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Research article

Normalized solutions to lower critical Choquard equation with mixed local-nonlocal operators

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Abstract: This paper investigates the existence and non-existence of normalized ground state solutions for the following Choquard equation with mixed local and nonlocal operators, involving the Hardy-Littlewood-Sobolev (HLS) lower critical exponent. For the critical case $p = 2 + \frac{4}{N}$, we employ fibering map analysis to establish the non-existence of solutions. In the subcritical regime $2 < p < 2 + \frac{4}{N}$, we utilize variational methods to prove the existence of normalized ground states, which are shown to be radially symmetric and strictly decreasing in $|x|$. For the supercritical case $2 + \frac{4}{N} < p < 2_s^*$, we introduce a homotopy-stable family to construct a Palais–Smale sequence with a negative Lagrange multiplier. By analyzing the compactness properties of this sequence, we demonstrate the existence of normalized ground state solutions in this regime as well.

Keywords: local and nonlocal operators; Choquard equation; normalized ground state; HLS lower critical exponent
Mathematics Subject Classification: 35R11, 49J35

1. Introduction

In this paper, we consider the normalized solutions to the following Hardy-Littlewood-Sobolev (HLS) lower critical Choquard equation with mixed local and non-local operators:

$$-\Delta u + (-\Delta)^s u = \beta u + (|x|^{\alpha-N} * |u|^{1+\frac{\alpha}{N}})|u|^{\frac{\alpha}{N}-1}u + \lambda|u|^{p-2}u, \text{ in } \mathbb{R}^N, \tag{1.1}$$

where $N \geqslant 3$, $0 < s < 1$, $\alpha \in (0, N)$, $\lambda > 0$, $2 < p < 2_s^* = \frac{2N}{N-2s}$. Here, $2_s^* = \frac{2N}{N-2s}$ is the fractional Sobolev critical exponent and $1 + \frac{\alpha}{N}$ is the lower critical exponent in the sense of the HLS inequality.

The fractional Laplacian operator $(-\Delta)^s$ is defined by

$$(-\Delta)^s u(x) = C_{N,s} P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy, \quad x \in \mathbb{R}^N,$$

where $C_{N,s}$ is an appropriate normalization constant, and $P.V.$ denotes the Cauchy principal value [2].

The equation

$$-\Delta u + u = (I_2 * |u|^2)u \quad \text{in } \mathbb{R}^3, \quad u \in H^1(\mathbb{R}^3), \tag{1.2}$$

often called the Choquard equation, was introduced in 1976 as a model arising from the Hartree-Fock theory of a one-component plasma [14]. It has important applications in quantum physics; for instance, it can describe the self-trapping of an electron in its own vacancy [25]. The term $(I_2 * |u|^2)u$ represents a nonlocal self-interaction, where I_2 is the Riesz potential, commonly appearing in Coulomb-type interactions (e.g., in Bose-Einstein condensates or plasmas). In Bose-Einstein condensation, such equations describe collective behavior of particles under long-range interactions. The nonlocal term captures pairwise particle interactions in a mean-field approximation.

The study of Choquard equations involving the HLS lower critical exponent has received widespread attention due to its unique mathematical challenges. A pivotal contribution was made by Moroz and Van Schaftingen [20], who provided a comprehensive analysis of the Choquard equation's ground states across the full range of exponents, including the HLS lower critical cases.

Building upon this foundational understanding, recent research has focused on more complex models that incorporate additional terms, often to investigate the interplay between nonlocal and local nonlinearities. In this direction, Yao et al. [30] studied the following Choquard equation with the HLS lower critical exponent

$$-\Delta u + \beta u = \lambda(|x|^{\alpha-N} * |u|^{1+\frac{\alpha}{N}})|u|^{\frac{\alpha}{N}-1}u + \mu|u|^{p-2}u, \text{ in } \mathbb{R}^N, \tag{1.3}$$

where λ and μ are given positive numbers. The equation is constrained on the mass manifold

$$\mathcal{H}_c := \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = c^2 \right\},$$

where c is a given positive number. Their work represents a significant step in analyzing problems with competing critical nonlinearities, where the concentration-compactness principle and careful energy level estimates are essential for overcoming the lack of compactness. Based on these, Li et al. [16] highlight the subtle balance between the nonlocal Choquard nonlinearity and local interactions under a constraint. Furthermore, Li [17] obtained the existence of normalized ground states in the Sobolev critical case by the Sobolev subcritical approximation method.

On the other hand, some authors considered the following general fractional Choquard equation

$$(-\Delta)^s u = \beta u + \lambda(|x|^{\alpha-N} * |u|^p)|u|^{p-2}u + \mu|u|^{q-2}u, \text{ in } \mathbb{R}^N, \tag{1.4}$$

where $0 < s < 1$, $\alpha \in (0, N)$, $N > 2s$, $\lambda, \mu > 0$, and β is an unknown Lagrange multiplier. Bhattarai [1], using the concentration compactness techniques, proved existence and stability of normalized solution when $2 \leq p < \frac{N+\alpha+2s}{N}$, and $2 < q < 2 + \frac{4s}{N}$. Feng et al. [9], by involving the profile decomposition of bounded sequences in $H^s(\mathbb{R}^N)$ and variational methods, extended the result of

Bhattarai to the case $1 + \frac{\alpha}{N} < q < 2 + \frac{4s}{N}$. Based on some suitable properties and the min-max principle, Yang [31] showed that the problem has a mountain type solution for cases $\frac{N+\alpha+2s}{N} \le p < 2^*_{\alpha,s} = \frac{N+\alpha}{N-2s}$ and $2 + \frac{4s}{N} \le q < 2^*_s$. Furthermore, they also obtain a ground state. A more interesting result of HLS critical Choquard equation can be seen in references [11, 18].

In recent years, the existence of equations with mixed operator $(-\Delta)^{s_1}u + (-\Delta)^{s_2}u$ has received widespread attention. Yang and Mao [32] studied the following problem:

$$(-\Delta)^{s_1}u + (-\Delta)^{s_2}u + \lambda u = f(u), \text{ in } \mathbb{R}^N,$$

where $N \geqslant 2$, $f \in C(\mathbb{R}, \mathbb{R})$, and $0 < s_1 < s_2 < 1$, where an energy state exists if $c \geqslant c_0$ and does not exist if $0 < c < c_0$. Here, c_0 is a positive number. Let $N \geqslant 1$ and $f(u) = |u|^p u$ in the above equation. Luo and Hajaiej [33] proved the existence of the energy ground state with $p \in (0, \frac{4s_1}{N})$ by using a scaling argument.

The study of mixed operators of the form $-\Delta u + (-\Delta)^s u$ brings a wide range of applications. Its significance stems from the superposition of two random processes with different scales (i.e., a classical random walk and a Lévy flight). Roughly speaking, low-order operators play a dominant role in large-scale time, while high-order operators guide diffusion in small-scale time. These mixed operators arise naturally in systems combining classical processes, and Lévy processes have been widely used in biomathematics and animal foraging during the theory of optimal searching; See [7, 8] and the references therein. See also [13, 22, 23, 27] and the references therein for further applications.

Some other results of mixed local and nonlocal operators can be found in [5, 28] and the references therein. The combined effects of mixed operators, HSL lower critical exponent, Choquard terms, and power-type nonlinearities under mass constraints require new analytical tools, particularly when p approaches critical exponents.

It is standard to observe that the normalized solutions of Eq (1.1) correspond to critical points of the energy functional

$$\begin{aligned} I_\lambda(u) :=& \frac{1}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x \\ &- \frac{N}{2(N+\alpha)}\int_{\mathbb{R}^N} (|x|^{\alpha-N} * |u|^{1+\frac{\alpha}{N}})|u|^{1+\frac{\alpha}{N}} \mathrm{d}x - \frac{\lambda}{p}\int_{\mathbb{R}^N} |u|^p \mathrm{d}x \end{aligned}$$

restricted to the (mass) constraint

$$\mathcal{H}_c := \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 \mathrm{d}x = c^2 \right\}.$$

For $s \in (0, 1)$, the space $H^1(\mathbb{R}^N)$ is continuously embedded in the fractional Sobolev space $H^s(\mathbb{R}^N)$, i.e.,

$$H^1(\mathbb{R}^N) \hookrightarrow H^s(\mathbb{R}^N).$$

This ensures that $\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 \mathrm{d}x$ is well-defined on $H^1(\mathbb{R}^N)$ for $0 < s < 1$.
We define

$$e_\lambda(c) := \inf_{u \in \mathcal{H}_c} I_\lambda(u). \tag{1.5}$$

A function $u \in H^1(\mathbb{R}^N)$ is called a normalized ground state solution of Eq (1.1) if it satisfies (1.1) for some $\beta \in \mathbb{R}$, belongs to the mass-constrained set $\mathcal{H}_c$, and attains the minimal energy among all critical points on $\mathcal{H}_c$. Moreover, a radial normalized ground state solution of Eq (1.1) is a normalized ground state solution that is additionally radially symmetric, meaning $u(x) = u(|x|)$. Our analysis will further make use of the fundamental Hardy-Littlewood-Sobolev inequality, stated as follows.

