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Research article

Coincidence among potential and axiomatic approaches for a weighted resolution

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Abstract: By simultaneously considering the maximal utilities across operational step vectors among constituents and weighting structures, we introduced the weighted max-value as a novel resolution concept under game-theoretical frameworks. The proposed formulation generalized traditional resolution by incorporating individual heterogeneity via weight distributions. To justify its theoretical soundness, we established a set of coincident relations that characterized the class of all resolutions admitting a potential function defined with weights. Additionally, we proposed the dividend approach as a dual perspective to the potential formulation and demonstrated the equivalence between these two through rigorous structural analysis. These coincident relations formed the basis for two axiomatic characterizations, effectiveness for multi-choice games combined with either weighted balanced contributions under multi-choice games or weighted path independence under multi-choice games, that uniquely identified the weighted max-value. The findings not only extended the theory of potential functions under game-theoretical settings but also offered a unifying framework for modeling weighted participatory behavior. These theoretical insights are applicable to real-world contexts involving differentiated participatory stakes, such as public finance and collaborative ventures.

Keywords: maximal-utility; weighted max-value; potential; dividend; axiomatic result
Mathematics Subject Classification: 91A06, 91B32

1. Introduction

In classical transferable-utility (TU) games, the characteristic mapping are always considered across possible coalitions formed by the constituents. Under this structure, each constituent is constrained to a binary mode of participation, either fully joining a coalition or remaining outside it. Nevertheless, such dichotomous behavior does not adequately capture the range of participatory intensities observed in real-world situations. To address this limitation, we adopt the setting of multi-choice TU games, which generalize the standard model by consenting each constituent to partake at distinct activity steps, thus accommodating a broader spectrum of involvement across coalitions. Hwang and Liao [7], Liao [8], and Nouweland et al. [11] considered distinct multi-choice generalizations and corresponding outcomes for the core, the EANSC, and the Shapley value, respectively, thereby presenting a more accurate and nuanced structure for investigating cooperation and fair distribution under diverse practical situations. Several multi-choice resolutions have also been proposed in Bednarczuk et al. [2], Guarini et al. [4], Mustakerov et al. [10], Wei et al. [16], and so on.

A vector field $\ddot{\mathbf{V}}$ is referred to as "conservative" if there exists a differentiable function $\vec{v}$ such that $\ddot{\mathbf{V}}$ coincides with the gradient of $\vec{v}$. In such a case, $\vec{v}$ is recognized as the potential function corresponding to $\ddot{\mathbf{V}}$. The concept of a conservative field implies that the total mechanical energy, combining kinetic and potential components, remains invariant for a particle traversing the field. The potential-based perspective has demonstrated wide applicability across theoretical domains. In particular, within game-theoretical settings, assigning a scalar potential to each economic configuration and deriving payoffs through marginal variations of this potential enables the interpretation of those marginal effects as the individual contributions of the constituents. Hart and Mas-Colell [6] introduced the potential approach to classical TU games. In consequence, they presented that the Shapley value [15] can be generalized as the marginal contributions vector due to a potential. Subsequent to the above result, Ortmann [12, 13] demonstrated an equivalent result due to a potential approach for a resolution, the axioms of balanced contributions and path independence. Calvo and Santos [3] presented that any resolution that admits a potential turns out to be the Shapley value [15] of an auxiliary game. In contrast to these classical potential-based characterizations of the (weighted) Shapley value, we introduce a resolution that is intrinsically tailored to multi-choice environments. Instead of aggregating marginal contributions along all coalition formation orders, the weighted max-value distributes the maximal-utilities generated by multi-choice step vectors and then scales the corresponding max-dividends by a weight distribution. Hence, the proposed resolution simultaneously internalizes **(i)** the multi-level activity of each constituent, encoded in the maximal-utility profile of the game, and **(ii)** heterogeneous influence or exposure, encoded in weight distributions. This structure fundamentally differs from the weighted Shapley value on a classical TU game, where weights distort marginal contributions in a binary participation setting and leads to a new form of potential-based representation adapted to multi-choice TU games.

In real-world collective decision environments, the assumption that constituents contribute identically or that their participatory intensities are uniform often fails to capture the intricacies of practical contexts. For instance, in public resource allocating mechanisms, constituents may engage with different levels of effort, priority, or entitlement, thereby necessitating the use of weight functions to reflect such heterogeneity. Under several environments, the notion of partial membership (e.g., via degrees of commitment or exposure to risk) has also been emphasized as a more realistic descriptor of coalition structures (Guarini et al. [4]). Similarly, in participatory investment situations such as crowdfunding models, the level of personal capital committed relative to an individual's total endowment represents a risk-weighted signal of involvement, as emphasized in Mohammed et al. [9]. These application domains suggest a strong need to develop resolution concepts that simultaneously incorporate both multi-choice operational steps and weighted participation differences.

Building on related streams of the research above, several motivations could be considered.

- By incorporating the possibility of graded participation, multi-choice models enable more realistic characterizations of agent involvement, which is essential in applications such as public budgeting, collaborative planning, and flexible task allocation. In particular, when such participation is further influenced by heterogeneous weights, reflecting risk exposure, capital commitment, or relative significance, the need arises for solution concepts that are sensitive to both multi-level activity and differential influence, which motivates the introduction of weighted formulations in multi-choice TU games.
- Furthermore, it is reasonable that a generalized potential framework can be developed within the setting of multi-choice TU games with weights, and further introduce the dividend approach as a dual interpretation. These developments not only extend the applicability of classical potential theory but also reveal deep coincident structures that underpin axiomatic characterizations of the proposed weighted resolution in this study.

Developments have further clarified the role of potentials and weighted allocations in cooperative games and related risk-sharing problems. For instance, Abe and Nakada [1] investigated how potential functions can be systematically related to solution concepts for TU games and provided new characterizations for such potential-based solutions. In parallel, several contributions in risk management and portfolio theory have applied (weighted) Shapley-type allocations to quantify heterogeneous risk exposures and capital contributions among agents. These studies underline the importance of combining structural properties of the underlying game with explicit weighting schemes. In this paper, we follow this line of research but extend it to a multi-choice environment where both maximal-utilities and weight distributions jointly determine the resulting resolution. Some more recent studies can be found in Hagan et al. [5], Shalit [14], and so on.

Different from other researchers, we propose novel outcomes under multi-choice TU games. Several principle results are as follows:

(1) In realistic environments, constituents may exhibit heterogeneous characteristics depending on the situational context. Accordingly, it is natural to incorporate weights to reflect the varying degrees of significance or influence attributed to each constituent under different conditions. In Section 2, we employ weight distributions to formulate the weighted max-value within multi-choice TU structures. The proposed resolution generalizes the unweighted resolution by incorporating the heterogeneity among constituents and provides a refined method to allocate the maximal gains under diverse engagement intensities. Accordingly, we construct a weighted multi-choice resolution as a mechanism that internalizes both structural and participatory heterogeneity under the game-theoretical framework.

(2) From the perspective of marginal contributions, we demonstrate that a unique weighted potential function exists, whose induced marginal payoff vector corresponds to the weighted max-value, as established in Section 3. Furthermore, adopting the viewpoint of cumulative marginal gains, we formulate the dividend approach with weights and examine its structural equivalence with the potential approach with weights. In particular, we present that any resolution admitting a potential with weights also admits a unique and effective dividend, and vice versa. This dual representation provides complementary interpretations for the weighted max-value analytically and axiomatically.

(3) To investigate the rationality of a resolution admitting a potential with weights, we first provide some coincident relations among the weighted multi-choice potential-formulation of a resolution, the weighted balanced contributions under multi-choice games and the weighted multi-choice path independence under multi-choice games in Section 4. Moreover, we utilize the concept of weighted potential to represent the weighted max-value within the setting of a measuration game. Building on these equivalence relations, two axiomatic formulations are established to support the logical foundation of the weighted max-value. These results establish that the weighted max-value is the unique resolution satisfying effectiveness for multi-choice games together with either of the axioms, weighted balanced contributions under multi-choice games, or weighted path independence under multi-choice games, thereby offering a robust axiomatic foundation for weighting mechanisms under multi-choice environments.

2. Preliminaries

Let $\mathbf{U\ddot{C}}$ be the universe of constituents. For $i \in \mathbf{U\ddot{C}}$ and $\vec{s}_i \in \mathbb{N}$, we set $\ddot{\mathbb{S}}_i = \{0, 1, \cdots, \vec{s}_i\}$ to be the operational step set of constituent i, where 0 means not partaking. For $\ddot{\mathbf{C}} \subseteq \mathbf{U\ddot{C}}$, let $\ddot{\mathbb{S}}^{\ddot{\mathbf{C}}} = \prod_{i \in \ddot{\mathbf{C}}} \ddot{\mathbb{S}}_i$ be the product set of the operational step sets for all constituents in $\ddot{\mathbf{C}}$, and $0_{\ddot{\mathbf{C}}}$ be the zero vector in $\ddot{\mathbb{S}}^{\ddot{\mathbf{C}}}$. Let $\ddot{\mathbf{C}} \subseteq \mathbf{U\ddot{C}}$, $\vec{\mu} \in \mathbb{R}^{\ddot{\mathbf{C}}}$, $\ddot{\mathbf{K}} \subseteq \ddot{\mathbf{C}}$ and $i \in \ddot{\mathbf{C}}$. Define that $\overline{NE}(\vec{\mu}) = \{i \in \ddot{\mathbf{C}} | \vec{\mu}_i \neq 0\}$, i.e., the set of total constituents using non-zero operational step. We also consider $\vec{\mu}_{\ddot{\mathbf{K}}}$ the restriction of $\vec{\mu}$ at $\ddot{\mathbf{K}}$, i.e., $\vec{\mu}_{\ddot{\mathbf{K}}} = (\vec{\mu}_i)_{i \in \ddot{\mathbf{K}}}$. Further, one can adopt $\vec{\mu}_{-i}$ to represent $\vec{\mu}_{\ddot{\mathbf{C}} \setminus \{i\}}$ and let $\vec{\iota} = (\vec{\mu}_{-i}, k)$ be defined by $\vec{\iota}_{-i} = \vec{\mu}_{-i}$ and $\vec{\iota}_i = k$. Besides, let $j \in \ddot{\mathbf{C}}$, $\vec{\mu}_{-ij}$ to represent $\vec{\mu}_{\ddot{\mathbf{C}} \setminus \{i,j\}}$ and $(\vec{\mu}_{-ij}, k, l)$ to represent $((\vec{\mu}_{-i}, k)_{-j}, l)$.

A multi-choice TU game is a triple $(\ddot{\mathbf{C}}, \vec{s}, \overline{U})$, where $\ddot{\mathbf{C}} \neq \emptyset$ is a finite set of constituents, $\vec{s} = (\vec{s}_i)_{i \in \ddot{\mathbf{C}}}$ represents the number of non-zero step for each constituent, and $\overline{U} : \ddot{\mathbb{S}}^{\ddot{\mathbf{C}}} \to \mathbb{R}$ is a function with $v(0_{\ddot{\mathbf{C}}}) = 0$, which renders to each $\vec{\mu} = (\vec{\mu}_i)_{i \in \ddot{\mathbf{C}}} \in \ddot{\mathbb{S}}^{\ddot{\mathbf{C}}}$ the merit that the constituents can generate if each constituent i partakes at step $\vec{\mu}_i$. Further, one could write $(\ddot{\mathbf{C}}, \vec{\mu}, \overline{U})$ for the multi-choice TU subgame considered by restricting $\overline{U}$ to $\{\vec{\xi} \in \ddot{\mathbb{S}}^{\ddot{\mathbf{C}}} \mid \vec{\xi}_i \leq \vec{\mu}_i \ \forall\, i \in \ddot{\mathbf{C}}\}$.

Denote the class of total multi-choice TU games as **GM**. A resolution on **GM** is a map ρ assigning to each $(\ddot{\mathbf{C}}, \vec{s}, \overline{U})$ a payoff vector

$$\rho(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) = \left(\rho_i(\ddot{\mathbf{C}}, \vec{s}, \overline{U})\right)_{i \in \ddot{\mathbf{C}}} \in \mathbb{R}^{\ddot{\mathbf{C}}}.$$

Here, $\rho_i(\ddot{\mathbf{C}}, \vec{s}, \overline{U})$ is the outcome of $i \in \ddot{\mathbf{C}}$ if i partakes under game $(\ddot{\mathbf{C}}, \vec{s}, \overline{U})$.

To present the main definition of this study, some more definitions and related results will be needed. For all $\ddot{\mathbf{H}} \subseteq \ddot{\mathbf{C}}$, $\overline{U}^m(\ddot{\mathbf{H}}) = \max_{\vec{\mu} \in \ddot{\mathbb{S}}^{\ddot{\mathbf{C}}}} \{\overline{U}(\vec{\mu}) | \overline{NE}(\vec{\mu}) = \ddot{\mathbf{H}}\}$ means the maximal-utility among total activity step vector with $\overline{NE}(\vec{\mu}) = \ddot{\mathbf{H}}$. We apply bounded multi-choice TU games, considered those games $(\ddot{\mathbf{C}}, \vec{s}, \overline{U})$, such that there exists $M_{\overline{U}} \in \mathbb{R}$, and such that $\overline{U}(\vec{\mu}) \leq M_{\overline{U}}$ for all $\vec{\mu} \in \ddot{\mathbb{S}}^{\ddot{\mathbf{C}}}$. We utilize it to assure that $\overline{U}^m(\ddot{\mathbf{H}})$ is well-defined. Let $\ddot{\mathbf{C}} \subseteq \mathbf{U\ddot{C}}$ and $d \in \ddot{\mathbb{S}}^{\ddot{\mathbf{C}}}$. The composition game for TU games, $(\ddot{\mathbf{C}}, \vec{s}, \overline{B}_{\ddot{\mathbf{H}}})$ with $\ddot{\mathbf{H}} \neq \emptyset$, is defined by

$$\overline{B}_{\ddot{\mathbf{H}}}(\ddot{\mathbf{K}}) = \begin{cases} 1 & \text{if } \ddot{\mathbf{H}} \subseteq \ddot{\mathbf{K}}, \\ 0 & \text{otherwise.} \end{cases}$$

Table 1 summarizes the major symbols used in this paper.

