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Research article

Geometric analysis of integral operators related to Touchard polynomials and generalized Bessel functions

Manas Kumar Giri[1], **Narjes Alabkary**[2,*], **Raghavendar Kondooru**[1] **and Saiful R. Mondal**[2]

[1] Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology Vellore, Vellore 632014, India

[2] Department of Mathematics and Statistics, College of Science, King Faisal University, Al-Ahsa 31982, Saudi Arabia

* **Correspondence:** Email: nalabkary@kfu.edu.sa.

Abstract: This article investigated the inclusion properties of Touchard polynomials $Y_m(p,z)$ and generalized Bessel functions of the first kind $\mathcal{G}_a(z)$, along with their associated convolution and integral operators, within the analytic function classes $\mathcal{R}^w_{s,d}(\delta)$ and $\mathcal{M}^w_{b,s}(\delta)$. Using coefficient bounds of certain function classes, we derived sufficient conditions for the convolution operators $\mathcal{Y}^p_m(\psi,z)$ and $\mathcal{E}_{l_a,c}(\psi,z)$, and the integral operators $\mathfrak{Y}_m(p,z)$ and $\mathfrak{G}_a(z)$ to belong to various subclasses of starlike and convex functions. Furthermore, inclusion results among these subclasses were obtained, thereby extending and unifying several existing results in geometric function theory.

Keywords: special functions; convex functions; starlike functions; spirallike functions; integral operators
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1. Introduction

The classification and behavior of subclasses such as starlike, convex, and close-to-convex functions, along with extremal problems and coefficient estimates for normalized analytic functions, play a central role in geometric function theory.

A normalized analytic function $\psi(z)$ is of the form

$$\psi(z) = z + \sum_{j=2}^{\infty} x_j z^j, \tag{1.1}$$

defined on the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$.

Let $\mathcal{A}$ be the class of normalized analytic functions and $\mathcal{S}$ be the class of univalent functions. The classes of starlike functions and convex functions are denoted by $\mathcal{S}^*$ and C, respectively (see [1]).

A normalized analytic function $\psi(z)$ is said to be k-uniformly convex of order r if and only if

$$\Re\left(1+\frac{z\psi''(z)}{\psi'(z)}\right)\geq k\left|\frac{z\psi''(z)}{\psi'(z)}\right|+r,$$

where $0 \leq k < \infty$ and $0 \leq r < 1$. This class of functions is denoted by $k - \mathcal{UCV}(r)$ (see [2]).

If $r = 0$, then $k - \mathcal{UCV}(0) \equiv k - \mathcal{UCV}$, and if $k = 1$, then $1 - \mathcal{UCV}(0) \equiv \mathcal{UCV}$. For more details, see [3–7].

A related class $k - \mathcal{S}_p(r)$ is defined as $\psi(z) \in k - \mathcal{UCV}(r) \iff z\psi'(z) \in k - \mathcal{S}_p(r)$.

If $r = 0$, then $k - \mathcal{S}_p(0) \equiv k - \mathcal{ST}$ (see [8]).

Again, $\psi(z)$ is said to be a k Janowski convex function if and only if

$$\Re\left(\frac{(X_2-1)\frac{(z\psi'(z))'}{\psi'(z)}-(X_1-1)}{(X_2+1)\frac{(z\psi'(z))'}{\psi'(z)}-(X_1+1)}\right)>k\left|\frac{(X_2-1)\frac{(z\psi'(z))'}{\psi'(z)}-(X_1-1)}{(X_2+1)\frac{(z\psi'(z))'}{\psi'(z)}-(X_1+1)}-1\right|,$$

where $k \geq 0$ and $-1 \leq X_2 < X_1 \leq 1$. This class of functions is denoted by $k-\mathcal{UCV}[X_1, X_2]$ (see [9,10]).

Again a related class $k - \mathcal{ST}[X_1, X_2]$ is defined as $\psi(z) \in k - \mathcal{UCV}[X_1, X_2] \iff z\psi'(z) \in k - \mathcal{ST}[X_1, X_2]$.

Note that if $X_1 = 1-2r$ and $X_2 = -1$, then $k-\mathcal{ST}[1-2r, -1] \equiv k-\mathcal{S}_p(r)$ and $k-\mathcal{UCV}[1-2r, -1] \equiv k - \mathcal{UCV}(r)$.

In recent years, several researchers have constructed convolution operators based on special functions because of their deep connections with mathematical analysis. In particular, operators associated with the Touchard polynomials and the generalized Bessel functions of the first kind provide a fruitful framework for analyzing various subclasses of analytic functions. Convolution operators transform one analytic function into another while preserving important geometric properties such as univalence, convexity, and starlikeness. For further details, see [15–17].

Motivated by these connections, the present work introduces and investigates new convolution operators derived from the Touchard polynomials and the generalized Bessel function of the first kind. These operators are studied in the context of the analytic function classes $\mathcal{R}^w_{s,d}(\delta)$ and $\mathcal{M}^w_{b,s}(\delta)$ to establish inclusion properties and sufficient conditions under which the operators preserve geometric characteristics. This approach not only unifies several existing results in the literature but also extends them to a more general framework involving special functions and probability-related structures.

Let us define convolution operators associated with the Touchard polynomials and the generalized Bessel function of the first kind. The Touchard polynomials (see [18]) arise naturally in the enumeration of set partitions and the study of Poisson-type distributions (see [19, 20]).

The Poisson distribution is defined for a random variable X and expected value p. The m^{th} moment $E(X_m) = \mathcal{T}_m(p)$ is defined as

$$\mathcal{T}_m(p) = e^{-p}\sum_{j=0}^{\infty}\frac{p^j j^m}{j!}. \tag{1.2}$$

The coefficients of the Touchard polynomials after the second force are defined as follows:

$$Y_m(p,z) = z + \sum_{j=2}^{\infty} \frac{p^{j-1}(j-1)^m}{(j-1)!} e^{-p} z^j. \tag{1.3}$$

$$Y'_m(p) = Y'_m(p,1) = 1 + e^{-p} \sum_{j=2}^{\infty} \frac{p^{j-1}(j-1)^m}{(j-1)!} j. \tag{1.4}$$

Now, we considered the linear operator $\mathcal{Y}_m^p : \mathcal{A} \to \mathcal{A}$, defined as

$$\mathcal{Y}_m^p(\psi, z) = Y_m(p,z) * \psi(z) = z + \sum_{j=2}^{\infty} \frac{p^{j-1}(j-1)^m}{(j-1)!} e^{-p} x_j z^j = z + \sum_{j=2}^{\infty} \Lambda_j z^j,$$

where $*$ is the Hadamard product or convolution and

$$\Lambda_j = \frac{p^{j-1}(j-1)^m}{(j-1)!} e^{-p} x_j. \tag{1.5}$$

The generalized Bessel function of the first kind (see [21, 22]) $\Phi_{a,b,c}(z)$ is the particular solution of the differential equation

$$z^2 \Phi''(z) + bz\Phi'(z) + (cz^2 - a^2 + (1-b)a)\Phi(z) = 0,$$

where $a, b \in \mathbb{R}$ and $c \in \mathbb{C}$. The a^{th} - order generalized Bessel function of the first kind is defined as

$$\Phi_a(z) = \Phi_{a,b,c}(z) = \sum_{j=0}^{\infty} \frac{(-1)^j c^j}{(j)!\Gamma\left(a + j + \frac{b+1}{2}\right)} \left(\frac{z}{2}\right)^{2j+a}, \ \forall z \in \mathbb{C}.$$

$\Phi_a(z)$ is normalized by the transformation $\mathcal{G}_a(z)$, which is defined as

$$\mathcal{G}_a(z) = [\alpha_0(p)]^{-1} z^{-\frac{p}{2}} \Phi_p(\sqrt{z}).$$

The series representation of $\mathcal{G}_a(z)$ is defined as

$$\mathcal{G}_a(z) = {}_0F_1\left(l_1; -\frac{cz}{4}\right) = \sum_{j=0}^{\infty} \frac{(-1)^j c^j}{4^j (l_a)_j (j)!} z^j \tag{1.6}$$

where $l_a = a + (b+1)/2 \neq 0, -1, -2, \cdots$. The first- and second-order derivatives of $\mathcal{G}_a(z)$ at $z = 1$ are given as follows:

$$\mathcal{G}'_a(1) = \sum_{j=1}^{\infty} \frac{(-c/4)^j}{(l_a)_j (j-1)!}. \tag{1.7}$$

$$\mathcal{G}''_a(1) = \sum_{j=2}^{\infty} \frac{(-c/4)^j}{(l_a)_j (j-2)!}. \tag{1.8}$$

The transformation also satisfies the following conditions (see [22]).

$$\sum_{n=0}^{\infty} \frac{(-c/4)^j}{(l_a)_j(1)_{j+1}} = \frac{-4(l_a-1)}{c}\left[\mathcal{G}_{a-1}(1)-1\right], \tag{1.9}$$

and

$$\mathcal{G}_{a+1}(z) = \frac{-4l_a}{c}\mathcal{G}'_a(z), \quad \forall z \in \mathbb{C}, \tag{1.10}$$

where $c < 0, l_a > 1$ and $l_a \neq 0, -1, -2, \cdots$.

Let $\psi(z) \in \mathcal{A}$, and the convolution operator $\mathcal{E}_{l_a,c}(\psi, z)$ is defined as (see [23])

$$\mathcal{E}_{l_a,c}(\psi, z) = z\mathcal{G}_a(z) * \psi(z) = z + \sum_{j=2}^{\infty} \Lambda_j z^j,$$

where $*$ denotes the Hadamard product or convolution and

$$\Lambda_j = \frac{(-c/4)^{j-1}}{(l_a)_{j-1}(j-1)!}x_j. \tag{1.11}$$

In this article, we investigate the inclusion properties of Touchard polynomials and the generalized Bessel function of the first kind, together with their associated convolution operators, within the function classes $\mathcal{R}^w_{s,d}(\delta)$ and $\mathcal{M}^w_{b,s}(\delta)$ (as introduced in Section 2). Specifically, we establish sufficient conditions, expressed in terms of the relevant parameters, that guarantee the convolution operators belong to various subclasses of univalent functions. The main theorems, along with their proofs, are presented in Section 3, which also explores several geometric properties of linear and integral operators. Moreover, certain special cases of our results are shown to reduce to well-known findings in the existing literature. For completeness, Section 2 recalls a few foundational results required for our analysis, while the concluding remarks are given in Section 4.

2. Preliminary results

In analytic and univalent function theory, the introduction of various subclasses of normalized analytic functions has proven useful for exploring the geometric aspects of complex mappings. This section presents a selection of these subclasses that form the foundation for the results developed in the later sections.

A function $\psi(z) \in \mathcal{S}$ is said to be spirallike if and only if

$$\Re\left(e^{-it}\frac{z\psi'(z)}{\psi(z)}\right) > 0$$

for some t with $|t| < \frac{\pi}{2}$ and for all $z \in \Delta$ (see [11, 12]).

A normalized analytic function $\psi(z)$ is said to be $k-$uniformly convex spirallike of order r if and only if

$$\Re\left\{e^{-it}\left(1+\frac{z\psi''(z)}{\psi'(z)}\right)\right\} \geq k\left|\frac{z\psi''(z)}{\psi'(z)}\right| + r, \tag{2.1}$$

where $k \geq 0$, $0 \leq r < 1$, and $t \leq 1$. This class of functions is denoted by $k - \mathcal{UCSP}(t,r)$ (see [13]).

Similarly if $\psi(z)$ satisfies the following condition for $k \geq 0$, $0 \leq r < 1$, and $t \leq 1$:

$$\Re\left\{e^{-\iota t}\frac{z\psi'(z)}{\psi(z)}\right\} \geq k\left|\frac{z\psi'(z)}{\psi(z)} - 1\right| + r, \tag{2.2}$$

then it is called k−uniformly spiralike of order r. This class of functions is denoted by $k - \mathcal{SP}_p(t,r)$. Note that if $k = 1$, then $1 - \mathcal{UCSP}(t,r) \equiv \mathcal{UCSP}(t,r)$ and $1 - \mathcal{SP}_p(t,r) \equiv \mathcal{SP}_p(t,r)$ (see [13]).

Similarly if $k = 1$ and $r = 0$, then $1 - \mathcal{UCSP}(t,0) \equiv \mathcal{UCSP}(t)$ and $1 - \mathcal{SP}_p(t,0) \equiv \mathcal{SP}_p(t)$ (see [14]).

Let us define a subclass $\mathcal{S}_{v,u}^{k,\varsigma}(t,r)$ of normalized analytic functions that generalizes $k - \mathcal{UCSP}(t,r)$, $k - \mathcal{SP}_p(t,r)$, $k - \mathcal{UCV}(r)$, $k - \mathcal{S}_p(r)$, and other subclasses.