Lemma 1.1. *[15] Assume that $1 \leq \gamma \leq \infty$, $g \in L^{\lambda_1}(\mathbb{R}^N)$, and $h \in L^{\lambda_2}(\mathbb{R}^N)$. Then there exists a constant $C > 0$ such that*

$$\|g * h\|_{L^\gamma(\mathbb{R}^N)} \leq C\|g\|_{L^{\lambda_1}(\mathbb{R}^N)}\|h\|_{L^{\lambda_2}(\mathbb{R}^N)},$$

where

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} = 1 + \frac{1}{\gamma}.$$

It follows from Lemma 1.1 that for any $v \in L^q(\mathbb{R}^N), q \in (1, \frac{N}{\alpha})$, $|x|^{\alpha-N} * v \in L^{\frac{Nq}{N-\alpha q}}(\mathbb{R}^N)$, and

$$\||x|^{\alpha-N} * v\|_{L^{\frac{Nq}{N-\alpha q}}(\mathbb{R}^N)} \leq C(\alpha, N, q)\|v\|_{L^q(\mathbb{R}^N)}. \tag{1.6}$$

Particularly, for any $u \in H^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} (|x|^{\alpha-N} * |u|^{1+\frac{\alpha}{N}})|u|^{1+\frac{\alpha}{N}} \mathrm{d}x \leqslant S_\alpha^{-(1+\frac{\alpha}{N})} \left(\int_{\mathbb{R}^N} |u|^2 \mathrm{d}x\right)^{\frac{N+\alpha}{N}}, \tag{1.7}$$

where

$$S_\alpha = \inf\left\{\int_{\mathbb{R}^N} |u|^2 \mathrm{d}x : u \in L^2(\mathbb{R}^N), \int_{\mathbb{R}^N} (|x|^{\alpha-N} * |u|^{1+\frac{\alpha}{N}})|u|^{1+\frac{\alpha}{N}} \mathrm{d}x = 1\right\} > 0. \tag{1.8}$$

It follows from [15, Theorem 4.3] that S_α is attained by

$$u(x) = Q_\epsilon(x) := C\left(\frac{\epsilon}{\epsilon^2 + |x-y|^2}\right)^{\frac{N+\alpha}{2}}, \tag{1.9}$$

for some $C \in \mathbb{R}$, $\epsilon > 0$, and $y \in \mathbb{R}^N$.

We now state our main results for Eq (1.1).

Theorem 1.2. *Assume that $N \geq 3$, $0 < s < 1$, $2 < p < 2 + \frac{4}{N}$, $\alpha \in (0, N)$, and $c > 0$. Then the following hold:*

1. *There exists $\bar{\lambda} > 0$ such that for any $\lambda > \bar{\lambda}$, the energy level satisfies*

$$e_\lambda(c) < -\frac{N}{2(N+\alpha)} S_\alpha^{-\left(1+\frac{\alpha}{N}\right)} c^{2\left(1+\frac{\alpha}{N}\right)}.$$

2. *The energy $e_\lambda(c)$ is attained by a function $\hat{u} \in \mathcal{H}_c$ with the following properties:*
   - *(a) $\hat{u}$ is radially symmetric and strictly decreasing in $|x|$.*
   - *(b) $\hat{u}$ is a solution of (1.1) with an associated Lagrange multiplier $\bar{\beta}$ satisfying*

$$\bar{\beta} < -\frac{N}{N+\alpha} S_\alpha^{-\left(1+\frac{\alpha}{N}\right)} c^{\frac{2\alpha}{N}}.$$

   Moreover, $\hat{u}$ is a normalized ground state solution of (1.1).

Remark 1.1. *For $2 < p < 2 + \frac{4}{N}$, Theorem 1.2 establishes the existence of normalized ground states with strictly negative energy and quantitatively sharp Lagrange multipliers ($\bar{\beta}$), improving the results in [16]. Two major difficulties arise in proving Theorem 1.2:*

(1) The combination of the HLS lower critical exponent and a local perturbation in the Choquard equation introduces inherent analytical challenges.

(2) The interplay between the local and nonlocal operators hinders a direct critical point analysis.

For $u \in \mathcal{H}_c$, $\theta \in \mathbb{R}$, and a.e. $x \in \mathbb{R}^N$, define

$$(\theta \star u)(x) := e^{\frac{N}{2}\theta} u(e^\theta x),$$

which ensures that $(\theta \star u)(x) \in \mathcal{H}_c$. With this scaling, we now introduce the fibering map

$$\begin{aligned} J_u^\lambda := & I_\lambda(\theta \star u) = \frac{e^{2s\theta}}{2}\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + \frac{e^{2\theta}}{2}\|u\|^2_{D^{1,2}(\mathbb{R}^N)} \\ & - \frac{N}{2(N+\alpha)}\int_{\mathbb{R}^N}(|x|^{\alpha-N} * |u|^{1+\frac{\alpha}{N}})|u|^{1+\frac{\alpha}{N}}\mathrm{d}x - \frac{\lambda e^{\frac{N(p-2)}{2}\theta}}{p}\|u\|^p_{L^p(\mathbb{R}^N)}. \end{aligned} \tag{1.10}$$

We first consider the case $p = 2 + \frac{4}{N}$. For every $u \in \mathcal{H}_c$, it follows from Lemma 2.4 that

$$\begin{aligned} (J_u^\lambda)'(\theta) = & s e^{2s\theta}\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + e^{2\theta}\|u\|^2_{D^{1,2}(\mathbb{R}^N)} - \frac{N\lambda(p-2)e^{2\theta}}{2p}\|u\|^p_{L^p(\mathbb{R}^N)} \\ \geqslant & s e^{2s\theta}\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + e^{2\theta}\left(1 - \frac{N\lambda c^{\frac{4}{N}}}{N+2}C_{N,p}\right)\|u\|^2_{D^{1,2}(\mathbb{R}^N)}. \end{aligned}$$

Under condition

$$0 < \lambda < \frac{N+2}{N c^{\frac{4}{N}} C_{N,p}}, \tag{1.11}$$

we find that $(J_u^\lambda)'(\theta) > 0$ for all $\theta \in \mathbb{R}$. Consequently, $J_u^\lambda(\theta)$ is strictly increasing on $(-\infty, +\infty)$, which leads to the following nonexistence result.

Theorem 1.3. *Assume that $N \geqslant 3$, $0 < s < 1$, $\alpha \in (0, N)$, $p = 2 + \frac{4}{N}$, and* (1.11) *hold. Then the functional $I_\lambda(u)$ has no critical point on $\mathcal{H}_c$.*

Now, we introduce the following Pohozaev identity. From reference [4, 19], the result of the next proposition can be easy obtained.

Proposition 1.4. *Let $u \in H^1(\mathbb{R}^N)$ be a weak solution of* (1.1)*, then u satisfies the Pohozaev identity:*

$$\begin{aligned} & \frac{N-2s}{2}\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + \frac{N-2}{2}\|u\|^2_{D^{1,2}(\mathbb{R}^N)} \\ = & \frac{\beta N}{2}\|u\|^2_{L^2(\mathbb{R}^N)} + \frac{N}{2}\int_{\mathbb{R}^N}(|x|^{\alpha-N} * |u|^{1+\frac{\alpha}{N}})|u|^{1+\frac{\alpha}{N}}\mathrm{d}x + \frac{\lambda N}{p}\int_{\mathbb{R}^N}|u|^p\mathrm{d}x. \end{aligned} \tag{1.12}$$

Lemma 1.5. *Let $u \in H^1(\mathbb{R}^N)$ be a weak solution of* (1.1)*, then u satisfies the following Nehair-Pohozaev manifold:*

$$\mathcal{M}_{c,\lambda} := \{u \in \mathcal{H}_c : E_\lambda(u) = 0\},$$

where the Nehair-Pohozaev functional is given by

$$E_\lambda(u) := s\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + \|u\|^2_{D^{1,2}(\mathbb{R}^N)} - \frac{\lambda N(p-2)}{2p}\|u\|^p_{L^p(\mathbb{R}^N)}.$$

Proof. By Proposition 1.4, we obtain that u satisfies the Pohozaev identity (1.12). Moreover, since u is the weak solution of (1.1), we have

$$\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + \|u\|^2_{D^{1,2}(\mathbb{R}^N)} = \beta\int_{\mathbb{R}^N}|u|^2dx + \int_{\mathbb{R}^N}(|x|^{\alpha-N}*|u|^{1+\frac{\alpha}{N}})|u|^{1+\frac{\alpha}{N}}\mathrm{d}x + \lambda\int_{\mathbb{R}^N}|u|^p\mathrm{d}x.$$

Therefore, we have

$$s\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + \|u\|^2_{D^{1,2}(\mathbb{R}^N)} - \frac{\lambda N(p-2)}{2p}\|u\|^p_{L^p(\mathbb{R}^N)} = 0,$$

which means that the proof is completed. □

In the following, we restrict our attention to the supercritical case $2+\frac{4}{N} < p < 2_s^*$, which is equivalent to $\frac{N}{N+2} < s < 1$.

By the Pohozaev identity, any critical point of the restricted functional $I_\lambda\big|_{\mathcal{H}_c}$ must lie in $\mathcal{M}_{c,\lambda}$. Moreover, following an argument analogous to that in [18, Lemmas 2.12 and 2.13], we conclude that $\mathcal{M}_{c,\lambda}$ is a natural constraint.