Table 1. Major notations used in this paper.

Symbol	Meaning / role in model
$\ddot{\mathbf{C}}$	Finite set of all constituents in multi-choice TU game
$\ddot{\mathbb{S}}_i$	Set of activity step vectors for constituent i
$\ddot{\mathbb{S}}^{\ddot{\mathbf{C}}}$	Set of activity step vectors for $\ddot{\mathbf{C}}$
$\overline{U}$	Characteristic mapping on $\ddot{\mathbb{S}}^{\ddot{\mathbf{C}}}$
$\overline{U}^m(\ddot{\mathbf{H}})$	Maximal-utility among step vectors with activity coalition $\ddot{\mathbf{H}}$
$\alpha_{\ddot{\mathbf{H}}}$	Max-dividend associated with coalition $\ddot{\mathbf{H}}$
κ	Weight distribution on $\ddot{\mathbf{C}}$
$\lvert\ddot{\mathbf{H}}\rvert_\kappa$	Weighted size of coalition $\ddot{\mathbf{H}}$ under κ
θ^κ	the weighted max-value resolution
$\overline{P}$	Potential function with weights
$\overline{E}$	Dividend function with weights
$\mathbf{M}_i^{\kappa}\overline{P}$	Weighted marginal effect of P for constituent i
$\mathbf{A}_i^{\kappa}\overline{E}$	Weighted dividend-based allocation for constituent i

In the following, it will be shown that each $(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) \in \mathbf{GM}$ can be constructed via composition games.

Lemma 2.1. *Let $\ddot{\mathbf{K}}$ be a finite set and $\ddot{\mathbf{R}} \subset \ddot{\mathbf{K}}$ with $\ddot{\mathbf{R}} \neq \ddot{\mathbf{K}}$. Then*

$$\sum_{\ddot{\mathbf{R}}\subseteq\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{K}}} (-1)^{|\ddot{\mathbf{H}}|-|\ddot{\mathbf{R}}|} = 0.$$

Proof. Let $\ddot{\mathbf{K}}$ be a finite set and $\ddot{\mathbf{R}} \subset \ddot{\mathbf{K}}$ with $\ddot{\mathbf{R}} \neq \ddot{\mathbf{K}}$.

$$\begin{aligned}
\sum_{\ddot{\mathbf{R}}\subseteq\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{K}}} (-1)^{|\ddot{\mathbf{H}}|-|\ddot{\mathbf{R}}|} &= \Big(\sum_{\substack{\ddot{\mathbf{R}}\subseteq\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{K}} \\ |\ddot{\mathbf{H}}|=|\ddot{\mathbf{R}}|}} + \sum_{\substack{\ddot{\mathbf{R}}\subseteq\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{K}} \\ |\ddot{\mathbf{H}}|=|\ddot{\mathbf{R}}|+1}} + \cdots + \sum_{\substack{\ddot{\mathbf{R}}\subseteq\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{K}} \\ |\ddot{\mathbf{H}}|=|\ddot{\mathbf{K}}|}} \Big)(-1)^{|\ddot{\mathbf{H}}|-|\ddot{\mathbf{R}}|} \\
&= \sum_{k=0}^{|\ddot{\mathbf{K}}|-|\ddot{\mathbf{R}}|} C_k^{|\ddot{\mathbf{K}}|-|\ddot{\mathbf{R}}|}(-1)^k \ \ (\textbf{let } |\ddot{\mathbf{H}}| - |\ddot{\mathbf{R}}| = k) \\
&= \sum_{k=0}^{|\ddot{\mathbf{K}}|-|\ddot{\mathbf{R}}|} C_k^{|\ddot{\mathbf{K}}|-|\ddot{\mathbf{R}}|}(-1)^k(1)^{|\ddot{\mathbf{K}}|-|\ddot{\mathbf{R}}|-k} \\
&= (-1+1)^{|\ddot{\mathbf{K}}|-|\ddot{\mathbf{R}}|} \\
&= 0. \ \ (\textbf{since } |\ddot{\mathbf{K}}| \neq |\ddot{\mathbf{R}}|).
\end{aligned}$$

□

Lemma 2.1 is a direct consequence of the binomial identity $(-1+1)^{|\ddot{\mathbf{K}}|-|\ddot{\mathbf{R}}|} = 0$ for $\ddot{\mathbf{R}} \neq \ddot{\mathbf{K}}$. The alternating sum over all supersets $\ddot{\mathbf{H}}$ of $\ddot{\mathbf{R}}$ simply encodes this identity on the lattice of coalitions.

The following lemma asserts that if a given game can be expressed as a composition (or linear combination) of elements from a prescribed collection $\{\overline{B}_{\ddot{\mathbf{H}}}\}_{\ddot{\mathbf{H}}\in 2^{\ddot{\mathbf{C}}}}$, then the associated coefficients are uniquely determined.

Lemma 2.2. *If for all $(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) \in \mathbf{GM}$, collection $\{\alpha_{\ddot{\mathbf{H}}}\}_{\ddot{\mathbf{H}}\in 2^{\ddot{\mathbf{C}}}}$ satisfies $\overline{U}^m = \sum_{\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{C}}} \alpha_{\ddot{\mathbf{H}}}\overline{B}_{\ddot{\mathbf{H}}}$, then for all $\ddot{\mathbf{H}} \in 2^{\ddot{\mathbf{C}}}$, $\alpha_{\ddot{\mathbf{H}}} = \sum_{\ddot{\mathbf{K}}\subseteq\ddot{\mathbf{H}}} (-1)^{|\ddot{\mathbf{H}}|-|\ddot{\mathbf{K}}|}\overline{U}^m(\ddot{\mathbf{K}})$.*

Proof. Assume that for all $(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) \in \mathbf{GM}$, collection $\{\alpha_{\ddot{\mathbf{H}}}\}_{\ddot{\mathbf{H}}\in 2^{\ddot{\mathbf{C}}}}$ satisfies $\overline{U}^m = \sum_{\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{C}}} \alpha_{\ddot{\mathbf{H}}}\overline{B}_{\ddot{\mathbf{H}}}$. Let $(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) \in \mathbf{GM}$. We complete this proof by induction on $|\ddot{\mathbf{H}}|$. Since

$$\overline{U}^m(\emptyset) = \sum_{\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{C}}} \alpha_{\ddot{\mathbf{H}}}\overline{B}_{\ddot{\mathbf{H}}}(\emptyset) = \sum_{\ddot{\mathbf{H}}=\emptyset} \alpha_{\ddot{\mathbf{H}}}\overline{B}_{\ddot{\mathbf{H}}}(\emptyset) + \sum_{\substack{\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{C}}\\ \ddot{\mathbf{H}}\neq\emptyset}} \alpha_{\ddot{\mathbf{H}}}\overline{B}_{\ddot{\mathbf{H}}}(\emptyset) = \alpha_\emptyset \cdot 1 + \sum_{\substack{\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{C}}\\ \ddot{\mathbf{H}}\neq\emptyset}} \alpha_{\ddot{\mathbf{H}}} \cdot 0 = \alpha_\emptyset,$$

it is clear that $\alpha_\emptyset = \overline{U}^m(\emptyset) = \sum_{\ddot{\mathbf{K}}\subseteq\emptyset} (-1)^{|\emptyset|-|\ddot{\mathbf{K}}|}\overline{U}^m(\ddot{\mathbf{K}})$, which shows that it holds if $|\ddot{\mathbf{H}}| = 0$. Suppose that it holds for all $\ddot{\mathbf{H}} \subseteq \ddot{\mathbf{C}}$ with $|\ddot{\mathbf{H}}| < t$ where $0 < t \leq |\ddot{\mathbf{C}}|$. Let $\ddot{\mathbf{K}} \subseteq \ddot{\mathbf{C}}$ with $|\ddot{\mathbf{K}}| = t$. Thus,

$$\overline{U}^m(\ddot{\mathbf{K}}) = \sum_{\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{C}}} \alpha_{\ddot{\mathbf{H}}}\overline{B}_{\ddot{\mathbf{H}}}(\ddot{\mathbf{K}}) = \sum_{\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{K}}} \alpha_{\ddot{\mathbf{H}}}\overline{B}_{\ddot{\mathbf{H}}}(\ddot{\mathbf{K}}) + \sum_{\ddot{\mathbf{H}}\not\subseteq\ddot{\mathbf{K}}} \alpha_{\ddot{\mathbf{H}}}\overline{B}_{\ddot{\mathbf{H}}}(\ddot{\mathbf{K}}) = \sum_{\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{K}}} \alpha_{\ddot{\mathbf{H}}} \cdot 1 + \sum_{\ddot{\mathbf{H}}\not\subseteq\ddot{\mathbf{K}}} \alpha_{\ddot{\mathbf{H}}} \cdot 0 = \sum_{\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{K}}} \alpha_{\ddot{\mathbf{H}}}.$$

Thus,

$$\alpha_{\ddot{\mathbf{K}}} = \sum_{\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{K}}} \alpha_{\ddot{\mathbf{H}}} - \sum_{\substack{\ddot{\mathbf{H}}\subset\ddot{\mathbf{K}}\\ \ddot{\mathbf{H}}\neq\ddot{\mathbf{K}}}} \alpha_{\ddot{\mathbf{H}}} = \overline{U}^m(\ddot{\mathbf{K}}) - \sum_{\substack{\ddot{\mathbf{H}}\subset\ddot{\mathbf{K}}\\ \ddot{\mathbf{H}}\neq\ddot{\mathbf{K}}}} \alpha_{\ddot{\mathbf{H}}}.$$

By induction hypothesis,

$$\begin{aligned}
\sum_{\substack{\ddot{\mathbf{H}}\subset\ddot{\mathbf{K}}\\ \ddot{\mathbf{H}}\neq\ddot{\mathbf{K}}}} \alpha_{\ddot{\mathbf{H}}} &= \sum_{\substack{\ddot{\mathbf{H}}\subset\ddot{\mathbf{K}}\\ \ddot{\mathbf{H}}\neq\ddot{\mathbf{K}}}} \sum_{\ddot{\mathbf{R}}\subseteq\ddot{\mathbf{H}}} (-1)^{|\ddot{\mathbf{H}}|-|\ddot{\mathbf{R}}|}\overline{U}^m(\ddot{\mathbf{R}}) \\
&= \sum_{\substack{\ddot{\mathbf{H}}\subset\ddot{\mathbf{K}}\\ \ddot{\mathbf{H}}\neq\ddot{\mathbf{K}}}} \sum_{\substack{\ddot{\mathbf{R}}\subset\ddot{\mathbf{K}}\\ \ddot{\mathbf{R}}\neq\ddot{\mathbf{K}}}} (-1)^{|\ddot{\mathbf{H}}|-|\ddot{\mathbf{R}}|}\overline{U}^m(\ddot{\mathbf{R}})\delta_{\ddot{\mathbf{R}}\ddot{\mathbf{H}}} \ \left(\textbf{where } \delta_{\ddot{\mathbf{R}}\ddot{\mathbf{H}}} = \begin{cases} 1 & \textbf{if } \ddot{\mathbf{R}} \subseteq \ddot{\mathbf{H}}, \\ 0 & \textbf{if } \ddot{\mathbf{R}} \not\subseteq \ddot{\mathbf{H}}. \end{cases}\right) \\
&= \sum_{\substack{\ddot{\mathbf{R}}\subset\ddot{\mathbf{K}}\\ \ddot{\mathbf{R}}\neq\ddot{\mathbf{K}}}} \sum_{\substack{\ddot{\mathbf{H}}\subset\ddot{\mathbf{K}}\\ \ddot{\mathbf{H}}\neq\ddot{\mathbf{K}}}} (-1)^{|\ddot{\mathbf{H}}|-|\ddot{\mathbf{R}}|}\overline{U}^m(\ddot{\mathbf{R}})\delta_{\ddot{\mathbf{R}}\ddot{\mathbf{H}}} \\
&= \sum_{\substack{\ddot{\mathbf{R}}\subset\ddot{\mathbf{K}}\\ \ddot{\mathbf{R}}\neq\ddot{\mathbf{K}}}} \Big(\sum_{\substack{\ddot{\mathbf{H}}\subset\ddot{\mathbf{K}}\\ \ddot{\mathbf{H}}\neq\ddot{\mathbf{K}}}} (-1)^{|\ddot{\mathbf{H}}|-|\ddot{\mathbf{R}}|}\delta_{\ddot{\mathbf{R}}\ddot{\mathbf{H}}} \Big)\overline{U}^m(\ddot{\mathbf{R}}) \\
&= \sum_{\substack{\ddot{\mathbf{R}}\subset\ddot{\mathbf{K}}\\ \ddot{\mathbf{R}}\neq\ddot{\mathbf{K}}}} \Big(\sum_{\substack{\ddot{\mathbf{R}}\subseteq\ddot{\mathbf{H}}\subset\ddot{\mathbf{K}}\\ \ddot{\mathbf{H}}\neq\ddot{\mathbf{K}}}} (-1)^{|\ddot{\mathbf{H}}|-|\ddot{\mathbf{R}}|} \Big)\overline{U}^m(\ddot{\mathbf{R}}) \\
&= \sum_{\substack{\ddot{\mathbf{R}}\subset\ddot{\mathbf{K}}\\ \ddot{\mathbf{R}}\neq\ddot{\mathbf{K}}}} \Big(-(-1)^{|\ddot{\mathbf{K}}|-|\ddot{\mathbf{R}}|} + \sum_{\ddot{\mathbf{R}}\subseteq\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{K}}} (-1)^{|\ddot{\mathbf{H}}|-|\ddot{\mathbf{R}}|} \Big)\overline{U}^m(\ddot{\mathbf{R}}).
\end{aligned}$$

