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Research article

Metrizability of 1-form projective deformation of sprays

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Abstract: In this paper, we consider a special case of the inverse problem of calculus of variations. For a given spray S on a manifold M, we investigate the projective deformation of S by a 1-form $\beta \in \Lambda^1(M)$ on M, precisely, $S_\beta = S - 2\beta C$. We show that, in general, the spray S_β is not Finsler metrizable. Moreover, the metrizability of S_β, when the background spray is flat, is characterized. In this case, we establish an explicit formula for the Finsler function whose geodesic spray is S_β. We conclude that this metric is a projectively flat metric of nonzero constant flag curvature; that is, the obtained metric is a solution for Hilbert's fourth problem.

Keywords: sprays; metrizability; projective deformation; Hilbert's fourth problem; projectively flat
Mathematics Subject Classification: 53B40, 53C60

1. Introduction

The concept of sprays, introduced by W. Ambrose et al. [1] in 1960, describes second-order vector fields that model systems of second-order ordinary differential equations with positively 2-homogeneous coefficient functions (SODEs). There exists a direct correspondence between sprays and SODEs: Every spray is associated with a SODE, and vice versa.

The inverse problem of the calculus of variations is a classical topic in differential geometry that seeks to characterize second-order ordinary differential equations. Or, more generally, sprays which can be derived from a variational principle. In the most interesting cases, the sprays can be derived from a variational principle. In particular, the geodesic structure of a Finsler metric is described by the geodesic spray of the Finsler metric. The Finsler metrizability problem for a spray S focuses on identifying a Finsler structure whose geodesic spray is S. This issue can be interpreted as a specific instance of the inverse problem of calculus of variations. Significant progress has been made in understanding the metrizability problem, with numerous important results documented in the literature; for example, see [5, 9, 12–14, 17]. It is well known that sprays with vanishing curvature are

Finsler metrizable. The case of nonzero curvature, however, is significantly more complex, and only partial results are currently available. Notably, the works [6] and [7] provide characterization of sprays that are metrizable by Finsler functions with nonzero constant flag curvature and nonzero scalar flag curvature, respectively. Based on these results, we refer to such sprays as having constant or scalar curvature, respectively. These notions appear also in [15] and [19], albeit in a slightly different context and meaning.

The Finsler metric F on an open subset $U \subset \mathbb{R}^n$ is called *projectively flat* if its geodesics are straight lines. A Finsler manifold (M, F) is called *locally projectively flat* if, for every point $x \in M$, there exists an adapted local coordinate system–that is, a mapping $(x^1, \ldots, x^n)$ on a neighborhood U of x into the Euclidean space $\mathbb{R}^n$–such that the straight lines of $\mathbb{R}^n$ correspond to the geodesics of (M, F). In the adapted coordinate system (x^i, y^i) on the tangent manifold TU, the geodesic spray coefficients are of the form

$$G^i = P(x, y)\, y^i, \quad i = 1, \ldots, n. \tag{1.1}$$

Characterizing all locally projectively flat Finsler metrics–that is, metrics whose geodesics are straight lines on an open subset of $\mathbb{R}^n$–is the typical case of Hilbert's fourth problem. Beltrami's theorem states that a Riemannian metric has constant sectional curvature if and only if it is locally projectively flat. In Finsler geometry, the notion of flag curvature serves as the counterpart to sectional curvature in Riemannian geometry. However, the above equivalence does not hold: There are Finsler metrics with constant flag curvature that are not projectively flat.

If two sprays on the same manifold share the same geodesics as point sets, then they are considered projectively related. For given sprays S and $\widetilde{S}$ on a manifold M, the two sprays are projectively related if there exists a positively 1-homogeneous function $\mathcal{P}\colon TM \to \mathbb{R}$ such that

$$\widetilde{S} = S - 2\mathcal{P}\cdot C, \tag{1.2}$$

where C is the Liouville vector field. The function $\mathcal{P}$ is called the *projective factor* of the projective deformation. In [18], Yang demonstrated that there exist sprays within the projective class of a projectively flat spray with constant flag curvature that are not Finsler metrizable. Building on this result, the authors in [4] showed that for any spray, its projective class includes non-metrizable sprays. They achieved this by examining the projective deformation of the geodesic spray S of a Finsler metric F, where the projective factor (1.2) has the form $\mathcal{P} = \lambda \cdot F$, a scalar multiple of the Finsler function F of S. They proved that for almost any $\lambda \in \mathbb{R}$, the deformed spray is not Finsler metrizable. Recently, this result was extended for the case of holonomy invariant projective deformations [10].

Determining the necessary and sufficient conditions for a projective deformation of a Finsler spray to be metrizable is an intriguing task in general. Despite its complexity, this problem includes Hilbert's fourth problem as a special case. Thus, partial findings that satisfy certain geometric or analytic features of the projective factor are of importance.