Proposition 1.6. *Assume that $N \geqslant 3$, $0 < s < 1$, and $2 + \frac{4}{N} < p < 2_s^*$. If $u \in \mathcal{M}_{c,\lambda}$ is a critical point of $I_\lambda|_{\mathcal{M}_{c,\lambda}}$, then u is a critical point of $I_\lambda|_{\mathcal{H}_c}$.*

We aim to show that $I_\lambda|_{\mathcal{M}_{c,\lambda}}$ is bounded from below. To this end, we analyze the structure of the Pohozaev manifold $\mathcal{M}_{c,\lambda}$, which is closely related to the monotonicity and convexity of the fibering map J_u^λ. Specifically, a direct calculation shows that

$$(J_u^\lambda)'(\theta) = se^{2s\theta}\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + e^{2\theta}\|u\|^2_{D^{1,2}(\mathbb{R}^N)} - \frac{\lambda N(p-2)e^{\frac{N(p-2)}{2}\theta}}{2p}\|u\|^p_{L^p(\mathbb{R}^N)} = E_\lambda(\theta\star u), \tag{1.13}$$

which yields that $\theta \in \mathbb{R}$ is a critical point of J_u^λ if and only if $\theta \star u \in \mathcal{M}_{c,\lambda}$. Furthermore, we have $(J_u^\lambda)'(0) = E_\lambda(u)$. To proceed, we decompose the Pohozaev manifold $\mathcal{M}_{c,\lambda}$ into three disjoint parts $\mathcal{M}_{c,\lambda} = \mathcal{M}^+_{c,\lambda} \cup \mathcal{M}^0_{c,\lambda} \cup \mathcal{M}^-_{c,\lambda}$, where

$$\mathcal{M}^+_{c,\lambda} = \left\{u \in \mathcal{M}_{c,\lambda} : (J_u^\lambda)''(0) > 0\right\} = \left\{u \in \mathcal{H}_c : (J_u^\lambda)'(0) = 0, (J_u^\lambda)''(0) > 0\right\},$$
$$\mathcal{M}^0_{c,\lambda} = \left\{u \in \mathcal{M}_{c,\lambda} : (J_u^\lambda)''(0) = 0\right\} = \left\{u \in \mathcal{H}_c : (J_u^\lambda)'(0) = 0, (J_u^\lambda)''(0) = 0\right\},$$

and

$$\mathcal{M}^-_{c,\lambda} = \left\{u \in \mathcal{M}_{c,\lambda} : (J_u^\lambda)''(0) < 0\right\} = \left\{u \in \mathcal{H}_c : (J_u^\lambda)'(0) = 0, (J_u^\lambda)''(0) < 0\right\}.$$

Therefore, for any $u \in \mathcal{M}_{c,\lambda}$, it follows that

$$\begin{aligned}(J_u^\lambda)''(0) =& 2s^2\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + 2\|u\|^2_{D^{1,2}(\mathbb{R}^N)} - \frac{N^2\lambda(p-2)^2}{4p}\|u\|^p_{L^p(\mathbb{R}^N)}\\ \leqslant& \frac{N\lambda(p-2)}{p}\|u\|^p_{L^p(\mathbb{R}^N)} - \frac{N^2\lambda(p-2)^2}{4p}\|u\|^p_{L^p(\mathbb{R}^N)}\\ =& \frac{N\lambda(p-2)}{p}\left(1 - \frac{N(p-2)}{4}\right)\|u\|^p_{L^p(\mathbb{R}^N)}.\end{aligned}$$

According to $2 + \frac{4}{N} < p < 2_s^*$, we have

$$(J_u^\lambda)''(0) < 0, \quad \text{for all } u \in \mathcal{M}_{c,\lambda}, \tag{1.14}$$

and

$$\mathcal{M}_{c,\lambda}^{+} = \mathcal{M}_{c,\lambda}^{0} = \emptyset.$$

From the preceding analysis of the Pohozaev manifold $\mathcal{M}_{c,\lambda}$, we now establish the following result.

Theorem 1.7. *Assume that $N \geqslant 3$, $0 < s < 1$, $\alpha \in (0, N)$,* and *$2 + \frac{4}{N} < p < 2_s^*$. Then there exists a constant $\lambda_0 > 0$ such that for any $\lambda > \lambda_0$,* (1.1) *admits a radial ground state solution $\tilde{u}$ and the corresponding Lagrange multiplier $\bar{\beta} < 0$.*

The rest of this paper is organized as follows. In Section 2, we collect the necessary preliminary results. We then prove Theorem 1.2 in Section 3 and proceed to establish Theorem 1.7 in Section 4.

2. Preliminaries

This section collects preliminary results needed for the proofs of our main theorems. We first recall the following compactness lemma from [24].

Lemma 2.1. *[24] For $N \geq 3$, there exists a constant $S = S(N)$ such that*

$$S = \inf_{u \in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\|\nabla u\|_{L^2(\mathbb{R}^N)}^2}{\|u\|_{L^{2^*}(\mathbb{R}^N)}^2}.$$

Furthermore, $H^1(\mathbb{R}^N)$ is continuously embedded into $L^p(\mathbb{R}^N)$ for all $2 \leqslant p \leqslant 2^*$ and compactly embedded into $L^p_{loc}(\mathbb{R}^N)$ for all $2 \leqslant p < 2^*$.

We will also use the Gagliardo–Nirenberg–Sobolev inequality from [3].

Lemma 2.2. *For any $u \in H^1(\mathbb{R}^N)$, and $p \in (2, 2^*)$, there exists a constant $C_{N,p} > 0$ such that*

$$\int_{\mathbb{R}^N} |u|^p \mathrm{d}x \leqslant C_{N,p} \left(\int_{\mathbb{R}^N} |\nabla u|^2 \mathrm{d}x \right)^{\frac{N(p-2)}{4}} \left(\int_{\mathbb{R}^N} |u|^2 \mathrm{d}x \right)^{\frac{2p-(p-2)N}{4}}. \tag{2.1}$$

We denote

$$\int_{\mathbb{R}^N} (|x|^{\alpha-N} * |u_n|^q)|u_n|^q \mathrm{d}x = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^q |u(y)|^q}{|x-y|^{N-\alpha}} dxdy.$$

We now introduce two lemmas that are essential for establishing the splitting property of the energy functional.

Lemma 2.3. *[21, Lemma 2.4] For $N \geq 3$, $\alpha \in (0, N)$,* and *$q \in [1, \frac{2N}{N+\alpha})$,* and *let $\{u_n\}$ be a bounded sequence in $L^{\frac{2Nq}{N+\alpha}}(\mathbb{R}^N)$ such that $u_n \to u$ a.e. on $\mathbb{R}^N$ as $n \to \infty$, then*

$$\lim_{n\to\infty} \left[\int_{\mathbb{R}^N} (|x|^{\alpha-N} * |u_n|^q)|u_n|^q \mathrm{d}x - \int_{\mathbb{R}^N} (|x|^{\alpha-N} * |u_n - u|^q)|u_n - u|^q \mathrm{d}x \right] = \int_{\mathbb{R}^N} (|x|^{\alpha-N} * |u|^q)|u|^q \mathrm{d}x.$$

Lemma 2.4. *[20] For $N \geqslant 3$, $\alpha \in (0, N)$, $q \in [1 + \frac{\alpha}{N}, \frac{N+\alpha}{N-2}]$,* and *$r \in [2, 2^*]$,* and *let $\{u_n\} \subset H^1(\mathbb{R}^N)$ be such that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$. Then, for all $v \in H^1(\mathbb{R}^N)$,*

$$\int_{\mathbb{R}^N} (|x|^{\alpha-N} * |u_n|^q)|u_n|^{q-2} u_n v \mathrm{d}x \to \int_{\mathbb{R}^N} (|x|^{\alpha-N} * |u|^q)|u|^{q-2} uv \mathrm{d}x \tag{2.2}$$

and

$$\int_{\mathbb{R}^N} |u_n|^{r-2} u_n v \mathrm{d}x \to \int_{\mathbb{R}^N} |u|^{r-2} uv \mathrm{d}x \tag{2.3}$$

as $n \to \infty$.

3. Proof of Theorem 1.2

This section is devoted to the existence of normalized ground state solutions.

Lemma 3.1. *Assume that $N \geq 3$, $0 < s < 1$, $2 < p < 2 + \frac{4}{N}$, $\alpha \in (0, N)$, and $c > 0$. Then the functional I_λ is bounded from below and coercive on $\mathcal{H}_c$.*

Proof. It follows from (2.2) and (1.7) that for each $u \in \mathcal{H}_c$,

$$\begin{aligned} I_\lambda(u) \geqslant & \frac{1}{2}\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + \frac{1}{2}\|u\|^2_{D^{1,2}(\mathbb{R}^N)} - \frac{N}{2(N+\alpha)} S_\alpha^{-(1+\frac{\alpha}{N})} c^{2(1+\frac{\alpha}{N})} \\ & - \frac{\lambda}{p} C_{N,p} \|u\|^{\frac{N(p-2)}{2}}_{D^{1,2}(\mathbb{R}^N)} c^{\frac{2p-N(p-2)}{2}}. \end{aligned} \tag{3.1}$$

Since $2 < p < 2 + \frac{4}{N}$, we have $0 < \frac{N(p-2)}{2} < 2$. It follows that I_λ is bounded from below and coercive on $\mathcal{H}_c$. □

Lemma 3.2. *Assume that $N \geq 3$, $0 < s < 1$, $2 < p < 2 + \frac{4}{N}$, $\alpha \in (0, N)$, and $c > 0$. Then, there exists $\bar{\lambda} > 0$ such that for any $\lambda > \bar{\lambda}$, the energy level defined in* (1.5) *satisfies*

$$e_\lambda(c) < -\frac{N}{2(N+\alpha)} S_\alpha^{-(1+\frac{\alpha}{N})} c^{2(1+\frac{\alpha}{N})} < 0. \tag{3.2}$$