Therefore, by applying Lemma 2.1,

$$\sum_{\substack{\ddot{\mathbf{H}}\subset\ddot{\mathbf{K}}\\ \ddot{\mathbf{H}}\neq\ddot{\mathbf{K}}}} \alpha_{\ddot{\mathbf{H}}} = -\sum_{\substack{\ddot{\mathbf{R}}\subset\ddot{\mathbf{K}}\\ \ddot{\mathbf{R}}\neq\ddot{\mathbf{K}}}} (-1)^{|\ddot{\mathbf{K}}|-|\ddot{\mathbf{R}}|}\overline{U}^m(\ddot{\mathbf{R}}).$$

Hence,

$$\alpha_{\ddot{\mathbf{K}}} = \overline{U}^m(\ddot{\mathbf{K}}) - \Big(-\sum_{\substack{\ddot{\mathbf{R}}\subset\ddot{\mathbf{K}}\\ \ddot{\mathbf{R}}\neq\ddot{\mathbf{K}}}} (-1)^{|\ddot{\mathbf{K}}|-|\ddot{\mathbf{R}}|}\overline{U}^m(\ddot{\mathbf{R}}) \Big) = \sum_{\ddot{\mathbf{R}}\subseteq\ddot{\mathbf{K}}} (-1)^{|\ddot{\mathbf{K}}|-|\ddot{\mathbf{R}}|}\overline{U}^m(\ddot{\mathbf{R}}).$$

The proof is completed. □

The following lemma states that for every game in the system, there exists a collection of associated coefficients such that the game can be constructed as a composition (or linear combination) of elements from the collection $\{\overline{B}_{\ddot{\mathbf{H}}}\}_{\ddot{\mathbf{H}}\in 2^{\ddot{\mathbf{C}}}}$.

Lemma 2.3. *If* $\alpha_{\ddot{\mathbf{H}}} = \sum_{\ddot{\mathbf{K}}\subseteq\ddot{\mathbf{H}}}(-1)^{|\ddot{\mathbf{H}}|-|\ddot{\mathbf{K}}|}\overline{U}^m(\ddot{\mathbf{K}})$ *for all* $(\ddot{\mathbf{C}},\vec{s},\overline{U})\in\mathbf{GM}$ *and for all* $\ddot{\mathbf{H}}\in 2^{\ddot{\mathbf{C}}}$*, then* $\overline{U}^m = \sum_{\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{C}}}\alpha_{\ddot{\mathbf{H}}}\overline{B}_{\ddot{\mathbf{H}}}$.

Proof. Assume that for all $(\ddot{\mathbf{C}},\vec{s},\overline{U})\in\mathbf{GM}$ and for all $\ddot{\mathbf{H}}\in 2^{\ddot{\mathbf{C}}}$, $\alpha_{\ddot{\mathbf{H}}} = \sum_{\ddot{\mathbf{K}}\subseteq\ddot{\mathbf{H}}}(-1)^{|\ddot{\mathbf{H}}|-|\ddot{\mathbf{K}}|}\overline{U}^m(\ddot{\mathbf{K}})$. We claim that $\overline{U}^m(\ddot{\mathbf{R}}) = \sum_{\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{C}}}\alpha_{\ddot{\mathbf{H}}}\overline{B}_{\ddot{\mathbf{H}}}(\ddot{\mathbf{R}})$ for all $\ddot{\mathbf{R}}\subseteq\ddot{\mathbf{C}}$. Let $\ddot{\mathbf{R}}=\emptyset$. Clearly,

$$\alpha_{\emptyset} = \sum_{\ddot{\mathbf{K}}\subseteq\emptyset}(-1)^{|\emptyset|-|\ddot{\mathbf{K}}|}\overline{U}^m(\ddot{\mathbf{K}}) = \overline{U}^m(\emptyset),$$

so that

$$\sum_{\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{C}}}\alpha_{\ddot{\mathbf{H}}}\overline{B}_{\ddot{\mathbf{H}}}(\emptyset) = \overline{U}^m(\emptyset),$$

and it holds if $\ddot{\mathbf{R}}=\emptyset$. Assume that $\ddot{\mathbf{R}}\neq\emptyset$. Similar to the previous lemma,

$$\begin{aligned}
\sum_{\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{C}}}\alpha_{\ddot{\mathbf{H}}}\overline{B}_{\ddot{\mathbf{H}}}(\ddot{\mathbf{R}}) &= \sum_{\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{R}}}\alpha_{\ddot{\mathbf{H}}}\\
&= \sum_{\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{R}}}\sum_{\ddot{\mathbf{K}}\subseteq\ddot{\mathbf{H}}}(-1)^{|\ddot{\mathbf{H}}|-|\ddot{\mathbf{K}}|}\overline{U}^m(\ddot{\mathbf{K}})\\
&= \sum_{\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{R}}}\sum_{\ddot{\mathbf{K}}\subseteq\ddot{\mathbf{R}}}(-1)^{|\ddot{\mathbf{H}}|-|\ddot{\mathbf{K}}|}\overline{U}^m(\ddot{\mathbf{K}})\delta_{\ddot{\mathbf{K}}\ddot{\mathbf{H}}} \quad \textbf{(by definition of } \delta_{\ddot{\mathbf{K}}\ddot{\mathbf{H}}}\textbf{)}\\
&= \sum_{\ddot{\mathbf{K}}\subseteq\ddot{\mathbf{R}}}\sum_{\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{R}}}(-1)^{|\ddot{\mathbf{H}}|-|\ddot{\mathbf{K}}|}\overline{U}^m(\ddot{\mathbf{K}})\delta_{\ddot{\mathbf{K}}\ddot{\mathbf{H}}}\\
&= \sum_{\ddot{\mathbf{K}}\subseteq\ddot{\mathbf{R}}}\Big(\sum_{\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{R}}}(-1)^{|\ddot{\mathbf{H}}|-|\ddot{\mathbf{K}}|}\delta_{\ddot{\mathbf{K}}\ddot{\mathbf{H}}}\Big)\overline{U}^m(\ddot{\mathbf{K}})\\
&= \sum_{\ddot{\mathbf{K}}\subseteq\ddot{\mathbf{R}}}\Big(\sum_{\ddot{\mathbf{K}}\subseteq\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{R}}}(-1)^{|\ddot{\mathbf{H}}|-|\ddot{\mathbf{K}}|}\Big)\overline{U}^m(\ddot{\mathbf{K}})\\
&= \Big(\sum_{\ddot{\mathbf{K}}=\ddot{\mathbf{R}}}+\sum_{\ddot{\mathbf{K}}\subseteq\ddot{\mathbf{R}}}\Big)\Big(\sum_{\ddot{\mathbf{K}}\subseteq\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{R}}}(-1)^{|\ddot{\mathbf{H}}|-|\ddot{\mathbf{K}}|}\Big)\overline{U}^m(\ddot{\mathbf{K}})\\
&= (-1)^{|\ddot{\mathbf{R}}|-|\ddot{\mathbf{R}}|}\overline{U}^m(\ddot{\mathbf{R}}) + \sum_{\substack{\ddot{\mathbf{K}}\subset\ddot{\mathbf{R}}\\ \ddot{\mathbf{K}}\neq\ddot{\mathbf{R}}}}\Big(\sum_{\ddot{\mathbf{K}}\subseteq\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{R}}}(-1)^{|\ddot{\mathbf{H}}|-|\ddot{\mathbf{K}}|}\Big)\overline{U}^m(\ddot{\mathbf{K}})\\
&= \overline{U}^m(\ddot{\mathbf{R}}). \quad \textbf{(by Lemma 2.1)}
\end{aligned}$$

The proof is completed. □

Remark 1. *Based on the two lemmas,* $\overline{U}^m = \sum_{\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{C}}}\alpha_{\ddot{\mathbf{H}}}\overline{B}_{\ddot{\mathbf{H}}}$ *with* $\alpha_{\ddot{\mathbf{H}}} = \sum_{\ddot{\mathbf{K}}\subseteq\ddot{\mathbf{H}}}(-1)^{|\ddot{\mathbf{H}}|-|\ddot{\mathbf{K}}|}\overline{U}^m(\ddot{\mathbf{K}})$ *for all* $\ddot{\mathbf{H}}\in 2^{\ddot{\mathbf{C}}}$. *From the mathematical expression for* $\alpha_{\ddot{\mathbf{H}}}$, $\alpha_{\ddot{\mathbf{H}}}$*represents the maximal effect when coalition* $\ddot{\mathbf{H}}$ *is formed, which is referred to as the max-dividend.*

As introduced in the Introduction, the use of weights reflects the heterogeneous characteristics of participants in multi-choice TU games, capturing variations in capacity, risk attitudes, or levels of involvement. These weights play a crucial role in tailoring resolutions to the profiles of each constituent. $\overline{\kappa}:\ddot{\mathbf{U}}\mathbf{C}\to\mathbb{R}^+$ is a weight distribution if it is a positive function. Given $(\ddot{\mathbf{C}},\vec{s},\overline{U})\in\mathbf{GM}$ and a weight distribution $\overline{\kappa}$, we define $|\ddot{\mathbf{H}}|_{\overline{\kappa}} = \sum_{i\in\ddot{\mathbf{H}}}\overline{\kappa}(i)$ for each $\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{C}}$. By allocating max-dividends with weight distributions, a different multi-choice resolution would be introduced as follows.

Definition 2.1. *The weighted max-value, denoted by $\theta^{\overline{\kappa}}$, is defined as for each $(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) \in \mathbf{GM}$, for each weight distribution $\overline{\kappa}$, and for each constituents $i \in \ddot{\mathbf{C}}$,*

$$\theta_i^{\overline{\kappa}}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) = \sum_{\substack{\ddot{\mathbf{H}} \subseteq \ddot{\mathbf{C}} \\ i \in \ddot{\mathbf{H}}}} \frac{\overline{\kappa}(i) \cdot \alpha_{\ddot{\mathbf{H}}}}{|\ddot{\mathbf{H}}|_{\overline{\kappa}}} .$$

The resolution $\theta^{\overline{\kappa}}$ represents that what each constituent receives is a weighted proportional distribution of relative max-dividend, collected from all coalitions they have participated in. W.L.O.G., one could consider that $\overline{NE}(\vec{s}) = \ddot{\mathbf{C}}$ for each $(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) \in \mathbf{GM}$. The weighted max-value provides a mechanism to allocate collective value in a way that reflects the maximal utilities and the personalized impact of each constituent's engagement. It directly connects with the motivation discussed in Introduction.

In the framework of multi-choice TU games, the notion of the following axiom plays a foundational role, ensuring that the resolution fully accounts for the total maximal utility achieved by the grand coalition under the considered multi-choice configurations. A resolution ρ satisfies effectiveness for multi-choice games (ETSMG) if $\sum_{i \in \ddot{\mathbf{C}}} \rho_i(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) = \overline{U}^m(\ddot{\mathbf{C}})$ for all $(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) \in \mathbf{GM}$. The significance of ETSMG embodies collective efficiency, guaranteeing that the entire value generated through multi-choice participation is allocated without artificial loss or surplus, thus preserving the integrity of cooperative decision-making. This ensures that all potential synergies, gains, or costs arising from multi-choice participation are captured and redistributed among the constituents.

