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Research article

Intuitive approximations for a residual waiting time process

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Abstract: The residual waiting time process, also known as the residual life process, represents the remaining time until the next renewal event, observed at an arbitrary moment. This process arises naturally in diverse areas such as queueing systems, reliability analysis, and inventory modeling. However, obtaining exact expressions for the expected residual waiting time is often analytically challenging, especially when the interarrival time distribution deviates from the Erlang distribution case. In this study, we propose intuitive approximations for the expected value of residual waiting time process, based on intuitive approximation of the renewal function. Two classes of interarrival distributions are examined: heavy-tailed distributions with regularly varying tails, and light-tailed distributions belonging to the special class of distributions denoted by $\Gamma(g)$, which naturally arises in extreme value theory. Using theoretical results from renewal theory and equilibrium distributions, intuitive approximation formulas are derived for both distributional settings. In particular, we investigate the Erlang distribution as a case study, comparing expected value of the residual waiting time computed via the exact renewal function with that obtained from the intuitive approximation. Moreover, for the Pareto and Burr XII distributions, we conduct case studies demonstrating how intuitive approximation closely matches asymptotic results for the expected value of the residual waiting time in the absence of exact formulas. This work provides a practical and mathematically grounded framework for analyzing systems involving stochastic arrivals, with potential extensions to higher-order moments.

Keywords: residual waiting time; renewal theory; equilibrium distribution; intuitive approximation; heavy-tailed distributions; regularly varying functions; class of $\Gamma(g)$; light-tailed distributions
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1. Introduction

Renewal processes are foundational in the modeling of stochastic systems subject to repeated events, and they find broad applications in reliability engineering, queueing systems, maintenance scheduling, and inventory control [1–3]. These models capture systems where events (e.g., failures, arrivals) occur repeatedly over time, and their analysis leads to important insights about the system's long-run behavior. As evidenced by numerous foundational and contemporary studies, the diverse applications of renewal and renewal-reward processes have given rise to a rich and growing literature. Comprehensive analyses of renewal and renewal-reward processes and their asymptotic applications to inventory and reliability models have been presented in several studies. Aliyev et al. [4] proposed an asymptotic approach for a renewal-reward process with general interference of chance, while Brown and Solomon [5] derived second-order approximations for the variance of such processes. Csenki [6] extended the classical framework by examining renewal-reward processes with retrospective reward structures. Hanalioglu and Khaniyev [7] investigated an (s, S)-type inventory model with asymmetric triangular interference of chance, and Kamışlık et al. [8] analyzed a similar model under heavy-tailed demand with infinite variance. Further developments were provided by Kamışlık et al. [9], who studied a semi-Markovian renewal-reward process with $\Gamma(g)$-distributed demand, and by Kamışlık et al. [10], who proposed moment-based approximations for stochastic control models of type (s, S). Khaniyev [11] contributed fundamental results on the moments of generalized renewal processes, while Khaniyev and Atalay [12] established weak convergence results for the ergodic distribution in inventory models. Khaniyev and Aksop [13] extended these findings to generalized beta interference of chance, and Khaniyev et al. [14] developed a semi-Markovian approach to the inventory model of type (s, S) framework. Most recently, Yazır et al. [15] introduced a moment-based approximation for a renewal-reward process with generalized gamma-distributed interference of chance, providing an important extension of earlier asymptotic models to the $\Gamma(g)$ class.

A central quantity in renewal theory is the residual waiting time process, which quantifies the time remaining until the next renewal event, conditional on an observation at a fixed time. Formally, let $\{X_i\}_{i\geq 1}$ be a sequence of independent and identically distributed (i.i.d.) non-negative interarrival times with distribution function F, and let

$$T_0 = 0,\ T_n = \sum_{i=1}^{n} X_i,\ n = 1, 2, ...$$

denote the time of the n-th renewal. Then the associated renewal counting process is

$$N(t) = \min\{n \geq 1 : T_n > t\},\ N(0) = 1.$$

The residual waiting time process $W(t)$ is defined by

$$W(t) = T_{N(t)} - t, \tag{1.1}$$

i.e., the remaining time until the next renewal after time t (see Feller [16]). This process is also referred to as the residual life process. The study of residual life and related processes has attracted significant attention due to their wide applicability in reliability engineering, queueing systems, and stochastic modeling. The mean residual life (MRL) function, which describes the expected remaining lifetime given survival up to time t, is a central object in renewal theory and survival analysis. Early contributions include the work of Cox [1], who provided the classical foundations of renewal theory, and Ross [17], whose textbook remains a standard reference for the renewal process and residual lifetime distributions. Later expositions such as Taylor and Karlin [18], Medhi [19], and Asmussen [20] expanded these analyses, offering both theoretical developments and applied perspectives.

A wide range of research has been devoted to the asymptotic and structural properties of the MRL function. Bradley and Gupta [21] studied the limiting behaviour of the mean residual life and derived asymptotic expansions that depend only on the failure rate and its derivatives, thereby extending earlier approaches without monotonicity assumptions. Lange et al. [22] developed Laplace transform representations for residual waiting times, enabling analytical investigation of distributional properties and moments. Further contributions by Raqaba and Asadi [23] introduced nonparametric statistical tests based on MRL functions, demonstrating their usefulness in comparing lifetime distributions. Nanda et al. [24] explored structural properties of the MRL function, introducing new stochastic orderings and linking them to notions of aging, convexity, and bathtub-shaped behaviours.

Recent works have also emphasized applications to maintenance and reliability. Ahmadi et al. [25] proposed the stochastic dynamic mean residual time for repairable systems (SD-MRTR) model for residual life analysis in repairable systems, incorporating both system age and operational environment to optimize maintenance scheduling. Morvai and Weiss [26] derived universal approximation rates for moments of the residual waiting time, including closed-form expressions for the second moment. Other contributions in queueing theory, such as Haviva and van der Wal [27], Van Houdt and Blondia [28], and Jagerman et al. [29], developed approximations for waiting times and queue length distributions that are closely related to residual lifetime analysis.

Beyond journal articles, several advanced monographs highlight the role of residual life within broader stochastic frameworks. Asmussen [20] discussed residual life in the context of age processes and renewal-reward processes. Wolff [30] analyzed residual waiting time distributions in queueing theory, while Gallager [31] treated residual and excess life distributions as part of a general theory of stochastic processes.

In summary, residual life analysis constitutes a fundamental topic that connects renewal theory, reliability, and queueing. The diverse approaches in the literature, from asymptotic expansions and transform methods to statistical tests and optimization models, demonstrate both the theoretical richness of the subject and its practical relevance in engineering and applied probability.