Let $\psi(z) \in \mathcal{A}$ and $\psi(z)$ will be in the class $\mathcal{S}_{v,u}^{k,\varsigma}(t,r)$ if and only if

$$\Re\left\{e^{-\iota t}\left(\varsigma + \frac{uz\psi'(z) + vz^2\psi''(z)}{u\psi(z) + vz\psi'(z)}\right)\right\} > k\left|\frac{uz\psi'(z) + vz^2\psi''(z)}{u\psi(z) + vz\psi'(z)} + \varsigma - 1\right| + r, \tag{2.3}$$

where $k \geq 0$, $0 \leq r < 1$, $t \leq 1$, and $0 \leq u, v, \varsigma \leq 1$.

Note that:

i. If $\varsigma = 0$, $v = 0$, and $u = 1$, then $\mathcal{S}_{v,u}^{k,\varsigma}(t,r) \equiv k - \mathcal{SP}_p(t,r)$, the class of k−uniformly spirallike functions of order r.

ii. If $\varsigma = 1$, $v = 1$, and $u = 0$, then $\mathcal{S}_{v,u}^{k,\varsigma}(t,r) \equiv k - \mathcal{UCSP}(t,r)$, the class of k−uniformly convex spirallike functions of order r.

iii. If $\varsigma = 0$, $v = 0$, $u = 1$, and $k = 1$, then $\mathcal{S}_{v,u}^{k,\varsigma}(t,r) \equiv \mathcal{SP}_p(t,r)$, the class of uniformly spirallike functions of order r (see [13]).

iv. If $\varsigma = 1$, $v = 1$, $u = 0$, and $k = 1$, then $\mathcal{S}_{v,u}^{k,\varsigma}(t,r) \equiv \mathcal{UCSP}(t,r)$, the class of uniformly convex spirallike functions of order r (see [13]).

v. If $\varsigma = 0$, $v = 0$, $u = 1$, and $t = 0$, then $\mathcal{S}_{v,u}^{k,\varsigma}(t,r) \equiv k - \mathcal{S}_p(r)$, the class of k−starlike functions of order r (see [2]).

vi. If $\varsigma = 1$, $v = 1$, $u = 0$, and $t = 0$, then $\mathcal{S}_{v,u}^{k,\varsigma}(t,r) \equiv k - \mathcal{UCV}(r)$, the class of k−uniformly convex functions of order r (see [2]).

vii. If $\varsigma = 0$, $v = 0$, $u = 1$, $r = 0$, and $t = 0$, then $\mathcal{S}_{v,u}^{k,\varsigma}(t,r) \equiv k - \mathcal{ST}$, the class of k−starlike functions (see [8]).

viii. If $\varsigma = 1$, $v = 1$, $u = 0$, $r = 0$, and $t = 0$, then $\mathcal{S}_{v,u}^{k,\varsigma}(t,r) \equiv k - \mathcal{UCV}$, the class of k−uniformly convex functions (see [6]).

ix. If $\varsigma = 0$, $v = 0$, $u = 1$, $k = 0$, $r = 0$, and $t = 0$, then $\mathcal{S}_{v,u}^{k,\varsigma}(t,r) \equiv \mathcal{S}^*$, the class of starlike functions (see [3]).

x. If $\varsigma = 1$, $v = 1$, $u = 0$, $k = 0$, $r = 0$, and $t = 0$, then $\mathcal{S}_{v,u}^{k,\varsigma}(t,r) \equiv C$, the class of convex functions (see [4]).

Definition 2.1. *[2] Consider a function $\psi(z) \in \mathcal{A}$, where $w \in \mathbb{C} \setminus 0$, $0 \leq s < 1$, and $0 \leq d \leq 1$. The function $\psi(z)$ is said to belong to the class $\mathcal{R}_{s,d}^{w}(\delta)$ if it satisfies the inequality*

$$\left|\frac{(1 - d + 2s)\frac{\psi(z)}{z} + (d - 2s)\psi'(z) + sz\psi''(z) - 1}{2w(1-\delta) + (1 - d + 2s)\frac{\psi(z)}{z} + (d - 2s)\psi'(z) + sz\psi''(z) - 1}\right| < 1.$$

For function $\psi(z) \in \mathcal{R}^{w}_{s,d}(\delta)$, the coefficients x_j satisfy the bound

$$|x_j| \leq \frac{2(1-\delta)|w|}{1+(j-1)(d-2s+js)}, \quad j = 2, 3, \cdots. \tag{2.4}$$

Definition 2.2. *[24] Consider a function $\psi(z) \in \mathcal{A}$, where $w \in \mathbb{C} \setminus 0$ and $\delta < 1$. For parameters $0 \leq b < 1$ and $0 \leq s < 1$, the function $\psi(z)$ is said to belong to the class $\mathcal{M}^{w}_{b,s}(\delta)$ if and only if*

$$\left|\frac{\psi'(z) + sz\psi''(z) + bz^2\psi'''(z) - 1}{2w(1-\delta) + \psi'(z) + sz\psi''(z) + bz^2\psi'''(z) - 1}\right| < 1. \tag{2.5}$$

For functions $\psi(z) \in \mathcal{M}^{w}_{b,s}(\delta)$, the coefficients x_j satisfy the inequality

$$|x_j| \leq \frac{2(1-\delta)|w|}{j + j(2b-s) + j^2(s-3b) + j^3 s}, \; j = 2, 3, \cdots. \tag{2.6}$$

The sufficient conditions for the classes $k - \mathcal{UCV}[X_1, X_2]$ and $k - \mathcal{ST}[X_1, X_2]$ are given in the following lemmas.

Lemma 2.1. *[10] Let $\psi(z) \in \mathcal{A}$. A sufficient condition for a function to belong to the class $k - \mathcal{UCV}[X_1, X_2]$ is the inequality*

$$\sum_{j=2}^{\infty} j[2(k+1)(j-1) + |j(X_2+1) - (X_1+1)|]|a_j| \leq |X_2 - X_1|. \tag{2.7}$$

Lemma 2.2. *[10] Let $\psi(z) \in \mathcal{A}$. A sufficient condition for a function to belong to the class $k - \mathcal{ST}[X_1, X_2]$ is the inequality*

$$\sum_{j=2}^{\infty} [2(k+1)(j-1) + |j(X_2+1) - (X_1+1)|]|a_j| \leq |X_2 - X_1|. \tag{2.8}$$

3. Main results

In the following lemmas, we establish sufficient conditions for the classes $\mathcal{S}^{k,\varsigma}_{v,u}(t,r)$, $k - \mathcal{UCSP}(t,r)$ and $k - \mathcal{SP}_p(t,r)$.

Lemma 3.1. *Let $\psi(z) \in \mathcal{A}$ be a function of the form* (1.1)*. Then, $\psi(z)$ belongs to the class $\mathcal{S}^{k,\varsigma}_{v,u}(t,r)$ provided that the following inequality is satisfied:*

$$\sum_{j=2}^{\infty}[(k+1)(j(j-1)v + j((\varsigma+r+1)v+u) + (\varsigma+r+1)u) + (1+r)(u+jv)]|x_j| < u - \varsigma(u+v). \tag{3.1}$$

Proof. Suppose the condition (2.3) holds, and then it suffices to show that

$$k\left|\frac{uz\psi'(z) + vz^2\psi''(z)}{u\psi(z) + vz\psi'(z)} + \varsigma - 1\right| + r - \Re\left\{e^{-\iota t}\left(\varsigma + \frac{uz\psi'(z) + vz^2\psi''(z)}{u\psi(z) + vz\psi'(z)}\right) - (1+r)\right\} < 1 + r. \tag{3.2}$$

From the left side of the above expression, we have

$$\begin{aligned}
&k\left|\frac{uz\psi'(z)+vz^2\psi''(z)}{u\psi(z)+vz\psi'(z)}+\varsigma-1\right|+r-\Re\left\{e^{-\iota t}\left(\varsigma+\frac{uz\psi'(z)+vz^2\psi''(z)}{u\psi(z)+vz\psi'(z)}\right)-(1+r)\right\}\\
&\leq (k+1)\left|\frac{uz\psi'(z)+vz^2\psi''(z)}{u\psi(z)+vz\psi'(z)}+\varsigma+r-1\right|\\
&=(k+1)\left|\frac{vz^2\sum_{j=2}^{\infty}j(j-1)x_jz^{j-2}+(u+(\varsigma+r-1)v)z(1+\sum_{j=2}^{\infty}jx_jz^{j-1})}{u(z+\sum_{j=2}^{\infty}x_jz^j)+vz(1+\sum_{j=2}^{\infty}jx_jz^{j-1})}\right.\\
&\quad\left.+\frac{(\varsigma+r-1)u(z+\sum_{j=2}^{\infty}x_jz^j)}{u(z+\sum_{j=2}^{\infty}x_jz^j)+vz(1+\sum_{j=2}^{\infty}jx_jz^{j-1})}\right|\\
&\leq (k+1)\frac{\sum_{j=2}^{\infty}[j(j-1)v+j(\varsigma+r)v+j(u+v)+(\varsigma+r)u+u]|x_j|}{(u+v)-\sum_{j=2}^{\infty}(u+jv)|x_j|}\\
&\quad+\frac{[(\varsigma+r)(u+v)+v]}{(u+v)-\sum_{j=2}^{\infty}(u+jv)|x_j|}.
\end{aligned}$$

The above expression is bounded above by 1 and applying it in (3.2), we get

$$\begin{aligned}
&(k+1)\sum_{j=2}^{\infty}[j(j-1)v+j(\varsigma+r)v+j(u+v)+(\varsigma+r)u+u]|x_j|+[(\varsigma r)(u+v)+v]\\
&<(1+r)(u+v)-(1+r)\sum_{j=2}^{\infty}(u+jv)|x_j|. \qquad (3.3)
\end{aligned}$$

From (3.3) we will get the stated condition. Thus, the proof is complete. □

Lemma 3.2. *Let $\psi(z)\in\mathcal{A}$ be a function of the form* (1.1). *Then, $\psi(z)$ belongs to the class $k-\mathcal{UCSP}(t,r)$ provided that the following inequality is satisfied:*

$$\sum_{j=2}^{\infty}(2k(j-1)+\cos t-r)j|x_j|\leq\cos t-r. \qquad (3.4)$$

Proof. Suppose the condition (2.1) given in the definition of $k-\mathcal{UCSP}(t,r)$ holds. Then it suffices to show that

$$k\left|\frac{z\psi''(z)}{\psi'(z)}\right|\leq\Re\left\{e^{-it}\left(1+\frac{z\psi''(z)}{\psi'(z)}\right)\right\}-r.$$

That is,

$$k\left|\frac{z\psi''(z)}{\psi'(z)}\right|-\Re\left\{e^{-it}\frac{z\psi''(z)}{\psi'(z)}\right\}\leq\cos t-r. \qquad (3.5)$$

From the left side of the above expression, we have

$$k\left|\frac{z\psi''(z)}{\psi'(z)}\right|-\Re\left\{e^{-it}\frac{z\psi''(z)}{\psi'(z)}\right\}\leq 2k\left|\frac{z\psi''(z)}{\psi'(z)}\right|\leq 2k\frac{\sum_{j=2}^{\infty}j(j-1)|x_j|}{1-\sum_{j=2}^{\infty}j|x_j|}.$$

The above expression is bounded above by 1 and applying it in (3.5), we get

$$\sum_{j=2}^{\infty} 2kj(j-1)|x_j| \le \cos t - r - \sum_{j=2}^{\infty}(\cos t - r)j|x_j|. \tag{3.6}$$

From (3.6) we will get the stated condition. Thus, the proof is complete. □

Lemma 3.3. *Let $\psi(z) \in \mathcal{A}$ be a function of the form* (1.1). *Then, ψ belongs to the class $k - \mathcal{SP}_p(t,r)$ provided that the following inequality is satisfied:*

$$\sum_{j=2}^{\infty}(2k(j-1) + \cos t - r)|x_j| \le \cos t - r. \tag{3.7}$$

Proof. By the Alexander-type theorem, $\psi(z) \in k - \mathcal{UCSP}(t,r)$ if and only if $\psi(z) \in k - \mathcal{SP}_p(t,r)$. □

Using the sufficient conditions established in the above lemmas, we obtain criteria ensuring that the linear operators $Y_m(p,z)$ and $\mathcal{G}_a(z)$, defined in the Introduction, belong to different subclasses of analytic functions.

3.1. Inclusion results for $Y_m(p,z)$ and $\mathcal{G}_a(z)$

The following results are established based on the coefficients of the linear operators together with the sufficient conditions for various subclasses of analytic functions.