Lemma 2.4. *The weighted max-value satisfies ETSMG.*

Proof. Let $(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) \in \mathbf{GM}$ and $\overline{\kappa}$ be a weight distribution. By Definition 2.1,

$$\begin{aligned}
\theta_i^{\overline{\kappa}}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) &= \sum_{\substack{\ddot{\mathbf{H}} \subseteq \ddot{\mathbf{C}} \\ i \in \ddot{\mathbf{H}}}} \tfrac{\overline{\kappa}(i) \cdot \alpha_{\ddot{\mathbf{H}}}}{|\ddot{\mathbf{H}}|_{\overline{\kappa}}} \\
&= \sum_{\substack{\ddot{\mathbf{H}} \subseteq \ddot{\mathbf{C}} \\ i \in \ddot{\mathbf{H}}}} \tfrac{\overline{\kappa}(i)}{|\ddot{\mathbf{H}}|_{\overline{\kappa}}} \sum_{\ddot{\mathbf{K}} \subseteq \ddot{\mathbf{H}}} (-1)^{|\ddot{\mathbf{H}}|-|\ddot{\mathbf{K}}|} \overline{U}^m(\ddot{\mathbf{K}}) \\
&= \sum_{\substack{\ddot{\mathbf{H}} \subseteq \ddot{\mathbf{C}} \\ i \in \ddot{\mathbf{H}}}} \tfrac{\overline{\kappa}(i)}{|\ddot{\mathbf{H}}|_{\overline{\kappa}}} \Big[\sum_{\substack{\ddot{\mathbf{K}} \subseteq \ddot{\mathbf{H}} \\ i \in \ddot{\mathbf{K}}}} (-1)^{|\ddot{\mathbf{H}}|-|\ddot{\mathbf{K}}|} \overline{U}^m(\ddot{\mathbf{K}}) + \sum_{\substack{\ddot{\mathbf{K}} \subseteq \ddot{\mathbf{H}} \\ i \notin \ddot{\mathbf{K}}}} (-1)^{|\ddot{\mathbf{H}}|-|\ddot{\mathbf{K}}|} \overline{U}^m(\ddot{\mathbf{K}}) \Big] \\
&= \sum_{\substack{\ddot{\mathbf{H}} \subseteq \ddot{\mathbf{C}} \\ i \in \ddot{\mathbf{H}}}} \tfrac{\overline{\kappa}(i)}{|\ddot{\mathbf{H}}|_{\overline{\kappa}}} \Big[\sum_{\substack{\ddot{\mathbf{K}} \subseteq \ddot{\mathbf{H}} \\ i \in \ddot{\mathbf{K}}}} (-1)^{|\ddot{\mathbf{H}}|-|\ddot{\mathbf{K}}|} \overline{U}^m(\ddot{\mathbf{K}}) + \sum_{\substack{\ddot{\mathbf{K}} \subseteq \ddot{\mathbf{H}} \\ i \in \ddot{\mathbf{K}}}} (-1)^{|\ddot{\mathbf{H}}|-|\ddot{\mathbf{K}}|-1} \overline{U}^m(\ddot{\mathbf{K}} \setminus \{i\}) \Big] \\
&= \sum_{\substack{\ddot{\mathbf{H}} \subseteq \ddot{\mathbf{C}} \\ i \in \ddot{\mathbf{H}}}} \tfrac{\overline{\kappa}(i)}{|\ddot{\mathbf{H}}|_{\overline{\kappa}}} \Big[\sum_{\substack{\ddot{\mathbf{K}} \subseteq \ddot{\mathbf{H}} \\ i \in \ddot{\mathbf{K}}}} (-1)^{|\ddot{\mathbf{H}}|-|\ddot{\mathbf{K}}|} \overline{U}^m(\ddot{\mathbf{K}}) - \sum_{\substack{\ddot{\mathbf{K}} \subseteq \ddot{\mathbf{H}} \\ i \in \ddot{\mathbf{K}}}} (-1)^{|\ddot{\mathbf{H}}|-|\ddot{\mathbf{K}}|} \overline{U}^m(\ddot{\mathbf{K}} \setminus \{i\}) \Big] \\
&= \sum_{\substack{\ddot{\mathbf{H}} \subseteq \ddot{\mathbf{C}} \\ i \in \ddot{\mathbf{H}}}} \tfrac{\overline{\kappa}(i)}{|\ddot{\mathbf{H}}|_{\overline{\kappa}}} \sum_{\substack{\ddot{\mathbf{K}} \subseteq \ddot{\mathbf{H}} \\ i \in \ddot{\mathbf{K}}}} (-1)^{|\ddot{\mathbf{H}}|-|\ddot{\mathbf{K}}|} [\overline{U}^m(\ddot{\mathbf{K}}) - \overline{U}^m(\ddot{\mathbf{K}} \setminus \{i\})].
\end{aligned} \tag{2.1}$$

Let $\ddot{\mathbf{K}} \subseteq \ddot{\mathbf{C}}$ be fixed. For each $k \in \ddot{\mathbf{C}}$,

(1) Let $\ddot{\mathbf{K}} \subseteq \ddot{\mathbf{H}} \subseteq \ddot{\mathbf{C}}$, $\ddot{\mathbf{K}} \neq \ddot{\mathbf{C}}$ and $k \in \ddot{\mathbf{H}}$. Related coefficient of $\overline{U}^m(\ddot{\mathbf{K}})$ in (2.1) is

$$\sum_{\substack{\ddot{\mathbf{H}} \subseteq \ddot{\mathbf{C}} \\ \ddot{\mathbf{K}} \subseteq \ddot{\mathbf{H}}}} \frac{\overline{\kappa}(k)}{|\ddot{\mathbf{H}}|_{\overline{\kappa}}} (-1)^{|\ddot{\mathbf{H}}|-|\ddot{\mathbf{K}}|} - \sum_{\substack{\ddot{\mathbf{H}} \subseteq \ddot{\mathbf{C}} \\ (\ddot{\mathbf{K}} \cup \{k\}) \subseteq \ddot{\mathbf{H}}}} \frac{\overline{\kappa}(k)}{|\ddot{\mathbf{H}}|_{\overline{\kappa}}} (-1)^{|\ddot{\mathbf{H}}|-|\ddot{\mathbf{K}} \cup \{k\}|}.$$

So,

$$\begin{aligned}
&\sum_{k\in\ddot{\mathbf{H}}}\sum_{\substack{\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{C}}\\ \ddot{\mathbf{K}}\subseteq\ddot{\mathbf{H}}}}\frac{\overline{\kappa}(k)}{|\ddot{\mathbf{H}}|_{\overline{\kappa}}}(-1)^{|\ddot{\mathbf{H}}|-|\ddot{\mathbf{K}}|}\\
=&\sum_{\substack{\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{C}}\\ \ddot{\mathbf{K}}\subseteq\ddot{\mathbf{H}}}}\frac{|\ddot{\mathbf{H}}|_{\overline{\kappa}}}{|\ddot{\mathbf{H}}|_{\overline{\kappa}}}(-1)^{|\ddot{\mathbf{H}}|-|\ddot{\mathbf{K}}|}\\
=&\sum_{\substack{\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{C}}\\ \ddot{\mathbf{K}}\subseteq\ddot{\mathbf{H}}}}(-1)^{|\ddot{\mathbf{H}}|-|\ddot{\mathbf{K}}|}\\
=&\Big[\sum_{\substack{\ddot{\mathbf{K}}\subseteq\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{C}}\\ |\ddot{\mathbf{H}}|-|\ddot{\mathbf{K}}|=0}}+\sum_{\substack{\ddot{\mathbf{K}}\subseteq\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{C}}\\ |\ddot{\mathbf{H}}|-|\ddot{\mathbf{K}}|=1}}+\cdots+\sum_{\substack{\ddot{\mathbf{K}}\subseteq\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{C}}\\ |\ddot{\mathbf{H}}|-|\ddot{\mathbf{K}}|=|\ddot{\mathbf{C}}|-|\ddot{\mathbf{K}}|}}\Big](-1)^{|\ddot{\mathbf{H}}|-|\ddot{\mathbf{K}}|}\\
=&\sum_{t=0}^{|\ddot{\mathbf{C}}|-|\ddot{\mathbf{K}}|}C_t^{|\ddot{\mathbf{C}}|-|\ddot{\mathbf{K}}|}(-1)^t\ \ \textbf{(Let } t=|\ddot{\mathbf{H}}|-|\ddot{\mathbf{K}}|\textbf{)}\\
=&\sum_{t=0}^{|\ddot{\mathbf{C}}|-|\ddot{\mathbf{K}}|}C_t^{|\ddot{\mathbf{C}}|-|\ddot{\mathbf{K}}|}(-1)^t(1)^{|\ddot{\mathbf{C}}|-|\ddot{\mathbf{K}}|-t}\\
=&(-1+1)^{|\ddot{\mathbf{C}}|-|\ddot{\mathbf{K}}|}\\
=&0.\ \ \textbf{(Since } \ddot{\mathbf{K}}\neq\ddot{\mathbf{C}}\textbf{)}.
\end{aligned}$$

Similarly, $\sum_{k\in\ddot{\mathbf{H}}}\sum_{\substack{\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{C}}\\ (\ddot{\mathbf{K}}\cup\{k\})\subseteq\ddot{\mathbf{H}}}}\frac{\overline{\kappa}(k)}{|\ddot{\mathbf{H}}|_{\overline{\kappa}}}(-1)^{|\ddot{\mathbf{H}}|-|\ddot{\mathbf{K}}\cup\{k\}|}=0.$

(2) Let $\ddot{\mathbf{K}}\subseteq\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{C}}$, $\ddot{\mathbf{K}}\neq\ddot{\mathbf{C}}$ and $k\notin\ddot{\mathbf{H}}$. Clearly, related coefficient of $\overline{U}^m(\ddot{\mathbf{K}})$ in (2.1) is 0.

(3) Let $\ddot{\mathbf{K}}=\ddot{\mathbf{C}}$. Related coefficient of $\overline{U}^m(\ddot{\mathbf{K}})$ in (2.1) is $\frac{\overline{\kappa}(k)}{|\ddot{\mathbf{C}}|_{\overline{\kappa}}}(-1)^{|\ddot{\mathbf{C}}|-|\ddot{\mathbf{C}}|}$. Thus,

$$\sum_{k\in\ddot{\mathbf{C}}}\frac{\overline{\kappa}(k)}{|\ddot{\mathbf{C}}|_{\overline{\kappa}}}(-1)^{|\ddot{\mathbf{C}}|-|\ddot{\mathbf{C}}|}=\frac{|\ddot{\mathbf{C}}|_{\overline{\kappa}}}{|\ddot{\mathbf{C}}|_{\overline{\kappa}}}(-1)^{|\ddot{\mathbf{C}}|-|\ddot{\mathbf{C}}|}=1.$$

Thus, we have that the coefficient $a_{\ddot{\mathbf{K}}}$ of $\overline{U}^m(\ddot{\mathbf{K}})$ in $\sum_{k\in\ddot{\mathbf{C}}}\theta_k^{\overline{\kappa}}(\ddot{\mathbf{C}},\vec{s},\overline{U})$ is

$$a_{\ddot{\mathbf{K}}}\ =\begin{cases}0 & \text{if } 0<|\ddot{\mathbf{K}}|<|\ddot{\mathbf{C}}|,\\ 1 & \text{if } |\ddot{\mathbf{K}}|=|\ddot{\mathbf{C}}|.\end{cases}$$

Therefore,

$$\sum_{k\in\ddot{\mathbf{C}}}\theta_k^{\overline{\kappa}}(\ddot{\mathbf{C}},\vec{s},\overline{U})=\sum_{\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{C}}}a_{\ddot{\mathbf{H}}}\cdot\overline{U}^m(\ddot{\mathbf{H}})=\overline{U}^m(\ddot{\mathbf{C}}).$$

Hence, the weighted max-value $\theta^{\overline{\kappa}}$ satisfies ETSMG. □

3. Potential approach and dividend approach

By integrating related works due to Hart and Mas-Colell [6], Ortmann [12, 13], and Calvo and Santos [3] under multi-choice structures, the potential approach is generalized in this section by simultaneously utilizing the maximal-utilities and the weight distributions.

Let $\overline{P}:\mathbf{GM}\longrightarrow\mathbb{R}$ be a function which appoints $\overline{P}(\ddot{\mathbf{C}},\vec{s},\overline{U})\in\mathbb{R}$ to each $(\ddot{\mathbf{C}},\vec{s},\overline{U})\in\mathbf{GM}$. For every weight distribution $\overline{\kappa}$ and for every $i\in\ddot{\mathbf{C}}$, one would define

$$\mathbf{M}_i^{\overline{\kappa}}\overline{P}(\ddot{\mathbf{C}},\vec{s},\overline{U})=\overline{\kappa}(i)\cdot[\overline{P}(\ddot{\mathbf{C}},\vec{s},\overline{U})-\overline{P}(\ddot{\mathbf{C}},(\vec{s}_{\ddot{\mathbf{C}}\setminus\{i\}},0),\overline{U})].$$

Definition 3.1. *A resolution* ρ *on* **GM** *admits a potential with weights if there exists a function* $\overline{P}$: $\mathbf{GM} \to \mathbb{R}$ *such that for every* $(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) \in \mathbf{GM}$*, for every weight distribution* $\overline{\kappa}$ *and for every* $i \in \ddot{\mathbf{C}}$*,*

$$\rho_i(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) = \mathbf{M}_i^{\overline{\kappa}} \overline{P}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}).$$

In the presence of a weighted potential, each resolution allocates a scalar value to the game, ensuring that every constituent's payoff corresponds to its marginal contribution weighted by the assigned distribution. A mapping $\overline{P}$: $\mathbf{GM} \longrightarrow \mathbb{R}$ *is 0-normalized if it satisfies* $\overline{P}(\emptyset, \vec{s}, \overline{U}) = 0$*. Moreover,* $\overline{P}$ *is termed effective when, for any given weight configuration* $\overline{\kappa}$ *and for all* $(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) \in \mathbf{GM}$*, the following holds*

$$\sum_{i \in \ddot{\mathbf{C}}} \mathbf{M}_i^{\overline{\kappa}} \overline{P}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) = \overline{U}^m(\ddot{\mathbf{C}}). \tag{3.1}$$

The function $\overline{P}^0(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) = \overline{P}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) - \overline{P}(\ddot{\mathbf{C}}, 0_{\ddot{\mathbf{C}}}, \overline{U})$ is 0-normalized absolutely if $\overline{P}$ is a potential. That is, the existence of a 0-normalized potential and a potential should be simultaneous. In addition, a resolution ρ admits one 0-normalized potential at most.