The statistical properties of $W(t)$, especially its expectation $\mathbb{E}[W(t)]$, are of central interest. A key identity that facilitates the analysis of $\mathbb{E}[W(t)]$ is derived from Wald's identity, which relates the sum of a number of i.i.d. random variables to their mean and the expectation of the counting process. Specifically, it yields:

$$\mathbb{E}[W(t)] = \mu_1 \cdot U(t) - t, \tag{1.2}$$

where $\mu_1 = \mathbb{E}[X_1]$ and $U(t) = \mathbb{E}[N(t)]$ are the renewal function, which gives the expected number of renewals in the interval $[0, t]$ [16, 17, 32]. This identity highlights the fact that accurate approximation of the renewal function directly translates into accurate estimates for the residual waiting time.

However, computing $U(t)$ exactly is often analytically intractable for general interarrival distributions, especially when Laplace transforms are not easily handled or closed-form expressions are unavailable. This limitation has led to the development of approximation methods for the renewal function and, consequently, for the expected residual waiting time. More recently, Mitov and Omey [33] developed an intuitive method for approximating the renewal function without resorting to convolutions or moment-generating functions. Their approach relies on asymptotic expansions and structural properties of the interarrival distribution, particularly within classes such as the generalized gamma family $\Gamma(g)$ and regularly varying distributions.

In this paper, we investigate the mean residual waiting time $\mathbb{E}[W(t)]$ for a broad class of interarrival distributions using the intuitive approximation framework introduced by Mitov and Omey [33]. Our goal is to derive accurate and simple approximations for $\mathbb{E}[W(t)]$ that remain effective even when the exact form of $U(t)$ is unavailable. In particular, we examine the Erlang distribution, Pareto distribution, and Burr XII distribution in detail as a case study, where both exact and approximated formulas are tractable and comparable. We demonstrate that the proposed intuitive approximation formulas yield results that are not only mathematically sound but also practically useful, especially in applications where computational simplicity is crucial.

The remainder of this paper is organized as follows: In Section 2, we present the essential preliminaries, notations, and theoretical background, including fundamental results on regularly varying functions and the class of $\Gamma(g)$ distributions. Section 3 develops the main approximation formulas for the expected residual waiting time process $W(t)$, with detailed analyses for both heavy-tailed and light-tailed interarrival time distributions. Subsections within Section 3 provide explicit asymptotic results for classical distributions such as Weibull, generalized extreme value, gamma-type, and logistic distributions, followed by numerical illustrations to assess the accuracy of the proposed approximations. Finally, Section 4 concludes the paper by summarizing the main findings and outlining possible directions for future research.

2. Preliminaries, theoretical framework, and existing theorems

In this section, we introduce the essential notations and provide a mathematical formulation of the model prior to addressing the main problem.

2.1. Theory of regular variation and important results

This section presents the key definitions and foundational results that will be used throughout the study. The well-established material is primarily drawn from [34–37].

Definition 2.1. *A positive measurable function $f : [0,\infty) \to [0,\infty)$ is regularly varying at ∞ with index α, $f \in \mathcal{RV}_\alpha$, if for each $\lambda > 0$*

$$\lim_{x\to\infty} \frac{f(\lambda x)}{f(x)} = \lambda^\alpha. \tag{2.1}$$

For $\alpha = 0$, i.e. when for $\lambda > 0$

$$\lim_{x\to\infty} \frac{L(\lambda x)}{L(x)} = 1,$$

then L is slowly varying, $L \in \mathcal{SV}$.

Remark 2.2. *Every regularly varying function f of index α can be represented as*

$$f(x) = x^\alpha L(x),$$

where L is some slowly varying function.

One of the fundamental theorems concerning regularly varying functions is Karamata's theorem. The theorem can be stated as follows:

Theorem 2.3 (Karamata's Theorem [37])**.** *Let L be slowly varying and locally bounded in* $[x_0, \infty)$ *for some* $x_0 \geq 0$. *Then*

(a) for $\alpha > -1$,

$$\int_{x_0}^{x} t^{\alpha} L(t)\, dt \sim (\alpha+1)^{-1} x^{\alpha+1} L(x), \quad x \to \infty;$$

(b) for $\alpha < -1$,

$$\int_{x}^{\infty} t^{\alpha} L(t)\, dt \sim -(\alpha+1)^{-1} x^{\alpha+1} L(x), \quad x \to \infty.$$

2.2. The class of $\Gamma(g)$ *distributions*

In this study, we propose intuitive approximation formulas for the expected residual waiting time, $\mathbb{E}[W(t)]$, based on structured expansions of the renewal function. Two classes of interarrival distributions are examined: heavy-tailed distributions with regularly varying tails, and light-tailed distributions belonging to the generalized class of $\Gamma(g)$.

While regularly varying functions provide a powerful framework for modeling heavy-tailed behavior, they do not capture the asymptotic properties of light-tailed distributions such as the exponential or gamma. To address this limitation, the class of $\Gamma(g)$, originally introduced in the context of extreme value theory, emerges as a natural and useful generalization.

A positive and measurable function h is said to belong to the class of $\Gamma(g)$ with auxiliary function g, if and only if for every fixed $y \in \mathbb{R}$,

$$\lim_{x\to\infty} \frac{h(x+yg(x))}{h(x)} = e^{y}. \tag{2.2}$$

This limiting relation characterizes h as an element of $\Gamma(g)$ class, and it extends the exponential-type behavior seen in light-tailed distributions. The function $g(x)$, referred to as the auxiliary function, governs the local scaling behavior of h at infinity.

Distributions in this class exhibit subtle asymptotic growth patterns and include many standard light-tailed distributions such as the exponential and gamma distributions, which fall outside the regularly varying class. The flexibility of the $\Gamma(g)$ class makes it particularly well-suited for approximation frameworks in renewal theory, where explicit forms of the renewal function are often unavailable.

The main objective of this study is to derive approximation formulas for the mean of the residual waiting time process $W(t)$, based on structured approximations of the renewal function. We adopt the intuitive approximation method developed by Mitov and Omey [33].

To this end, we consider two broad classes of interarrival time distributions: heavy-tailed distributions with regularly varying tails and light-tailed distributions that belong to the generalized class of $\Gamma(g)$. For each class, we provide tailored approximation formulas for $\mathbb{E}[W(t)]$ expressed in terms of the approximate renewal function $U(t)$. These results serve as practical and analytically tractable alternatives for evaluating the behavior of $W(t)$ when exact computations are infeasible. To proceed, we first formalize the structure of the approximation through the following propositions, which will serve as the basis for estimating the mean of $W(t)$ under different distributional assumptions.

Proposition 2.4 (Mitov and Omey [33]). *Assume that the tail distribution function $\overline{F}(x) = 1 - F(x)$ belongs to the class of regularly varying functions with index $-\alpha$, that is,*

$$\overline{F}(x) \in RV_{-\alpha}, \quad \text{with } \alpha > 2.$$

Then the interarrival distribution F has finite mean and variance:

$$\mathbb{E}[X_1] < \infty, \quad Var(X_1) < \infty.$$

Under this assumption, the expected number of renewals $U(x) = \mathbb{E}[N(x)]$ satisfies the following intuitive approximation as $x \to \infty$:

$$U(x) \sim \frac{x}{m_1} + \frac{1}{m_1} \int_0^x F_e(y)\, dy + 2m_e \overline{F}_e(x), \tag{2.3}$$

where $m_1 = \mathbb{E}[X_1]$, $F_e(x)$ is the equilibrium distribution defined by

$$F_e(x) = \frac{1}{\mu_1} \int_0^x \overline{F}(y)\, dy, \quad x \geq 0,$$

and $m_e = \int_0^\infty \overline{F}_e(y)\, dy$ is the mean of the equilibrium distribution.