In this paper, we introduce and investigate the 1-form projective deformation of a spray: For a given spray S on a manifold M, we define the 1-form deformation as the projective deformation of S by a 1-form $\beta \in \Lambda^1(M)$ on M:

$$\widetilde{S}_\beta := S - 2\beta\cdot C. \tag{1.3}$$

We prove that, in general, the spray $\widetilde{S}_\beta$ is not Finsler metrizable. We characterize the metrizability property of this deformation when the background spray is flat. Let S_0 denote the flat spray. Using adapted coordinates, the spay coefficients of S_0 are identically zero, and (2.1) gives the local expression

$$S_0 = y^i \frac{\partial}{\partial x^i}, \tag{1.4}$$

and its 1-form deformation with $\beta \in \Lambda^1(M)$ is

$$S_\beta := S_0 - 2\beta \cdot C. \tag{1.5}$$

In Theorem 3.6, we show that the 1-form deformation (1.5) of the spray (1.4) is Finsler metrizable if and only if the coefficients of $\beta = \beta_k(x)dx^k$ satisfy (3.9) where $a_{ij} = a_{ji}$, and c are constants and $b = (b_1, \ldots, b_n)$ is a constant vector.

We remark that formula (3.9) can be found in G. Yang's paper [19]. However, his method is different from ours. As a by-product, we provide a family of projectively flat metrics of nonzero constant flag curvature, and we get solutions for Hilbert's fourth problem. Namely, we establish the following family of projectively flat metrics:

$$F = \sqrt{\frac{4q(x)a_{ij}y^i y^j - 4(a_{ij}x^i y^j)^2 - 4\langle b, y\rangle a_{ij}x^i y^j - \langle b, y\rangle^2}{[2q(x)]^2}} \tag{1.6}$$

where $q(x) := a_{ij}x^i x^j + \langle b, x\rangle + c$, $a_{ij} = a_{ji}$. The aforementioned family represents projectively flat Riemannian metrics with a nonzero constant flag curvature. Additionally, the 1-form with components described in (3.9) serves as a Hamel function associated with S_0, making it a notable case of interest.

With an appropriate choice of the constants, the Klein metric can be seen as a special case of (1.6), namely $a_{ij} = \mu\delta_{ij}$, $b_i = 0$, and $c = 1$.

2. Preliminaries

Let M be an n-dimensional manifold, with its tangent bundle denoted by (TM, π_M, M), and let $(\mathcal{T}M, \pi, M)$ represent the subbundle consisting of nonzero tangent vectors. The local coordinates on the manifold M are written as $\left(x^i\right)$, while the corresponding coordinates on TM are given by $\left(x^i, y^i\right)$, where y^i are the components of the tangent vectors. The tangent bundle TM is equipped with a natural almost-tangent structure J, which is locally given by $J = \frac{\partial}{\partial y^i} \otimes dx^i$. The canonical (or Liouville) vector field C on TM is a vertical vector field defined by $C = y^i \frac{\partial}{\partial y^i}$, where the Einstein summation convention is used throughout.

A spray S is a vector field $S \in \mathfrak{X}(\mathcal{T}M)$ where $JS = C$ and $[C, S] = S$. Locally, a spray S can be written as follows:

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i}, \tag{2.1}$$

where $G^i = G^i(x, y)$, called the spray coefficients, are 2-homogeneous functions in the y-variable.

A nonlinear connection is an n-dimensional distribution $H(\mathcal{T}M)$, which is supplementary to the vertical distribution $V(\mathcal{T}M) := \operatorname{Ker} \pi_*$. In other words, for each $z \in \mathcal{T}M$, the tangent space at z satisfies the direct sum decomposition

$$T_z(\mathcal{T}M) = H_z(\mathcal{T}M) \oplus V_z(\mathcal{T}M). \tag{2.2}$$

Each spray S provides a canonical nonlinear connection, where the horizontal and vertical projectors are

$$h = \frac{1}{2}(Id + [J, S]), \qquad v = \frac{1}{2}(Id - [J, S]). \tag{2.3}$$

The projectors h and v have the following local formulae:

$$h = \frac{\delta}{\delta x^i} \otimes dx^i, \qquad v = \frac{\partial}{\partial y^i} \otimes \delta y^i, \tag{2.4}$$

where we have

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j(x, y)\frac{\partial}{\partial y^j}, \quad \delta y^i = dy^i + N_j^i(x, y)dx^j, \quad N_i^j(x, y) = \frac{\partial G^j}{\partial y^i}. \tag{2.5}$$

Let K be a vector k-form on M; then, we define the graded derivations i_K and d_K on the Grassmann algebra of M, as follows:

$$i_K f = 0, \quad i_K df = df \circ K, \tag{2.6}$$

where $f \in C^\infty(M)$ and df represent the exterior derivative of f. Additionally, the derivation d_K is given by:

$$d_K := [i_K, d] = i_K \circ d - (-1)^{k-1} d i_K. \tag{2.7}$$

The Jacobi endomorphism (or, Riemann curvature in [16]) Φ is defined by

$$\Phi = v \circ [S, h] = R^i{}_j \frac{\partial}{\partial y^i} \otimes dx^j = \left(2\frac{\partial G^i}{\partial x^j} - S(N_j^i) - N_k^i N_j^k\right)\frac{\partial}{\partial y^i} \otimes dx^j. \tag{2.8}$$