Theorem 3.1. *Let $\psi(z) \in \mathcal{A}$. If $\psi(z)$ satisfies the following condition, then $Y_m(p,z) \in \mathcal{S}_{v,u}^{k,\varsigma}(t,r)$.*

$$\begin{aligned}&(k+1)vY'_{m+1}(p) + [(k+1)((\varsigma+r+1)v+u) + (1+r)v]Y'_m(p) + [(k+1)(\varsigma+r+1)u + (1+r)u]Y_m(p)\\ &\quad < u + (k+1)[(\varsigma+r+2)(v+u)] + (1+r-\varsigma)(u+v).\end{aligned}$$

Proof. Substituting Eq (1.3) in (3.1), as the sufficient condition for the class $\mathcal{S}_{v,u}^{k,\varsigma}(t,r)$, we get the following expression:

$$\begin{aligned}&\sum_{j=2}^{\infty}[(k+1)(j(j-1)v + j((\varsigma+r+1)v+u) + (\varsigma+r+1)u) + (1+r)u + (1+r)vj]\frac{p^{j-1}(j-1)^m}{(j-1)!}e^{-p}\\ &\quad < u - \varsigma(u+v).\end{aligned}$$

This leads to the following expression:

$$\begin{aligned}&(k+1)v\sum_{j=2}^{\infty} j\frac{p^{j-1}(j-1)^{m+1}}{(j-1)!}e^{-p} + (k+1)((\varsigma+r+1)v+u)\sum_{j=2}^{\infty} j\frac{p^{j-1}(j-1)^m}{(j-1)!}e^{-p}\\ &+ (k+1)(\varsigma+r+1)u\sum_{j=2}^{\infty}\frac{p^{j-1}(j-1)^m}{(j-1)!}e^{-p} + (1+r)u\sum_{j=2}^{\infty}\frac{p^{j-1}(j-1)^m}{(j-1)!}e^{-p} + (1+r)v\sum_{j=2}^{\infty} j\frac{p^{j-1}(j-1)^m}{(j-1)!}e^{-p}\\ &< u - \varsigma(u+v).\end{aligned}$$

Applying the identity (1.4) to simplify the above, we obtain

$$\begin{aligned}&(k+1)v\left(Y'_{m+1}(p)-1\right) + (k+1)((\varsigma+r+1)v+u)\left(Y'_m(p)-1\right) + (k+1)(\varsigma+r+1)u\left(Y_m(p)-1\right)\\ &\quad + (1+r)u\left(Y_m(p)-1\right) + (1+r)v\left(Y'_m(p)-1\right) < u - \varsigma(u+v).\end{aligned} \tag{3.8}$$

Hence, the required result is obtained from (3.8), completing the proof. □

Theorem 3.2. *Let a function $\psi(z) \in \mathcal{A}$ and of the form* (1.1)*. If $\psi(z)$ satisfies the following condition, then $\mathcal{G}_a(z) \in \mathcal{S}_{v,u}^{k,\varsigma}(t,r)$.*

$$(k+1)v\mathcal{G}_a''(1) + [(k+1)((\varsigma+r+1)v+u) + (1+r)v]\mathcal{G}_a'(1) + [(k+1)(\varsigma+r+1)u + (1+r)u]\mathcal{G}_a(1)$$
$$+ [(k+1)((\varsigma+r+1)(v+u)+u) + (1+r)(v+u)]\frac{c}{4l_a} < (2+r)u - \varsigma(u+v) + (k+1)(\varsigma+r+1)u.$$

Proof. Substituting Eq (1.6) in (3.1), as the sufficient condition for the class $\mathcal{S}_{v,u}^{k,\varsigma}(t,r)$, we get the following expression:

$$\sum_{j=2}^{\infty}[(k+1)(j(j-1)v + j(\varsigma+r)v + j(u+v) + (\varsigma+r)u + u) + (1+r)(u+jv)]\frac{(-c/4)^j}{(l_a)_j(j)!}$$
$$< u - \varsigma(u+v).$$

Applying (1.7) and (1.8) in the above expression, we obtain the following result:

$$(k+1)v\mathcal{G}_a''(1) + [(k+1)((\varsigma+r)v + (u+v)) + (1+r)v]\left(\mathcal{G}_a'(1) + \frac{c}{4l_a}\right)$$
$$+ [(k+1)((\varsigma+r)u+u) + (1+r)u]\left(\mathcal{G}_a(1) + \frac{c}{4l_a} - 1\right)$$
$$< u - \varsigma(u+v). \tag{3.9}$$

We will get the required result from (3.9). □

Theorem 3.3. *Let $\psi(z) \in \mathcal{A}$. If $\psi(z)$ satisfies the following condition, then $Y_m(p,z) \in k - \mathcal{UCSP}(t,r)$.*

$$2kY_{m+1}'(p) + (\cos t - r)Y_m'(p) \le 2(k - r + \cos t).$$

Proof. Substituting Eq (1.3) in (3.4), as the sufficient condition for the class $k - \mathcal{UCSP}(t,r)$, we get the following expression:

$$\sum_{j=2}^{\infty}(2k(j-1) + \cos t - r)j\frac{p^{j-1}(j-1)^m}{(j-1)!}e^{-p} \le \cos t - r.$$

This leads to the following expression:

$$2k\sum_{j=2}^{\infty} j\frac{p^{j-1}(j-1)^{m+1}}{(j-1)!}e^{-p} + (\cos t - r)\sum_{j=2}^{\infty} j\frac{p^{j-1}(j-1)^m}{(j-1)!}e^{-p} \le \cos t - r.$$

Applying the identity (1.4) to simplify the above, we obtain

$$2k\left(Y_{m+1}'(p) - 1\right) + (\cos t - r)\left(Y_m'(p) - 1\right) \le \cos t - r. \tag{3.10}$$

The desired result follows directly from (3.10), which completes the proof. □

Theorem 3.4. *Let $\psi(z) \in \mathcal{A}$. If $\psi(z)$ satisfies the following condition, then $Y_m(p,z) \in k - \mathcal{SP}_p(t,r)$.*

$$2kY_m'(p) + (\cos t - 2k - r)Y_m(p) \le 2(\cos t - r).$$

Proof. Substituting Eq (1.3) in (3.7), as the sufficient condition for the class $k-\mathcal{SP}_p(t,r)$, we get the following expression:

$$\sum_{j=2}^{\infty}(2kj+\cos t-2k-r)\frac{p^{j-1}(j-1)^m}{(j-1)!}e^{-p}\leq \cos t-r.$$

This leads to the following expression:

$$2k\sum_{j=2}^{\infty}j\frac{p^{j-1}(j-1)^m}{(j-1)!}e^{-p}+(\cos t-2k-r)\sum_{j=2}^{\infty}\frac{p^{j-1}(j-1)^m}{(j-1)!}e^{-p}\leq \cos t-r.$$

Applying the identity (1.4) to simplify the above, we obtain

$$2k\left(Y_m'(p)-1\right)+(\cos t-r)\left(Y_m(p)-1\right)\leq \cos t-r. \tag{3.11}$$

From (3.11), we obtain the required result. □

Theorem 3.5. *Let a function* $\psi(z)\in\mathcal{A}$ *and of the form* (1.1). *If* $\psi(z)$ *satisfies the following condition, then* $\mathcal{G}_a(z)\in k-\mathcal{UCSP}(t,r)$.

$$2k\mathcal{G}_a''(1)+(\cos t-r)\mathcal{G}_a'(1)\leq(\cos t-r)\left(1-\frac{c}{4l_a}\right).$$

Proof. Substituting Eq (1.6) in (3.4), as the sufficient condition for the class $k-\mathcal{UCSP}(t,r)$, we get the following expression:

$$\sum_{j=2}^{\infty}(2k(j-1)+\cos t-r)j\frac{(-c/4)^j}{(l_a)_j(j)!}\leq \cos t-r.$$

Simplifying the above expression, we have

$$2k\sum_{j=2}^{\infty}\frac{(-c/4)^j}{(l_a)_j(j-2)!}+(\cos t-r)\sum_{j=2}^{\infty}\frac{(-c/4)^j}{(l_a)_j(j-1)!}\leq \cos t-r.$$

Applying (1.7) and (1.8) in the above expression, we obtain the following result:

$$2k\mathcal{G}_a''(1)+(\cos t-r)\left(\mathcal{G}_a'(1)+\frac{c}{4l_a}\right)\leq \cos t-r. \tag{3.12}$$

Hence, the required result is obtained from (3.12), completing the proof. □

Theorem 3.6. *Let a function* $\psi(z)\in\mathcal{A}$ *and of the form* (1.1). *If* $\psi(z)$ *satisfies the following condition, then* $\mathcal{G}_a(z)\in k-\mathcal{SP}_p(t,r)$.

$$8kl_a\mathcal{G}_a'(1)+4l_a(\cos t-2k-r)\mathcal{G}_a(1)+c(\cos t-r)\leq 8l_a(\cos t-k-r).$$

Proof. Substituting Eq (1.6) in (3.7), as the sufficient condition for the class $k-\mathcal{UCSP}(t,r)$, we get the following expression:

$$\sum_{j=2}^{\infty}(2kj+\cos t-2k-r)\frac{(-c/4)^j}{(l_a)_j(j)!}\leq \cos t-r.$$

Simplifying the above expression, we have

$$2k\sum_{j=2}^{\infty}\frac{(-c/4)^j}{(l_a)_j(j-1)!}+(\cos t-2k-r)\sum_{j=2}^{\infty}\frac{(-c/4)^j}{(l_a)_j(j)!}\leq \cos t-r.$$

Applying (1.6) and (1.7) in the above expression, we obtain the following result:

$$2k\left(\mathcal{G}_a'(1)+\frac{c}{4l_a}\right)+(\cos t-2k-r)\left(\mathcal{G}_a(1)+\frac{c}{4l_a}-1\right)\leq \cos t-r. \tag{3.13}$$

We will get the required result from (3.13). □

Using linear operators $Y_m(p,z)$ and $\mathcal{G}_a(z)$, the convolution operators $\mathcal{Y}_m^p(\psi,z)$ and $\mathcal{E}_{l_a,c}(\psi,z)$ were introduced in the Introduction. Based on the coefficient bounds of the classes $\mathcal{M}_{b,s}^w(\delta)$ and $\mathcal{R}_{s,d}^w(\delta)$, we establish sufficient conditions ensuring that these convolution operators belong to different subclasses of analytic functions.

3.2. *Results on the convolution operators $\mathcal{Y}_m^p(\psi,z)$ and $\mathcal{E}_{l_a,c}(\psi,z)$ associated with the class $\mathcal{M}_{b,s}^w(\delta)$*

The following theorems provide conditions under which the operators $\mathcal{Y}_m^p(\psi,z)$ and $\mathcal{E}_{l_a,c}(\psi,z)$ belong to the classes $\mathcal{S}_{v,u}^{k,\varsigma}(t,r)$, $k-\mathcal{UCSP}(t,r)$, $k-\mathcal{SP}_p(t,r)$, $k-\mathcal{UCV}[X_1,X_2]$, and $k-\mathcal{ST}[X_1,X_2]$.