We now turn to the second case, where the interarrival time distribution has a light tail and belongs to the generalized class of $\Gamma(g)$. In this setting, Mitov and Omey [33] provided an alternative approximation for the renewal function, which is summarised in the following proposition.

Proposition 2.5 (Mitov and Omey [33]). *Assume that the tail distribution function $\overline{F}(x) = 1 - F(x)$ belongs to the generalized class of $\Gamma(g)$, where $g(x)$ is an auxiliary function associated with $\overline{F}(x)$. Then the renewal function $U(x) = \mathbb{E}[N(x)]$ admits the following asymptotic approximation as $x \to \infty$:*

$$U(x) \sim \frac{x}{m_1} + \frac{m_e}{m_1} - \frac{g^2(x)\, \overline{F}(x)}{m_1^2}, \tag{2.4}$$

where $m_1 = \mathbb{E}[X_1]$ and $m_e = \int_0^\infty \overline{F}_e(y)\, dy$ is the mean of the equilibrium distribution. The function $g(x)$ characterizes the local scaling behavior of $\overline{F}(x)$ at infinity and satisfies

$$\lim_{x \to \infty} \frac{\overline{F}(x + yg(x))}{\overline{F}(x)} = e^{-y}, \quad \text{for all } y \in \mathbb{R}.$$

3. Approximation formulas for the expected value of the residual waiting time process W(t)

Before proceeding with approximation formulas for the expected value of the residual waiting time process $W(t)$, we present the following theorem, which provides an explicit approximation for the expected value of the residual waiting time process when the interarrival time distribution has a regularly varying tail.

Theorem 3.1. *Let X_n be a random variable whose tail distribution satisfies $\overline{F}(x) \in RV_{-\alpha}$ with $\alpha > 2$. Then the following approximation formula for the expected value of the residual waiting time process $W(t)$ is valid:*

$$\mathbb{E}(W(t)) \sim m_e + \frac{1}{(\alpha-1)(2-\alpha)}\, t^{2-\alpha}L(t) + 2m_e m_1\, t^{1-\alpha}L(t) \tag{3.1}$$

where $m_i = \mathbb{E}(X_i)$, for $i = 1, 2, 3$ and $m_e = \frac{m_2}{2m_1}$.

Proof of Theorem 3.1. Let $\{X_i\}_{i\geq 1}$ be an i.i.d. sequence of interarrival times with finite mean $m_1 = \mathbb{E}[X_1]$. Let $N(t)$ be the associated renewal process and define the residual waiting time process as

$$W(t) := T_{N(t)} - t,$$

where

$$T_{N(t)} = \sum_{i=1}^{N(t)} X_i.$$

Taking expectation and applying Wald's identity yields

$$\mathbb{E}[W(t)] = \mathbb{E}[T_{N(t)}] - t = \mathbb{E}\left[\sum_{i=1}^{N(t)} X_i\right] - t = m_1 \cdot \mathbb{E}[N(t)] - t = m_1 U(t) - t. \tag{3.2}$$

We aim to find an approximation for $\mathbb{E}[W(t)]$, where $U(t)$ is the renewal function. Suppose the interarrival distribution F satisfies

$$\overline{F}(x) = 1 - F(x) \in \mathrm{RV}_{-\alpha}, \quad \text{with } \alpha > 2.$$

Then both $\mathbb{E}[X] < \infty$ and $\mathrm{Var}(X) < \infty$. The equilibrium distribution associated with F is defined by

$$F_e(x) = \frac{1}{m_1}\int_0^x \overline{F}(y)\,dy, \quad x \geq 0.$$

Its tail is given by

$$\overline{F}_e(x) = \frac{1}{m_1}\int_x^\infty \overline{F}(u)\,du.$$

Assume that $\overline{F}(x) = x^{-\alpha}L(x)$, where $L(x)$ is slowly varying and $\alpha > 2$. By Karamata's theorem, we have

$$\int_x^\infty t^{-\alpha}L(t)\,dt \sim \frac{x^{1-\alpha}L(x)}{\alpha - 1}, \quad \text{as } x \to \infty,$$

which implies

$$\overline{F}_e(x) \sim \frac{1}{m_1} \cdot \frac{x^{1-\alpha}L(x)}{\alpha - 1}. \tag{3.3}$$

Let the mean of the equilibrium distribution be

$$m_e := \int_0^\infty \overline{F}_e(x)\,dx < \infty.$$

Then, using integration by parts, we write

$$\int_0^x \overline{F}_e(y)\,dy = m_e - \int_x^\infty \overline{F}_e(y)\,dy.$$

From the previous result in Eq (3.3), using Karamata's theorem we obtain

$$\int_x^\infty \overline{F}_e(y)\,dy \sim \frac{1}{m_1}\int_x^\infty \frac{t^{1-\alpha}L(t)}{\alpha-1}\,dt \sim \frac{1}{m_1(\alpha-1)(2-\alpha)}x^{2-\alpha}L(x). \tag{3.4}$$

Hence,

$$\frac{1}{m_1}\int_0^x \overline{F}_e(y)\,dy \sim \frac{m_e}{m_1} - \frac{1}{m_1^2(\alpha-1)(2-\alpha)}x^{2-\alpha}L(x). \tag{3.5}$$

Substituting Eq (3.5) into the Approximation (2.3) for $U(t)$, we obtain

$$U(t) \sim \frac{t}{m_1} + \frac{m_e}{m_1} - \frac{1}{m_1^2(\alpha-1)(2-\alpha)}t^{2-\alpha}L(t) + \frac{2m_e}{m_1}\cdot\frac{t^{1-\alpha}L(t)}{\alpha-1}. \tag{3.6}$$

Therefore, by Wald's identity,

$$\mathbb{E}[W(t)] = m_1U(t) - t \sim m_e + \frac{1}{(\alpha-1)(2-\alpha)}t^{2-\alpha}L(t) + 2m_e\cdot t^{1-\alpha}L(t).$$

If the second moment $m_2 := \mathbb{E}[X^2] < \infty$, then $m_e = \frac{m_2}{2m_1}$, and the result simplifies to

$$\mathbb{E}[W(t)] \sim \frac{m_2}{2m_1} + \frac{1}{(\alpha-1)(2-\alpha)}t^{2-\alpha}L(t) + m_2\cdot t^{1-\alpha}L(t).$$