Theorem 3.1. *There exists a uniquely 0-normalized and effective potential* $\overline{P}$ *such that for every* $(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) \in \mathbf{GM}$*, for every weight distribution* $\overline{\kappa}$ *and for every* $i \in \ddot{\mathbf{C}}$*,*

$$\theta_i^{\overline{\kappa}}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) = \mathbf{M}_i^{\overline{\kappa}} \overline{P}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}).$$

Proof. The proof proceeds by recursively relying on the decomposition of each multi-choice TU game into a finite combination of composition games. The combinatorial identities in Lemmas 2.1–2.3 guarantee that the resulting potential is uniquely determined by the maximal-utility profile of the game. Let $(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) \in \mathbf{GM}$ and $\overline{\kappa}$ be a weight distribution. Thus (3.1) can be reformulated as

$$\overline{P}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) = \frac{1}{|\ddot{\mathbf{C}}|_{\overline{\kappa}}} \cdot [\overline{U}^m(\ddot{\mathbf{C}}) + \sum_{i \in \ddot{\mathbf{C}}} \overline{\kappa}(i) \cdot \overline{P}(\ddot{\mathbf{C}}, (\vec{s}_{\ddot{\mathbf{C}} \setminus \{i\}}, 0), \overline{U})]. \tag{3.2}$$

Starting with $\overline{P}(\ddot{\mathbf{C}}, 0_{\ddot{\mathbf{C}}}, \overline{U}) = 0$, it resolves $\overline{P}(\ddot{\mathbf{C}}, \vec{s}, \overline{U})$ recursively. This affirms the existence of $\overline{P}$, and moreover that $\overline{P}(\ddot{\mathbf{C}}, \vec{s}, \overline{U})$ is uniquely generated from (3.2) adopted to $(\ddot{\mathbf{C}}, \vec{\mu}, \overline{U})$ for every $\vec{\mu} \in \mathbb{S}^{\ddot{\mathbf{C}}}$. Let

$$\overline{P}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) = \sum_{\ddot{\mathbf{H}} \subseteq \ddot{\mathbf{C}}} \frac{\alpha_{\ddot{\mathbf{H}}}}{|\ddot{\mathbf{H}}|_{\overline{\kappa}}}. \tag{3.3}$$

Based on Lemma 2.4, (3.1) is matched by this $\overline{P}$; thereby, (3.3) generates the unique potential. Integrating Definitions 2.1 and 3.1 with (3.3), the result now follows since

$$\overline{\kappa}(i) \cdot [\overline{P}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) - \overline{P}(\ddot{\mathbf{C}}, (\vec{s}_{\ddot{\mathbf{C}} \setminus \{i\}}, 0), \overline{U})] = \theta_i^{\overline{\kappa}}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}),$$

for all $i \in \ddot{\mathbf{C}}$. □

Different from the notion of marginal concept, one would further apply the notion of accumulation to define the dividend approach with weights. Let $\overline{E}$: $\mathbf{GM} \longrightarrow \mathbb{R}$ be a function that appoints $\overline{E}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) \in \mathbb{R}$ to each $(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) \in \mathbf{GM}$. For every weight distribution $\overline{\kappa}$ and for every $i \in \ddot{\mathbf{C}}$, one could consider

$$\mathbf{A}_i^{\overline{\kappa}} \overline{E}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) = \overline{\kappa}(i) \cdot \sum_{\substack{\ddot{\mathbf{H}} \subseteq \ddot{\mathbf{C}} \\ i \in \ddot{\mathbf{H}}}} \overline{E}(\ddot{\mathbf{C}}, (\vec{s}_{\ddot{\mathbf{H}}}, 0_{\ddot{\mathbf{C}} \setminus \ddot{\mathbf{H}}}), \overline{U}).$$

Definition 3.2. *A resolution ρ on* **GM** *admits a dividend with weights if there exists a function* $\overline{E}$: **GM** $\rightarrow \mathbb{R}$*, such that for every* $(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) \in$ **GM***, for every weight distribution* $\overline{\kappa}$*, and for every* $i \in \ddot{\mathbf{C}}$*,*

$$\rho_i(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) = \mathbf{A}_i^{\overline{\kappa}}\overline{E}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}).$$

Resolutions that incorporate weighted dividends associate a scalar measure to each game, such that the payoff assigned to each constituent reflects its weighted marginal accumulation in relation to this measure. A mapping $\overline{E}$: **GM** $\longrightarrow \mathbb{R}$ is referred to as 0-normalized if it holds that $\overline{E}(\ddot{\mathbf{C}}, 0_{\ddot{\mathbf{C}}}, \overline{U}) = 0$ for every $\ddot{\mathbf{C}} \subseteq \mathbf{U}\ddot{\mathbf{C}}$. Furthermore, $\overline{E}$ is effective, provided it meets the following requirement: For every $(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) \in$ **GM**, and for every weight distribution $\overline{\kappa}$,

$$\sum_{i\in\ddot{\mathbf{C}}} \mathbf{A}_i^{\overline{\kappa}}\overline{E}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) = \overline{U}^m(\ddot{\mathbf{C}}).$$

Theorem 3.2. *A resolution ρ on* **GM** *admits a potential with weights if and only if ρ admits a dividend with weights.*

Proof. Let ρ be a resolution admitting a dividend with weights $\overline{E}$. Define $\overline{P}$: **GM** $\longrightarrow \mathbb{R}$ as $\overline{P}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) = \sum_{\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{C}}} \overline{E}(\ddot{\mathbf{C}}, (\vec{s}_{\ddot{\mathbf{H}}}, 0_{\ddot{\mathbf{C}}\setminus\ddot{\mathbf{H}}}), \overline{U})$ for each $(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) \in$ **GM**. Since ρ admits the dividend with weights $\overline{E}$, for every $(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) \in$ **GM**, for every weight distribution $\overline{\kappa}$, and for every $i \in \ddot{\mathbf{C}}$,

$$\begin{aligned}
\rho_i(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) &= \mathbf{A}_i^{\overline{\kappa}}\overline{E}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) \\
&\textbf{(since } \rho \textbf{ admits the dividend with weights } \overline{E}\textbf{)} \\
&= \overline{\kappa}(i) \sum_{\substack{\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{C}} \\ i\in\ddot{\mathbf{H}}}} \overline{E}(\ddot{\mathbf{C}}, (\vec{s}_{\ddot{\mathbf{H}}}, 0_{\ddot{\mathbf{C}}\setminus\ddot{\mathbf{H}}}), \overline{U}) \\
&\textbf{(by definition of } \mathbf{A}_i^{\overline{\kappa}}\overline{E}\textbf{)} \\
&= \overline{\kappa}(i) \cdot \Big[\sum_{\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{C}}} \overline{E}(\ddot{\mathbf{C}}, (\vec{s}_{\ddot{\mathbf{H}}}, 0_{\ddot{\mathbf{C}}\setminus\ddot{\mathbf{H}}}), \overline{U}) - \sum_{\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{C}}\setminus\{i\}} \overline{E}(\ddot{\mathbf{C}}, (\vec{s}_{\ddot{\mathbf{H}}}, 0_{\ddot{\mathbf{C}}\setminus\ddot{\mathbf{H}}}), \overline{U}) \Big] \\
&= \overline{\kappa}(i) \cdot [\overline{P}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) - \overline{P}(\ddot{\mathbf{C}}, (\vec{s}_{\ddot{\mathbf{H}}}, 0_{\ddot{\mathbf{C}}\setminus\ddot{\mathbf{H}}}), \overline{U})] \\
&\textbf{(by definition of } \overline{P}\textbf{))} \\
&= \mathbf{M}_i^{\overline{\kappa}}\overline{P}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}). \\
&\textbf{(by definition of } \mathbf{M}_i^{\overline{\kappa}}\overline{P}\textbf{)}.
\end{aligned}$$

Hence, ρ admits the potential with weights $\overline{P}$. Observe that subtracting the sum over all coalitions $\ddot{\mathbf{H}} \subseteq \ddot{\mathbf{C}} \setminus \{i\}$ from the sum over all coalitions $\ddot{\mathbf{H}} \subseteq \ddot{\mathbf{C}}$ removes precisely those terms that do not contain i. Hence, the remaining expression coincides with the aggregation over all coalitions $\ddot{\mathbf{H}}$, such that $i \in \ddot{\mathbf{H}}$. In this way, the dividend-based representation collects exactly the weighted contributions of max-dividends associated with coalitions containing i, which matches the structure of the potential-based marginal effect $\mathbf{M}_i^{\overline{\kappa}}\overline{P}$.

Let ρ be a resolution admitting a potential with weights $\overline{P}$. Define $\overline{E}$: **GM** $\longrightarrow \mathbb{R}$ as $\overline{E}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) = \sum_{\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{C}}} (-1)^{|\ddot{\mathbf{C}}|-|\ddot{\mathbf{H}}|}\overline{P}(\ddot{\mathbf{C}}, (\vec{s}_{\ddot{\mathbf{H}}}, 0_{\ddot{\mathbf{C}}\setminus\ddot{\mathbf{H}}}), \overline{U})$ for all $(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) \in$ **GM**. It is easy to check that

$$\overline{P}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) = \sum_{\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{C}}} \overline{E}(\ddot{\mathbf{C}}, (\vec{s}_{\ddot{\mathbf{H}}}, 0_{\ddot{\mathbf{C}}\setminus\ddot{\mathbf{H}}}), \overline{U}).$$

Since ρ admits the potential with weights $\overline{P}$, for every $(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) \in \mathbf{GM}$, for every weight distribution $\overline{\kappa}$, and for every $i \in \ddot{\mathbf{C}}$,

$$\begin{aligned}
& \rho_i(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) \\
= \; & \mathbf{M}_i^{\overline{\kappa}}\overline{P}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) \\
= \; & \overline{\kappa}(i) \cdot [\overline{P}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) - \overline{P}(\ddot{\mathbf{C}}, (\vec{s}_{\ddot{\mathbf{C}}\setminus\{i\}}, 0), \overline{U})] \\
= \; & \overline{\kappa}(i) \cdot [\sum_{\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{C}}} \overline{E}(\ddot{\mathbf{C}}, (\vec{s}_{\ddot{\mathbf{H}}}, 0_{\ddot{\mathbf{C}}\setminus\ddot{\mathbf{H}}}), \overline{U}) - \sum_{\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{C}}\setminus\{i\}} \overline{E}(\ddot{\mathbf{C}}, (\vec{s}_{\ddot{\mathbf{H}}}, 0_{\ddot{\mathbf{C}}\setminus\ddot{\mathbf{H}}}), \overline{U})] \\
= \; & \overline{\kappa}(i) \sum_{\substack{\ddot{\mathbf{H}}\subseteq\ddot{\mathbf{C}} \\ i\in\ddot{\mathbf{H}}}} \overline{E}(\ddot{\mathbf{C}}, (\vec{s}_{\ddot{\mathbf{H}}}, 0_{\ddot{\mathbf{C}}\setminus\ddot{\mathbf{H}}}), \overline{U}) \\
= \; & \mathbf{A}_i^{\overline{\kappa}}\overline{E}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}).
\end{aligned}$$

Hence, ρ admits the dividend with weights $\overline{E}$. Conversely, starting from the potential-based marginal representation, one can recover the dividend by a Möbius-type inversion over the lattice of coalitions. The key point is that, for each $\ddot{\mathbf{H}} \subseteq \ddot{\mathbf{C}}$, the max-dividend $\alpha_{\ddot{\mathbf{H}}}$ is determined by alternating sums of maximal-utilities $\overline{U}^m(\ddot{\mathbf{K}})$ for $\ddot{\mathbf{K}} \subseteq \ddot{\mathbf{H}}$. This ensures that every coalition's contribution is counted exactly once in the reconstruction of $\overline{P}$, so that the potential and dividend approaches encode equivalent information about the underlying weighted max-value. □

This duality reveals that whether one interprets individual payoff allocation as marginal contributions (potential) or cumulative accumulations (dividend), the fundamental allocating logic remains consistent under weighted schemes.

Theorem 3.3. *There exists a uniquely 0-normalized and effective dividend $\overline{E}$ such that for every $(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) \in \mathbf{GM}$, for every weight distribution $\overline{\kappa}$, and for every $i \in \ddot{\mathbf{C}}$,*

$$\theta_i^{\overline{\kappa}}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) = \mathbf{A}_i^{\overline{\kappa}}\overline{E}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}).$$

Proof. This theorem can be derived by Theorems 3.1 and 3.2. □

Remark 2. *Based on Theorem 3.2, it is known that the notions of weighted potential and weighted dividend provide dual mechanisms for characterizing specific resolutions. That is, the dividend approach serves as a complementary perspective to the potential approach. In essence, when axioms are reformulated to accommodate resolution schemes centered on "dividends" with weights, the procedures for deriving characterizations in multi-choice TU games remain analogous. Such dividend-based representations not only furnish alternative interpretations for resolution rules but also underpin the development of corresponding axiomatic frameworks.*

4. Coincident relations and relative characterizations

In this section, we provide several coincident relations to describe the class of whole resolutions admitting a potential with weights. To present the rationality of the weighted max-value, the proposed coincident relations would be utilized to axiomatize the weighted max-value.