Proof. Equation (1.9) shows that

$$\int_{\mathbb{R}^N} (|x|^{\alpha-N} * |Q_\epsilon|^{1+\frac{\alpha}{N}}) |Q_\epsilon|^{1+\frac{\alpha}{N}} \mathrm{d}x = S_\alpha^{-(1+\frac{\alpha}{N})} \left(\int_{\mathbb{R}^N} |Q_\epsilon|^2 \mathrm{d}x \right)^{\frac{N+\alpha}{N}}.$$

Based on the above equality, we define

$$\psi := \frac{Q_\epsilon c}{\|Q_\epsilon\|_{L^2(\mathbb{R}^N)}} \ \text{ and } \ (\theta \star \psi)(x) := e^{\frac{N\theta}{2}} \psi(e^\theta x), \ \text{ for } x \in \mathbb{R}^N.$$

It is clear that $\psi \in \mathcal{H}_c$ and $(\theta \star \psi) \in \mathcal{H}_c$. A direct computation shows that

$$\begin{aligned} I_\lambda(\theta \star \psi) =& \frac{e^{2s\theta}}{2} \|\psi\|^2_{D^{s,2}(\mathbb{R}^N)} + \frac{e^{2\theta}}{2} \|\psi\|^2_{D^{1,2}(\mathbb{R}^N)} \\ & - \frac{N}{2(N+\alpha)} \int_{\mathbb{R}^N} (|x|^{\alpha-N} * |\psi|^{1+\frac{\alpha}{N}}) |\psi|^{1+\frac{\alpha}{N}} \mathrm{d}x - \frac{\lambda e^{\frac{N(p-2)}{2}\theta}}{p} \|\psi\|^p_{L^p(\mathbb{R}^N)} \\ =& \frac{e^{2s\theta}}{2} \|\psi\|^2_{D^{s,2}(\mathbb{R}^N)} + \frac{e^{2\theta}}{2} \|\psi\|^2_{D^{1,2}(\mathbb{R}^N)} - \frac{N}{2(N+\alpha)} S_\alpha^{-(1+\frac{\alpha}{N})} c^{2(1+\frac{\alpha}{N})} \\ & - \frac{\lambda e^{\frac{N(p-2)}{2}\theta}}{p} \|\psi\|^p_{L^p(\mathbb{R}^N)}. \end{aligned}$$

Hence, for $2 < p < 2 + \frac{4s}{N}$, it follows that there exists $\theta_0 \ll -1$ such that

$$e_\lambda(c) < -\frac{N}{2(N+\alpha)} S_\alpha^{-(1+\frac{\alpha}{N})} c^{2(1+\frac{\alpha}{N})}.$$

For $p = 2 + \frac{4s}{N}$, if

$$\lambda > \lambda_1 = \frac{p\|\psi\|^2_{D^{1,2}(\mathbb{R}^N)}}{2\|\psi\|^p_{L^p(\mathbb{R}^N)}},$$

then, there exists $\theta_0 \ll -1$ such that (3.2) holds.

For $2 + \frac{4s}{N} < p < 2 + \frac{4}{N}$, let $\gamma_1 = 2s$, and $\gamma_2 = \frac{N(p-2)}{2}$. We have

$$0 < \gamma_1 < \gamma_2 < 2.$$

Set $t = e^\theta > 0$. Define

$$f(t) = At^{\gamma_1} + Bt^2 - Ct^{\gamma_2},\ t > 0,$$

where $A = \frac{\|\psi\|^2_{D^{s,2}(\mathbb{R}^N)}}{2} > 0,\ B = \frac{\|\psi\|^2_{D^{1,2}(\mathbb{R}^N)}}{2} > 0,\ C = \frac{\lambda}{p}\|\psi\|^p_{L^p(\mathbb{R}^N)} > 0.$

Consider the function

$$g(t) = At^{\gamma_1} - Ct^{\gamma_2}.$$

Since $\gamma_2 > \gamma_1$, for small $t > 0$, $g(t) > 0$. Let

$$t_1 = \left(\frac{A}{C}\right)^{\frac{1}{\gamma_2-\gamma_1}}.$$

Then $g(t_1) = 0$, and for $t > t_1$, we have $g(t) < 0$.

Now, since g is continuous and $g(t) < 0$ for $t > t_1$, there exists $\delta > 0$ such that for all $t \in (t_1, t_1 + \delta)$,

$$g(t) \le -\eta \quad \text{for some } \eta > 0.$$

Now examine $f(t) = g(t) + Bt^2$. Since Bt^2 is continuous and $t_1 > 0$, we can choose δ small enough and $\lambda > \lambda_2$ large enough so that for $t \in (t_1, t_1 + \delta)$,

$$Bt^2 \le B(t_1 + \delta)^2 < \eta.$$

Then for such t,

$$f(t) \le -\eta + Bt^2 < -\eta + \eta = 0.$$

Thus, $f(t) < 0$ for $t \in (t_1, t_1 + \delta)$. Taking $\bar{\lambda} = \max\{\lambda_1, \lambda_2\}$, we obtain the desired results. □

Lemma 3.3. *Assume that $N \ge 3$, $0 < s < 1$, $2 < p < 2+\frac{4}{N}$, $\alpha \in (0, N)$, and $c > 0$, and let $\{u_n\} \subset \mathcal{H}_c$ be a minimizing sequence of $e_\lambda(c)$. Then, there exists a subsequence, still denoted by $\{u_n\}$, a sequence $\{z_n\} \subset \mathbb{R}^N$ and $\hat{u} \in \mathcal{H}_c$ such that $u_n(\cdot + z_n) \to \hat{u}$ strongly in $H^1(\mathbb{R}^N)$.*

Proof. It follows easily from (3.1) and $2 < p < 2+\frac{4}{N}$ that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Thus, there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) and $\hat{u} \in H^1(\mathbb{R}^N)$ such that

$$u_n \rightharpoonup \hat{u} \ \text{ in } H^1(\mathbb{R}^N),\ \ u_n \to \hat{u} \ \text{ in } L^2_{loc}(\mathbb{R}^N),\ \ u_n \to \hat{u} \ \ a.e. \text{ on } \mathbb{R}^N.$$

We claim that $\hat{u} \neq 0$. Suppose, for contradiction, that $\hat{u} = 0$. Then $u_n \rightharpoonup 0$ in $H^1(\mathbb{R}^N)$. Applying Lemma 2.4, we find that

$$\int_{\mathbb{R}^N} (|x|^{\alpha-N} * |u_n|^{1+\frac{\alpha}{N}})|u_n|^{1+\frac{\alpha}{N}}\mathrm{d}x = o_n(1), \quad \int_{\mathbb{R}^N} |u_n|^p\mathrm{d}x = o_n(1).$$

Hence,

$$e_\lambda(c) = I_\lambda(u_n) + o_n(1) = \frac{1}{2}\|u_n\|^2_{D^{s,2}(\mathbb{R}^N)} + \frac{1}{2}\|u_n\|^2_{D^{1,2}(\mathbb{R}^N)} \geqslant 0.$$

This contradiction with Lemma 3.2 implies that $\hat{u} \neq 0$. Consequently, there exists a sequence $\{z_n\}$ such that $\hat{u}_n := u_n(\cdot + z_n)$ converges weakly to $\hat{u} \neq 0$ in $H^1(\mathbb{R}^N)$. By applying Brezis-Lieb Lemma and Lemma 2.3, we derive that

$$c^2 = \|u_n\|^2_{L^2(\mathbb{R}^N)} = \|u_n - \hat{u}\|^2_{L^2(\mathbb{R}^N)} + \|\hat{u}\|^2_{L^2(\mathbb{R}^N)} + o_n(1). \tag{3.3}$$

$$I_\lambda(u_n) = I_\lambda(u_n - \hat{u}) + I_\lambda(\hat{u}) + o_n(1). \tag{3.4}$$

If $\|\hat{u}\|^2_{L^2(\mathbb{R}^N)} < c^2$, then by setting $t := \frac{c}{\|\hat{u}\|_{L^2(\mathbb{R}^N)}}$, we have $t > 1, t\hat{u} \in \mathcal{H}_c$ and

$$\begin{aligned} I_\lambda(t\hat{u}) =& \frac{t^2}{2}\|\hat{u}\|^2_{D^{s_1,2}(\mathbb{R}^N)} + \frac{t^2}{2}\|\hat{u}\|^2_{D^{1,2}(\mathbb{R}^N)} - \frac{Nt^{2(1+\frac{\alpha}{N})}}{2(N+\alpha)}\int_{\mathbb{R}^N}(|x|^{\alpha-N} * |\hat{u}|^{1+\frac{\alpha}{N}})|\hat{u}|^{1+\frac{\alpha}{N}}\mathrm{d}x \\ &- \frac{\lambda t^p}{p}\int_{\mathbb{R}^N}|\hat{u}|^p\mathrm{d}x, \end{aligned}$$

which yields that

$$I_\lambda(\hat{u}) = \frac{1}{t^2}I_\lambda(t\hat{u}) + \frac{N(t^{\frac{2\alpha}{N}} - 1)}{2(N+\alpha)}\int_{\mathbb{R}^N}(|x|^{\alpha-N} * |\hat{u}|^{1+\frac{\alpha}{N}})|\hat{u}|^{1+\frac{\alpha}{N}}\mathrm{d}x + \frac{\lambda(t^{p-2} - 1)}{p}\int_{\mathbb{R}^N}|\hat{u}|^p\mathrm{d}x. \tag{3.5}$$