□

Remark 3.2 (Comparison with the classical renewal approximation (heavy-tailed case))**.** *Recall that the classical renewal asymptotic (non–arithmetic, $m_2 < \infty$) gives*

$$U_{\rm cl}(t) = \frac{t}{m_1} + \frac{m_2}{2m_1^2} + o(1), \qquad \mathbb{E}_{\rm cl}[W(t)] = m_1U(t) - t \to m_e = \frac{m_2}{2m_1}.$$

In contrast, the heavy-tailed (regular variation) refinement above provides two explicit, polynomially decaying corrections:

$$\mathbb{E}[W(t)] - m_e = \underbrace{\frac{1}{(\alpha-1)(2-\alpha)}\,t^{2-\alpha}L(t)}_{\textit{order } t^{2-\alpha}} \;+\; \underbrace{m_2\,t^{1-\alpha}L(t)}_{\textit{order } t^{1-\alpha}}.$$

Since $\alpha > 2$, we have $(\alpha-1) > 0$ and $(2-\alpha) < 0$, hence

$$\frac{1}{(\alpha-1)(2-\alpha)} = -\frac{1}{(\alpha-1)(\alpha-2)} < 0.$$

Therefore, the leading correction is negative and of order $t^{2-\alpha}L(t)$, while the second correction is positive and of strictly smaller order $t^{1-\alpha}L(t)$ (because $1-\alpha < 2-\alpha$). Consequently,

$$\mathbb{E}[W(t)] - m_e \;=\; -\frac{1}{(\alpha-1)(\alpha-2)}\,t^{2-\alpha}L(t) \;+\; m_2\,t^{1-\alpha}L(t) \;=\; -\frac{1}{(\alpha-1)(\alpha-2)}\,t^{2-\alpha}L(t)\,(1+o(1)),$$

and, for all sufficiently large t,

$$\mathbb{E}[W(t)] \; < \; m_e.$$

We can conclude that the heavy-tailed process approaches the classical constant m_e from below at a polynomial rate $t^{2-\alpha}L(t)$. The classical form $\mathbb{E}[W(t)] = m_e + o(1)$ is asymptotic but non-quantitative; it does not tell how fast (or from which side) $\mathbb{E}[W(t)]$ approaches m_e. Our formula provides (i) the rate ($t^{2-\alpha}L(t)$), (ii) the sign (negative), and (iii) the next order $t^{1-\alpha}L(t)$ term. Because heavy tails converge only polynomially, capturing these terms yields substantially more accurate finite-t approximations than the classical $o(1)$ statement, and it precisely explains the slower approach to m_e in the heavy-tailed regime.

Now we will illustrate the approximation formula for the expected residual waiting time,

$$\mathbb{E}[W(t)] \sim m_e + \frac{1}{(\alpha-1)(2-\alpha)} t^{2-\alpha} L(t) + 2m_e t^{1-\alpha} L(t),$$

by applying it to two different interarrival time distributions with regularly varying tails: the Pareto distribution and the Burr XII distribution.

3.1. Comparison of Pareto and Burr XII distributions for $\mathbb{E}[W(t)]$

Note that the Burr XII distribution is a flexible family commonly used to model heavy-tailed phenomena. Its survival function is given by

$$\overline{F}(x) = \left(1 + \left(\frac{x}{x_0}\right)^c\right)^{-k}, \quad x \geq 0,$$

where $c > 0$ and $k > 0$ are shape parameters, and $x_0 > 0$ is a scale parameter. The probability density function (pdf) is

$$f(x) = \frac{ck}{x_0}\left(\frac{x}{x_0}\right)^{c-1}\left(1 + \left(\frac{x}{x_0}\right)^c\right)^{-(k+1)}.$$

The Pareto distribution is a well-known heavy-tailed distribution whose survival function follows a power-law decay. The Burr XII distribution includes the Pareto as a special case when $c = 1$. In these distributions:

- α is the tail index (shape) for Pareto,
- k governs tail heaviness for Burr XII and,
- c adjusts the shape profile in Burr XII.

Below, we compare the asymptotic behavior of the residual waiting time mean ($\mathbb{E}[W(t)]$) under each distribution, expressing $\overline{F}(x)$ in the form $x^{-\alpha}L(x)$.

Case 1: Pareto distribution

Let the interarrival times follow a Pareto distribution with tail index $\alpha = 3$ and $x_0 = 1$. Then the survival function is

$$\overline{F}(x) = \left(\frac{1}{x}\right)^3 = x^{-3}, \quad x \geq 1.$$

This directly gives the form

$$\overline{F}(x) = x^{-3}L(x), \quad \text{where } L(x) = 1.$$

Moments

$$\mathbb{E}[X] = \frac{3}{2}, \quad \mathbb{E}[X^2] = 3, \quad m_e = \frac{3}{3} = 1.$$

Thus, the intuitive approximation formula

$$\mathbb{E}[W(t)] \sim 1 - \frac{1}{2t} + \frac{2}{t^2}, \quad \text{as } t \to \infty.$$

Case 2: Burr XII distribution with k = 9

Take $c = 1$, $k = 9$ and $x_0 = 1$. Then the survival function is

$$\overline{F}(x) = (1 + x)^{-9}, \quad x \geq 0.$$

We write

$$\overline{F}(x) = x^{-9}L(x), \quad \text{with } L(x) = \left(1 + \frac{1}{x}\right)^{-9}.$$

Again, $L(x) \to 1$ as $x \to \infty$.

Moments

$$\mathbb{E}[X] = \frac{1}{8}, \quad \mathbb{E}[X^2] = \frac{1}{7}, \quad m_e = \frac{1/7}{2 \cdot (1/8)} = \frac{8}{14} = \frac{4}{7} \approx 0.571.$$

Thus

$$\mathbb{E}[W(t)] \sim 0.571 - \frac{1}{2t} + \frac{0.3265}{t^2}, \quad \text{as } t \to \infty.$$

Table 1. Intuitive approximation of $\mathbb{E}[W(t)]$ for Pareto and Burr XII distributions.

Distribution	**Parameters**	$\overline{F}(x)$	$L(x)$	m_e	$\mathbb{E}[W(t)] \sim$
Pareto	$\alpha = 3, x_0 = 1$	x^{-3}	1	1	$1 - \frac{1}{2t} + \frac{2}{t^2}$
Burr XII	$c = 1, k = 9$	$x^{-9}\left(1 + \frac{1}{x}\right)^{-9}$	$\left(1 + \frac{1}{x}\right)^{-9}$	$\frac{4}{7} \approx 0.571$	$0.571 - \frac{1}{2t} + \frac{0.3265}{t^2}$

Tables 2 and 3 provide a comparative analysis of the results obtained through the asymptotic method and the intuitive approximation method for the expected value of the residual waiting time processes generated by the Pareto and Burr XII distributions, respectively. Here, $\hat{\mathbb{E}}(W(t))$ denotes the limiting value approached by the asymptotic expansion of the expected value of the residual waiting time processes, whereas $\tilde{\mathbb{E}}(W(t))$ corresponds to the value obtained through the intuitive approximation approach. Moreover, denote

$$\Delta = \left|\hat{\mathbb{E}}(W(t)) - \tilde{\mathbb{E}}(W(t))\right|, \qquad \delta = \frac{\Delta}{\hat{\mathbb{E}}(W(t))} \times 100\%.$$

Here, Δ is absolute errors of approximate formulas, and δ represents relative errors. Corresponding numerical results are summarised in Tables 2 and 3. It is observed that, within the interval $0 \leq t \leq 3$, the intuitive approximation method provides better results compared to the asymptotic approach. For $4 \leq t \leq 10$, the intuitive approximation method yields an improvement of approximately 2-3%.