Theorem 3.7. *Let a function $\psi(z)\in\mathcal{A}$ and of the form* (1.1). *Again if $\psi(z)\in\mathcal{M}_{b,s}^w(\delta)$ and satisfies the following condition, then $\mathcal{Y}_m^p(\psi,z)\in\mathcal{S}_{v,u}^{k,\varsigma}(t,r)$.*

$$\begin{aligned}&(k+1)vY_{m-1}'(p)+[(k+1)((\varsigma+r+1)v+u)+(1+r)v]Y_{m-2}'(p)+[(k+1)(\varsigma+r+1)u+(1+r)u]Y_{m-2}(p)\\&\quad<(1+r+(k+1)(\varsigma+r+2))(v+u)+(u-\lambda(u+v))\frac{(s-2b)}{2(1-\delta)|w|}.\end{aligned}$$

Proof. Substituting Eq (1.5) in (3.1), as the sufficient condition for the class $\mathcal{S}_{v,u}^{k,\varsigma}(t,r)$, we get the following expression:

$$\begin{aligned}&\sum_{j=2}^{\infty}[(k+1)(j(j-1)v+j((\varsigma+r+1)v+u)+(\varsigma+r+1)u)+(1+r)(u+jv)]\frac{p^{j-1}(j-1)^m}{(j-1)!}e^{-p}|x_j|\\&\quad<u-\varsigma(u+v).\end{aligned}\tag{3.14}$$

From the denominator of (2.6), $j+j(2b-s)+j^2(s-3b)+j^3b\geq (j-1)^2(s-2b)$, and then (2.6) is

$$|x_j|\leq\frac{2(1-\delta)|w|}{(j-1)^2(s-2b)},\quad j=2,3,\cdots. \tag{3.15}$$

Substituting this upper bound from (3.15) into the inequality (3.14), we have

$$\begin{aligned}
&\sum_{j=2}^{\infty}[(k+1)(j(j-1)v+j((\varsigma+r+1)v+u)+(\varsigma+r+1)u)\\
&\quad+(1+r)(u+jv)]\frac{p^{j-1}(j-1)^m}{(j-1)!}e^{-p}|x_j|\\
&\leq \sum_{j=2}^{\infty}[(k+1)(j(j-1)v+j((\varsigma+r+1)v+u)+(\varsigma+r+1)u)\\
&\quad+(1+r)(u+jv)]\frac{p^{j-1}(j-1)^m}{(j-1)!}e^{-p}\frac{2(1-\delta)|w|}{(j-1)^2(s-2b)}\\
&=\frac{2(1-\delta)|w|}{(s-2b)}\Bigg((k+1)v\sum_{j=2}^{\infty}\frac{p^{j-1}(j-1)^{m-1}}{(j-1)!}e^{-p}j\\
&\quad+[(k+1)((\varsigma+r+1)v+u)+(1+r)v]\sum_{j=2}^{\infty}\frac{p^{j-1}(j-1)^{m-2}}{(j-1)!}e^{-p}j\\
&\quad+[(k+1)(\varsigma+r+1)u+(1+r)u]\sum_{j=2}^{\infty}\frac{p^{j-1}(j-1)^{m-2}}{(j-1)!}e^{-p}\Bigg).
\end{aligned} \tag{3.16}$$

Now, applying (1.3) and (1.4) into Eq (3.16), and then using the result in (3.14), we conclude

$$\begin{aligned}
&(k+1)v(Y'_{m-1}(p)-1)+[(k+1)((\varsigma+r+1)v+u)+(1+r)v](Y'_{m-2}(p)-1)\\
&\quad+[(k+1)(\varsigma+r+1)u+(1+r)u](Y_{m-2}(p)-1)\\
&<(u-\varsigma(u+v))\frac{(s-2b)}{2(1-\delta)|w|}.
\end{aligned} \tag{3.17}$$

Hence, the required result is obtained from (3.17), completing the proof. □

Remark 3.1. *If we take $b=0$, $s=0$, $\varsigma=1$, $v=0$, $u=1$, $k=0$, $r=0$, and $t=0$ in Theorem 3.7, this result reduces to Theorem 4.3 of [25] when $\lambda=0$, $B=1$, and $A=1+\frac{2\omega(1-\delta)}{\tau}$.*

Theorem 3.8. *Let a function $\psi(z)\in\mathcal{A}$ and of the form* (1.1). *Again if $\psi(z)\in\mathcal{M}^{w}_{b,s}(\delta)$ and satisfies the following condition, then $\mathcal{Y}^{p}_{m}(\psi,z)\in k-\mathcal{UCSP}(t,r)$.*

$$2kY'_{m-1}(p)+(\cos t-r)Y'_{m-2}(p)\leq 2k+(\cos t-r)\left(1+\frac{(s-2b)}{2(1-\delta)|w|}\right).$$

Proof. Substituting Eq (1.5) in (3.4), as the sufficient condition for the class $k-\mathcal{UCSP}(t,r)$, we get the following expression:

$$\sum_{j=2}^{\infty}(2k(j-1)+\cos t-r)j\frac{p^{j-1}(j-1)^m}{(j-1)!}e^{-p}|x_j|\leq\cos t-r. \tag{3.18}$$

Applying (3.15) in (3.18), we have

$$\sum_{j=2}^{\infty}(2k(j-1)+\cos t-r)j\frac{p^{j-1}(j-1)^m}{(j-1)!}e^{-p}\frac{2(1-\delta)|w|}{(j-1)^2(s-2b)}\leq \cos t-r.$$

Further simplifying, we get

$$2k\sum_{j=2}^{\infty}j\frac{p^{j-1}(j-1)^{m-1}}{(j-1)!}e^{-p}+(\cos t-r)\sum_{j=2}^{\infty}j\frac{p^{j-1}(j-1)^{m-2}}{(j-1)!}e^{-p}\leq(\cos t-r)\frac{(s-2b)}{2(1-\delta)|w|}. \tag{3.19}$$

Now, applying (1.4) into Eq (3.19), we conclude

$$2k(Y'_{m-1}(p)-1)+(\cos t-r)(Y'_{m-2}(p)-1)\leq(\cos t-r)\frac{(s-2b)}{2(1-\delta)|w|}. \tag{3.20}$$

Hence, the required result is obtained from (3.20), completing the proof. □

Theorem 3.9. *Let a function $\psi(z)\in\mathcal{A}$ and of the form* (1.1). *Again if $\psi(z)\in\mathcal{M}^{w}_{b,s}(\delta)$ and satisfies the following condition, then $\mathcal{Y}^{p}_{m}(\psi,z)\in k-\mathcal{SP}_p(t,r)$.*

$$2kY'_{m-2}(p)+(\cos t-2k-r)Y_{m-2}(p)\leq(\cos t-r)\left(1+\frac{(s-2b)}{2(1-\delta)|w|}\right).$$

Proof. Substituting Eq (1.5) in (3.7), as the sufficient condition for the class $k-\mathcal{SP}_p(t,r)$, we get the following expression:

$$\sum_{j=2}^{\infty}(2k(j-1)+\cos t-r)\frac{p^{j-1}(j-1)^m}{(j-1)!}e^{-p}|x_j|\leq\cos t-r. \tag{3.21}$$

Applying (3.15) in (3.21), we have

$$\sum_{j=2}^{\infty}(2kj+\cos t-2k-r)\frac{p^{j-1}(j-1)^m}{(j-1)!}e^{-p}\frac{2(1-\delta)|w|}{(j-1)^2(s-2b)}\leq\cos t-r.$$

Further simplifying, we get

$$2k\sum_{j=2}^{\infty}j\frac{p^{j-1}(j-1)^{m-2}}{(j-1)!}e^{-p}+(\cos t-2k-r)\sum_{j=2}^{\infty}\frac{p^{j-1}(j-1)^{m-2}}{(j-1)!}e^{-p}\leq(\cos t-r)\frac{(s-2b)}{2(1-\delta)|w|}. \tag{3.22}$$

Now, applying (1.4) into Eq (3.22), we conclude

$$2k(Y'_{m-2}(p)-1)+(\cos t-2k-r)(Y_{m-2}(p)-1)\leq(\cos t-r)\frac{(s-2b)}{2(1-\delta)|w|}. \tag{3.23}$$

The desired result follows directly from (3.23), which completes the proof. □

Theorem 3.10. *Let $\psi(z) \in \mathcal{A}$. Again if $\psi(z) \in \mathcal{M}^{w}_{b,s}(\delta)$ and satisfies the following condition, then $\mathcal{Y}^{p}_{m}(\psi, z) \in k - \mathcal{UCV}[X_1, X_2]$.*

$$
\begin{aligned}
&(2(k+1) + |X_2 + 1|)\, Y'_{m-1}(p) + (|X_1 + 1| + |X_2 + 1|)\, Y'_{m-2}(p) \\
&\quad \le 2(k+1) + 2|X_2 + 1| + |X_1 + 1| + \frac{(s-2b)|X_2 - X_1|}{2(1-\delta)|w|}.
\end{aligned}
$$

Proof. Substituting Eq (1.5) in (2.7), applying (3.15), and following as in the above theorem, we get the following expression:

$$
\begin{aligned}
&\sum_{j=2}^{\infty} j[2(k+1)(j-1) + j|(X_2+1)| + |(X_1+1)|]\frac{p^{j-1}(j-1)^m}{(j-1)!}e^{-p}\frac{2(1-\delta)|w|}{(j-1)^2(s-2b)} \\
&\quad \le |X_2 - X_1|.
\end{aligned} \tag{3.24}
$$

Now, applying (1.3) in (3.24), we conclude

$$
(2(k+1) + |(X_2+1)|)\,(Y'_{m-1}(p) - 1) + (|(X_1+1)| + |(X_2+1)|)\,(Y'_{m-2}(p) - 1) \le \frac{(s-2b)|X_2 - X_1|}{2(1-\delta)|w|}. \tag{3.25}
$$

From (3.25), we obtain the required result. □

Theorem 3.11. *Let $\psi(z) \in \mathcal{A}$. Again if $\psi(z) \in \mathcal{M}^{w}_{b,s}(\delta)$ and satisfies the following condition, then $\mathcal{Y}^{p}_{m}(\psi, z) \in k - \mathcal{ST}[X_1, X_2]$.*

$$
2(k+1)Y_{m-1}(p) + |X_2+1|Y'_{m-1}(p) + |X_1+1|Y_{m-2}(p) \le 2(k+1) + |X_2+1| + |X_1+1| + |X_2 - X_1|\frac{(s-2b)}{2(1-\delta)|w|}.
$$

Proof. Substituting Eq (1.5) in (2.8), applying (3.15), and following as in the above theorem, we get the following expression:

$$
\begin{aligned}
&\sum_{j=2}^{\infty} [2(k+1)(j-1) + j|(X_2+1)| + |(X_1+1)|]\frac{p^{j-1}(j-1)^m}{(j-1)!}e^{-p}\frac{2(1-\delta)|w|}{(j-1)^2(s-2b)} \\
&\quad \le |X_2 - X_1|.
\end{aligned} \tag{3.26}
$$

Now applying (1.3) and (1.4) in (3.26), we get

$$
2(k+1)(Y_{m-1}(p) - 1) + |(X_2+1)|(Y'_{m-1}(p) - 1) + |(X_1+1)|(Y_{m-2}(p) - 1) \le |X_2 - X_1|\frac{(s-2b)}{2(1-\delta)|w|}. \tag{3.27}
$$

Hence, the required result is obtained from (3.27), completing the proof. □

Theorem 3.12. *Let $\psi(z) \in \mathcal{A}$. Again, if $\psi(z) \in \mathcal{M}^{w}_{b,s}(\delta)$ and the following condition is met, then $\mathcal{E}_{l_a,c}(\psi, z) \in \mathcal{S}^{k,\varsigma}_{v,u}(t, r)$.*

$$
(k+1)v\mathcal{G}_a(1) + [(k+1)((\varsigma + r - 1)v + u) + (1+r)v]\left(\frac{4(1-l_a)}{c}\right)\mathcal{G}_{a-1}(1)
$$

$$+ [(k+1)(2v-u)+(1+r)(u-v)]\left(\frac{16(l_a-2)_2}{c^2}\right)\mathcal{G}_{a-2}(1)$$
$$+ [(k+1)((\varsigma+r-1)v+u)+(1+r)v]\left(\frac{4(l_a-1)}{c}-1\right)$$
$$+ [(k+1)(2v-u)+(1+r)(u-v)]\left(\frac{16(l_a-2)_2}{c^2}\right)\left(\frac{c}{4(l_a-2)}-1\right)$$
$$< (k+1)(3v-u)+(1+r)(u-v)+[u-\varsigma(u+v)]\frac{(s-3b)}{2(1-\delta)|w|}.$$

Proof. Substituting Eq (1.11) in (3.1), as the sufficient condition for the class $\mathcal{S}^{k,\varsigma}_{v,u}(t,r)$, we get the following expression:

$$\sum_{j=2}^{\infty}[(k+1)(j(j-1)v+j((\varsigma+r+1)v+u)+(\varsigma+r+1)u) \ +(1+r)(u+jv)]\frac{(-c/4)^{j-1}}{(l_a)_{j-1}(j-1)!}|x_j|$$
$$< u-\varsigma(u+v). \tag{3.28}$$

From the denominator of (2.6), $j+j(2b-s)+j^2(s-3b)+j^3\rho \geq j(j+1)(s-3b)$, and then (2.6) can be written as

$$|x_j| \leq \frac{2(1-\delta)|w|}{j(j+1)(s-3b)}. \tag{3.29}$$

Substituting this upper bound from (3.29) into the inequality (3.28), we get

$$\sum_{j=2}^{\infty}[(k+1)vj(j+1)+(j+1)[(k+1)((\varsigma+r-1)v+u)+(1+r)v]$$
$$+ (k+1)(2v-u)+(1+r)(u-v)]\frac{(-c/4)^{j-1}}{(l_a)_{j-1}(j-1)!}\frac{2(1-\delta)|w|}{j(j+1)(s-3b)}$$
$$< u-\varsigma(u+v). \tag{3.30}$$