Similarly, setting $t_n := \frac{c}{\|u_n - \hat{u}\|_{L^2(\mathbb{R}^N)}} \geqslant 1$, then, $t_n(u_n - \hat{u}) \in \mathcal{H}_c$, it follows that

$$\begin{aligned} I_\lambda(u_n - \hat{u}) =& \frac{1}{t_n^2}I_\lambda(t_n(u_n - \hat{u})) + \frac{N(t_n^{\frac{2\alpha}{N}} - 1)}{2(N+\alpha)}\int_{\mathbb{R}^N}(|x|^{\alpha-N} * |u_n - \hat{u}|^{1+\frac{\alpha}{N}})|u_n - \hat{u}|^{1+\frac{\alpha}{N}}\mathrm{d}x \\ &+ \frac{\lambda(t_n^{p-2} - 1)}{p}\int_{\mathbb{R}^N}|u_n - \hat{u}|^p\mathrm{d}x \\ \geqslant& \frac{1}{t_n^2}I_\lambda(t_n(u_n - \hat{u})). \end{aligned} \tag{3.6}$$

In view of (1.7), and (3.3)–(3.6), we deduce that

$$\begin{aligned} e_\lambda(c) =& I_\lambda(u_n) + o_n(1) = I_\lambda(\hat{u}) + I_\lambda(u_n - \hat{u}) + o_n(1) \\ \geqslant& \frac{1}{t^2}I_\lambda(t\hat{u}) + \frac{1}{t_n^2}I(t_n(u_n - \hat{u})) + \frac{N(t^{\frac{2\alpha}{N}} - 1)}{2(N+\alpha)}\int_{\mathbb{R}^N}(|x|^{\alpha-N} * |\hat{u}|^{1+\frac{\alpha}{N}})|\hat{u}|^{1+\frac{\alpha}{N}}\mathrm{d}x \\ &+ \frac{\lambda(t^{p-2} - 1)}{p}\int_{\mathbb{R}^N}|\hat{u}|^p\mathrm{d}x + o_n(1) \\ >& \frac{1}{t^2}e_\lambda(c) + \frac{1}{t_n^2}e_\lambda(c) = e_\lambda(c), \end{aligned}$$

which is a contradiction. Hence, $\|\hat{u}\|^2_{L^2(\mathbb{R}^N)} = c^2$. Consequently, the sequence $\hat{u}_n := u_n(\cdot + z_n)$ converges strongly to $\hat{u}$ in $L^2(\mathbb{R}^N)$. Hence, by [21, Lemma 2.4], we obtain

$$\int_{\mathbb{R}^N}(|x|^{\alpha-N} * |\hat{u}_n|^{1+\frac{\alpha}{N}})|\hat{u}_n|^{1+\frac{\alpha}{N}}\mathrm{d}x = \int_{\mathbb{R}^N}(|x|^{\alpha-N} * |\hat{u}|^{1+\frac{\alpha}{N}})|\hat{u}|^{1+\frac{\alpha}{N}}\mathrm{d}x + o_n(1). \tag{3.7}$$

By the interpolation inequality and the Sobolev embedding theorem, we deduce that

$$\|\hat{u}_n - \hat{u}\|_{L^p(\mathbb{R}^N)} \leqslant \|\hat{u}_n - \hat{u}\|^{\theta}_{L^2(\mathbb{R}^N)} \|\hat{u}_n - \hat{u}\|^{1-\theta}_{L^{2^*}(\mathbb{R}^N)} \leqslant C\|\hat{u}_n - \hat{u}\|^{\theta}_{L^2(\mathbb{R}^N)} \to 0, \tag{3.8}$$

as $n \to \infty$, where $p \in (2, 2^*)$ and $\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{2^*}$. From (3.7) and (3.8), and the weakly lower semicontinuity of norm, we conclude that

$$e_\lambda(c) \leqslant I_\lambda(\hat{u}) \leqslant \liminf_{n\to\infty} I_\lambda(\hat{u}_n) = \liminf_{n\to\infty} I_\lambda(u_n) = e_\lambda(c),$$

which implies that $\|\hat{u}_n\|_{D^{s,2}(\mathbb{R}^N)} \to \|\hat{u}\|_{D^{s,2}(\mathbb{R}^N)}$ and $\|\hat{u}_n\|_{D^{1,2}(\mathbb{R}^N)} \to \|\hat{u}\|_{D^{1,2}(\mathbb{R}^N)}$ as $n \to \infty$. □

Proof of Theorem 1.2. Lemma 3.3 guarantees the existence of a minimizer $\hat{u}$ for I_λ on $\mathcal{H}_c$. Consider the symmetric decreasing rearrangement $|\hat{u}|^*$ of $\hat{u}$. Clearly,

$$\||\hat{u}|\|_{L^2(\mathbb{R}^N)} = \||\hat{u}|^*\|_{L^2(\mathbb{R}^N)}, \quad \||\hat{u}|\|_{L^p(\mathbb{R}^N)} = \||\hat{u}|^*\|_{L^p(\mathbb{R}^N)}. \tag{3.9}$$

Combining the classical and fractional Polya-Szegö inequality [26] with (A.11) from [29], we conclude that

$$\||\hat{u}|^*\|^2_{D^{s,2}(\mathbb{R}^N)} \leqslant \||\hat{u}|\|^2_{D^{s,2}(\mathbb{R}^N)} \leqslant \|\hat{u}\|^2_{D^{s,2}(\mathbb{R}^N)}, \tag{3.10}$$

$$\||\hat{u}|^*\|^2_{D^{1,2}(\mathbb{R}^N)} \leqslant \||\hat{u}|\|^2_{D^{1,2}(\mathbb{R}^N)} \leqslant \|\hat{u}\|^2_{D^{1,2}(\mathbb{R}^N)}. \tag{3.11}$$

By the Riesz rearrangement inequality [15, Theorem 3.4], we have

$$\int_{\mathbb{R}^N} (|x|^{\alpha-N} * |\hat{u}|^{1+\frac{\alpha}{N}})|\hat{u}|^{1+\frac{\alpha}{N}} \mathrm{d}x \leqslant \int_{\mathbb{R}^N} (|x|^{\alpha-N} * (|\hat{u}|^*)^{1+\frac{\alpha}{N}})(|\hat{u}|^*)^{1+\frac{\alpha}{N}} \mathrm{d}x. \tag{3.12}$$

From (3.9)–(3.11), we obtain $|\hat{u}|^* \in \mathcal{H}_c$ and $I_\lambda(|\hat{u}|^*) \leqslant I_\lambda(|\hat{u}|) = e_\lambda(c)$. Therefore, the minimizer $|\hat{u}|^*$, which is radially symmetric and decreasing, attains $e_\lambda(c)$. For simplicity, we continue to denote this minimizer by $\hat{u}$. Furthermore, there exists a corresponding Lagrange multiplier $\bar{\beta}$ such that

$$\begin{aligned}
\bar{\beta}c^2 =& \|\hat{u}\|^2_{D^{s,2}(\mathbb{R}^N)} + \|\hat{u}\|^2_{D^{1,2}(\mathbb{R}^N)} - \int_{\mathbb{R}^N} (|x|^{\alpha-N} * |\hat{u}|^{1+\frac{\alpha}{N}})|\hat{u}|^{1+\frac{\alpha}{N}} \mathrm{d}x - \lambda\|\hat{u}\|^p_{L^p(\mathbb{R}^N)} \\
=& 2e_\lambda(c) - \frac{\alpha}{N+\alpha} \int_{\mathbb{R}^N} (|x|^{\alpha-N} * |\hat{u}|^{1+\frac{\alpha}{N}})|\hat{u}|^{1+\frac{\alpha}{N}} \mathrm{d}x - \frac{(p-2)\lambda}{p}\|\hat{u}\|^p_{L^p(\mathbb{R}^N)} \\
\leqslant& 2e_\lambda(c) < -\frac{N}{N+\alpha} S_\alpha^{-(1+\frac{\alpha}{N})} c^{2(1+\frac{\alpha}{N})} < 0,
\end{aligned}$$

recalling Lemma 3.2, which implies that

$$\bar{\beta} < -\frac{N}{N+\alpha} S_\alpha^{-(1+\frac{\alpha}{N})} c^{\frac{2\alpha}{N}} < 0.$$

This completes the proof. □

4. Proof of Theorem 1.7

In this section, we shall prove Theorem 1.7.

Lemma 4.1. *Assume that $N \geqslant 3$, $0 < s < 1$, $\alpha \in (0, N)$, and $2 + \frac{4}{N} < p < 2_s^*$. For each $u \in \mathcal{H}_c$, J_u^λ admits a unique critical point $\theta_u \in \mathbb{R}$ such that*

$$I_\lambda(\theta_u \star u) = \max_{t\in\mathbb{R}} I_\lambda(\theta \star u), \quad (\theta_u \star u) \in \mathcal{M}_{c,\lambda}. \tag{4.1}$$

Particularly, the map $u \in \mathcal{H}_c \mapsto \theta_u \in \mathbb{R}$ is of class C^1.

Proof. For any $u \in \mathcal{H}_c$, we have

$$\begin{aligned}&(J_u^\lambda)'(\theta)\\ =&se^{2s\theta}\left(\|u\|^2_{D^{s,2}(\mathbb{R}^N)} + \frac{1}{s}e^{2(1-s)\theta}\|u\|^2_{D^{1,2}(\mathbb{R}^N)} - \frac{N\lambda(p-2)}{2p}e^{(\frac{N(p-2)}{2}-2s)\theta}\|u\|^p_{L^p(\mathbb{R}^N)}\right).\end{aligned} \tag{4.2}$$

Thanks to $1 > s > 0$ and $2 + \frac{N}{4} < p$, we can derive that $(J_u^\lambda)'(\theta) \to 0^+$ as $t \to -\infty$ and $(J_u^\lambda)'(\theta) \to -\infty$ as $\theta \to +\infty$. Furthermore, from (1.10), we conclude that $(J_u^\lambda)'(\theta)$ has a unique zero point θ_u, which is the unique maximum point of $J_u^\lambda(\theta)$. Together with (1.10) and (1.13), (4.1) holds.