When $t > 10$, no significant difference can be observed between the formula obtained by the intuitive approximation method and that derived from the asymptotic approach.

Table 2. Comparison results for $\mathbb{E}(W(t))$ under the Pareto distribution.

t	$\hat{\mathbb{E}}(W(t))$	$\tilde{\mathbb{E}}(W(t))$	Δ	$\delta(\%)$
10	1.000000000	0.970000000	0.030000000	3.000000000
9	1.000000000	0.969136000	0.030864000	3.086419753
8	1.000000000	0.968750000	0.031250000	3.125000000
7	1.000000000	0.969388000	0.030612000	3.061224490
6	1.000000000	0.972222000	0.027778000	2.777777778
5	1.000000000	0.980000000	0.020000000	2.000000000
4	1.000000000	1.000000000	0.000000000	0.000000000
3	1.000000000	1.055556000	0.055556000	5.555555556
2	1.000000000	1.250000000	0.250000000	25.000000000
1	1.000000000	2.500000000	1.500000000	150.000000000

Table 3. Comparison results for $\mathbb{E}(W(t))$ under the Burr XII distribution.

t	$\hat{\mathbb{E}}(W(t))$	$\tilde{\mathbb{E}}(W(t))$	Δ	$\delta(\%)$
70	0.571	0.563924	0.007076	1.239269
60	0.571	0.562757	0.008243	1.443544
50	0.571	0.561131	0.009869	1.728441
40	0.571	0.558704	0.012296	2.153404
30	0.571	0.554696	0.016304	2.855322
20	0.571	0.546816	0.024184	4.235333
15	0.571	0.539118	0.031882	5.583577
10	0.571	0.524265	0.046735	8.184764
9	0.571	0.519475	0.051525	9.023589
8	0.571	0.513602	0.057398	10.05227
7	0.571	0.506235	0.064765	11.34244
6	0.571	0.496736	0.074264	13.00594
5	0.571	0.484060	0.086940	15.22592
4	0.571	0.466406	0.104594	18.31764
3	0.571	0.440611	0.130389	22.83518
2	0.571	0.402625	0.168375	29.48774
1	0.571	0.397500	0.173500	30.38529

It is observed from the Table 3 that, within the interval $0 \leq t \leq 10$, the intuitive approximation method provides better results compared to the asymptotic approach. For $15 \leq t \leq 40$, the intuitive approximation method yields an improvement of around 2-5.5%. When $t > 40$, no significant difference can be observed between the formula obtained by the intuitive approximation method and that derived from the asymptotic approach.

In order to further analyze the behavior of the expected residual waiting time, we next investigate the effect of the regular variation index α within the framework of the Burr XII and Pareto distributions.

3.2. Effect of regular variation index k on $\mathbb{E}[W(t)]$ under Burr XII and Pareto distributions

Let the interarrival times follow the Burr XII distribution with fixed $x_0 = 1$, shape parameters $c = 1$, and index of regular variation $k \in \{2, 3, 5, 7, 9\}$. Then the survival function becomes

$$\overline{F}(x) = (1 + x)^{-k}.$$

This yields

$$\overline{F}(x) = x^{-k}L(x), \quad \text{where } L(x) = \left(1 + \frac{1}{x}\right)^{-k}, \quad L(x) \to 1 \text{ as } x \to \infty.$$

The moments for this distribution are

$$\mathbb{E}[X] = \frac{1}{k-1}, \quad \mathbb{E}[X^2] = \frac{2}{k-2}, \quad m_e = \frac{\mathbb{E}[X^2]}{2\mathbb{E}[X]} = \frac{2}{k-2} \cdot \frac{k-1}{2} = \frac{k-1}{k-2}.$$

Thus, the asymptotic form of $\mathbb{E}[W(t)]$ is

$$\mathbb{E}[W(t)] \sim \frac{k-1}{k-2} - \frac{1}{2t} + \frac{(k-1)^2}{(k-2)^2 t^2}, \quad \text{as } t \to \infty.$$

Table 4. Effect of tail index α on $\mathbb{E}[W(t)]$ for Burr XII distribution ($c = 1, x_0 = 1$).

k	$\mathbb{E}[X]$	$\mathbb{E}[X^2]$	m_e	$\mathbb{E}[W(t)] \sim$
2	1	–	–	Not finite
3	$\frac{1}{2}$	1	1	$1 - \frac{1}{2t} + \frac{1}{t^2}$
5	$\frac{1}{4}$	$\frac{2}{3}$	$\frac{2}{3}$	$0.667 - \frac{1}{2t} + \frac{0.444}{t^2}$
7	$\frac{1}{6}$	$\frac{2}{5}$	$\frac{2}{5}$	$0.4 - \frac{1}{2t} + \frac{0.16}{t^2}$
9	$\frac{1}{8}$	$\frac{2}{7}$	$\frac{4}{7}$	$0.571 - \frac{1}{2t} + \frac{0.3265}{t^2}$

As the tail index α increases, the equilibrium mean m_e (and hence the leading term in $\mathbb{E}[W(t)]$) decreases, indicating faster decay of the tail and reduced long-term residual waiting times.

Now, let the interarrival times follow a Pareto distribution with scale parameter $x_0 = 1$ and shape parameter $\alpha > 2$. Its survival function is

$$\overline{F}(x) = \left(\frac{x_0}{x}\right)^{\alpha} = x^{-\alpha}, \quad x \geq x_0 = 1.$$

So, it follows the regularly varying form

$$\overline{F}(x) = x^{-\alpha}L(x), \quad \text{where } L(x) = 1.$$

The moments are

$$\mathbb{E}[X] = \frac{\alpha x_0}{\alpha - 1}, \quad \mathbb{E}[X^2] = \frac{\alpha x_0^2}{\alpha - 2},$$

and the equilibrium mean is

$$m_e = \frac{\mathbb{E}[X^2]}{2\mathbb{E}[X]} = \frac{1}{2} \cdot \frac{\alpha}{\alpha-2} \cdot \frac{\alpha-1}{\alpha} = \frac{\alpha-1}{2(\alpha-2)}.$$

Hence, the asymptotic approximation becomes

$$\mathbb{E}[W(t)] \sim m_e - \frac{1}{2t} + \frac{m_e^2}{t^2}, \quad \text{as } t \to \infty.$$

Table 5. Effect of tail index α on $\mathbb{E}[W(t)]$ for Pareto distribution ($x_0 = 1$).