4.1. Coincidences and relative characterizations among potential approaches and axioms

To present the major results of this section, the following properties are needed. Let ρ be a resolution. ρ satisfies weighted balanced contributions under multi-choice games (WBCMG) if for all $(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) \in \mathbf{GM}$, for each weight distribution $\overline{\kappa}$, and for every $i, j \in \ddot{\mathbf{C}}$,

$$\tfrac{1}{\overline{\kappa}(i)}\Big[\rho_i(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) - \rho_i(\ddot{\mathbf{C}}, (\vec{s}_{\ddot{\mathbf{C}}\setminus\{j\}}, 0), \overline{U})\Big] = \tfrac{1}{\overline{\kappa}(j)}\Big[\rho_j(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) - \rho_j(\ddot{\mathbf{C}}, (\vec{s}_{\ddot{\mathbf{C}}\setminus\{i\}}, 0), \overline{U})\Big].$$

The theoretical framework of this study emphasizes that resolutions in the weighted max-value setting are inherently shaped by potential differences. In particular, the weighted balanced contributions under the multi-choice games (WBCMG) condition reflects a fairness criterion where differences in payoffs due to the inclusion or exclusion of a constituent are proportionally adjusted by their relative weights. This relation encapsulates the idea that the marginal influence a participant exerts on others' payoffs must be weighted appropriately, preserving a symmetric and equitable perspective even when participants possess heterogeneous capacities or risk profiles. Such symmetry is crucial in multi-choice environments, where varying levels of participation introduce complex interdependencies among constituents. This ensures that relative differences in payoffs arising from coalition modifications are proportionally attributed, thus reflecting fairness in environments with heterogeneous participation intensities.

Further, a multi-choice type of the path independence axiom due to Ortmann [12, 13] would be introduced with weights. A ranking for $(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) \in \mathbf{GM}$ is a 1-1 and onto mapping $\lambda : \ddot{\mathbf{C}} \to \ddot{\mathbf{C}}$. Let λ, λ' be two rankings for $(\ddot{\mathbf{C}}, \vec{s}, \overline{U})$. λ' is a conversion of λ if there exist $i, j \in \ddot{\mathbf{C}}$ with $i \neq j$ and $\lambda(j) = \lambda(i) + 1$, such that $\lambda'(i) = \lambda(j)$, $\lambda'(j) = \lambda(i)$ and $\lambda'(q) = \lambda(q)$ for all $q \in \ddot{\mathbf{C}} \setminus \{i, j\}$. Let λ be a ranking. The step vector that is presented after t-th constituent due to λ, denoted as $\overline{\pi}^{\lambda,t}$, is considered as for all $i \in \ddot{\mathbf{C}}$,

$$\overline{\pi}_i^{\lambda,t} = \begin{cases} \vec{s}_i & \lambda(i) \leq t; \\ 0 & \text{o.w.} \end{cases}$$

Each ranking can be transformed to another ranking by utilizing conversions. ρ satisfies weighted path independence under multi-choice games (WPIMG) if for all $(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) \in \mathbf{GM}$, for each weight distribution $\overline{\kappa}$, and for every rankings λ, λ',

$$\sum_{i\in\ddot{\mathbf{C}}} \frac{1}{\overline{\kappa}(i)} \cdot \rho_i(\ddot{\mathbf{C}}, \overline{\pi}^{\lambda,\lambda(i)}, \overline{U}) = \sum_{i\in\ddot{\mathbf{C}}} \frac{1}{\overline{\kappa}(i)} \cdot \rho_i(\ddot{\mathbf{C}}, \overline{\pi}^{\lambda',\lambda'(i)}, \overline{U}).$$

The axiom of WPIMG embodies the principle that the final cumulative outcome assigned to all constituent should depend solely on the final outcome, not on the sequence or path of partial coalition formations that led to it. By requiring that different stepwise rankings result in identical aggregated outcome for all constituents, WPIMG ensures that the resolution remains consistent regardless of the order in which constituents join the coalition. This further guarantees that the final cumulative outcome remains invariant regardless of the order in which constituents sequentially join, aligning with the fundamental concept of path independence.

Subsequently, a multi-choice generalization of a specific game due to Calvo and Santos [3] is introduced. Let $(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) \in \mathbf{GM}$ and ρ be a resolution. The mensuration game $(\ddot{\mathbf{C}}, \vec{s}, \overline{U}_\rho)$ is defined as follows: For all $\vec{\mu} \in \ddot{\mathbb{S}}^{\ddot{\mathbf{C}}}$,

$$\overline{U}_\rho(\vec{\mu}) = \sum_{i\in\overline{NE}(\vec{\mu})} \rho_i\Big(\ddot{\mathbf{C}}, (\vec{s}_{\overline{NE}(\vec{\mu})}, \vec{0}_{\ddot{\mathbf{C}}\setminus\overline{NE}(\vec{\mu})}), \overline{U}\Big),$$

where $\overline{U}^{m}_{\rho}(\ddot{\mathbf{H}}) = \max_{\vec{\mu}\in\mathbb{S}^{\ddot{\mathbf{C}}}}\{\overline{U}_{\rho}(\vec{\mu})|\overline{NE}(\vec{\mu}) = \ddot{\mathbf{H}}\}$ for all $\ddot{\mathbf{H}} \subseteq \ddot{\mathbf{C}}$. Note that if ρ satisfies the effectiveness for multi-choice games, then $\overline{U}^{m} = \overline{U}^{m}_{\rho}$. The definition of the mensuration game extends this framework by constructing a resolution tailored to capture the maximal utility arising from subsets of active constituents in any given multi-choice step vector. The mensuration game formalizes the idea that the realized utility should reflect only those constituents that are effectively participating in the coalition under multi-choice step vectors.

Further, the main outcome of this study is introduced.

Theorem 4.1. *Let ρ be a resolution. The following are coincident:*

(1) ρ admits a potential with weights;
(2) ρ fits WBCMG;
(3) ρ matches WPIMG;
*(4) $\rho(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) = \theta^{\bar{\kappa}}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}_{\rho})$ for every $(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) \in$ **GM** and for each weight distribution $\bar{\kappa}$.*

Proof. Let ρ be a resolution on **GM**. To analyze (1) $\Rightarrow$ (2), assume that ρ admits a potential with weights $\overline{P}$. For each $(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) \in$ **GM**, for each weight distribution $\bar{\kappa}$ and for every $i, j \in \ddot{\mathbf{C}}$, $i \neq j$,

$$\begin{aligned}
& \tfrac{1}{\bar{\kappa}(i)} \cdot \Big[\rho_i(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) - \rho_i(\ddot{\mathbf{C}}, (\vec{s}_{\ddot{\mathbf{C}}\setminus\{j\}}, 0), \overline{U})\Big] \\
= \ & \tfrac{1}{\bar{\kappa}(i)} \cdot \Big[\bar{\kappa}(i) \cdot [\overline{P}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) - \overline{P}(\ddot{\mathbf{C}}, (\vec{s}_{\ddot{\mathbf{C}}\setminus\{i\}}, 0), \overline{U})] \\
& \quad - \bar{\kappa}(i) \cdot [\overline{P}(\ddot{\mathbf{C}}, (\vec{s}_{\ddot{\mathbf{C}}\setminus\{j\}}, 0), \overline{U}) - \overline{P}(\ddot{\mathbf{C}}, (\vec{s}_{\ddot{\mathbf{C}}\setminus\{i,j\}}, 0, 0), \overline{U})]\Big] \\
= \ & [\overline{P}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) - \overline{P}(\ddot{\mathbf{C}}, (\vec{s}_{\ddot{\mathbf{C}}\setminus\{i\}}, 0), \overline{U})] \\
& \quad - [\overline{P}(\ddot{\mathbf{C}}, (\vec{s}_{\ddot{\mathbf{C}}\setminus\{j\}}, 0), \overline{U}) - \overline{P}(\ddot{\mathbf{C}}, (\vec{s}_{\ddot{\mathbf{C}}\setminus\{i,j\}}, 0, 0), \overline{U})] \\
= \ & [\overline{P}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) - \overline{P}(\ddot{\mathbf{C}}, (\vec{s}_{\ddot{\mathbf{C}}\setminus\{j\}}, 0), \overline{U})] \\
& \quad - [\overline{P}(\ddot{\mathbf{C}}, (\vec{s}_{\ddot{\mathbf{C}}\setminus\{i\}}, 0), \overline{U}) - \overline{P}(\ddot{\mathbf{C}}, (\vec{s}_{\ddot{\mathbf{C}}\setminus\{i,j\}}, 0, 0), \overline{U})] \\
= \ & \tfrac{1}{\bar{\kappa}(j)} \cdot \Big[\bar{\kappa}(j) \cdot [\overline{P}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) - \overline{P}(\ddot{\mathbf{C}}, (\vec{s}_{\ddot{\mathbf{C}}\setminus\{j\}}, 0), \overline{U})] \\
& \quad - \bar{\kappa}(j) \cdot [\overline{P}(\ddot{\mathbf{C}}, (\vec{s}_{\ddot{\mathbf{C}}\setminus\{i\}}, 0), \overline{U}) - \overline{P}(\ddot{\mathbf{C}}, (\vec{s}_{\ddot{\mathbf{C}}\setminus\{i,j\}}, 0, 0), \overline{U})]\Big] \\
= \ & \tfrac{1}{\bar{\kappa}(j)} \cdot \Big[\rho_j(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) - \rho_j(\ddot{\mathbf{C}}, (\vec{s}_{\ddot{\mathbf{C}}\setminus\{i\}}, 0), \overline{U})\Big].
\end{aligned}$$

Hence, ρ satisfies WBCMG.

To present (2) $\Rightarrow$ (3), assume that ρ fits WBCMG. Let $(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) \in$ **GM** and λ, λ' be two rankings for $(\ddot{\mathbf{C}}, \vec{s}, \overline{U})$. Since each ranking can be transformed into another ranking by utilizing conversions, one can suppose that λ' is a conversion of λ. Let $i, j \in \ddot{\mathbf{C}}$, $i \neq j$, and $\lambda(j) = \lambda(i) + 1$, such that $\lambda'(i) = \lambda(j)$, $\lambda'(j) = \lambda(i)$, and $\lambda'(q) = \lambda(q)$ for all $q \in \ddot{\mathbf{C}} \setminus \{i, j\}$. Since λ' is a conversion of λ, for each $t \in \ddot{\mathbf{C}} \setminus \{i, j\}$,

$$\rho_t(\ddot{\mathbf{C}}, \overline{\pi}^{\lambda,\lambda(t)}, \overline{U}) = \rho_t(\ddot{\mathbf{C}}, \overline{\pi}^{\lambda',\lambda'(t)}, \overline{U}). \tag{4.1}$$

Since λ' is a conversion of λ, based on (4.1),

$$\begin{aligned}
& \sum_{p\in\ddot{\mathbf{C}}} \tfrac{1}{\bar{\kappa}(p)} \cdot \rho_p(\ddot{\mathbf{C}}, \overline{\pi}^{\lambda',\lambda'(p)}, \overline{U}) - \sum_{p\in\ddot{\mathbf{C}}} \tfrac{1}{\bar{\kappa}(p)} \cdot \rho_p(\ddot{\mathbf{C}}, \overline{\pi}^{\lambda,\lambda(p)}, \overline{U}) \\
= \ & \tfrac{1}{\bar{\kappa}(j)} \cdot \rho_j(\ddot{\mathbf{C}}, (\vec{s}_{\ddot{\mathbf{C}}\setminus\{i\}}, 0), \overline{U}) + \tfrac{1}{\bar{\kappa}(i)} \cdot \rho_i(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) - \tfrac{1}{\bar{\kappa}(i)} \cdot \rho_i(\ddot{\mathbf{C}}, (\vec{s}_{\ddot{\mathbf{C}}\setminus\{j\}}, 0), \overline{U}) - \tfrac{1}{\bar{\kappa}(j)} \cdot \rho_j(\ddot{\mathbf{C}}, \vec{s}, \overline{U}).
\end{aligned} \tag{4.2}$$

Since ρ satisfies WBCMG,

$$\frac{1}{\bar{\kappa}(i)} \cdot [\rho_i(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) - \rho_i(\ddot{\mathbf{C}}, (\vec{s}_{\ddot{\mathbf{C}}\setminus\{j\}}, 0), \overline{U})] = \frac{1}{\bar{\kappa}(j)} \cdot [\rho_j(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) - \rho_j(\ddot{\mathbf{C}}, (\vec{s}_{\ddot{\mathbf{C}}\setminus\{i\}}, 0), \overline{U})]. \tag{4.3}$$

Based on (4.2) and (4.3),

$$\sum_{p\in\ddot{\mathbf{C}}}\frac{1}{\overline{\kappa}(p)}\cdot\rho_p(\ddot{\mathbf{C}},\overline{\pi}^{\lambda,\lambda(p)},\overline{U})-\sum_{p\in\ddot{\mathbf{C}}}\frac{1}{\overline{\kappa}(p)}\cdot\rho_p(\ddot{\mathbf{C}},\overline{\pi}^{\lambda',\lambda'(p)},\overline{U})=0.$$

Thus, $\sum_{p\in\ddot{\mathbf{C}}}\frac{1}{\overline{\kappa}(p)}\cdot\rho_p(\ddot{\mathbf{C}},\overline{\pi}^{\lambda,\lambda(p)},\overline{U})=\sum_{p\in\ddot{\mathbf{C}}}\frac{1}{\overline{\kappa}(p)}\cdot\rho_p(\ddot{\mathbf{C}},\overline{\pi}^{\lambda',\lambda'(p)},\overline{U})$. Hence, ρ satisfies WPIMG.