The curvature of a spray S, denoted by R, is the Nijenhuis torsion of the horizontal projection, that is

$$R = \frac{1}{2}[h, h] = R_{ij}^\ell \frac{\partial}{\partial y^\ell} \otimes dx^i \otimes dx^j,$$

where the components R_{ij}^ℓ are given by

$$R_{ij}^\ell = \frac{\delta N_i^\ell}{\delta x^j} - \frac{\delta N_j^\ell}{\delta x^i}. \tag{2.9}$$

The two curvatures Φ and R are related by

$$3R = [J, \Phi], \quad \Phi = i_S R. \tag{2.10}$$

The Ricci curvature, Ric, and the Ricci scalar, $\rho \in C^\infty(\mathcal{T}M)$ (see [2] and [16]), are defined by

$$\text{Ric} = (n-1)\rho = R^i{}_i = \text{Tr}(\Phi). \tag{2.11}$$

Definition 2.1. A spray S is called *isotropic* if the Jacobi endomorphism Φ has the form

$$\Phi = \rho J - \alpha \otimes C, \tag{2.12}$$

where $\alpha = \alpha_i(x, y)dx^i \in \Lambda^1(\mathcal{T}M)$ is a semi-basic 1-form on $\mathcal{T}M$.

Due to the homogeneity property, for isotropic sprays, the Ricci scalar satisfies

$$\rho = i_S \alpha. \tag{2.13}$$

Definition 2.2. A pair (M, F) is called a Finsler manifold of dimension n where M is an n-dimensional differentiable manifold, and F is a real-valued function $F : TM \longrightarrow \mathbb{R}$ such that

a) F is smooth and strictly positive on $\mathcal{T}M$,
b) F is positively homogeneous of degree 1 in y, i.e., $\mathcal{L}_C F = F$,
c) The Hessian matrix $g_{ij} = \frac{\partial^2 E}{\partial y^i \partial y^j}$ has rank n on $\mathcal{T}M$, where $E := \frac{1}{2}F^2$ is the energy function.

From condition c), one can get that the 2-form $dd_J E$ is nondegenerate, and we conclude that the Euler-Lagrange equation

$$i_S dd_J E - d(E - \mathcal{L}_C E) = 0,$$

identifies a unique spray S on TM. This spray is known as the *geodesic spray* of F.

Definition 2.3. A spray S on a manifold M is called *Finsler metrizable* if there exists a Finsler metric F whose geodesic spray is S.

Definition 2.4. The Finsler metric F is called of scalar flag curvature μ if the Jacobi endomorphism (2.8) satisfies the equation

$$\Phi = \mu(F^2 J - F d_J F \otimes C),$$

where $\mu \in C^\infty(\mathcal{T}M)$. Moreover, when μ is a constant function, then the Finsler metric is called of constant flag curvature.

We remark that any spray metrizable by a Finsler function with constant or scalar flag curvature is necessarily isotropic. Characterizations of sprays metrizable by Finsler functions with nonzero constant flag curvature and nonzero scalar flag curvature are provided in [6, Theorem 4.1] and [7, Theorem 3.1], respectively.

Theorem 2.5. *[6, 7] Consider S a spray with non-vanishing Ricci curvature.*

1) The spray S is metrizable by a Finsler function of nonzero constant flag curvature if and only if it is isotropic, and its Jacobi endomorphism (2.12) *satisfies the following equations (CFC test):*

(i) $2(n-1)\Phi - 2(\operatorname{Tr}\Phi)J + d_J(\operatorname{Tr}\Phi) \otimes C = 0$*;*

(ii) $d_h(\operatorname{Tr}\Phi) = 0$*;*

(iii) $\operatorname{rank} dd_J(\operatorname{Tr}\Phi) = 2n$.

2) The spray S is metrizable by a Finsler function of nonzero scalar flag curvature if and only if it is isotropic, and its Jacobi endomorphism (2.12) *satisfies the following conditions (SFC test):*

(i) $d_J(\alpha/\rho) = 0$*;*

(ii) $D_{hX}(\alpha/\rho) = 0$*, for all* $X \in \mathfrak{X}(\mathcal{T}M)$*;*

(iii) $d(\alpha/\rho) + 2i_{\mathbb{F}}\alpha/\rho \wedge \alpha/\rho$ *is a symplectic form on* $\mathcal{T}M$.

Based on these results, we use the following definition:

Definition 2.6. A spray S is said to have *constant curvature*, respectively *scalar curvature*, if its curvature vanishes, or if its Jacobi endomorphism Φ is isotropic and satisfies the constant flag curvature (CFC), respectively scalar flag curvature (SFC) test.