α	$\mathbb{E}[X]$	$\mathbb{E}[X^2]$	m_e	$\mathbb{E}[W(t)] \sim$
2.5	$\frac{5}{3}$	5	$\frac{3}{2}$	$1.5 - \frac{1}{2t} + \frac{2.25}{t^2}$
3	$\frac{3}{2}$	3	1	$1 - \frac{1}{2t} + \frac{1}{t^2}$
5	$\frac{5}{4}$	$\frac{5}{3}$	$\frac{2}{3}$	$0.667 - \frac{1}{2t} + \frac{0.444}{t^2}$
7	$\frac{7}{6}$	$\frac{7}{5}$	$\frac{5}{6}$	$0.833 - \frac{1}{2t} + \frac{0.694}{t^2}$
9	$\frac{9}{8}$	$\frac{9}{7}$	$\frac{7}{8}$	$0.875 - \frac{1}{2t} + \frac{0.766}{t^2}$

As α increases (i.e., the distribution becomes lighter-tailed), the equilibrium mean m_e and the leading term in $\mathbb{E}[W(t)]$ decrease. This reflects shorter residual waiting times in systems with less extreme variability.

Another objective of this study is to derive intuitive approximation formulas for the expected value $\mathbb{E}(W(t))$ of the residual waiting time process $W(t)$, when the random variables X_i generating $W(t)$ belong to the $\Gamma(g)$ class of distributions. For this purpose, we employ the intuitive approximation formula for the renewal function generated by random variables from the $\Gamma(g)$ class, as established in the article of Mitov and Omey [33] (see Proposition 2.5). In this framework, the following theorem provides an explicit approximation formula for $\mathbb{E}[W(t)]$ when the interarrival times belong to the class $\Gamma(g)$.

Theorem 3.3. *Let X_n be a random variable whose tail distribution satisfies $\overline{F}(x) \in \Gamma(g)$. Then the following intuitive approximation for the expected value of the residual waiting time process $W(t)$ is valid:*

$$\mathbb{E}[W(t)] \sim m_e - \frac{1}{m_1} g^2(t)\overline{F}(t), \tag{3.7}$$

where $m_i = \mathbb{E}(X_i)$, for $i = 1, 2, 3$ and $m_e = \frac{m_2}{2m_1}$.

Proof of Theorem 3.3. The result follows immediately from the substitution of the renewal function Approximation (2.4) into Eq (3.2). □

Now we will illustrate the intuitive approximation for the expected residual waiting time process under several lifetime distributions. Throughout, we use the general formula

$$\mathbb{E}[W(t)] \sim m_e - \frac{1}{m_1} g^2(t)\, \overline{F}(t), \qquad t \to \infty, \tag{3.8}$$

where $\overline{F}(t)$ denotes the survival function of the interarrival distribution.

Case 1: Standard Weibull distribution

Consider the standard Weibull distribution with survival function

$$\overline{F}(x) = \exp(-\lambda x^{\alpha}), \qquad x \geq 0, \tag{3.9}$$

where $\lambda > 0$ and $\alpha > 0$. In this case, we have $\overline{F}(x) \in \Gamma(g)$ with

$$g(x) = \frac{x^{1-\alpha}}{\lambda}. \tag{3.10}$$

Substituting Eq (3.10) into Eq (3.8) yields

$$\mathbb{E}[W(t)] \sim m_e - \frac{1}{m_1}\frac{t^{2-2\alpha}}{\lambda^2}\exp(-\lambda t^{\alpha}). \tag{3.11}$$

This gives the intuitive approximation of the residual waiting time process under the Weibull law.

Case 2: Generalized extreme value distribution (EVD)

Next, consider the generalized extreme value-type distribution with survival function

$$\overline{F}(x) = 1 - \exp(-e^{-x}). \tag{3.12}$$

Since

$$\lim_{x\to\infty}\overline{F}(x) \sim e^{-x}, \quad \text{and} \quad \lim_{x\to\infty}\frac{\overline{F}(x+y)}{\overline{F}(x)} = e^{-y},$$

we conclude that $\overline{F}(x) \in \Gamma(g)$ with $g(x) = 1$. Substituting into the renewal approximation, we obtain

$$U(x) \sim \frac{x}{m_1} + \frac{m_e}{m_1} - \frac{1}{m_1^2}\overline{F}(x). \tag{3.13}$$

Consequently, the expected value of residual waiting time process satisfies

$$\begin{aligned}\mathbb{E}[W(t)] &\sim m_e + \frac{1}{m_1}\left(\frac{1}{e^{e^{-t}}} - \frac{e^{e^{-t}}}{e^{e^{-t}}}\right)\\ &\sim m_e + \frac{1}{m_1}\left(\frac{1-e^{e^{-t}}}{e^{e^{-t}}}\right).\end{aligned} \tag{3.14}$$

By expanding $e^{e^{-t}}$ in a Taylor series, it follows that

$$\mathbb{E}[W(t)] \to m_e, \qquad t \to \infty.$$

Case 3: Gamma type tail

Let the density function be

$$f(x) = \frac{x^{\beta-1}e^{-x}}{\Gamma(\beta)}, \qquad x \geq 0,\ \beta > 0. \tag{3.15}$$

The corresponding survival function is

$$\overline{F}(x) = \frac{\Gamma(\beta, x)}{\Gamma(\beta)}, \tag{3.16}$$

where $\Gamma(\beta, x)$ is the incomplete gamma function. Using the following asymptotic expansion (see for example [38]):

$$\Gamma(\beta, x) \sim x^{\beta-1} e^{-x}, \qquad x \to \infty,$$

it follows that

$$\lim_{x\to\infty} \frac{\overline{F}(x+y)}{\overline{F}(x)} = e^{-y},$$

and hence, $\overline{F}(x) \in \Gamma(1)$. Substituting into (3.8) gives

$$\mathbb{E}[W(t)] \sim m_e - \frac{1}{m_1} \cdot \frac{\Gamma(\beta, t)}{\Gamma(\beta)}. \tag{3.17}$$

Applying the approximation for $\Gamma(\beta, x)$, we obtain

$$\mathbb{E}[W(t)] \sim m_e - \frac{1}{m_1 \Gamma(\beta)} t^{\beta-1} e^{-t}. \tag{3.18}$$

Case 4: Logistic distribution

Finally, consider the logistic distribution with survival function

$$\overline{F}(x) = \frac{C}{1+e^x}, \qquad C > 0,\ x \in \mathbb{R}. \tag{3.19}$$

Since $\overline{F} \in \Gamma(1)$, the residual waiting time approximation becomes

$$\mathbb{E}[W(t)] \sim m_e - \frac{1}{m_1} \cdot \frac{C}{1+e^t}. \tag{3.20}$$

This completes the analysis for the logistic case.

In summary, the obtained results for distributions with $\overline{F} \in \Gamma(g)$ are collected in the following table. For each case, the equilibrium mean m_e is reported together with the intuitive approximation of the expected value of the residual waiting time process $\mathbb{E}[W(t)]$ as $t \to \infty$.