Now applying (1.6) in (3.30), we get

$$(k+1)v(\mathcal{G}_a(1)-1)$$
$$+ [(k+1)((\varsigma+r-1)v+u)+(1+r)v]\left(\frac{4(1-l_a)}{c}\right)\left(\mathcal{G}_{a-1}(1)-1+\frac{c}{4(l_a-1)}\right)$$
$$+ [(k+1)(2v-u)+(1+r)(u-v)]\left(\frac{16(l_a-2)_2}{c^2}\left(\mathcal{G}_{a-2}(1)-1+\frac{c}{4(l_a-2)}\right)-1\right)$$
$$< [u-\varsigma(u+v)]\frac{(s-3b)}{2(1-\delta)|w|}. \tag{3.31}$$

The desired result follows directly from (3.31), which completes the proof. □

Theorem 3.13. *Let a function $\psi(z) \in \mathcal{A}$ and of the form* (1.1). *Again if $\psi(z) \in \mathcal{M}^{w}_{b,s}(\delta)$ and satisfies the following condition, then $\mathcal{E}_{l_a,c}(\psi,z) \in k-\mathcal{UCSP}(t,r)$.*

$$2kc\mathcal{G}_a(1)-4(l_a-1)(\cos t-2k-r)\mathcal{G}_{a-1}(1)-8k(l_a-1) \leq (\cos t-r)\left(c-4(l_a-1)+\frac{c(s-3b)}{2(1-\delta)|w|}\right).$$

Proof. Substituting Eq (1.11) in (3.4), as the sufficient condition for the class $k-\mathcal{UCSP}(t,r)$, we get the following expression:

$$\sum_{j=2}^{\infty}(2k(j-1)+\cos t-r)j\frac{(-c/4)^{j-1}}{(l_a)_{j-1}(j-1)!}|x_j| \leq \cos t-r. \tag{3.32}$$

From the denominator of (2.6), $j+j(2b-s)+j^2(s-3b)+j^3b \geq j^2(s-3b)$, and then (2.6) can be written as

$$|x_j| \leq \frac{2(1-\delta)|w|}{j^2(s-3b)}, \quad j=2,3,\cdots. \tag{3.33}$$

Applying (3.33) in (3.32), we have

$$\sum_{j=2}^{\infty}(2kj+\cos t-2k-r)j\frac{(-c/4)^{j-1}}{(l_a)_{j-1}(j-1)!}\frac{2(1-\delta)|w|}{j^2(s-3b)} \leq \cos t-r. \tag{3.34}$$

Simplifying (3.34), we get

$$2k\sum_{j=1}^{\infty}\frac{(-c/4)^j}{(l_a)_j(j)!}+(\cos t-2k-r)\frac{-4(l_a-1)}{c}\sum_{j=2}^{\infty}\frac{(-c/4)^j}{(l_a-1)_j(j)!} \leq (\cos t-r)\frac{(s-3b)}{2(1-\delta)|w|}. \tag{3.35}$$

Now, applying (1.6) into equation (3.35), we conclude

$$2k(\mathcal{G}_a(1)-1)-(\cos t-2k-r)\frac{4(l_a-1)}{c}\left(\mathcal{G}_{a-1}(1)-1+\frac{(c/4)}{(l_a-1)}\right) \leq (\cos t-r)\frac{(s-3b)}{2(1-\delta)|w|}. \tag{3.36}$$

From (3.36), we obtain the required result. □

Theorem 3.14. *Let a function $\psi(z)\in\mathcal{A}$ and of the form* (1.1). *Again if $\psi(z)\in\mathcal{M}_{b,s}^{w}(\delta)$ and satisfies the following condition, then $\mathcal{E}_{l_a,c}(\psi,z)\in k-\mathcal{SP}_p(t,r)$.*

$$\begin{aligned}&8kc(1-l_a)\mathcal{G}_{a-1}(1)+16(\cos t-4k-r)(l_a-2)_2\mathcal{G}_{a-2}(1)\\&\quad+16(\cos t-4k-r)(l_a-2)_2\left(\frac{c}{4(l_a-2)}-1\right)+8kc(l_a-1)+2kc^2\\&\leq(\cos t-r)c^2\left(1+\frac{(s-3b)}{2(1-\delta)|w|}\right).\end{aligned}$$

Proof. Substituting Eq (1.11) in (3.7), as the sufficient condition for the class $k-\mathcal{SP}_p(t,r)$, we get the following expression:

$$\sum_{j=2}^{\infty}(2k(j-1)+\cos t-r)\frac{(-c/4)^{j-1}}{(l_a)_{j-1}(j-1)!}|x_j| \leq \cos t-r. \tag{3.37}$$

Applying (3.29) in (3.37), we have

$$\sum_{j=2}^{\infty}(2k(j+1)+\cos t-4k-r)\frac{(-c/4)^{j-1}}{(l_a)_{j-1}(j-1)!}\frac{2(1-\delta)|w|}{j(j+1)(s-3b)} \leq \cos t-r.$$

Further simplifying, we get

$$2k\frac{-4(l_a-1)}{c}\sum_{j=2}^{\infty}\frac{(-c/4)^j}{(l_a-1)_j(j)!}+(\cos t-4k-r)\frac{4^2(l_a-2)_2}{c^2}\sum_{j=2}^{\infty}\frac{(-c/4)^{j+1}}{(l_a-2)_{j+1}(j+1)!}$$
$$\leq(\cos t-r)\frac{(s-3b)}{2(1-\delta)|w|}. \tag{3.38}$$

Now, applying (1.6) into equation (3.38), we conclude

$$2k\frac{4(l_a-1)}{c}\left(\mathcal{G}_{a-1}(1)-1+\frac{(c/4)}{(l_a-1)}\right)$$
$$+(\cos t-4k-r)\frac{4^2(l_a-2)_2}{c^2}\left(\mathcal{G}_{a-2}(1)-1+\frac{c}{4(l_a-2)}-\frac{c^2}{16(l_a-2)_2}\right)$$
$$\leq(\cos t-r)\frac{(s-3b)}{2(1-\delta)|w|}. \tag{3.39}$$

Hence, the required result is obtained from (3.39), completing the proof. □

Theorem 3.15. *Let $\psi(z)\in\mathcal{A}$. Again if $\psi(z)\in\mathcal{M}_{b,s}^{w}(\delta)$ and satisfies the following condition, then $\mathcal{E}_{l_a,c}(\psi,z)\in k-\mathcal{UCV}[X_1,X_2]$.*

$$(2(k+1)+|X_2+1|)\,\mathcal{G}_a(1)+[|X_1+1|-2(k+1)]\left(\frac{4(1-l_a)}{c}\right)\mathcal{G}_{a-1}(1)$$
$$+[|X_1+1|-2(k+1)]\left(\frac{4(1-l_a)}{c}\right)\left(\frac{c}{4(l_a-1)}-1\right)$$
$$\leq 2(k+1)+|X_2+1|+|X_2-X_1|\frac{(s-3b)}{2(1-\delta)|w|}.$$

Proof. Substituting Eq (1.11) in (2.7), applying (3.33), and following as in the above theorem, we get the following expression:

$$\sum_{j=2}^{\infty}j[2(k+1)(j-1)+j|X_2+1|+|X_1+1|]\frac{(-c/4)^{j-1}}{(l_a)_{j-1}(j-1)!}\frac{2(1-\delta)|w|}{j^2(s-3b)}$$
$$\leq|X_2-X_1|. \tag{3.40}$$

Now applying (1.7) and (1.6) in (3.40), we get

$$(2(k+1)+|X_2+1|)\,(\mathcal{G}_a(1)-1)$$
$$+[|X_1+1|-2(k+1)]\left(\frac{4(1-l_a)}{c}\right)\left(\mathcal{G}_{a-1}(1)-1+\frac{(c/4)}{(l_a-1)}\right)$$
$$\leq|X_2-X_1|\frac{(s-3b)}{2(1-\delta)|w|}. \tag{3.41}$$

The desired result follows directly from (3.41), which completes the proof. □

Theorem 3.16. *Let a function* $\psi(z) \in \mathcal{A}$ *and of the form* (1.1). *Again if* $\psi(z) \in \mathcal{M}^{w}_{b,s}(\delta)$ *and satisfies the following condition, then* $\mathcal{E}_{l_a,c}(\psi, z) \in k - \mathcal{ST}[X_1, X_2]$.

$$\begin{aligned}&(2(k+1) + |X_2+1|)\left(\frac{4(1-l_a)}{c}\right)\mathcal{G}_{a-1}(1)\\&\quad + (|X_1+1| - |X_2+1| - 4(k+1))\left(\frac{16(l_a-1)_2}{c^2}\right)\mathcal{G}_{a-2}(1)\\&\quad + (2(k+1) + |X_2+1|)\left(\frac{4(1-l_a)}{c}\right)\left(\frac{c}{4(l_a-1)} - 1\right)\\&\quad + (|X_1+1| - |X_2+1| - 4(k+1))\left(\frac{16(l_a-1)_2}{c^2}\right)\left(\frac{c}{4(l_a-2)} - 1\right)\\&\leq |X_2 - X_1|\frac{(s-3b)}{2(1-\delta)|w|}.\end{aligned}$$

Proof. Substituting Eq (1.11) in (2.8), applying (3.29), and following as in the above theorem, we get the following expression:

$$\sum_{j=2}^{\infty}[2(k+1)j - 2(k+1) + |X_2+1|j + |X_1+1|]\frac{(-c/4)^{j-1}}{(l_a)_{j-1}(j-1)!}\frac{2(1-\delta)|w|}{j(j+1)(s-3b)} \leq |X_2 - X_1|. \quad (3.42)$$

Now applying (1.6) in (3.42), we get

$$\begin{aligned}&(2(k+1) + |X_2+1|)\left(\frac{4(1-l_a)}{c}\right)\left(\mathcal{G}_{a-1}(1) + \frac{c}{4(l_a-1)} - 1\right)\\&\quad + (|X_1+1| - |X_2+1| - 4(k+1))\left(\frac{16(l_a-1)_2}{c^2}\right)\left(\mathcal{G}_{a-2}(1) + \frac{c}{4(l_a-2)} - 1\right)\\&\leq |X_2 - X_1|\frac{(s-3b)}{2(1-\delta)|w|}.\end{aligned} \quad (3.43)$$

We will get the required result from (3.43). This completes the proof. □

3.3. Results on the convolution operators $\mathcal{Y}^{p}_{m}(\psi, z)$ *and* $\mathcal{E}_{l_a,c}(\psi, z)$ *associated with the class* $\mathcal{R}^{w}_{s,d}(\delta)$

Theorem 3.17. *Let* $\psi(z) \in \mathcal{A}$. *Again if* $\psi(z) \in \mathcal{R}^{w}_{s,d}(\delta)$ *and satisfies the following condition, then* $\mathcal{Y}^{p}_{m}(\psi, z) \in \mathcal{S}^{k,\varsigma}_{v,u}(t, r)$.

$$\begin{aligned}&(k+1)vY'_m(p) + [(k+1)((\varsigma+r+1)v+u) + (1+r)v]Y'_{m-1}(p)\\&\quad + [(k+1)(\varsigma+r+1)u + (1+r)u]Y_{m-1}(p)\\&\quad - (k+1)((\varsigma+r+2)(u+v) - (1+r)(u+v)\\&< [(k+1)((\varsigma+r+2) + (1+r)](u+v) + \frac{(d-2s)}{2(1-\delta)|w|}(u - \varsigma(u+v)).\end{aligned}$$

Proof. Substituting Eq (1.5) in (3.1), as the sufficient condition for the class $\mathcal{S}^{k,\varsigma}_{v,u}(t, r)$, we get the following expression:

$$\sum_{j=2}^{\infty}[(k+1)(j(j-1)v + j((\varsigma+r+1)v+u) + (\varsigma+r+1)u) + (1+r)(u+jv)]\frac{p^{j-1}(j-1)^m}{(j-1)!}e^{-p}|x_j|$$

$$< u - \varsigma(u+v). \tag{3.44}$$

Since $1 + (j-1)(d - 2s + js) \geq (j-1)(d-2s)$, the inequality from (2.4) can be written as

$$|x_j| \leq \frac{2(1-\delta)|w|}{(j-1)(d-2s)}, \quad j = 2, 3, \cdots . \tag{3.45}$$

Using the bound from (3.45) in the inequality (3.44), we obtain

$$\begin{aligned} &\sum_{j=2}^{\infty}[(k+1)(j(j-1)v + j((\varsigma + r + 1)v + u) + (\varsigma + r + 1)u) \\ &\quad + (1+r)(u + jv)]\frac{p^{j-1}(j-1)^m}{(j-1)!}e^{-p}\frac{2|w|(1-\delta)}{(j-1)(d-2s)} \\ &< u - \varsigma(u+v). \end{aligned} \tag{3.46}$$