We denote by $\Psi : \mathbb{R} \times \mathcal{H}_c \mapsto \mathbb{R}$ the function $\Psi(\theta, u) = (J_u^\lambda)'(\theta)$. Applying the implicit functon theorem to the C^1 function Ψ, we can complete the proof. □

Lemma 4.2. *Assume that $N \geqslant 3$, $0 < s < 1$, $\alpha \in (0, N)$, and $2 + \frac{4}{N} < p < 2_s^*$. Then I_λ is coercive on $\mathcal{M}_{c,\lambda}$.*

Proof. For each $u \in \mathcal{M}_{c,\lambda}$, from (1.7) and Lemma 1.5, we observe that

$$\begin{aligned}I_\lambda(u) =&\frac{1}{2}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\\ &- \frac{N}{2(N+\alpha)}\int_{\mathbb{R}^N}(|x|^{\alpha-N} * |u|^{1+\frac{\alpha}{N}})|u|^{1+\frac{\alpha}{N}}\mathrm{d}x - \frac{\lambda}{p}\int_{\mathbb{R}^N}|u|^p\mathrm{d}x\\ \geqslant&\frac{N(p-2)-4s}{2N(p-2)}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2\mathrm{d}x + \frac{N(p-2)-4}{2N(p-2)}\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\\ &- \frac{N}{2(N+\alpha)}S_\alpha^{-(1+\frac{\alpha}{N})}c^{\frac{2(N+\alpha)}{N}},\end{aligned} \tag{4.3}$$

which completes the proof. □

We now introduce the following key definitions

$$\begin{aligned}H^1_{rad}(\mathbb{R}^N) :=&\{u \in H^1(\mathbb{R}^N) : u \text{ is radially symmetric}\},\\ \mathcal{H}_c^{rad} :=&\mathcal{H}_c \cap H^1_{rad}(\mathbb{R}^N),\\ \mathcal{M}_{c,\lambda}^{rad} :=&\mathcal{M}_{c,\lambda} \cap H^1_{rad}(\mathbb{R}^N),\\ m(c) :=& \inf_{u\in\mathcal{M}_{c,\lambda}} I_\lambda(u).\end{aligned}$$

Lemma 4.3. *Assume that $N \geqslant 3$, $0 < s < 1$, $\alpha \in (0, N)$, and $2 + \frac{4}{N} < p < 2_s^*$. Then,*

$$\inf_{u \in \mathcal{M}_{c,\lambda}} I_\lambda(u) = \inf_{u \in \mathcal{M}_{c,\lambda}^{rad}} I_\lambda(u).$$

Proof. From the embedding $\mathcal{M}_{c,\lambda}^{rad} \subset \mathcal{M}_{c,\lambda}$, it follows that

$$\inf_{u \in \mathcal{M}_{c,\lambda}} I_\lambda(u) \leqslant \inf_{u \in \mathcal{M}_{c,\lambda}^{rad}} I_\lambda(u).$$

Thus, the proof reduces to show that

$$\inf_{u \in \mathcal{M}_{c,\lambda}} I_\lambda(u) \geqslant \inf_{u \in \mathcal{M}_{c,\lambda}^{rad}} I_\lambda(u). \tag{4.4}$$

To this end, let $|u|^*$ represent the symmetric decreasing rearrangement of $|u|$. Through applications of (3.9)–(3.11), we obtain $|u|^* \in \mathcal{H}_c^{rad}$ and

$$J_{|u|^*}^\lambda(\theta) = I_\lambda(\theta \star |u|^*) \leqslant I_\lambda(\theta \star u) = J_u^\lambda(\theta).$$

From (3.9), (3.10) and (4.2), we obtain $-\infty < \theta_{|u|^*} \leqslant \theta_u$. Combining this with Lemma 4.1 yields

$$J_u^\lambda(\theta_u) \geqslant J_u^\lambda(\theta_{|u|^*}) \geqslant J_{|u|^*}^\lambda(\theta_{|u|^*}).$$

Since $u \in \mathcal{M}_{c,\lambda}$, it follows that $\theta_u = 0$, and consequently

$$I_\lambda(u) = J_u^\lambda(0) \geqslant J_{|u|^*}^\lambda(\theta_{|u|^*}) = I_\lambda(\theta_{|u|^*} \star |u|^*).$$

Observing that $\theta_{|u|^*} \star |u|^* \in \mathcal{M}_{c,\lambda}^{rad}$, we deduce that

$$\inf_{u \in \mathcal{M}_{c,\lambda}} I_\lambda(u) \geqslant \inf_{u \in \mathcal{M}_{c,\lambda}} I_\lambda(\theta_{|u|^*} \star |u|^*) \geqslant \inf_{u \in \mathcal{M}_{c,\lambda}^{rad}} I_\lambda(u),$$

from which we conclude that (4.4) holds. □

Lemma 4.4. *Let $u \in \mathcal{H}_c^{rad}$ and $\theta \in \mathbb{R}$. The mapping*

$$T_u\mathcal{H}_c^{rad} \to T_{\theta_u \star u}\mathcal{H}_c^{rad}, \quad \phi \mapsto \theta_u \star \phi \tag{4.5}$$

is a linear isomorphism, with its inverse given by

$$\varphi \mapsto (-\theta) \star \varphi.$$

Here $T_u\mathcal{H}_c^{rad}$ denotes the tangent space to $\mathcal{H}_c^{rad}$ at u.

Proof. The proof is standard, see [30, Lemma 5.5]. □

Next, we introduce the functional $\bar{I}_\lambda : \mathcal{H}_c^{rad} \mapsto \mathbb{R}$ defined by

$$\bar{I}_\lambda = I_\lambda(\theta_u \star u).$$

It follows from Lemma 4.1 that $\bar{I}_\lambda$ is of class C^1 on $\mathcal{H}_c^{rad}$. Moreover, adapting the methods of [6, Lemma 3.15], we obtain the following result.

Lemma 4.5. *For each $u \in \mathcal{H}_c^{rad}$ and $\psi \in T_u\mathcal{H}_c^{rad}$, the following identity holds*

$$\bar{I}'_\lambda(u)[\psi] = \bar{I}'_\lambda(\theta_u \star u)[\theta_u \star \psi]. \tag{4.6}$$

Analogously to [6, Lemma 3.16], the existence of Palais-Smale sequences holds for any general homotopy-stable family of symmetric subsets of $\mathcal{H}_c^{rad}$. This follows from Lemmas 4.4 and 4.5.

Lemma 4.6. *Let K be a homotopy-stable family of compact subsets of $\mathcal{H}_c^{rad}$ with closed boundary $\mathcal{D}$ and define*

$$\sigma_K := \inf_{B\in K} \max_{u\in B} \bar{I}_\lambda(u).$$

Suppose the following assumptions hold:

(i) $\mathcal{D}$ is contained in a connected component of $\mathcal{M}_{c,\lambda}^{rad}$.

(ii) The min-max level σ_K satisfies the strict inequality

$$\max\{\sup \bar{I}_\lambda(\mathcal{D}), 0\} < \sigma_K < \infty.$$

Then, there exists a Palais-Smale sequence $\{u_n\} \subset \mathcal{M}_{c,\lambda}^{rad}$ of $\bar{I}_\lambda$ restricted to $\mathcal{H}_c^{rad}$ at level σ_K.

By Lemma 4.6, there exists a Palais-Smale sequence $\{u_n\} \subset \mathcal{M}_{c,\lambda}^{rad}$ for the restricted functional $\bar{I}_\lambda|_{\mathcal{H}_c^{rad}}$ at the level $m(c) \neq 0$.

Lemma 4.7. *Assume that $N \geqslant 3$, and $2+\frac{4}{N} < p < 2_s^*$. Then, there exists a Palais-Smale sequence $\{u_n\} \subset \mathcal{M}_{c,\lambda}^{rad}$ for the restricted functional $I_\lambda|_{\mathcal{H}_c^{rad}}$ at the level $m(c) \neq 0$.*

Proof. Let $\bar{K}$ denote the family of all singletons with $u \in \mathcal{H}_c^{rad}$. Since $\bar{K}$ consists only of single-element sets, its boundary $\mathcal{D}$ is trivially empty. Following the framework established in [10, Definition 3.1], we observe that $\bar{K}$ forms a homotopy-stable family of compact subset in $\mathcal{H}_c^{rad}$ without boundary. Combining this structural property with Lemma 4.3, we obtain

$$\sigma_{\bar{K}} = \inf_{B\in\bar{K}} \max_{u\in B} \bar{I}_\lambda(u) = \inf_{u\in\mathcal{H}_c^{rad}} \bar{I}_\lambda(u) = \inf_{u\in\mathcal{M}_c^{rad}} I_\lambda(u) = \inf_{u\in\mathcal{M}_{c,\lambda}} I_\lambda(u) = m(c).$$

Therefore, applying Lemma 4.6, we conclude the proof.

□

Next, we analyze the convergence of special Palais-Smale sequences satisfying additional structural conditions. Our approach follows the pioneering framework introduced by Jeanjean in [12].