Going forward and for simplicity, we denote the partial differentiation with regard to x^i by ∂_i, while we denote the partial differentiation with regard to y^i by $\dot{\partial}_i$.

Remark 2.7. Let F be Finsler metric F on an open subset U of $\mathbb{R}^n$. By making use of [8], F is projectively flat if and only if it satisfies the system:

$$y^j \dot{\partial}_i \partial_j F - \partial_i F = 0. \tag{2.14}$$

In this case, the geodesic coefficients satisfy (1.1) where the projective factor $P(x, y)$ can be calculated as

$$P = \frac{y^k \partial_k F}{2F}. \tag{2.15}$$

3. 1-form deformation of sprays

A curve $\gamma\colon I \to M$ is called regular if its tangent lift $\gamma'\colon I \to TM$ exists. A regular curve γ on M is termed a geodesic of a spray S if the condition $S \circ \gamma' = \gamma''$ holds. Locally, for $\gamma(t) = \big(x^i(t)\big)$, this implies that γ satisfies the equation:

$$\frac{d^2 x^i}{dt^2} + 2G^i\Big(x, \frac{dx}{dt}\Big) = 0. \tag{3.1}$$

An orientation-preserving reparameterization $t \to \tilde{t}(t)$ of the system (3.1) yields a new spray $\widetilde{S}$ satisfying (1.2), where the $\mathcal{P} \in C^\infty(\mathcal{T}M)$, called the projective factor, is a positively 1-homogeneous scalar function. Moreover, $\mathcal{P}$ and the new parameter satisfy the property

$$\frac{d^2\tilde{t}}{dt^2} = 2\mathcal{P}\Big(x^i(t), \frac{dx^i}{dt}\Big)\frac{d\tilde{t}}{dt}, \qquad \frac{d\tilde{t}}{dt} > 0.$$

Two sprays, S and $\widetilde{S}$, are said to be projectively related if their geodesics coincide, modulo an orientation-preserving reparameterization. When S and $\widetilde{S}$ are projectively related, one can be considered as the projective deformation of the other. Regarding the metrizability problem, one has to remark that, in general, the projective deformation of a metrizable spray is not metrizable. For further details and examples, see [4, 10].

In this work, we introduce the concept of a 1-form deformation (1.3) for a given spray S on a manifold M, where $\beta \in \Lambda^1(M)$ is a 1-form on the base manifold M. We remark, that β can be considered as a linear function on the tangent manifold TM, and its local expression can be considered as

$$\beta = \beta_k(x) dx^k = \beta_k(x) y^k. \tag{3.2}$$

Generally, the deformation (1.3) is not metrizable even if S is metrizable. Indeed, if, for example, β is a holonomy invariant form preserved by the parallel translation associated with the spray S, then, by making use of the results of [10], we get that $\widetilde{S}$ is not Finsler metrizable.

The concept of first integral of a spray (or equivalently, geodesic-invariant function) was studied in [11]. Let S be a spray. A function f on TM (or $\mathcal{T}M$) is called a first integral of S, if it is S-invariant, that is, it is constant along the geodesics of S. We have the following theorem:

Theorem 3.1. *Let $\widetilde{S}_\beta = S - 2\lambda\beta C$ be a 1-form deformation of a metrizable spray S, where $\lambda \in \mathbb{R}$. Then, if β is closed and S-invariant, then $\widetilde{S}_\beta$ is not Finsler metrizable for almost all values of λ.*

Proof. Assume that $\beta = \beta_i(x)y^i$ is a first integral of S. Then, we have

$$\mathcal{L}_S(\beta) = y^i\partial_i\beta - 2G^i\dot{\partial}_i\beta = y^i\partial_i\beta - 2G^i\beta_i = 0. \tag{3.3}$$

Taking the derivative of the aforementioned equation with regard to y^j and y^k, we get

$$\partial_j\beta_k + \partial_k\beta_j - 2G^i_{jk}\beta_i = 0. \tag{3.4}$$

Since β is closed, that implies $\partial_j\beta_k = \partial_k\beta_j$; therefore,

$$\partial_k\beta_j - G^i_{jk}\beta_i = 0. \tag{3.5}$$

Then, contracting the equation mentioned above by y^j, we have

$$\partial_k\beta - N^i_k\dot{\partial}_i\beta = 0. \tag{3.6}$$

Then, the left side of the above equation is the horizontal covariant derivative of β with respect to the induced Berwald connection. That is, $d_h\beta = 0$, i.e., β is a holonomy invariant function, then, by making use of [10], $\widetilde{S}_\beta$ is not Finsler metrizable for almost every value of λ. □

Remark 3.2. Recall that, for a given spray S, a positively 1-homogeneous smooth function P on $\mathcal{T}M$ is called a Hamel function if $d_h d_J P = 0$. Based on the concept of Hamel functions, one can read the above theorem as follows: Since the 1-form β is closed, then it is a Hamel function. But, Hamel function that S-invariant is holonomy invariant (see [3]).