Now applying (1.3) and (1.4) in (3.46), and then using the result in (3.44), we get

$$\begin{aligned} &(k+1)v(Y_m'(p) - 1) + (k+1)((\varsigma + r + 1)v + u)(Y_{m-1}'(p) - 1) \\ &\quad + (k+1)(\varsigma + r + 1)u(Y_{m-1}(p) - 1) \\ &\quad + (1+r)u(Y_{m-1}(p) - 1) + (1+r)v(Y_{m-1}'(p) - 1) \\ &< \frac{(d-2s)}{2|w|(1-\delta)}(u - \varsigma(u+v)). \end{aligned} \tag{3.47}$$

From (3.47), we obtain the required result. □

Remark 3.2. *If we take $d = 1$, $s = 0$, $\varsigma = 1$, $v = 0$, $u = 1$, $k = 0$, $r = 0$, and $t = 0$ in Theorem 3.17, then it reduces to Theorem 4.3 of [25] for $\lambda = 0$, $B = 1$, and $A = 1 + \frac{2\omega(1-\delta)}{\tau}$.*

Theorem 3.18. *Let a function $\psi(z) \in \mathcal{A}$ and of the form* (1.1). *Again if $\psi(z) \in \mathcal{R}_{s,d}^{w}(\delta)$ and satisfies the following condition, then $\mathcal{Y}_m^p(\psi, z) \in k - \mathcal{UCSP}(t, r)$.*

$$2kY_m'(p) + (\cos t - r)Y_{m-1}'(p) \leq 2k + (\cos t - r)\left(1 + \frac{(d-2s)}{2(1-\delta)|w|}\right).$$

Proof. Substituting Eq (1.5) in (3.4), as the sufficient condition for the class $k - \mathcal{UCSP}(t, r)$, we get the following expression:

$$\sum_{j=2}^{\infty}(2k(j-1) + \cos t - r)j\frac{p^{j-1}(j-1)^m}{(j-1)!}e^{-p}|x_j| \leq \cos t - r. \tag{3.48}$$

Applying (3.45) in (3.48), we have

$$\sum_{j=2}^{\infty}(2k(j-1) + \cos t - r)j\frac{p^{j-1}(j-1)^m}{(j-1)!}e^{-p}\frac{2(1-\delta)|w|}{(j-1)(d-2s)} \leq \cos t - r.$$

Further simplifying, we get

$$2k\sum_{j=2}^{\infty} j\frac{p^{j-1}(j-1)^m}{(j-1)!}e^{-p} + (\cos t - r)\sum_{j=2}^{\infty} j\frac{p^{j-1}(j-1)^{m-1}}{(j-1)!}e^{-p} \leq (\cos t - r)\frac{(d-2s)}{2(1-\delta)|w|}. \tag{3.49}$$

Now, applying (1.4) into Eq (3.49), we conclude

$$2k(Y'_m(p) - 1) + (\cos t - r)(Y'_{m-1}(p) - 1) \leq (\cos t - r)\frac{(d-2s)}{2(1-\delta)|w|}. \tag{3.50}$$

Hence, the required result is obtained from (3.50), completing the proof. □

Theorem 3.19. *Let a function $\psi(z) \in \mathcal{A}$ and of the form* (1.1). *Again if $\psi(z) \in \mathcal{R}^w_{s,d}(\delta)$ and satisfies the following condition, then $\mathcal{Y}^p_m(\psi, z) \in k - \mathcal{SP}_p(t, r)$.*

$$2kY'_{m-1}(p) + (\cos t - 2k - r)Y_{m-1}(p) \leq (\cos t - r)\left(1 + \frac{d-2s}{2(1-\delta)|w|}\right).$$

Proof. Substituting Eq (1.5) in (3.7), as the sufficient condition for the class $k - \mathcal{SP}_p(t, r)$, we get the following expression:

$$\sum_{j=2}^{\infty}(2k(j-1) + \cos t - r)\frac{p^{j-1}(j-1)^m}{(j-1)!}e^{-p}|x_j| \leq \cos t - r. \tag{3.51}$$

Applying (3.15) in (3.51), we have

$$\sum_{j=2}^{\infty}(2kj + \cos t - 2k - r)\frac{p^{j-1}(j-1)^m}{(j-1)!}e^{-p}\frac{2(1-\delta)|w|}{(j-1)(d-2s)} \leq \cos t - r.$$

Further simplifying, we get

$$\begin{aligned} &2k\sum_{j=2}^{\infty} j\frac{p^{j-1}(j-1)^{m-1}}{(j-1)!}e^{-p} + (\cos t - 2k - r)\sum_{j=2}^{\infty} \frac{p^{j-1}(j-1)^{m-1}}{(j-1)!}e^{-p} \\ &\leq (\cos t - r)\frac{(d-2s)}{2(1-\delta)|w|}. \end{aligned} \tag{3.52}$$

Now, applying (1.4) into Eq (3.52), we conclude

$$2k(Y'_{m-1}(p) - 1) + (\cos t - 2k - r)(Y_{m-1}(p) - 1) \leq (\cos t - r)\frac{(d-2s)}{2(1-\delta)|w|}. \tag{3.53}$$

By virtue of (3.53), the desired conclusion is achieved, and the proof is thereby finalized. □

Theorem 3.20. *Let $\psi(z) \in \mathcal{A}$. Again if $\psi(z) \in \mathcal{R}^w_{s,d}(\delta)$ and satisfies the following condition, then $\mathcal{Y}^p_m(\psi, z) \in k - \mathcal{UCV}[X_1, X_2]$.*

$$\begin{aligned} &(2(k+1) + |X_2 + 1|)\, Y_{m+1}(p) + (|X_1 + 1| + |X_2 + 1|)\, Y_m(p) \\ &\leq 2(k+1) + 2|X_2 + 1| + |X_1 + 1| + \frac{(d-3s)|X_2 - X_1|}{2(1-\delta)|w|}. \end{aligned}$$

Proof. From the denominator of (2.4), $1+(j-1)(d-2s+js) \geq j(d-3s)$, and then (2.4) can be written as

$$|x_j| \leq \frac{2(1-\delta)|w|}{j(d-3s)}, \quad j = 2, 3, \cdots . \tag{3.54}$$

Substituting Eq (1.5) in (2.7), applying (3.54), and following as in the above theorem, we get the following expression:

$$\sum_{j=2}^{\infty} j[2(k+1)(j-1) + j|X_2+1| + |X_1+1|]\frac{p^{j-1}(j-1)^m}{(j-1)!}e^{-p}\frac{2(1-\delta)|w|}{j(d-3s)}|X_2 - X_1|. \tag{3.55}$$

Now applying (1.3) in (3.55), we get

$$(2(k+1) + |X_2+1|)(Y_{m+1}(p) - 1) + (|X_1+1| + |X_2+1|)(Y_m(p) - 1) \leq \frac{(d-3s)|X_2 - X_1|}{2(1-\delta)|w|}. \tag{3.56}$$

We will get the required result from (3.56). This completes the proof. □

Theorem 3.21. *Let $\psi(z) \in \mathcal{A}$. Again if $\psi(z) \in \mathcal{R}^w_{s,d}(\delta)$ and satisfies the following condition, then $\mathcal{Y}^p_m(\psi, z) \in k - \mathcal{ST}[X_1, X_2]$.*

$$\begin{aligned} &2(k+1)Y_m(p) + |X_2+1|Y'_{m-1}(p) + |X_1+1|Y_{m-1}(p) \\ &\leq 2(k+1) + |X_2+1| + |X_1+1| + \frac{(d-2s)|X_2 - X_1|}{2(1-\delta)|w|}. \end{aligned}$$

Proof. Substituting Eq (1.5) in (2.8), applying (3.45), and following as in the above theorem, we get the following expression:

$$\sum_{j=2}^{\infty} [2(k+1)(j-1) + j|X_2+1| + |X_1+1|]\frac{p^{j-1}(j-1)^m}{(j-1)!}e^{-p}\frac{2(1-\delta)|w|}{(j-1)(d-2s)} \leq |X_2 - X_1|. \tag{3.57}$$

Now applying (1.3) and (1.4) in (3.57), we get

$$2(k+1)(Y_m(p) - 1) + |X_2+1|(Y'_{m-1}(p) - 1) + |X_1+1|(Y_{m-1}(p) - 1) \leq \frac{(d-2s)|X_2 - X_1|}{2(1-\delta)|w|}. \tag{3.58}$$

From (3.58), we obtain the required result. □

Theorem 3.22. *Let $\psi(z) \in \mathcal{A}$. Again if $\psi(z) \in \mathcal{R}^w_{s,d}(\delta)$ and satisfies the following condition, then $\mathcal{E}_{l_a,c}(\psi, z) \in \mathcal{S}^{k,\varsigma}_{v,u}(t, r)$.*

$$\begin{aligned} &[(k+1)((\varsigma + r + 1)v + u) + (1+r)v]\mathcal{G}_a(1) - \frac{c(k+1)v}{4l_a}\mathcal{G}_{a+1}(1) \\ &- \left(\frac{4u(l_a - 1)[(k+1)(\varsigma + r + 1) + 1 + r]}{c}\right)\mathcal{G}_{a-1}(1) \\ &- \left(\frac{4u(l_a - 1)[(k+1)(\varsigma + r + 1) + 1 + r]}{c}\right)\left(\frac{c}{4(l_a - 1)} - 1\right) \\ &< [(k+1)((\varsigma + r + 1)v + u) + (1+r)v] + [u - \varsigma(u+v)]\frac{(d-3s)}{2(1-\delta)|w|}. \end{aligned}$$

Proof. Substituting Eq (1.11) in (3.1), as the sufficient condition for the class $\mathcal{S}^{k,\varsigma}_{v,u}(t,r)$, we get the following expression:

$$\begin{aligned}
&\sum_{j=2}^{\infty}[(k+1)\big(j(j-1)v + j((\varsigma+r+1)v+u) + (\varsigma+r+1)u\big)\\
&\qquad + (1+r)(u+jv)]\frac{(-c/4)^{j-1}}{(l_a)_{j-1}(j-1)!}|x_j|\\
&\quad < u - \varsigma(u+v).
\end{aligned} \tag{3.59}$$

Applying (3.54) on the left side of (3.59), we have

$$\begin{aligned}
&\sum_{j=2}^{\infty}[(k+1)\big(j(j-1)v + j((\varsigma+r+1)v+u) + (\varsigma+r+1)u\big)\\
&\qquad + (1+r)(u+jv)]\frac{(-c/4)^{j-1}}{(l_a)_{j-1}(j-1)!}\frac{2(1-\delta)|w|}{j(d-3s)}\\
&\quad < u - \varsigma(u+v).
\end{aligned} \tag{3.60}$$

Now applying (1.6) in (3.60), we get

$$\begin{aligned}
&(k+1)v\left(\frac{-c}{4l_a}\right)\mathcal{G}_{a+1}(1)\\
&\quad + [(k+1)((\varsigma+r+1)v+u) + (1+r)v](\mathcal{G}_a(1)-1)\\
&\quad + [(k+1)(\varsigma+r+1)u + (1+r)u]\left(\frac{-4(l_a-1)}{c}\right)\left(\mathcal{G}_{a-1}(1) - 1 + \frac{c}{4(l_a-1)}\right)\\
&< [u-\varsigma(u+v)]\frac{(d-3s)}{2(1-\delta)|w|}.
\end{aligned} \tag{3.61}$$

Hence, the required result is obtained from (3.61), completing the proof. □

Theorem 3.23. *Let a function $\psi(z) \in \mathcal{A}$ and of the form* (1.1). *Again if $\psi(z) \in \mathcal{R}^{w}_{s,d}(\delta)$ and satisfies the following condition, then $\mathcal{E}_{l_a,c}(\psi, z) \in k - \mathcal{UCSP}(t,r)$.*

$$2k\mathcal{G}'_a(1) - (\cos t - r)\mathcal{G}_a(1) \leq (\cos t - r)\left(\frac{d-3s}{2(1-\delta)|w|} - 1\right).$$

Proof. Substituting Eq (1.11) in (3.4), as the sufficient condition for the class $k - \mathcal{UCSP}(t,r)$, we get the following expression:

$$\sum_{j=2}^{\infty}(2k(j-1) + \cos t - r)j\frac{(-c/4)^{j-1}}{(l_a)_{j-1}(j-1)!}|x_j| \leq \cos t - r. \tag{3.62}$$

Applying (3.54) in (3.62), we have

$$\sum_{j=2}^{\infty}(2k(j-1) + \cos t - r)j\frac{(-c/4)^{j-1}}{(l_a)_{j-1}(j-1)!}\frac{2(1-\delta)|w|}{j(d-3s)} \leq \cos t - r. \tag{3.63}$$

Simplifying (3.63), we get

$$2k\sum_{j=1}^{\infty}\frac{(-c/4)^j}{(l_a)_j(j-1)!}+(\cos t-r)\sum_{j=1}^{\infty}\frac{(-c/4)^j}{(l_a)_j(j)!}\leq(\cos t-r)\frac{(d-3s)}{2(1-\delta)|w|}. \tag{3.64}$$