Lemma 4.8. *Assume $N \geqslant 3$, and $2+\frac{4}{N} < p < 2_s^*$. Let $\{u_n\} \subset \mathcal{M}_{c,\lambda}^{rad}$ be a bounded Palais-Smale sequence for $I_\lambda|_{\mathcal{H}_c^{rad}}$ at level $m(c) \neq 0$ in $H_{rad}^1(\mathbb{R}^N)$. Then, there eixsts $\lambda_0 > 0$ such that for each $\lambda > \lambda_0$, up to a subsequence, $u_n \to \tilde{u}$ strongly in $H_{rad}^1(\mathbb{R}^N)$.*

Proof. The argument proceeds in four steps:

Step 1. Since $\{u_n\} \subset \mathcal{M}_{c,\lambda}^{rad}$ is bounded in $H_{rad}^1(\mathbb{R}^N)$ and the embedding $H_{rad}^1(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is compact for all $q \in (2, 2_s^*)$, there exists $\tilde{u} \in H_{rad}^1(\mathbb{R}^N)$ such that, up to a subsequence,

$$u_n \rightharpoonup \tilde{u} \text{ in } H_{rad}^1(\mathbb{R}^N), \ \ u_n \to \tilde{u} \text{ in } L^q(\mathbb{R}^N), \text{ for } q \in (2, 2_s^*) \text{ and a.e. in } \mathbb{R}^N. \tag{4.7}$$

Furthermore, there exists a sequence $\{\beta_n\} \subset \mathbb{R}$ such that for any $v \in H^1_{rad}(\mathbb{R}^N)$,

$$\begin{aligned}&\int_{\mathbb{R}^N} (-\Delta)^{\frac{s}{2}} u_n (-\Delta)^{\frac{s}{2}} v \mathrm{d}x + \int_{\mathbb{R}^N} \nabla u_n \nabla v \mathrm{d}x - \beta_n \int_{\mathbb{R}^N} u_n v \mathrm{d}x \\ &- \int_{\mathbb{R}^N} (|x|^{\alpha-N} * |u_n|^{1+\frac{\alpha}{N}})|u_n|^{\frac{\alpha}{N}-1} u_n v \mathrm{d}x - \lambda \int_{\mathbb{R}^N} |u_n|^{p-2} u_n v \mathrm{d}x = o_n(1)\|v\|.\end{aligned} \tag{4.8}$$

Taking $v = u_n$ in (4.8), we observe that

$$-\beta_n c^2 = \lambda \|u_n\|^p_{L^p(\mathbb{R}^N)} + \int_{\mathbb{R}^N} (|x|^{\alpha-N} * |u_n|^{1+\frac{\alpha}{N}})|u_n|^{\frac{\alpha}{N}+1} \mathrm{d}x - \|u_n\|^2_{D^{s,2}(\mathbb{R}^N)} - \|u_n\|^2_{D^{1,2}(\mathbb{R}^N)} + o_n(1),$$

which yields that β_n is bounded. Then, up to a subsequence, there exists $\bar{\beta} \in \mathbb{R}$ such that $\beta_n \to \bar{\beta}$ as $n \to \infty$.

Step 2. $\bar{\beta} < 0$ and $\tilde{u} \neq 0$. From $2 + \frac{4}{N} < p < 2^*_s$ and the fact that $\{u_n\} \subset \mathcal{M}^{rad}_{c,\lambda}$, we deduce that

$$\begin{aligned}-\beta_n c^2 =& \left(\frac{2ps - N(p-2)}{N(p-2)}\right) \|u_n\|^2_{D^{s,2}(\mathbb{R}^N)} + \left(\frac{2p - N(p-2)}{N(p-2)}\right) \|u_n\|^2_{D^{1,2}(\mathbb{R}^N)} \\ &+ \int_{\mathbb{R}^N} (|x|^{\alpha-N} * |u_n|^{1+\frac{\alpha}{N}})|u_n|^{\frac{\alpha}{N}+1} \mathrm{d}x \geqslant 0,\end{aligned} \tag{4.9}$$

which leads to $\bar{\beta} \leqslant 0$ with equality if and only if $\tilde{u} = 0$. We will show that $\bar{\beta} \neq 0$; if not, due to (4.9) and $E_\lambda(u_n) = o_n(1)$, we can see that $m(c) + o_n(1) = I_\lambda(u_n) = o_n(1)$, which contradicts Lemma 4.2. Thus, $\bar{\beta} < 0$ and $\tilde{u} \neq 0$.

Step 3. The upper bound of $m(c) - \frac{\bar{\beta}}{2}c^2$. From (1.8) and (1.9), we obtain

$$\int_{\mathbb{R}^N} (|x|^{\alpha-N} * |Q_\epsilon|^{1+\frac{\alpha}{N}})|Q_\epsilon|^{1+\frac{\alpha}{N}} \mathrm{d}x = S_\alpha^{-(1+\frac{\alpha}{N})} \left(\int_{\mathbb{R}^N} |Q_\epsilon|^2 \mathrm{d}x\right)^{\frac{N+\alpha}{N}}.$$

Let

$$\phi := \frac{cQ_\epsilon}{\|Q_\epsilon\|_{L^2(\mathbb{R}^N)}} \text{ and } (\theta \star \phi)(x) := e^{\frac{N}{2}\theta}\phi(e^\theta x),$$

for a.e. $x \in \mathbb{R}^N$. Obviously, $\phi \in \mathcal{H}_c$ and $(\theta \star \phi)(x) \in \mathcal{H}_c$. By Lemma 4.1, there exists a unique $\theta_\phi \in \mathbb{R}$ such that

$$I_\lambda(\theta_\phi \star \phi) = \max_{\theta \in \mathbb{R}} I_\lambda(\theta \star \phi), \;\; \theta_\phi \star \phi \in \mathcal{M}_{c,\lambda}.$$

Lemma 4.3 yields that

$$m(c) \leqslant I_\lambda(\theta_\phi \star \phi).$$

A direct computation yields

$$\begin{aligned}m(c) \leqslant& I_\lambda(\theta_\phi \star \phi) \\ =& \frac{e^{2s\theta_\phi}}{2} \|\phi\|^2_{D^{s,2}(\mathbb{R}^N)} + \frac{e^{2\theta_\phi}}{2} \|\phi\|^2_{D^{1,2}(\mathbb{R}^N)} - \frac{N}{2(N+\alpha)} \int_{\mathbb{R}^N} (|x|^{\alpha-N} * |\phi|^{1+\frac{\alpha}{N}})|\phi|^{1+\frac{\alpha}{N}} \mathrm{d}x \\ &- \frac{\lambda e^{\frac{N(p-2)}{2}\theta_\phi}}{p} \|\phi\|^p_{L^p(\mathbb{R}^N)} \\ =& \frac{e^{2s\theta_\phi}}{2} \|\phi\|^2_{D^{s,2}(\mathbb{R}^N)} + \frac{e^{2\theta_\phi}}{2} \|\phi\|^2_{D^{1,2}(\mathbb{R}^N)} - \frac{\lambda e^{\frac{N(p-2)}{2}\theta_\phi}}{p} \|\phi\|^p_{L^p(\mathbb{R}^N)} \\ &- \frac{N}{2(N+\alpha)} S_\alpha^{-(1+\frac{\alpha}{N})} c^{2(1+\frac{\alpha}{N})}.\end{aligned}$$

Consequently, taking

$$\lambda_0 := \left(\frac{e^{2s\theta_\phi}}{2}\|\phi\|^2_{D^{s,2}(\mathbb{R}^N)} + \frac{e^{2\theta_\phi}}{2}\|\phi\|^2_{D^{1,2}(\mathbb{R}^N)}\right)\frac{pe^{-\frac{N(p-2)}{2}\theta_\phi}}{\|\phi\|^p_{L^p(\mathbb{R}^N)}},$$

we conclude that for any $\lambda > \lambda_0$,

$$m(c) < -\frac{N}{2(N+\alpha)}S_\alpha^{-(1+\frac{\alpha}{N})}c^{2(1+\frac{\alpha}{N})}. \tag{4.10}$$

Therefore, applying (4.10), we conclude that

$$m(c) - \frac{\bar{\beta}}{2}c^2 < -\frac{N}{2(N+\alpha)}S_\alpha^{-(1+\frac{\alpha}{N})}c^{2(1+\frac{\alpha}{N})} - \frac{\bar{\beta}}{2}c^2.$$

Now, we define a function $h : \mathbb{R}^+ \to \mathbb{R}$

$$h(c) := -\frac{N}{2(N+\alpha)}S_\alpha^{-(1+\frac{\alpha}{N})}c^{2(1+\frac{\alpha}{N})} - \frac{\bar{\beta}}{2}c^2.$$

Obviously, there exists a unique critical point

$$c_0 = \left(-\bar{\beta}S_\alpha^{1+\frac{\alpha}{N}}\right)^{\frac{N}{2\alpha}},$$

and

$$h(c_0) = \frac{\alpha}{2(N+\alpha)}\left(-\bar{\beta}S_\alpha^{1+\frac{\alpha}{N}}\right)^{\frac{N+\alpha}{\alpha}}$$

is the maximum of g. Hence, it holds that

$$m(c) - \frac{\bar{\beta}}{2}c^2 < \frac{\alpha}{2(N+\alpha)}\left(-\bar{\beta}S_\alpha\right)^{\frac{N+\alpha}{\alpha}}. \tag{4.11}$$

