 0.25\mathbf{j} + 0.125\mathbf{k} \\ 1.5 + 1.5\mathbf{i} & -0.25 + \mathbf{i} - \mathbf{j} + 1.125\mathbf{k} \end{pmatrix},\\
&D_1 = \begin{pmatrix} 0 & -1.5 - \mathbf{i} - 3\mathbf{j} \\ 0 & -1.5 + 3.125\mathbf{i} - 1.125\mathbf{j} + 2.75\mathbf{k} \end{pmatrix},\ B_1 = \begin{pmatrix} 0 & \mathbf{i} - \mathbf{j} + \mathbf{k} \\ 0 & 0.5 + \mathbf{j} + 0.5\mathbf{k} \end{pmatrix},\\
&F_1 = \begin{pmatrix} 1 + 2.5\mathbf{i} + 2.5\mathbf{j} + \mathbf{k} & -1.25 + 1.5\mathbf{i} + 1.5\mathbf{j} + 2\mathbf{k} \\ 0.5 + \mathbf{i} + \mathbf{k} & 0.5 + 0.5\mathbf{i} - 0.5\mathbf{j} \end{pmatrix},\\
&D_2 = \begin{pmatrix} 3.375 + 3.25\mathbf{i} - 2.75\mathbf{j} & -4.125 + 6\mathbf{j} \\ -0.75 - 0.25\mathbf{j} + 2.5\mathbf{k} & 1 + \mathbf{i} - 0.5\mathbf{j} - \mathbf{k} \end{pmatrix},\ B_5 = \begin{pmatrix} 0.5\mathbf{i} & \mathbf{j} \\ 0.5\mathbf{k} & 0.5\mathbf{i} \end{pmatrix},\\
&F_2 = \begin{pmatrix} -1.625 + 3.0625\mathbf{i} - 1.375\mathbf{j} + 1.5\mathbf{k} & 2.375 + 0.25\mathbf{i} + 0.375\mathbf{j} - 1.0625\mathbf{k} \\ -2.875 + 0.75\mathbf{i} - 1.875\mathbf{j} + \mathbf{k} & 1.875 + \mathbf{i} + 3.75\mathbf{j} - \mathbf{k} \end{pmatrix}.
\end{aligned}
$$

Taking

$$
\ddot{X} = \begin{pmatrix} 2 + \mathbf{i} + \mathbf{j} & \mathbf{i} + \mathbf{j} + \mathbf{k} \\ 1 + \mathbf{i} - \mathbf{j} & 0.5\mathbf{j} \end{pmatrix},\ \ddot{Y} = \begin{pmatrix} 0.5 + \mathbf{i} + \mathbf{k} & \mathbf{i} - \mathbf{k} \\ 1 + \mathbf{j} & 0.25\mathbf{i} + \mathbf{k} \end{pmatrix},\ \ddot{Z} = \begin{pmatrix} 1 - \mathbf{i} + \mathbf{k} & 0.25\mathbf{i} \\ 0.5\mathbf{j} & \mathbf{k} \end{pmatrix}.
$$

Let

$$
\begin{gathered}
\Gamma_{A_1} G(\ddot{X}) = \Gamma_{C_1},\ \Gamma_{\ddot{Y}} G(B_1) = \Gamma_{D_1},\\
\Gamma_{A_2} G(\ddot{Z}) = \Gamma_{C_2},\ \Gamma_{\ddot{Z}} G(B_2) = \Gamma_{D_2},\\
\Gamma_{A_3} G(\ddot{X}) + \Gamma_{\ddot{Z}} G(B_3) = \Gamma_{F_1},\\
\Gamma_{A_4} G(\ddot{Z}) G(B_4) + \Gamma_{A_5} G(\ddot{Y}) G(B_5) = \Gamma_{F_2}.
\end{gathered}
$$

Leveraging Algorithm 1 and MATLAB (R2021b), we get

$$
r\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = 44 < 12n^2 = 48,\ \gamma_2 = 2.6796 \times 10^{-14}.
$$

Since the given coefficient matrices satisfy condition (3.10), the system (1.1) is solvable. The condition $\mathrm{r}\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} < 48$ implies that the system (1.1) has infinitely many solutions in Θ_3. Additionally, we get $\gamma_1 = 2.2837 \times 10^{-14}$ and $\gamma_3 = 2.7979 \times 10^{-14}$.

7. Conclusions

This paper investigates, via two distinct approaches, the necessary and sufficient conditions for the solvability of the system (1.1) over $\mathbb{Q}_c$ and derives its general solution sets. As a key application, we prove that the optimization problem (4.1) admits a unique minimum solution. Furthermore, the solvability condition and general solution set for the Hermitian solutions of the system (1.1) over $\mathbb{Q}_c$ are characterized. Finally, we provide an algorithm and a numerical example to validate our main results.

Author contributions

Xiao-Xiao Ma, Long-Sheng Liu, and Xiao-Quan Chen were involved in all stages of the work, including conceptualization, formal analysis, methodology, software, validation, and the writing of the manuscript from initial draft to review and editing. All authors contributed equally to this article and have approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors disclose no competing interests.

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