Proposition 3.3. *Let $\widetilde{S}_\beta$ be the 1-form deformation* (1.3) *where β is closed. Then, $\widetilde{S}_\beta$ is of constant curvature if and only if S is of constant curvature.*

Proof. We have

$$d_h d_J\beta(\partial_i, \partial_j) = \partial_i\dot{\partial}_j\beta - \partial_j\dot{\partial}_i\beta = \partial_i\beta_j - \partial_j\beta_i. \tag{3.7}$$

The fact that β is closed implies that $\partial_i\beta_j = \partial_j\beta_i$. That is, the projective factor β is a Hamel function, and by the Finslerian version of Beltrami's Theorem [3, Theorem 3.2], the result follows. □

1-form projective deformation of a flat spray

Let us consider the case when the background spray, denoted by S_0, is flat. In an adapted coordinate system, its spray coefficients are identically zero, and (2.1) gives the local expression (1.4). In what follows, we use this adapted coordinate system.

Its 1-form projective deformation (1.5) by β takes the form

$$S_\beta = y^i\frac{\partial}{\partial x^i} - 2\beta_j(x)y^j y^i\frac{\partial}{\partial y^i}, \tag{3.8}$$

where $\beta = \beta_i(x)dx^i$. The following lemma, based on [4, Proposition 4.4], will be helpful for further use.

Lemma 3.4. *For the deformation spray $S_\beta = S_0 - 2\beta C$ of a flat spray S_0, the corresponding horizontal projectors h_β and Jacobi endomorphisms Φ_β of the two sprays are related as follows:*

(a) $h_\beta = h_0 - \beta J - d_J\beta \otimes C$,

(b) $\Phi_\beta = \rho J - \alpha \otimes C$,

where h_β and Φ_β (respectively h_0 and Φ_0) are the horizontal projector and the Jacobi endomorphism of S_β (resp. S_0), $\rho := \beta^2 - S_0(\beta)$, and $\alpha := \beta d_J\beta + d_J(S_0\beta) - 3d_{h_0}\beta$.

Consider the following special case for the 1-form components $\beta_i(x)$, precisely

$$\beta_k(x) = -\frac{2a_{ik}x^i + b_k}{2q(x)}, \tag{3.9}$$

where $q(x) := a_{ij}x^ix^j + \langle b, x\rangle + c$, $a_{ij} = a_{ji}$, c are constants and $b = (b_1, b_2, ..., b_n)$ is an arbitrary but fixed vector. From Proposition 3.3 we have the following corollary:

Corollary 3.5. *Let S_β be the spray* (1.5) *such that the components of β are given by* (3.9). *Then, β is closed and hence a Hamel function associated with S_0. Moreover, S_0 is of vanishing flag curvature, so by Proposition 3.3, the spray S_β is of constant curvature.*

It is known that the projective deformation preserves the Jacobi endomorphism if and only if the projective factor is a Funk function. That is, the spray (3.8) is of zero curvature if and only if β is a funk function since S_0 is flat. In this case, the spray (3.8) is locally metrizable.

Theorem 3.6. *The* 1*-form deformation $S_\beta = S_0 - 2\beta C$ of the flat spray S_0 with $\beta = \beta_k dx^k$ is Finsler metrizable by a Finsler function of nonzero constant curvature if and only if β_k is given by the formula* (3.9) *with* (3.31) *satisfied.*

Lemma 3.7. *[6, Theorem 4.1] Consider an isotropic spray S with Jacobi endomorphism having the expression* (2.12). *Then, the metrizability of S by a Finsler metric of constant flag curvature is characterized by the following conditions:*

(i) $d_J\alpha = 0$,

(ii) $d_h\rho = 0$,

(iii) $\mathrm{rank}(dd_J\rho) = 2n$.

Now, we are ready to prove Theorem 3.6 as follows.

Proof of Theorem 3.6. We are going to show that the conditions (i)–(iii) in Lemma 3.7 are satisfied if and only if $\beta_k(x)$ is given by the formula (3.9).

First, let the spray S_β be metrizable by a Finsler metric of nonzero constant flag curvature. Then, the conditions (i)–(iii) in Lemma 3.7 are satisfied, and our goal is to get an explicit formula for $\beta_k(x)$. Since the condition (i) is satisfied, then $d_J\alpha = 0$. Since the spray S_0 is flat, then the curvature $R_0 = \frac{1}{2}[h_0, h_0] = 0$ and, therefore, $d^2_{h_0} = 0$. Moreover, the 2-form $d_J\alpha$ has been calculated in [7, Eq (4.4)] as follows:

$$d_J\alpha = 3d_Jd_{h_0}\beta = -3d_{h_0}d_J\beta. \tag{3.10}$$

That is, $d_J\alpha = -3d_{h_0}d_J\beta = 0$ if and only if $d_J\beta = d_{h_0}g$, for some smooth function g on an open subset of TM. Then, we have

$$d_J\beta(\partial_i) = d_{h_0}g(\partial_i) \quad \Rightarrow \quad \dot{\partial}_i(\beta_j y^j) = \partial_i g \quad \Rightarrow \quad \beta_i = \partial_i g. \tag{3.11}$$