Step 4. $u_n \to \tilde{u}$ in $H^1_{rad}(\mathbb{R}^N)$. Since $u_n \rightharpoonup \tilde{u}$ in $H^1_{rad}(\mathbb{R}^N)$, according to (4.8) and Lemma 2.4, we deduce that $\tilde{u}$ is a weak solution of

$$-\Delta\tilde{u} + (-\Delta)^s\tilde{u} = \bar{\beta}\tilde{u} + (|x|^{\alpha-N} * |\tilde{u}|^{1+\frac{\alpha}{N}})|\tilde{u}|^{\frac{\alpha}{N}-1} + \lambda|\tilde{u}|^{p-2}u, \text{ in } \mathbb{R}^N. \tag{4.12}$$

Thus, we derive that

$$E_\lambda(\tilde{u}) = s\|\tilde{u}\|^2_{D^{s,2}(\mathbb{R}^N)} + \|\tilde{u}\|^2_{D^{1,2}(\mathbb{R}^N)} - \frac{\lambda N(p-2)}{2p}\|\tilde{u}\|^p_{L^p(\mathbb{R}^N)} = 0.$$

Let $v_n = u_n - \tilde{u}$, then $v_n \rightharpoonup 0$ in $H^1_{rad}(\mathbb{R}^N)$. Thus,

$$\|u_n\|^2_{D^{s,2}(\mathbb{R}^N)} = \|\tilde{u}\|^2_{D^{s,2}(\mathbb{R}^N)} + \|v_n\|^2_{D^{s,2}(\mathbb{R}^N)}.$$

$$\|u_n\|^2_{D^{1,2}(\mathbb{R}^N)} = \|\tilde{u}\|^2_{D^{1,2}(\mathbb{R}^N)} + \|v_n\|^2_{D^{1,2}(\mathbb{R}^N)}.$$

By (4.7) and Lemma 2.3, we conclude that

$$\begin{aligned}\int_{\mathbb{R}^N}(|x|^{\alpha-N}*|u_n|^{1+\frac{\alpha}{N}})|u_n|^{1+\frac{\alpha}{N}}\mathrm{d}x=&\int_{\mathbb{R}^N}(|x|^{\alpha-N}*|v_n|^{1+\frac{\alpha}{N}})|v_n|^{1+\frac{\alpha}{N}}\mathrm{d}x\\&+\int_{\mathbb{R}^N}(|x|^{\alpha-N}*|\tilde{u}|^{1+\frac{\alpha}{N}})|\tilde{u}|^{1+\frac{\alpha}{N}}\mathrm{d}x+o_n(1),\end{aligned}\tag{4.13}$$

and

$$\int_{\mathbb{R}^N}|u_n|^p\mathrm{d}x=\int_{\mathbb{R}^N}|\tilde{u}|^p\mathrm{d}x+o_n(1).\tag{4.14}$$

Combining $E_\lambda(u_n)=0$ and $E_\lambda(\tilde{u})=0$, we derive that

$$\|u_n\|^2_{D^{s,2}(\mathbb{R}^N)}=\|\tilde{u}\|^2_{D^{s,2}(\mathbb{R}^N)},\quad \|v_n\|^2_{D^{s,2}(\mathbb{R}^N)}=o_n(1).\tag{4.15}$$

$$\|u_n\|^2_{D^{1,2}(\mathbb{R}^N)}=\|\tilde{u}\|^2_{D^{1,2}(\mathbb{R}^N)},\quad \|v_n\|^2_{D^{1,2}(\mathbb{R}^N)}=o_n(1).\tag{4.16}$$

On the other hand, by (4.12), we have for all $v\in H^1_{rad}(\mathbb{R}^N)$ that

$$I'_\lambda(\tilde{u})v-\beta\int_{\mathbb{R}^N}\tilde{u}v\mathrm{d}x=0.\tag{4.17}$$

Taking $v=u_n-\tilde{u}$ as a test function in (4.8) and (4.17), we deduce that

$$\begin{aligned}\|v_n\|^2_{D^{s,2}(\mathbb{R}^N)}+\|v_n\|^2_{D^{1,2}(\mathbb{R}^N)}=&\bar{\beta}\|v_n\|^2_{L^2(\mathbb{R}^N)}+\int_{\mathbb{R}^N}(|x|^{\alpha-N}*|v_n|^{1+\frac{\alpha}{N}})|v_n|^{1+\frac{\alpha}{N}}\mathrm{d}x\\&+\lambda\|v_n\|^p_{L^p(\mathbb{R}^N)}+o_n(1).\end{aligned}$$

Using (4.14)–(4.16), we obtain

$$d:=-\bar{\beta}\|v_n\|^2_{L^2(\mathbb{R}^N)}=\int_{\mathbb{R}^N}(|x|^{\alpha-N}*|v_n|^{1+\frac{\alpha}{N}})|v_n|^{1+\frac{\alpha}{N}}\mathrm{d}x.\tag{4.18}$$

Recalling (1.8), we see that

$$d=0 \text{ or } d\geqslant(-\bar{\beta}S_\alpha)^{\frac{N+\alpha}{\alpha}}.$$

We distinguish two cases:

Case 1. If $d=0$, then $u_n\to\tilde{u}$ in $H^1_{rad}(\mathbb{R}^N)$, and the proof concludes since $\tilde{u}$ achieves the infimum $m(c)$.

Case 2. If $d\geqslant(-\bar{\beta}S_\alpha)^{\frac{N+\alpha}{\alpha}}$, by (4.13)–(4.16), we derive the strict inequality

$$I_\lambda(\tilde{u})>\lim_{n\to\infty}I_\lambda(u_n)=m(c).$$

Together with (4.13)–(4.16) and (4.18), recalling that $\bar{\beta} < 0$, we have that

$$\begin{aligned} m(c) - \frac{\bar{\beta}}{2}c^2 =& m(c) - \frac{\bar{\beta}}{2} \lim_{n\to\infty} \|u_n\|^2_{L^2(\mathbb{R}^N)} \\ \geqslant& m(c) - \frac{\bar{\beta}}{2} \lim_{n\to\infty} \|v_n\|^2_{L^2(\mathbb{R}^N)} \\ =& I_\lambda(\tilde{u}) + \lim_{n\to\infty} \left(I_\lambda(v_n) - \frac{\bar{\beta}}{2}\|v_n\|^2_{L^2(\mathbb{R}^N)} \right) \\ =& I_\lambda(\tilde{u}) + \lim_{n\to\infty} \left(-\frac{N}{2(N+\alpha)} \int_{\mathbb{R}^N} (|x|^{\alpha-N} * |v_n|^{1+\frac{\alpha}{N}})|v_n|^{1+\frac{\alpha}{N}} \mathrm{d}x - \frac{\bar{\beta}}{2}\|v_n\|^2_{L^2(\mathbb{R}^N)} \right) \\ =& I_\lambda(\tilde{u}) + \frac{\alpha}{2(N+\alpha)} \lim_{n\to\infty} \int_{\mathbb{R}^N} (|x|^{\alpha-N} * |v_n|^{1+\frac{\alpha}{N}})|v_n|^{1+\frac{\alpha}{N}} \mathrm{d}x \\ \geqslant& I_\lambda(\tilde{u}) + \frac{\alpha}{2(N+\alpha)}(-\bar{\beta}S_\alpha)^{\frac{N+\alpha}{\alpha}} \\ >& \frac{\alpha}{2(N+\alpha)}(-\bar{\beta}S_\alpha)^{\frac{N+\alpha}{\alpha}}, \end{aligned} \tag{4.19}$$

which contradicts (4.11), ruling out this possibility. Then, we complete the proof. □

Proof of Theorem 1.7. By Lemma 4.7, there exists a Palais-Smale sequence $\{u_n\} \subset \mathcal{M}^{rad}_{c,\lambda}$ for $I_\lambda|_{\mathcal{H}_c}$ at the level $m(c) \neq 0$. From Lemma 4.2, $\{u_n\}$ is bounded in $H^1_{rad}(\mathbb{R}^N)$. Then, from Lemma 4.8, there eixsts $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$, up to a subsequence,

$$u_n \to \tilde{u} \text{ strongly in} H^1_{rad}(\mathbb{R}^N).$$

By Lemma 4.3, $\tilde{u}$ is a radial minimizer of I_λ on $\mathcal{M}_{c,\lambda}$ and solves (1.1) with $\bar{\beta} < 0$. Finally, recalling Lemma 4.1, we conclude that $\tilde{u}$ is a ground state solution of I_λ on $\mathcal{H}_c$. This completes the proof. □

5. Conclusions

In this paper, we study the existence and non-existence of normalized ground state solutions for the Choquard equation with mixed operators and the Hardy-Littlewood-Sobolev lower critical exponent. By employing the fibering map and the Gagliardo-Nirenberg-Sobolev inequality, we obtain the non-existence of solutions for the critical case $p = 2 + \frac{4}{N}$. By utilizing variational method, we get the existence of normalized ground states for he subcritical regime $2 < p < 2 + \frac{4}{N}$. By constructing a Palais–Smale sequence, we obtain the existence of normalized ground state solutions for the supercritical case $2 + \frac{4}{N} < p < 2^*_s$.

Our study extends previous result for Choquard equation with local or non-local operators, solving significant technical challenges due to the combinations of mixed operators, the Hardy-Littlewood-Sobolev lower critical exponent, and a local perturbation in the Choquard equation. These findings contribute to the broader understanding of local and non-local partial differential equations and have potential applications fields such as physics, optimization, and phase transitions. Future research could explore more general potentials, multi-peak solutions, and further applications in stochastic processes.

Author contributions

Chun Qin: Conceptualization, writing-original draft preparation; Jie Yang: Conceptualization, writing-review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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