Now, applying (1.6) and (1.7) into Eq (3.35), we conclude

$$2k\mathcal{G}_a'(1)-(\cos t-r)\left(\mathcal{G}_a(1)-1\right)\leq(\cos t-r)\frac{(d-3s)}{2(1-\delta)|w|}. \tag{3.65}$$

The desired result follows directly from (3.65), which completes the proof. □

Theorem 3.24. *Let a function* $\psi(z)\in\mathcal{A}$ *and of the form* (1.1). *Again if* $\psi(z)\in\mathcal{R}_{s,d}^{w}(\delta)$ *and satisfies the following condition, then* $\mathcal{E}_{l_a,c}(\psi,z)\in k-\mathcal{SP}_p(t,r)$.

$$\begin{aligned}&kc\mathcal{G}_a(1)-2(\cos t-2k-r)(l_a-1)\mathcal{G}_{a-1}(1)+2(\cos t-2k-r)(l_a-1)\\&\leq 2kc+(\cos t-r)\frac{c}{2}\left(1+\frac{(d-3s)}{2(1-\delta)|w|}\right).\end{aligned}$$

Proof. Substituting Eq (1.11) in (3.7), as the sufficient condition for the class $k-\mathcal{SP}_p(t,r)$, we get the following expression:

$$\sum_{j=2}^{\infty}(2k(j-1)+\cos t-r)\frac{(-c/4)^{j-1}}{(l_a)_{j-1}(j-1)!}|x_j|\leq\cos t-r. \tag{3.66}$$

Applying (3.54) in (3.66), we have

$$\sum_{j=2}^{\infty}(2kj+\cos t-2k-r)\frac{(-c/4)^{j-1}}{(l_a)_{j-1}(j-1)!}\,\frac{2(1-\delta)|w|}{j(d-3s)}\leq\cos t-r.$$

Further simplifying, we get

$$2k\sum_{j=1}^{\infty}\frac{(-c/4)^j}{(l_a)_j(j)!}+(\cos t-2k-r)\frac{-4(l_a-1)}{c}\sum_{j=2}^{\infty}\frac{(-c/4)^j}{(l_a-1)_j(j)!}\leq(\cos t-r)\frac{(d-3s)}{2(1-\delta)|w|}. \tag{3.67}$$

Now, applying (1.6) into Eq (3.67), we conclude

$$2k(\mathcal{G}_a(1)-1)+(\cos t-2k-r)\frac{-4(l_a-1)}{c}\left(\mathcal{G}_{a-1}(1)-1+\frac{c}{4(l_a-1)}\right)\leq(\cos t-r)\frac{(d-3s)}{2(1-\delta)|w|}. \tag{3.68}$$

From (3.68), we obtain the required result. □

Theorem 3.25. *Let* $\psi(z)\in\mathcal{A}$. *Again if* $\psi(z)\in\mathcal{R}_{s,d}^{w}(\delta)$ *and satisfies the following condition, then* $\mathcal{E}_{l_a,c}(\psi,z)\in k-\mathcal{UCV}[X_1,X_2]$.

$$\begin{aligned}&(2(k+1)+|X_2+1|)\,\mathcal{G}_a'(1)+(|X_1+1|+|X_2+1|)\,\mathcal{G}_a(1)\\&\leq|X_1+1|+|X_2+1|+|X_2-X_1|\frac{(d-3s)}{2(1-\delta)|w|}.\end{aligned}$$

Proof. Substituting Eq (1.11) in (2.7), applying (3.54), and following as in the above theorem, we get the following expression:

$$\sum_{j=2}^{\infty} j[2(k+1)(j-1)+j|(X_2+1)|+|(X_1+1)|]\frac{(-c/4)^{j-1}}{(l_a)_{j-1}(j-1)!}\frac{2(1-\delta)|w|}{j(d-3s)} \leq |X_2-X_1|. \tag{3.69}$$

Now applying (1.7) and (1.6) in (3.69), we get

$$(2(k+1)+|X_2+1|)\,\mathcal{G}'_a(1)+(|X_1+1|+|X_2+1|)\,(\mathcal{G}_a(1)-1) \leq |X_2-X_1|\frac{(d-3s)}{2(1-\delta)|w|}. \tag{3.70}$$

Hence, the required result is obtained from (3.70), completing the proof. □

Theorem 3.26. *Let $\psi(z) \in \mathcal{A}$. Again if $\psi(z) \in \mathcal{R}^{w}_{s,d}(\delta)$ and satisfies the following condition, then $\mathcal{E}_{l_a,c}(\psi,z) \in k-\mathcal{ST}[X_1,X_2]$.*

$$\begin{aligned}&(2(k+1)+|X_2+1|)\,\mathcal{G}_a(1)+(|X_1+1|-2(k+1))\,\mathcal{G}_{a-1}(1)\\ &\leq |X_2+1|+|X_1+1|+|X_2-X_1|\frac{(d-3s)}{2(1-\delta)|w|}.\end{aligned}$$

Proof. Substituting Eq (1.11) in (2.8), applying (3.54), and following as in the above theorem, we get the following expression:

$$\sum_{j=2}^{\infty}[2(k+1)j+j|X_2+1|+|X_1+1|-2(k+1)]\frac{(-c/4)^{j-1}}{(l_a)_{j-1}(j-1)!}\frac{2(1-\delta)|w|}{j(d-3s)} \leq |X_2-X_1|. \tag{3.71}$$

Now applying (1.6) in (3.71), we get

$$(2(k+1)+|X_2+1|)\,(\mathcal{G}_a(1)-1)+(|X_1+1|-2(k+1))\,(\mathcal{G}_{a-1}(1)-1) \leq |X_2-X_1|\frac{(d-3s)}{2(1-\delta)|w|}. \tag{3.72}$$

The desired result follows directly from (3.72), which completes the proof. □

3.4. Geometric properties of the integral operator $\mathfrak{Y}_m(p,z)$

The integral operator $\mathfrak{Y}_m(p,z)$ is defined, for all $z \in \mathbb{D}$, by

$$\mathfrak{Y}_m(p,z) = \int_0^z \frac{Y_m(p,\theta)}{\theta}d\theta = z+\sum_{j=2}^{\infty}\frac{p^{j-1}(j-1)^m}{(j)!}e^{-p}z^j. \tag{3.73}$$

Theorem 3.27. *Let $\psi(z) \in \mathcal{A}$. If $\psi(z)$ satisfies the following condition:*

$$2kY'_m(p)+(\cos t-2k-r)Y_m(p) \leq 2(\cos t-r),$$

then $\mathfrak{Y}_m(p,z) \in k-\mathcal{UCSP}(t,r)$.

Proof. The proof is similar to the preceding theorems. In particular, the sufficient condition in Theorem 3.27 is identical to that given in Theorem 3.4. □

A convolution operator $\mathfrak{Y}_m^p(\psi, z)$ is defined, for $\psi(z) \in \mathcal{A}$ and $z \in \mathbb{D}$, by

$$\mathfrak{Y}_m^p(\psi, z) = \mathfrak{Y}_m(p, z) * \psi(z) = z + \sum_{j=2}^{\infty} \frac{p^{j-1}(j-1)^m}{(j)!} e^{-p} x_j z^j, \tag{3.74}$$

where $*$ denotes the Hadamard product.

Theorem 3.28. *Let a function $\psi(z) \in \mathcal{A}$ and of the form* (1.1)*. Again if $\psi(z) \in \mathcal{M}_{b,s}^w(\delta)$ and satisfies the following condition, then $\mathfrak{Y}_m^p(\psi, z) \in k - \mathcal{UCSP}(t, r)$.*

$$2kY'_{m-2}(p) + (\cos t - 2k - r)Y_{m-2}(p) \leq (\cos t - r)\left(1 + \frac{(s-2b)}{2(1-\delta)|w|}\right).$$

Theorem 3.29. *Let $\psi(z) \in \mathcal{A}$. Again if $\psi(z) \in \mathcal{M}_{b,s}^w(\delta)$ and satisfies the following condition, then $\mathfrak{Y}_m^p(\psi, z) \in k - \mathcal{UCV}[X_1, X_2]$.*

$$\begin{aligned} &2(k+1)Y_{m-1}(p) + |X_2 + 1|Y'_{m-1}(p) + |X_1 + 1|Y_{m-2}(p) \\ &\quad \leq 2(k+1) + |X_2 + 1| + |X_1 + 1| + |X_2 - X_1|\frac{(s-2b)}{2(1-\delta)|w|}. \end{aligned}$$

Theorem 3.30. *Let a function $\psi(z) \in \mathcal{A}$ and of the form* (1.1)*. Again if $\psi(z) \in \mathcal{R}_{s,d}^w(\delta)$ and satisfies the following condition, then $\mathfrak{Y}_m^p(\psi, z) \in k - \mathcal{UCSP}(t, r)$.*

$$2kY'_{m-1}(p) + (\cos t - 2k - r)Y_{m-1}(p) \leq (\cos t - r)\left(1 + \frac{d-2s}{2(1-\delta)|w|}\right).$$

Theorem 3.31. *Let $\psi(z) \in \mathcal{A}$. Again if $\psi(z) \in \mathcal{R}_{s,d}^w(\delta)$ and satisfies the following condition, then $\mathfrak{Y}_m^p(\psi, z) \in k - \mathcal{UCV}[X_1, X_2]$.*

$$\begin{aligned} &2(k+1)Y_m(p) + |X_2 + 1|Y'_{m-1}(p) + |X_1 + 1|Y_{m-1}(p) \\ &\quad \leq 2(k+1) + |X_2 + 1| + |X_1 + 1| + \frac{(d-2s)|X_2 - X_1|}{2(1-\delta)|w|}. \end{aligned}$$

Remark 3.3. *The conditions stated in Theorems 3.28–3.31 are similar to the sufficient conditions given in Theorem 3.9, Theorem 3.11, Theorem 3.19, and Theorem 3.21, respectively.*

3.5. Geometric properties of the integral operator $\mathfrak{G}_a(z)$

An integral operator $\mathfrak{G}_a(z)$ is defined, for all $z \in \mathbb{D}$, by

$$\mathfrak{G}_a(z) = \int_0^z \mathcal{G}_a(\theta)d\theta = z + \sum_{j=2}^{\infty} \frac{(-c/4)^{j-1}}{(l_a)_{j-1}(j)!} z^j. \tag{3.75}$$

Theorem 3.32. *Let $\psi(z) \in \mathcal{A}$ be a function of the form* (1.1)*. If $\psi(z)$ satisfies the following condition:*

$$\begin{aligned} &[(k+1)((\varsigma + r + 1)v + u) + (1+r)v]\mathcal{G}_a(1) - \frac{(k+1)vc}{4l_a}\mathcal{G}_{a+1}(1) \\ &\quad + [(k+1)(\varsigma + r + 1)u + (1+r)u]\mathfrak{G}_a(1) \\ &\quad < (k+2)u + ((k+1)(\varsigma + r + 1) + 1 + r - \varsigma)(v+u), \end{aligned}$$

then $\mathfrak{G}_a(z) \in \mathcal{S}_{v,u}^{k,\varsigma}(t, r)$.

Proof. Substituting (3.75) in (3.1), we obtain

$$\sum_{j=2}^{\infty}[(k+1)(j(j-1)v+j(\varsigma+r)v+j(u+v)+(\varsigma+r)u+u)+(1+r)(u+jv)]\frac{(-c/4)^{j-1}}{(l_a)_{j-1}(j)!}$$
$$< u-\varsigma(u+v).$$

Applying (1.7) and (1.8) in the above expression, we obtain

$$\begin{aligned}(k+1)v\frac{-c}{4l_a}\mathcal{G}_{a+1}(1)+[(k+1)((\varsigma+r+1)v+u))+(1+r)v]\left(\mathcal{G}_a(1)-1\right)\\ +[(k+1)(\varsigma+r+1)u+(1+r)u]\left(\mathfrak{G}_a(1)-1\right)\\ < u-\varsigma(u+v).\end{aligned} \tag{3.76}$$

The required result follows directly from (3.76). □

Theorem 3.33. *Let $\psi(z)\in\mathcal{A}$ be a function of the form* (1.1)*. If $\psi(z)$ satisfies the following condition:*

$$2k\mathcal{G}_a'(1)+(\cos t-r)\mathcal{G}_a(1)\leq 2(\cos t-r),$$

then $\mathfrak{G}_a(z)\in k-\mathcal{UCSP}(t,r)$.