So, we can write $\beta = y^i\beta_i(x) = S_0(g)$ and $\beta_i(x) = \partial_i g$. Using Lemma 3.4 **(a)**, we have $d_h\rho = d_{h_0}\rho - \beta d_J\rho - 2\rho d_J\beta$. That is, the condition (ii) is satisfied if and only if

$$d_{h_0}\rho - S_0(g)d_J\rho - 2\rho d_{h_0}g = 0. \tag{3.12}$$

Applying the above equation on ∂_i and using the facts that $S_0(g) = \beta$ and $\rho = \beta^2 - S_0(\beta)$, we have

$$\partial_i\rho - \beta\dot{\partial}_i\rho - 2\rho\partial_i g = 0. \tag{3.13}$$

Making use of the facts that $\beta = y^i\beta_i$ and $\rho = \beta^2 - S_0\beta$ and by substituting into (3.13), we get

$$4\beta\partial_i\beta - 4\beta^2\beta_i - y^r\partial_r\partial_i\beta + 2y^r\partial_r\beta\beta_i = 0, \tag{3.14}$$

which can be rewritten in the form

$$y^r y^s(\partial_r\partial_i\beta_s - 4\beta_r\partial_i\beta_s - 2\partial_r\beta_s\beta_i + 4\beta_r\beta_s\beta_i) = 0. \tag{3.15}$$

By differentiation with respect to y^j and y^k, the above equation takes the form

$$\partial_j\partial_i\beta_k - 4\beta_j\partial_i\beta_k - 2\partial_j\beta_k\beta_i + 4\beta_j\beta_k\beta_i = 0. \tag{3.16}$$

Since $\beta_i = \partial_i g$, we get

$$\partial_i\partial_j\partial_k g - 4\partial_j g\partial_i\partial_k g - 2\partial_j\partial_k g\partial_i g + 4\partial_i g\partial_j g\partial_k g = 0. \tag{3.17}$$

Now, using the substitution $g = \ln\frac{1}{\sqrt{f}}$, where $f(x)$ is a locally positive smooth function on M, the above PDE transforms to

$$2f\partial_i\partial_j\partial_k f = \partial_j\partial_k f\partial_i f - \partial_i\partial_j f\partial_k f. \tag{3.18}$$

One can notice that the left side of the above equation is symmetric, especially in i and k, and the right side is anti-symmetric in i and k. Therefore, we must have

$$\partial_i\partial_j\partial_k f = 0, \tag{3.19}$$

from which the function f must be given by the form

$$f(x) = c_{jk}x^j x^k + c_i x^i + c. \tag{3.20}$$

Consequently, the solution of (3.13) is given by

$$g(x, y) = -\frac{1}{2}\ln\left(a_{ij}x^i x^j + \langle b, x\rangle + c\right). \tag{3.21}$$

Differentiating (3.21) with respect to ∂_k we have

$$\beta_k(x) = \partial_k g = -\frac{2a_{ik}x^i + b_k}{2(a_{ij}x^i x^j + \langle b, x\rangle + c)}. \tag{3.22}$$

Conversely, assume that $S = S_0 - 2\beta C$, where $\beta = \beta_k(x)y^k$ and $\beta_k(x)$ is given by (3.9). The 2-form $d_J\alpha$, as we discussed before, is given by

$$d_J\alpha = 3d_J d_{h_0}\beta = -3d_{h_0}d_J\beta. \tag{3.23}$$

Then we have

$$d_J\alpha(\partial_i, \partial_j) = -3d_{h_0}d_J\beta(\partial_i, \partial_j) = -3(\partial_i\dot{\partial}_j\beta - \partial_j\dot{\partial}_i\beta) = -3(\partial_i\beta_j - \partial_j\beta_i). \tag{3.24}$$

Using the property that a_{ij} is symmetric and β is closed ($\partial_i\beta_j = \partial_j\beta_i$), then $d_J\alpha = 0$.

Now, let us calculate ρ by making use of its definition in Lemma 3.4. First, we have to compute $S_0(\beta) = y^k\partial_k\beta$ as follows

$$S_0(\beta) = -\frac{2q(x)a_{ij}y^i y^j - \left(2a_{ij}x^i y^j + \langle b, y\rangle\right)^2}{2(q(x))^2}. \tag{3.25}$$

Using the formula of β together with (3.25), we have

$$\rho = \frac{4q(x)a_{ij}y^i y^j - 4(a_{ij}x^i y^j)^2 - 4\langle b, y\rangle a_{ij}x^i y^j - \langle b, y\rangle^2}{(2q(x))^2}. \tag{3.26}$$