To analyze (3) $\Rightarrow$ (2), assume that ρ matches WPIMG. Let $(\ddot{\mathbf{C}},\vec{s},\overline{U})\in\mathbf{GM}$. It is trivial if $|\ddot{\mathbf{C}}|=1$. Let $|\ddot{\mathbf{C}}|\geq 2$, $i,j\in\ddot{\mathbf{C}}$, and λ_1, λ_2 be two rankings, where $\lambda_1(i)=\lambda_2(j)=|\ddot{\mathbf{C}}|$, $\lambda_1(j)=\lambda_2(i)=|\ddot{\mathbf{C}}|-1$, and $\lambda_1(q)=\lambda_2(q)$ for all $q\notin\{i,j\}$. Since ρ matches WPIMG,

$$\begin{aligned}0=&\sum_{p\in\ddot{\mathbf{C}}}\tfrac{1}{\overline{\kappa}(p)}\cdot\rho_p(\ddot{\mathbf{C}},\overline{\pi}^{\lambda_1,\lambda_1(p)},\overline{U})-\sum_{p\in\ddot{\mathbf{C}}}\tfrac{1}{\overline{\kappa}(p)}\cdot\rho_p(\ddot{\mathbf{C}},\overline{\pi}^{\lambda_2,\lambda_2(p)},\overline{U})\\=&\tfrac{1}{\overline{\kappa}(j)}\cdot\rho_j(\ddot{\mathbf{C}},(\vec{s}_{\ddot{\mathbf{C}}\setminus\{i\}},0),\overline{U})+\tfrac{1}{\overline{\kappa}(i)}\cdot\rho_i(\ddot{\mathbf{C}},\vec{s},\overline{U})-\tfrac{1}{\overline{\kappa}(i)}\cdot\rho_i(\ddot{\mathbf{C}},(\vec{s}_{\ddot{\mathbf{C}}\setminus\{j\}},0),\overline{U})-\tfrac{1}{\overline{\kappa}(j)}\cdot\rho_j(\ddot{\mathbf{C}},\vec{s},\overline{U}).\end{aligned}$$

Therefore,

$$\frac{1}{\overline{\kappa}(i)}[\rho_i(\ddot{\mathbf{C}},\vec{s},\overline{U})-\rho_i(\ddot{\mathbf{C}},(\vec{s}_{\ddot{\mathbf{C}}\setminus\{j\}},0),\overline{U})]=\frac{1}{\overline{\kappa}(j)}[\rho_j(\ddot{\mathbf{C}},\vec{s},\overline{U})-\rho_j(\ddot{\mathbf{C}},(\vec{s}_{\ddot{\mathbf{C}}\setminus\{i\}},0),\overline{U})].$$

Thus, ρ satisfies WBCMG.

To present (2) $\Rightarrow$ (4), assume that ρ fits WBCMG. Let $(\ddot{\mathbf{C}},\vec{s},\overline{U})\in\mathbf{GM}$. By Theorem 3.1 and "(1) $\Rightarrow$ (2)" of this theorem, the resolution $\theta^{\overline{\kappa}}$ matches WBCMG. The rest of proof proceeds by induction on $|\ddot{\mathbf{C}}|$. Let $|\ddot{\mathbf{C}}|=1$ and $\ddot{\mathbf{C}}=\{i\}$. By ETSMG of $\theta^{\overline{\kappa}}$ and definition of $\overline{U}_\rho$,

$$\theta_i^{\overline{\kappa}}(\ddot{\mathbf{C}},\vec{s},\overline{U}_\rho)=\overline{U}_\rho^m(\{i\})=\rho_i(\ddot{\mathbf{C}},\vec{s},\overline{U}).$$

Assume that $\rho(\ddot{\mathbf{C}},\vec{s},\overline{U})=\theta^{\overline{\kappa}}(\ddot{\mathbf{C}},\vec{s},\overline{U})$ if $|\ddot{\mathbf{C}}|\leq r-1$, where $r\geq 2$. The condition $|\ddot{\mathbf{C}}|=r$: By induction hypotheses, and WBCMG of ρ and $\theta^{\overline{\kappa}}$, for $i,j\in\ddot{\mathbf{C}}$,

$$\begin{aligned}&\tfrac{1}{\overline{\kappa}(i)}\cdot\rho_i(\ddot{\mathbf{C}},\vec{s},\overline{U})-\tfrac{1}{\overline{\kappa}(j)}\cdot\rho_j(\ddot{\mathbf{C}},\vec{s},\overline{U})\\=&\tfrac{1}{\overline{\kappa}(i)}\cdot\rho_i(\ddot{\mathbf{C}},(\vec{s}_{\ddot{\mathbf{C}}\setminus\{j\}},0),\overline{U})-\tfrac{1}{\overline{\kappa}(j)}\cdot\rho_j(\ddot{\mathbf{C}},(\vec{s}_{\ddot{\mathbf{C}}\setminus\{i\}},0),\overline{U})\quad\textbf{(by WBCMG of }\rho\textbf{)}\\=&\tfrac{1}{\overline{\kappa}(i)}\cdot\theta_i^{\overline{\kappa}}(\ddot{\mathbf{C}},(\vec{s}_{\ddot{\mathbf{C}}\setminus\{j\}},0),\overline{U})-\tfrac{1}{\overline{\kappa}(j)}\cdot\theta_j^{\overline{\kappa}}(\ddot{\mathbf{C}},(\vec{s}_{\ddot{\mathbf{C}}\setminus\{i\}},0),\overline{U})\quad\textbf{(by induction hypotheses)}\\=&\tfrac{1}{\overline{\kappa}(i)}\cdot\theta_i^{\overline{\kappa}}(\ddot{\mathbf{C}},\vec{s},\overline{U}_\rho)-\tfrac{1}{\overline{\kappa}(j)}\cdot\theta_j^{\overline{\kappa}}(\ddot{\mathbf{C}},\vec{s},\overline{U}_\rho).\quad\textbf{(by WBCMG of }\theta^{\overline{\kappa}}\textbf{)}\end{aligned}$$

That is, $\overline{\kappa}(j)\cdot[\rho_i(\ddot{\mathbf{C}},\vec{s},\overline{U})-\theta_i^{\overline{\kappa}}(\ddot{\mathbf{C}},\vec{s},\overline{U}_\rho)]=\overline{\kappa}(i)\cdot[\rho_j(\ddot{\mathbf{C}},\vec{s},\overline{U})-\theta_j^{\overline{\kappa}}(\ddot{\mathbf{C}},\vec{s},\overline{U}_\rho)]$ for every $i,j\in\ddot{\mathbf{C}}$. Based on ETSMG of $\theta^{\overline{\kappa}}$ and the definition of $\overline{U}_\rho$,

$$\begin{aligned}&|\ddot{\mathbf{C}}|_{\overline{\kappa}}\cdot[\rho_i(\ddot{\mathbf{C}},\vec{s},\overline{U})-\theta_i^{\overline{\kappa}}(\ddot{\mathbf{C}},\vec{s},\overline{U}_\rho)]\\=&\sum_{j\in\ddot{\mathbf{C}}}\overline{\kappa}(j)\cdot[\rho_i(\ddot{\mathbf{C}},\vec{s},\overline{U})-\theta_i^{\overline{\kappa}}(\ddot{\mathbf{C}},\vec{s},\overline{U}_\rho)]\\=&\sum_{j\in\ddot{\mathbf{C}}}\overline{\kappa}(i)\cdot[\rho_j(\ddot{\mathbf{C}},\vec{s},\overline{U})-\theta_j^{\overline{\kappa}}(\ddot{\mathbf{C}},\vec{s},\overline{U}_\rho)]\\=&\overline{\kappa}(i)\cdot[\sum_{j\in\ddot{\mathbf{C}}}\rho_j(\ddot{\mathbf{C}},\vec{s},\overline{U})-\sum_{j\in\ddot{\mathbf{C}}}\theta_j^{\overline{\kappa}}(\ddot{\mathbf{C}},\vec{s},\overline{U}_\rho)]\\=&\overline{\kappa}(i)\cdot[\overline{U}_\rho^m(\ddot{\mathbf{C}})-\overline{U}_\rho^m(\ddot{\mathbf{C}})]\\=&0.\end{aligned}$$

Since $|\ddot{\mathbf{C}}|_{\overline{\kappa}} \neq 0$, we have that $\rho_i(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) - \theta_i^{\overline{\kappa}}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}_\rho) = 0$ for all $i \in \ddot{\mathbf{C}}$, i.e., $\rho_i(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) = \theta_i^{\overline{\kappa}}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}_\rho)$ for all $i \in \ddot{\mathbf{C}}$.

To show (4) $\Rightarrow$ (1), assume that $\rho(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) = \theta^{\overline{\kappa}}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}_\rho)$ for all $(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) \in \mathbf{GM}$ and for each weight distribution $\overline{\kappa}$. By Theorem 3.1, the resolution $\theta^{\overline{\kappa}}$ admits a potential with weights $\overline{P}_{\theta^{\overline{\kappa}}}$. Consider a function related to ρ as $\overline{P}_\rho(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) = \overline{P}_{\theta^{\overline{\kappa}}}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}_\rho)$ for all $(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) \in \mathbf{GM}$. Then for all $i \in \ddot{\mathbf{C}}$,

$$\begin{aligned} & \rho_i(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) \\ = \; & \theta_i^{\overline{\kappa}}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}_\rho) \\ = \; & \overline{\kappa}(i) \cdot [\overline{P}_{\theta^{\overline{\kappa}}}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}_\rho) - \overline{P}_{\theta^{\overline{\kappa}}}(\ddot{\mathbf{C}}, (\vec{s}_{\ddot{\mathbf{C}}\setminus\{i\}}, 0), \overline{U}_\rho)] \\ = \; & \overline{\kappa}(i) \cdot [\overline{P}_\rho(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) - \overline{P}_\rho(\ddot{\mathbf{C}}, (\vec{s}_{\ddot{\mathbf{C}}\setminus\{i\}}, 0), \overline{U})]. \end{aligned}$$

That is, ρ admits the potential with weights $\overline{P}_\rho$. □

In the following, two characterizations of the weighted max-value can be presented by utilizing Theorems 3.1 and 4.1.

Theorem 4.2. *(1) A resolution ρ satisfies ETSMG and WBCMG if and only if $\rho = \theta^{\overline{\kappa}}$.*
(2) A resolution ρ satisfies ETSMG and WPIMG if and only if $\rho = \theta^{\overline{\kappa}}$.

Proof. The proof follows by Theorems 3.1 and 4.1, and the definition of the measuration game. □

The following resolutions verify that each of the applied axioms of Theorem 4.2 is logically independent of the remaining axioms.

Example 1. *Define a resolution ρ as $\rho_i(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) = 0$ for all $(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) \in \mathbf{GM}$, for each weight distribution $\overline{\kappa}$, and for every $i \in \ddot{\mathbf{C}}$. Thus, ρ satisfies WBCMG and WPIMG, but it does not satisfy ETSMG.*

Example 2. *Define a resolution ρ by for all $(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) \in \mathbf{GM}$, for each weight distribution $\overline{\kappa}$, and for every $i \in \ddot{\mathbf{C}}$,*

$$\rho_i(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) = \frac{\overline{\kappa}(i) \cdot \overline{U}^m(\ddot{\mathbf{C}})}{|\ddot{\mathbf{C}}|_{\overline{\kappa}}}.$$

Thus, ρ satisfies ETSMG, but it does not satisfy WBCMG and WPIMG.

As discussed in the introduction, the weighted max-value framework is introduced to overcome limitations of traditional TU games by capturing heterogeneity in participatory behavior and incorporating differences in constituents' capacities and risk exposure. This approach emphasizes that participants in multi-choice settings may engage at varying intensity levels and that their contributions are not simply additive, but weighted according to their unique profiles. The role of weights is crucial in determining how individual payoffs are adjusted based on the extent of participation and the relative significance of each constituent within the coalition. This weighting mechanism links directly to the notion of resolutions formulated through potential differences, where the allocation among participants reflects not only aggregate utilities but also the personalized marginal impacts driven by these weights. Such connections also underlie the coincident properties discussed in earlier sections, which demonstrate how weighted resolutions can be equivalently characterized via potential differences, balanced contributions, and path independence.

4.2. *Practical applications*

Such a setting naturally captures situations in public finance or crowdfunding, where each constituent corresponds to an agency or an investor, and the activity level reflects the proportion of its budget or capital committed to a joint project. To concretely illustrate how these theoretical constructs operate in practice, an applied instance is offered below to demonstrate the computation and interpretation of the weighted max-value. Let $\ddot{\mathbf{C}}$ be a set of constituents engaged in a multi-level collaborative project $(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) \in \mathbf{GM}$, where each $i \in \ddot{\mathbf{C}}$ possesses a given capability or resource amount c_i. In this setting, the activity level $\vec{\mu}_i$ of each constituent i in a given step vector $\vec{\mu} \in \ddot{\mathbb{S}}^{\ddot{\mathbf{C}}}$ can be interpreted as the degree to which the constituent is mobilizing their available resources or effort toward the collective outcome. The value c_i may be represented as the case where constituent i requires external support or generates a net burden on the coalition. The interpretation of $\vec{\mu}_i$ as a relative engagement level, specifically, the ratio related to resource c_i actually committed by i to their total availability, reflects the intensity of participation and the personal stake involved in the joint decision process.

In this model, each $\vec{\mu} \in \ddot{\mathbb{S}}^{\ddot{\mathbf{C}}}$ represents a multi-choice configuration in which constituents engage at discrete but heterogeneous levels. Unlike traditional models where engagement is binary or scalar without context, the formulation here emphasizes the personalized impact of each action by embedding individual weight distributions derived from c_i. Notably, if two constituents commit the same absolute quantity to the coalition under $\vec{\mu}$, the constituent with lower total capacity c_i bears proportionally greater risk and demonstrates higher commitment. This aligns with the weighted max-value framework, where weights can be used to adjust the max-dividend contributions proportionally to reflect relative involvement and sensitivity to coalition structure.