Table 6. Comparison of special distributions and the corresponding intuitive approximations for $\mathbb{E}[W(t)]$.

Distribution	$\overline{F}(x)$	$g(x)$	$\mathbb{E}[W(t)] \sim$
Weibull ($\alpha > 0,\ \lambda > 0$)	$e^{-\lambda x^\alpha}$	$\frac{x^{1-\alpha}}{\lambda}$	$m_e - \frac{1}{m_1} \frac{t^{2-2\alpha}}{\lambda^2} e^{-\lambda t^\alpha}$
Generalized EVD	$1 - e^{-e^{-x}}$	1	$m_e + \frac{1}{m_1} \left(\frac{1 - e^{e^{-t}}}{e^{e^{-t}}} \right)$
Gamma-type tail ($\beta > 0$)	$\Gamma(\beta, x)/\Gamma(\beta)$	1	$m_e - \frac{1}{m_1 \Gamma(\beta)} t^{\beta-1} e^{-t}$
Logistic	$C/(1+e^x)$	1	$m_e - \frac{1}{m_1} \frac{C}{1+e^t}$

Remark 3.4 (Comparison with the classical renewal approximation (light-tailed case))**.** *For non-arithmetic interarrival laws with $m_2 < \infty$, the classical statement asserts only*

$$U_{\mathrm{cl}}(t) = \frac{t}{m_1} + \frac{m_2}{2m_1^2} + o(1), \quad \mathbb{E}_{\mathrm{cl}}[W(t)] \; = \; m_e + o(1), \qquad m_e = \frac{m_2}{2m_1}.$$

In contrast, for $F \in \Gamma(g)$ our intuitive *approximation specifies the remainder and its sign:*

$$\boxed{\mathbb{E}[W(t)] \;\sim\; m_e \;-\; \frac{1}{m_1}\, g(t)^2\, \overline{F}(t), \qquad t \to \infty.}$$

Thus, in all light-tailed cases discussed below, $E[W(t)]$ approaches m_e at an exponentially (or faster) decaying rate. Moreover, since the correction term $-\frac{1}{m_1} g(t)^2 \overline{F}(t)$ is always negative, $E[W(t)]$ converges to m_e from below, *and the explicit form of the remainder yields sharper finite t accuracy than the unspecified classical $o(1)$.*

Weibull case. *If $\overline{F}(t) = e^{-\lambda t^{\alpha}}$ and $g(t) = t^{1-\alpha}/\lambda$, then*

$$\mathbb{E}[W(t)] \sim m_e - \frac{1}{m_1} \frac{t^{2-2\alpha}}{\lambda^2}\, e^{-\lambda t^{\alpha}}.$$

Because the factor $e^{-\lambda t^{\alpha}}$ decays super–exponentially when $\alpha > 1$, the difference $\mathbb{E}[W(t)] - m_e$ shrinks extremely rapidly, and the approximation becomes accurate even for relatively small values of t.

Generalized EVD case. *Here, $g(t) = 1$ and $\overline{F}(t) \sim e^{-t}$. Then*

$$\mathbb{E}[W(t)] \sim m_e - \frac{1}{m_1}\, \overline{F}(t).$$

Since the tail is exponentially small, the remainder term also decays exponentially, implying a rapid convergence of $\mathbb{E}[W(t)]$ to m_e.

Gamma-type tail. *If $\overline{F}(t) = \Gamma(\beta, t)/\Gamma(\beta)$ and $g(t) \equiv 1$, then*

$$\mathbb{E}[W(t)] \sim m_e - \frac{1}{m_1 \Gamma(\beta)}\, t^{\beta-1} e^{-t}.$$

The factor $t^{\beta-1}e^{-t}$ shows that the error decreases at an exponentially fast rate, with the shape parameter β determining the precise form of the decay.

Logistic case. *If $\overline{F}(t) = \dfrac{C}{1+e^t}$ and $g(t) \equiv 1$, then*

$$\mathbb{E}[W(t)] \sim m_e - \frac{1}{m_1} \frac{C}{1+e^t}.$$

Since $\frac{C}{1+e^t}$ behaves like Ce^{-t} for large t, the correction term again decays exponentially, giving a smooth and fast convergence to m_e.

As seen in the four cases above in detail, the classical result identifies only the limit m_e via $o(1)$, without rate or sign information. In contrast, intuitive approximation formulas provide (i) the sign of the error term (always negative in this setting), (ii) order of magnitude, given by $g(t)^2\overline{F}(t)$, and (iii) the distribution-specific shape of the error (e.g., $t^{\beta-1}e^{-t}$ for gamma/Erlang, $e^{-\lambda t^{\alpha}}$ for Weibull).

In order to provide more concrete illustrations of the approximation formula, we now compute explicit expressions for special parameter choices of the Weibull and gamma-type distributions. In particular, for the Weibull distribution with $\alpha = 2$ and $\lambda = 1$, the model reduces to the well-known Rayleigh distribution, while for the gamma-type distribution with $\beta = 2$, it corresponds to the Erlang distribution with shape parameter two and unit scale. The resulting equilibrium means and intuitive approximations for the expected value of the residual waiting time process are presented in Table 7.

Table 7. Special examples with concrete parameter choices: m_e and intuitive approximation for $\mathbb{E}[W(t)]$.

Distribution	**Parameters**	m_e	$\mathbb{E}[W(t)] \sim$
Weibull	$\alpha = 2,\ \lambda = 1$	$\frac{\Gamma(2)}{2\,\Gamma(1.5)} = \frac{1}{2} \cdot \frac{1}{\Gamma(1.5)} \approx 0.564$	$0.564 - \frac{e^{-t^2}}{\Gamma(1.5)\,t^2}$
Gamma-type tail	$\beta = 2$	$\frac{3}{2}$	$\frac{3}{2} - \frac{1}{2}\,t\,e^{-t}$

For the generalized extreme value (GEV) and logistic distributions, exact closed-form expressions for the equilibrium mean m_e could not be obtained in this setting. This is due to the fact that these distributions, as formulated here, are supported on the entire real line rather than $[0, \infty)$, which prevents a direct computation of the renewal-theoretic quantities without introducing a shift or truncation. Consequently, only the asymptotic forms of $\mathbb{E}[W(t)]$ can be reported for those cases.