Differentiating (3.26) with respect to y^k and x^k respectively, we get

$$\dot{\partial}_k\rho = \frac{8q(x)a_{ik}y^i - 8a_{ij}x^i y^j a_{kr}x^r - 4b_k a_{ij}x^i y^j - 4\langle b, y\rangle a_{kj}x^j - 2\langle b, y\rangle b_k}{(2q(x))^2}, \tag{3.27}$$

$$\begin{aligned}\partial_k\rho = &\frac{4a_{ij}y^i y^j\,(2a_{rk}x^r + b_k) - 8a_{ij}x^i y^j a_{rk}y^r - 4\langle b, y\rangle a_{ik}y^i}{(2q(x))^2}\\ &- \frac{4\,(2ac_{rk}x^r + b_k)\left(4q(x)a_{ij}y^i y^j - 4(a_{ij}x^i y^j)^2 - 4\langle b, y\rangle a_{ij}x^i y^j - \langle b, y\rangle^2\right)}{(2q(x))^3}.\end{aligned} \tag{3.28}$$

By using Lemma 3.4 (a), we have $d_h\rho = d_{h_0}\rho - \beta d_J\rho - 2\rho d_J\beta$. That is, we obtain

$$d_h\rho(\partial_i) = \partial_i\rho - \beta\dot{\partial}_i\rho - 2\rho\beta_i. \tag{3.29}$$

Substituting from (3.26)–(3.28) into the above equation, we get $d_h\rho(\partial_i) = 0$, and condition (ii) of the Lemma 3.7 is satisfied.

To check the condition (iii), we have

$$dd_J\rho = (\partial_i\dot{\partial}_j\rho - \partial_j\dot{\partial}_i\rho)\,dx^i \wedge dx^j - \dot{\partial}_i\dot{\partial}_j\rho\,dx^i \wedge dy^j, \tag{3.30}$$

therefore, the 2-form $dd_J\rho$ has maximal rank if and only if

$$\det(\rho_{ij}) \neq 0, \tag{3.31}$$

where we set

$$\rho_{ij} := \dot{\partial}_i\dot{\partial}_j\rho = \frac{4a_{ij}q(x) - 4(a_{ir}x^k)(a_{jk}x^k) - 2b_ia_{jk}x^k - 2b_ja_{ik}x^k - b_ib_j}{(2q(x))^2}. \tag{3.32}$$

Consequently, for almost all constants a_{ij}, b_i and c, the condition (iii) is satisfied. This completes the proof. □

By making use of Theorem 3.3 together with Theorem 3.6, we have the following corollary:

Corollary 3.8. *Let S_β be the* 1*-form deformation* (1.5) *of a flat spray. Then, the following assertions are equivalent:*

(a) S_β is Finsler metrizable.
(b) S_β is Finsler metrizable by a Finsler function of constant curvature.
(c) S_β is the geodesic spray of a projectively flat Berwald metric of isotropic curvature.
(d) In an adapted coordinate system, the 1-form $\beta = \beta_i dx^i$ satisfies (3.9).

According to the formula of the Jacobi endomorphism Φ_β given in Lemma 3.4 **(b)**, the spray S_β is isotropic. Hence, by [6], we have the following lemma which is required to prove Theorem 3.6.

Remark 3.9. It is worthy to derive the Ricci scalar, as follows:

$$R^i{}_i = (n-1)\left(\frac{4q(x)a_{rs}y^ry^s - (2a_{rs}x^ry^s + \langle b, y\rangle)^2}{(2q(x))^2}\right) = (n-1)\rho. \tag{3.33}$$

Indeed: since the coefficients of the flat spray S_0 vanish in the adapted coordinate system, we have

$$G^i = -\frac{2a_{jk}x^jy^k + \langle b, y\rangle}{(2q(x))}y^i. \tag{3.34}$$

Since the Jacobi endomorphism $R^i{}_j$ has the form

$$R^i{}_j = 2\partial_jG^i - y^k\partial_kN^i_j + 2G^kG^i_{jk} - N^i_kN^k_j \tag{3.35}$$

where $N^i_j = \dot{\partial}_jG^i$ and $G^i_{jk} = \dot{\partial}_kN^i_j$, then by using the formula of G^i, we have

$$\partial_jG^i = -\frac{(2q(x))2ac_{js}y^s - (2a_{rs}x^ry^s + \langle b, y\rangle)(2(2a_{rj}x^r + b_j))}{(2q(x))^2}y^i, \tag{3.36}$$

$$N^i_j = -\frac{(2a_{rj}x^r + b_j)y^i + (2a_{rs}x^ry^s + \langle b, y\rangle)\delta^i_j}{(2q(x))}, \tag{3.37}$$

$$G^i_{jk} = -\frac{(2a_{rj}x^r + b_j)\delta^i_k + (2a_{rk}x^r + b_k)\delta^i_j}{(2q(x))}. \tag{3.38}$$