Proof. Applying (3.75) in (3.4), we obtain

$$\sum_{j=2}^{\infty}(2k(j-1)+\cos t-r)j\frac{(-c/4)^{j-1}}{(l_a)_{j-1}(j)!}\leq\cos t-r.$$

Simplifying, we get

$$2k\sum_{j=2}^{\infty}\frac{(-c/4)^{j-1}}{(l_a)_{j-1}(j-2)!}+(\cos t-r)\sum_{j=2}^{\infty}\frac{(-c/4)^{j-1}}{(l_a)_{j-1}(j-1)!}\leq\cos t-r.$$

Using (1.7) and (1.6) in the above expression, we obtain

$$2k\mathcal{G}_a'(1)+(\cos t-r)\left(\mathcal{G}_a(1)-1\right)\leq\cos t-r. \tag{3.77}$$

Hence, the required result is obtained from (3.77), completing the proof. □

Theorem 3.34. *Let $\psi(z)\in\mathcal{A}$ be a function of the form* (1.1)*. If $\psi(z)$ satisfies the following condition:*

$$2k\mathcal{G}_a(1)+(\cos t-2k-r)\mathfrak{G}_a(1)\leq 2(\cos t-r),$$

then $\mathfrak{G}_a(z)\in k-\mathcal{SP}_p(t,r)$.

Proof. Applying (3.75) in (3.7), we obtain

$$\sum_{j=2}^{\infty}(2kj+\cos t-2k-r)\frac{(-c/4)^{j-1}}{(l_a)_{j-1}(j)!}\leq\cos t-r.$$

Simplifying, we have

$$2k\sum_{j=2}^{\infty}\frac{(-c/4)^{j-1}}{(l_a)_{j-1}(j-1)!}+(\cos t-2k-r)\sum_{j=2}^{\infty}\frac{(-c/4)^{j-1}}{(l_a)_{j-1}(j)!}\leq\cos t-r.$$

Using (1.6) and (3.75) in the above expression, we get

$$2k\left(\mathcal{G}_a(1)-1\right)+(\cos t-2k-r)\left(\mathfrak{G}_a(1)-1\right)\leq\cos t-r. \tag{3.78}$$

It follows from (3.78) that the required result holds true, which concludes the proof. □

A convolution operator $\mathfrak{E}_{l_a,c}(\psi,z)$ is defined by

$$\mathfrak{E}_{l_a,c}(\psi,z)=\mathfrak{G}_a(z)*\psi(z)=z+\sum_{j=2}^{\infty}\frac{(-c/4)^{j-1}}{(l_a)_{j-1}(j)!}x_jz^j, \tag{3.79}$$

where $*$ denotes the Hadamard product.

Theorem 3.35. *Let $\psi(z)\in\mathcal{A}$. Suppose $\psi(z)\in\mathcal{M}_{b,s}^{w}(\delta)$ and satisfies the following condition:*

$$\begin{aligned}&8kc(1-l_a)\mathcal{G}_{a-1}(1)+16(\cos t-4k-r)(l_a-2)_2\mathcal{G}_{a-2}(1)\\&\quad+16(\cos t-4k-r)(l_a-2)_2\left(\frac{c}{4(l_a-2)}-1\right)+8kc(l_a-1)\\&\leq\frac{(\cos t-r)c^2}{2}\left(1+\frac{(s-3b)}{(1-\delta)|w|}\right).\end{aligned}$$

Then $\mathfrak{E}_{l_a,c}(\psi,z)\in k-\mathcal{UCSP}(t,r)$.

Proof. Substituting (3.79) and (3.29) into (3.4), we obtain

$$\sum_{j=2}^{\infty}(2k(j+1)+\cos t-4k-r)j\frac{(-c/4)^{j-1}}{(l_a)_{j-1}(j)!}\frac{2(1-\delta)|w|}{j(j+1)(s-3b)}\leq\cos t-r. \tag{3.80}$$

Simplifying (3.80), we get

$$\begin{aligned}&2k\frac{-4(l_a-1)}{c}\sum_{j=2}^{\infty}\frac{(-c/4)^{j}}{(l_a-1)_j(j)!}+(\cos t-4k-r)\frac{16(l_a-2)_2}{c^2}\sum_{j=3}^{\infty}\frac{(-c/4)^{j}}{(l_a-2)_j(j)!}\\&\leq(\cos t-r)\frac{(s-3b)}{2(1-\delta)|w|}.\end{aligned} \tag{3.81}$$

Applying (1.6) to (3.81), we get

$$\begin{aligned}&2k\frac{-4(l_a-1)}{c}\left(\mathcal{G}_{a-1}(1)-1+\frac{c}{4(l_a-1)}\right)\\&\quad+(\cos t-4k-r)\frac{16(l_a-2)_2}{c^2}\left(\mathcal{G}_{a-2}(1)-1+\frac{c}{4(l_a-2)}-\frac{c^2}{32(l_a-2)_2}\right)\\&\leq(\cos t-r)\frac{(s-3b)}{2(1-\delta)|w|}.\end{aligned} \tag{3.82}$$

Hence, the required result is obtained from (3.82), completing the proof. □

Theorem 3.36. *Let $\psi(z) \in \mathcal{A}$. Suppose $\psi(z) \in \mathcal{M}^{w}_{b,s}(\delta)$ and satisfies the following condition:*

$$\begin{aligned}
&(4(1-l_a)(2(k+1)+|X_2+1|))\, c\mathcal{G}_{a-1}(1)\\
&\quad + 16(|X_1+1|-|X_2+1|-4(k+1))(l_a-2)_2\mathcal{G}_{a-2}(1)\\
&\quad + 16(|X_1+1|-|X_2+1|-4(k+1))(l_a-2)_2\left(\frac{c}{4(l_a-2)}-1\right)\\
&\quad + (2(k+1)+|X_2+1|)c^2\left(\frac{4(l_a-1)}{c}-1\right)-(|X_1+1|-|X_2+1|-4(k+1))\frac{1}{2}\\
&\le |X_2-X_1|\frac{(s-3b)}{2(1-\delta)|w|}.
\end{aligned}$$

Then $\mathfrak{E}_{l_a,c}(\psi,z) \in k-\mathcal{UCV}[X_1,X_2]$.

Proof. Substituting (3.79) in (2.7), and then applying (3.29), we obtain

$$\begin{aligned}
&\sum_{j=2}^{\infty} j[(2(k+1)+|X_2+1|)(j+1)]\frac{(-c/4)^{j-1}}{(l_a)_{j-1}(j)!}\frac{2(1-\delta)|w|}{j(j+1)(s-3b)}\\
&\qquad \sum_{j=2}^{\infty} j[|X_1+1|-|X_2+1|-4(k+1)]\frac{(-c/4)^{j-1}}{(l_a)_{j-1}(j)!}\frac{2(1-\delta)|w|}{j(j+1)(s-3b)}\\
&\qquad \le |X_2-X_1|.
\end{aligned} \tag{3.83}$$

Now applying (1.7) and (1.6) to (3.83), we get

$$\begin{aligned}
&[2(k+1)+|X_2+1|]\left(\frac{4(1-l_a)}{c}\right)\left(\mathcal{G}_{a-1}(1)-1+\frac{c}{4(l_a-1)}\right)\\
&\quad + [|X_1+1|-|X_2+1|-4(k+1)]\left(\frac{16(l_a-2)_2}{c^2}\left(\mathcal{G}_{a-2}(1)-1+\frac{c}{4(l_a-2)}\right)-\frac{1}{2}\right)\\
&\le |X_2-X_1|\frac{(s-3b)}{2(1-\delta)|w|}.
\end{aligned} \tag{3.84}$$

The desired result follows directly from (3.84), which completes the proof. □

Theorem 3.37. *Let $\psi(z) \in \mathcal{A}$. Suppose $\psi(z) \in \mathcal{R}^{w}_{s,d}(\delta)$ and satisfies the following condition:*

$$\begin{aligned}
&2kc\mathcal{G}_a(1)+4(\cos t-2k-r)(l_a-1)\mathcal{G}_{a-1}(1)-4(\cos t-2k-r)(l_a-1)\\
&\quad \le 4kc+(\cos t-r)c\left(\frac{(d-3s)}{2(1-\delta)|w|}-1\right).
\end{aligned}$$

Then $\mathfrak{E}_{l_a,c}(\psi,z) \in k-\mathcal{UCSP}(t,r)$.

Proof. Substituting (3.79) and (3.54) into (3.4), we obtain

$$\sum_{j=2}^{\infty}(2kj+\cos t-2k-r)j\frac{(-c/4)^{j-1}}{(l_a)_{j-1}(j)!}\frac{2(1-\delta)|w|}{j(d-3s)} \le \cos t - r. \tag{3.85}$$

Simplifying (3.85), we get

$$2k\sum_{j=1}^{\infty}\frac{(-c/4)^j}{(l_a)_j(j)!}+(\cos t-2k-r)\sum_{j=2}^{\infty}\frac{(-c/4)^{j-1}}{(l_a)_{j-1}(j)!}\leq(\cos t-r)\frac{(d-3s)}{2(1-\delta)|w|}. \tag{3.86}$$

Applying (1.6) and (1.7) to (3.86), we get

$$2k(\mathcal{G}_a(1)-1)+(\cos t-2k-r)\frac{4(l_a-1)}{c}\left(\mathcal{G}_{a-1}(1)-1+\frac{c}{4(l_a-1)}\right)\leq(\cos t-r)\frac{(d-3s)}{2(1-\delta)|w|}. \tag{3.87}$$

From (3.87), we obtain the required result. □

Theorem 3.38. *Let $\psi(z)\in\mathcal{A}$. Suppose $\psi(z)\in\mathcal{R}^w_{s,d}(\delta)$ and satisfies the following condition:*

$$\begin{aligned}&(2(k+1)+|X_2+1|)\mathcal{G}_a(1)+4(|X_1+1|-2(k+1))\frac{(1-l_a)}{c}\mathcal{G}_{a-1}(1)\\&\quad+(|X_1+1|-2(k+1))\,\frac{4(l_a-1)}{c}-(|X_1+1|+|X_2+1|)\\&\leq|X_2-X_1|\frac{(d-3s)}{2(1-\delta)|w|}.\end{aligned}$$

Then $\mathfrak{E}_{l_a,c}(\psi,z)\in k-\mathcal{UCV}[X_1,X_2]$.

Proof. Substituting Eq (1.11) in (2.7), applying (3.54), and proceeding as in the previous theorem, we obtain

$$\sum_{j=2}^{\infty}j[(2(k+1)+|X_2+1|)j+|X_1+1|-2(k+1)]\frac{(-c/4)^{j-1}}{(l_a)_{j-1}(j)!}\frac{2(1-\delta)|w|}{j(d-3s)}\leq|X_2-X_1|. \tag{3.88}$$

Now, applying (1.7) and (1.6) to (3.88), we get

$$\begin{aligned}&(2(k+1)+|X_2+1|)\,(\mathcal{G}_a(1)-1)+4(|X_1+1|-2(k+1))\frac{(1-l_a)}{c}\mathcal{G}_{a-1}(1)\\&\quad+(|X_1+1|-2(k+1))\,\frac{4(1-l_a)}{c}\left(\frac{c}{4(l_a-1)}-1\right)\\&\leq|X_2-X_1|\frac{(d-3s)}{2(1-\delta)|w|}.\end{aligned} \tag{3.89}$$

Consequently, by using (3.89), we arrive at the required result, thereby concluding the proof. □

4. Conclusions

The findings of this study provide a comprehensive framework for analyzing the geometric behavior of operators associated with Touchard polynomials and generalized Bessel functions of the first kind. In this work, we have examined the inclusion properties of the Touchard polynomials $Y_m(p,z)$ and the generalized Bessel functions of the first kind $\mathcal{G}_a(z)$, along with their associated convolution and integral operators, within the analytic function classes $\mathcal{R}^w_{s,d}(\delta)$ and $\mathcal{M}^w_{b,s}(\delta)$. By employing coefficient bounds and appropriate parameter constraints, we have derived sufficient conditions ensuring that the considered operators belong to various subclasses of starlike and convex functions. These findings not only unify and extend several existing results in geometric function theory but also highlight new interconnections between special functions and operator theory.

The results presented in this study provide a deeper understanding of how linear operators are generated by special functions. The inclusion results established among different subclasses contribute to the broader framework of geometric function theory by offering generalized criteria that can be applied to a wide range of operators. Overall, this work enriches the theory of analytic and univalent functions, providing the way for future investigations involving more generalized operators, fractional calculus approaches, and other well-known special functions such as Struve functions, error functions, hypergeometric functions, and $q-$analogues of special functions.

Author contributions

M. K. G., N. A., R. K. and S. R. M.: Conceptualization, methodology, writing—review and editing; M. K. G. and N. A.: Formal analysis, writing—original draft preparation; S. R. M.: Validation; R. K: Validation, supervision; N. A.: Funding. All authors of this article have contributed equally. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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