3.3. Numerical calculations

In this section, our objective is to assess the accuracy of the intuitive approximation $\widetilde{\mathbb{E}}(W(t))$ relative to the exact expression $\mathbb{E}(W(t))$ for the expected value of the residual waiting time process. To this end, we focus on a special case in which the random variable X_n follows an Erlang distribution with parameters $(2, \lambda = 1)$. In this case, it is well known that

$$f(x) = xe^{-x}, \quad x > 0.$$

The exact formula for the renewal function generated by X_n can be derived as follows:

$$U(t) = \frac{t}{2} + \frac{3}{4} + \frac{1}{4}e^{-2t}. \tag{3.21}$$

Moreover, in the paper of Mitov and Omey [33], the following intuitive approximation formula is obtained for renewal function:

$$\widetilde{U}(t) \approx \frac{t}{2} + \frac{3}{4} + \frac{1}{2}e^{-t} - \frac{1}{4}\overline{F}(t). \tag{3.22}$$

On the other hand, $\overline{F}(x) \to xe^{-x}$, as $x \to \infty$. Hence, following results are found for $\mathbb{E}(W(t))$ and $\widetilde{\mathbb{E}}(W(t))$ by using Eqs (3.21) and (3.22) in Eq (1.2), respectively:

$$\mathbb{E}(W(t)) = \frac{3}{2} + \frac{1}{2}e^{-2t} \tag{3.23}$$

and

$$\widetilde{\mathbb{E}}(W(t)) \approx \frac{3}{2} - \frac{1}{2}te^{-t}. \tag{3.24}$$

Our aim is to compare the exact expression Eq (3.23) with the intuitive approximation given in Eq (3.24). For this purpose, we introduce the following error metrics:

$$\Delta = \left|\mathbb{E}(W(t)) - \widetilde{\mathbb{E}}(W(t))\right|, \qquad \delta = \frac{\Delta}{\mathbb{E}(W(t))} \times 100\%, \qquad AP = 100 - \delta.$$

Here, Δ, δ, and AP denote the absolute error, the relative error, and the accuracy percentage, respectively, between the exact mean and its approximation. The corresponding numerical results are summarized in Table 8.

Table 8. Numerical results of $\mathbb{E}(W(t))$, $\widetilde{\mathbb{E}}(W(t))$, Δ, $\delta(\%)$, and $AP(\%)$.

t	$\mathbb{E}(W(t))$	$\widetilde{\mathbb{E}}(W(t))$	Δ	$\delta(\%)$	$AP(\%)$
10	1.50000001	1.499773	0.000227000675	0.01513337862	99.98486662
9	1.500000008	1.499444656	0.000555351733	0.0370234487	99.96297655
8	1.500000056	1.498658149	0.00134190677	0.08946044859	99.91053955
7	1.500000416	1.496808413	0.00319200264	0.2128001173	99.78719988
6	1.500003072	1.492563743	0.00743932863	0.4959542267	99.50404577
5	1.5000227	1.483155133	0.01686756746	1.12448748	98.87551252
4	1.500167731	1.463368722	0.036799009	2.45292977	97.54707023
3	1.501239376	1.425319397	0.075919787	5.05715343	94.94284657
2	1.509157819	1.36464717	0.144493103	9.574419641	90.42558036
1	1.567667642	1.316060279	0.251670362	16.04978986	83.95021014

The results presented in Table 8 compare the exact values of $\mathbb{E}(W(t))$ obtained for the Erlang distribution with those derived from the intuitive approximation method $\widetilde{\mathbb{E}}(W(t))$. Although the intuitive approximation has been originally derived under the assumption of large t, it is remarkable that it also provides a good level of accuracy for values of t that are not large. In particular, for $t \geq 6$, the accuracy exceeds 99%, and the error margin becomes negligible. While the error rate is relatively higher for small t, it still remains within acceptable bounds and captures the short-term behavior of the process satisfactorily. This demonstrates that the intuitive approximation can be reliably applied not only in long-run asymptotic regimes but also in short- and medium-term settings. In summary, the findings of Table 8 confirm that although the method is theoretically derived for large t, it also yields good approximations for values of t that are not large, thereby establishing its role as a robust and practical alternative in both theoretical and applied contexts.

4. Conclusions

In this work, we developed intuitive approximation formulas for the expected value of the residual waiting time process $W(t)$, drawing upon intuitive approximation of the renewal function. Two distributional classes were examined: heavy-tailed interarrival times with regularly varying tails and light-tailed distributions belonging to the generalized class of $\Gamma(g)$. For both cases, explicit approximation formulas were derived, expressed in terms of equilibrium quantities such as the mean of the equilibrium distribution.

Applications to classical distributions, including Pareto, Burr XII, Weibull, gamma-type tails, GEV, and logistic distributions, illustrate the broad applicability of the proposed framework. A detailed case study on the Erlang distribution further confirm the high accuracy of the formulas, with numerical results showing agreement above 99% across a range of values of t. For the Pareto and Burr XII distributions, where no closed-form exact expressions are available, comparisons with asymptotic expansions demonstrate the robustness of the intuitive approach.

To conclude, and to provide context for future studies, the main motivation of the present work, its current limitations, and prospective research directions can be summarized as follows.

4.1. Motivation and limitations

The main motivation of this study is to develop a practically accurate framework for approximating the mean residual waiting time $\mathbb{E}[W(t)]$ in renewal processes when the renewal function $U(t)$ is not available in closed form. Classical asymptotic results describe only the limiting behavior, $\mathbb{E}[W(t)] = m_e + o(1)$, without providing quantitative information about the convergence rate or its dependence on the interarrival distribution. In many real-world applications such as reliability analysis, queueing systems, and inventory models, the time horizon t is moderate rather than asymptotically large, and interarrival distributions often exhibit heavy- or light-tailed behavior outside the scope of regular variation. These settings motivate the use of the intuitive approximation approach, which provides explicit, distribution-dependent correction terms and maintains high accuracy across both heavy- and light-tailed regimes.

A central contribution of this study lies in the observation that, although the proposed intuitive approximations are formally derived under the assumption of large values of t, they continue to yield reliable and accurate results even for small values of t, as confirmed by the results in Tables 2, 3, and 8. This finding emphasizes the practical strength of the method, extending its attantion beyond asymptotic regimes and making it suitable for a wider class of applications.

The scope of the present study is subject to a few natural restrictions. First, our results focus on the first moment $\mathbb{E}[W(t)]$ under the standard assumption of independent and identically distributed interarrival times with finite mean. Second, the approximation formulas are derived in an asymptotic setting; although the numerical results show good performance for moderate and even small values of t, a fully rigorous analysis of the small-t regime lies beyond the present scope. Third the numerical illustrations are based on representative heavy- and light-tailed families, and the treatment of more complex or hybrid interarrival structures is deferred to future work.

4.2. Future research

Future work may extend the intuitive approximation framework in several directions. One promising line is the derivation of analogous formulas for higher-order moments of the residual waiting time process or related functionals such as the cumulative reward process. Another potential extension involves relaxing the i.i.d. assumption to include semi-Markov, compound renewal, or renewal-reward systems with dependence between interarrivals and rewards. Furthermore, exploring numerical schemes or simulation-based corrections for non-stationary environments could bridge the gap between asymptotic theory and real-time applications. Such developments would enhance the theoretical depth and practical scope of the intuitive approximation method.

Author contributions

All authors contributed equally to the conception, methodology, analysis, and writing of this manuscript. All authors have read and approved the final version of the paper.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflict of interest.

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