Moreover, one can calculate the quantities $\partial_kN^i_j$, $N^i_kN^k_j$ and $G^kG^i_{jk}$ as follows:

$$\partial_kN^i_j = -\frac{2q(x)(2a_{jk}y^i + 2a_{ks}y^s\delta^i_j) - 2(2a_{sk}x^s + b_k)((2a_{rj}x^r + b_j)y^i + (2a_{rs}x^ry^s + \langle b, y\rangle)\delta^i_j)}{(2q(x))^2}, \tag{3.39}$$

$$N^i_k N^k_j = \frac{(2a_{rs}x^r y^s + \langle b, y\rangle)(3(2a_{rj}x^r + b_j)y^i + (2a_{rs}x^r y^s + \langle b, y\rangle)\delta^i_j)}{(2q(x))^2}, \tag{3.40}$$

$$G^k G^i_{jk} = \frac{(2a_{rs}x^r y^s + \langle b, y\rangle)((2a_{rj}x^r + b_j)y^i + (2a_{rs}x^r y^s + \langle b, y\rangle)\delta^i_j)}{(2q(x))^2}. \tag{3.41}$$

By plugging the above quantities into the formula of $R^i{}_j$, we obtain

$$R^i{}_j = \frac{4q(x)c_{rs}y^r y^s - (2a_{rs}x^r y^s + \langle b, y\rangle)^2}{(2q(x))^2}\delta^i_j - \frac{4q(x)a_{rj}y^r - (2a_{rs}x^r y^s + 4\langle b, y\rangle)(2a_{rj}x^r + b_j)}{(2q(x))^2}y^i. \tag{3.42}$$

Contracting the indices i and j in the aforementioned equation, we get the required formula of the Ricci scalar.

Remark 3.10. Since the spray (3.8) is isotropic and metrizable by a Finsler function F of a nonzero constant flag curvature μ, then $\rho = \mu F^2$. Since ρ is given by (3.26), F is given by formula (1.6).

In fact, with appropriate constants a_{ij}, c, $b_1, b_2, ..., b_n$, the family (1.6) is a family of projectively flat Riemannian metrics of constant flag curvature $\mu \neq 0$ with the projective factor

$$P(x, y) = -\frac{2a_{ij}x^i y^j + \langle b, y\rangle}{2q(x)}. \tag{3.43}$$

Using the formula (2.15) of the projective factor and by straightforward calculations, one can verify the above formula. Moreover, since it is known that the Hilbert's fourth problem is to find (or, characterize) all locally projectively flat Finsler spaces-that is, the metrics whose geodesics are straight lines on an open subset of $\mathbb{R}^n$-then the family (1.6) is a non trivial solutions for Hilbert's fourth problem.

Let us conclude this work with the following two examples of projectively flat metrics.

Example 1. As a special case, we recover the classical Klein metric. By choosing

$$a_{ij} = \mu\delta_{ij}, \qquad b_i = 0, \qquad c = 1,$$

we obtain

$$F_\mu = \sqrt{\frac{(1 + \mu|x|^2)\,|y|^2 - \mu\langle x, y\rangle^2}{(1 + \mu|x|^2)^2}}, \qquad G^i = -\frac{\mu\langle x, y\rangle}{1 + \mu|x|^2}\,y^i.$$

The next example provides a more general projectively flat metric, extending the Klein case.

Example 2. Let $a_{ij} = \delta_{ij}$, and let b_i and c be arbitrary constants. Then, the corresponding Finsler function is

$$F = \sqrt{\frac{4(|x|^2 + \langle b, x\rangle + c)\,|y|^2 - 4\langle x, y\rangle^2 - 4\langle b, y\rangle\langle x, y\rangle - \langle b, y\rangle^2}{4(|x|^2 + \langle b, x\rangle + c)^2}},$$

and the spray coefficients are given by

$$G^i = -\frac{2\langle x, y\rangle + \langle b, y\rangle}{2(|x|^2 + \langle b, x\rangle + c)}\,y^i.$$

4. Conclusions

In this paper, we investigated the metrizability of a spray under a 1-form projective deformation. We first showed that, in general, the deformation $S_\beta = S - 2\beta C$ of a metrizable spray is not Finsler metrizable. When the background spray is flat, however, we obtained a complete characterization of the metrizability of the deformed spray. Specifically, we proved that S_β is Finsler metrizable if and only if the components of the 1-form β satisfy the condition (3.9).

For such 1-forms, we derived an explicit expression for the corresponding Finsler function and demonstrated that the resulting metric is projectively flat with nonzero constant flag curvature. Since projectively flat Finsler metrics are precisely those solving the local form of Hilbert's fourth problem, then the obtained family provides a constructive family of solutions to Hilbert's fourth problem, which includes, as a special case, the Klein metric.

These results show that while 1-form projective deformations typically destroy metrizability, in the flat case they give rise to a rich and explicit class of projectively flat Finsler metrics of constant curvature.

Author contributions

All authors contributed equally to the preparation of this manuscript. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflict of interest in this paper.

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