Consequently, the weighted max-value $\theta^{\overline{\kappa}}$ provides a resolution that internalizes the structural configuration of the multi-choice vector $\vec{\mu}$ and the personalized contribution patterns through weight-adjusted allocation. This reinforces the interpretation of coalition formation as a risk-weighted and individually-modulated collective decision mechanism, consistent with the generalized potential and dividend framework introduced in this study. To concretely illustrate the theoretical constructs and axiomatic underpinnings introduced earlier, the following numerical example demonstrates how the weighted max-value is computed and interpreted under a multi-choice TU setting.

Let $(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) \in \mathbf{GM}$ with $\ddot{\mathbf{C}} = \{x, y, z\}$, where each $i \in \ddot{\mathbf{C}}$ has capability or resource c_i. Consider $\vec{s} = (1, 2, 1)$, indicating constituent x acts at levels $0, 1_x$, y acts at levels $0, 1_y, 2_y$, and z acts at levels $0, 1_z$. Suppose $c_x = 4$, $c_y = 8$, and $c_z = 6$. Relative weights of $i \in \ddot{\mathbf{C}}$ can be computed by

$$\overline{\kappa}(i) = \frac{c_i}{c_x + c_y + c_z}, \quad i \in \ddot{\mathbf{C}}.$$

Thus,

$$\overline{\kappa}(x) = \frac{2}{9}, \quad \overline{\kappa}(y) = \frac{4}{9}, \quad \overline{\kappa}(z) = \frac{1}{3}.$$

The utility function θ^t are determined from situation-based operations as follows:

$$\begin{array}{llll}
\overline{U}(1,2,1) = 12, & \overline{U}(1,1,1) = 18, & \overline{U}(1,2,0) = 6, & \overline{U}(0,2,1) = -2, \\
\overline{U}(1,1,0) = 14, & \overline{U}(1,0,1) = 13, & \overline{U}(0,1,1) = 16, & \overline{U}(0,2,0) = 10, \\
\overline{U}(1,0,0) = 5, & \overline{U}(0,1,0) = -8, & \overline{U}(0,0,1) = 8, & \overline{U}(0,0,0) = 0.
\end{array}$$

Furthermore, relative maximal-utilities can also be verified.

$$\begin{array}{lclllclllclllcl} \overline{U}^m(\ddot{\mathbf{C}}) & = & 18, & \overline{U}^m(\{x,y\}) & = & 14, & \overline{U}^m(\{x,z\}) & = & 13, & \overline{U}^m(\{y,z\}) & = & 16, \\ \overline{U}^m(\{x\}) & = & 5, & \overline{U}^m(\{y\}) & = & 10, & \overline{U}^m(\{z\}) & = & 8, & \overline{U}^m(\emptyset) & = & 0. \end{array}$$

Therefore, related max-dividends α_H of coalition $\ddot{\mathbf{H}}$ can be computed as follows. For $\ddot{\mathbf{H}} = \emptyset$,

$$\alpha_\emptyset = \overline{U}^m(\emptyset) = 0.$$

For $\ddot{\mathbf{H}} = \{x\}$,

$$\alpha_{\{x\}} = \overline{U}^m(\{x\}) - \alpha_\emptyset = 5 - 0 = 5.$$

For $\ddot{\mathbf{H}} = \{y\}$,

$$\alpha_{\{y\}} = \overline{U}^m(\{y\}) - \alpha_\emptyset = 10 - 0 = 10.$$

For $\ddot{\mathbf{H}} = \{z\}$,

$$\alpha_{\{z\}} = \overline{U}^m(\{z\}) - \alpha_\emptyset = 8 - 0 = 8.$$

For $\ddot{\mathbf{H}} = \{x, y\}$,

$$\alpha_{\{x,y\}} = \overline{U}^m(\{x,y\}) - (\alpha_{\{x\}} + \alpha_{\{y\}} + \alpha_\emptyset) = 14 - (5 + 10 + 0) = -1.$$

For $\ddot{\mathbf{H}} = \{x, z\}$,

$$\alpha_{\{x,z\}} = \overline{U}^m(\{x,z\}) - (\alpha_{\{x\}} + \alpha_{\{z\}} + \alpha_\emptyset) = 13 - (5 + 8 + 0) = 0.$$

For $\ddot{\mathbf{H}} = \{y, z\}$,

$$\alpha_{\{y,z\}} = \overline{U}^m(\{y,z\}) - (\alpha_{\{y\}} + \alpha_{\{z\}} + \alpha_\emptyset) = 16 - (10 + 8 + 0) = -2.$$

For $\ddot{\mathbf{H}} = \{x, y, z\}$,

$$\begin{aligned} \alpha_{\{x,y,z\}} &= \overline{U}^m(\ddot{\mathbf{C}}) - (\alpha_{\{x\}} + \alpha_{\{y\}} + \alpha_{\{z\}} + \alpha_{\{x,y\}} + \alpha_{\{x,z\}} + \alpha_{\{y,z\}} + \alpha_\emptyset) \\ &= 18 - (5 + 10 + 8 + (-1) + 0 + (-2) + 0) \\ &= 18 - 20 = -2. \end{aligned}$$

Next, one would compute $|\ddot{\mathbf{H}}|_{\overline{\kappa}}$ of coalition $\ddot{\mathbf{H}}$. For example,

$$|\{x,y\}|_{\overline{\kappa}} = \overline{\kappa}(x) + \overline{\kappa}(y) = \frac{2}{9} + \frac{4}{9} = \frac{2}{3}.$$

By Definition 2.1, for all $i \in \ddot{\mathbf{C}}$,

$$\theta_i^{\overline{\kappa}}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) = \sum_{\substack{\ddot{\mathbf{H}} \subseteq \ddot{\mathbf{C}}, \\ i \in \ddot{\mathbf{H}}}} \frac{\overline{\kappa}(i)}{|\ddot{\mathbf{H}}|_{\overline{\kappa}}} \cdot \alpha_{\ddot{\mathbf{H}}}.$$

Subsequently, the weighted max-value for each constituent can be computed as follows: For constituent x,

$$\theta_x^{\overline{\kappa}}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) = \frac{\frac{2}{9}}{\frac{2}{9}} \cdot 5 + \frac{\frac{2}{9}}{\frac{2}{3}} \cdot (-1) + \frac{\frac{2}{9}}{\frac{5}{9}} \cdot 0 + \frac{\frac{2}{9}}{1} \cdot (-2) = 5 - \frac{1}{3} - \frac{4}{9} = \frac{266}{63}.$$

For constituent y,

$$\theta_y^{\bar{\kappa}}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) = \frac{\frac{4}{9}}{\frac{4}{9}} \cdot 10 + \frac{\frac{4}{9}}{\frac{2}{3}} \cdot (-1) + \frac{\frac{4}{9}}{\frac{7}{9}} \cdot (-2) + \frac{\frac{4}{9}}{1} \cdot (-2) = 10 - \frac{2}{3} - \frac{8}{7} - \frac{8}{9} = \frac{460}{63}.$$

For constituent z,

$$\theta_z^{\bar{\kappa}}(\ddot{\mathbf{C}}, \vec{s}, \overline{U}) = \frac{\frac{1}{3}}{\frac{1}{3}} \cdot 8 + \frac{\frac{1}{3}}{\frac{5}{9}} \cdot 0 + \frac{\frac{1}{3}}{\frac{7}{9}} \cdot (-2) + \frac{\frac{1}{3}}{1} \cdot (-2) = 8 - \frac{6}{7} - \frac{2}{3} = \frac{408}{63}.$$

The sum of all the weighted max-values for all constituents is

$$\frac{266}{63} + \frac{460}{63} + \frac{408}{63} = \frac{1134}{63} = 18.$$

Based on the above situation, each constituent's weighted max-value differs, reflecting how its maximal-utilities and relative capacities shape the payoff vector. For example, although y has higher capacity, the negative dividends in larger coalitions reduce its payoff relative to x and z. This concretely demonstrates how the weighted max-value $\theta^{\bar{\kappa}}$ internalizes multi-choice participation and heterogeneous risk distribution, consistent with the generalized potential and dividend framework established in this study. This numerical example thus validates the theoretical predictions and underscores the operational feasibility of the weighted max-value resolution within multi-choice TU situations. Interpreted in public finance or crowdfunding contexts, the resulting weighted max-values indicate how the maximal joint surplus is redistributed once the discrete engagement levels and heterogeneous budget or risk capacities of the participants are taken into account.

Remark 3. Discussion: Uncertainty and fuzzy environments. *The analysis in this paper has been carried out under the assumption that both weight distributions and related characteristic mappings are deterministic. In many practical situations, however, participation levels, capacities, and payoffs are subject to uncertainty, ambiguity, or incomplete information. For instance, the effective capacity of an agency in a public finance problem or the realized contribution of an investor in a crowdfunding platform may fluctuate over time or be only partially observable. In such contexts, it is natural to consider random or fuzzy variants of weight distributions and related characteristic mappings.*

One possible extension is to model the weights as random variables or fuzzy numbers, reflecting uncertain risk exposures or imprecise importance weights. Similarly, the multi-choice utilities could be defined in terms of expected or possibilistic assessments of collective outcomes under different activity profiles. Within this perspective, recent advances in uncertainty modeling and expert judgment for complex systems provide useful guidance on how to incorporate probabilistic, fuzzy, and expert-based information into risk-aware decision models. In particular, Yazdi et al. [17] and Zarei et al. [18] discussed uncertainty modeling in digitalized process systems and sociotechnical environments, highlighting how probability, fuzzy logic, and expert judgment can be combined in a unified risk assessment framework. Adapting the weighted max-value to such settings, for example by defining stochastic or fuzzy versions of max-dividends and potentials, constitutes a promising direction for future research.

5. Concluding remarks

(1) In this study, we establish a unified resolution concept, the weighted max-value, for multi-choice transferable-utility games that accounts for multi-choice participatory configurations and heterogeneous constituents' weights. Several principal contributions are summarized as follows.

- We introduce a general formulation of the weighted max-value by incorporating a weight distribution into related max-dividends distributing generated multi-choice frameworks, enabling the model to reflect constituent-specific influence and varying steps of engagement.
- We develop a potential approach by defining a weight-based potential function and proves that the corresponding marginal vector uniquely coincides with the weighted max-value, thereby extending the classical characterization results in standard TU games.
- A dual representation via the dividend approach is also proposed and shown to be structurally equivalent to the potential formulation. This equivalence provides alternative interpretative foundation for the resolution.
- To justify related rationality due to the proposed resolution, this study demonstrates that the weighted max-value is the unique resolution satisfying effectiveness for multi-choice games in combination with either weighted balanced contributions under multi-choice games or weighted path independence under multi-choice games. These axiomatic results extend well-known classical characterizations to the multi-choice and weighted setting.

(2) One would compare the results of this study with existing studies. There are several major differences:

(a) Differing from numerous multi-choice generalizations of standard TU resolutions, we propose the weighted max-value, potential, and dividend approaches with weights and related outcomes by simultaneously utilizing maximal-utilities among multi-choice activity step vectors and relative weights among constituents.

(b) We propose some weighted coincident relations to axiomatize the family of total weighted multi-choice resolutions that admit a potential with weights. Further, we adopt these coincident relations to axiomatize the weighted max-value. The proposed weighted coincident relations and corresponding characterizations do not appear in studies under multi-choice TU games.

(c) Considering how to assign a corresponding overall outcome to each constituent within a multi-choice framework, there are several generalizations of standard TU resolutions in the literature. Focusing on the concepts of marginal contributions and duplicated behavior, Nouweland et al. [11] proposed a multi-choice Shapley value. Building on the concepts of replicated behavior and unit-level-payoff, Hwang and Liao [7] introduced a multi-choice analogue of the core, along with its non-emptiness and related characterizations. Additionally, by simultaneously considering weighting distribution and replicated behavior, Liao [8] presented a generalized EANSC.

  - While we, in this study, and Nouweland et al. [11] propose extensions of the Shapley value within a multi-choice framework, Nouweland et al.'s [11] study is based on marginal contributions and duplicated behavior, whereas this study is based on maximal-utilities among multi-choice step vectors and relative weights.

- In distinct contrast to Hwang and Liao [7] and Liao [8], who focus on the extended core and the extended EANSC, respectively, within a multi-choice framework, we concentrate on the Shapley value.

(d) As mentioned in the Introduction, we integrate the concept of weighting distribution within a multi-choice framework to synthesize and extend the equivalent results between the Shapley value and potential approaches, as presented by Hart and Mas-Colell [6], Ortmann [12, 13], and Calvo and Santos [3]. The primary difference between this study and these studies is its consideration of the concepts of weights and the multi-choice framework.

(3) The advantages of proposed approaches in this study are that the weighted max-value always exists and enables one to determine a type of global outcome for a given constituent by simultaneously utilizing maximal-utilities among multi-choice step vectors and relative weights related to constituents.

(4) From an axiomatic perspective, the weighted max-value is distinct from the weighted Shapley value and from core-type concepts. The weighted Shapley value extends the classical Shapley value by distorting marginal contributions through exogenous weights but remains anchored in a binary participation framework. In contrast, the weighted max-value is driven by maximal-utilities in a multi-choice setting and aggregates max-dividends through weight distributions. This leads to a resolution that is sensitive to discrete engagement levels and heterogeneous risk or capacity profiles. On the other hand, core and extended-core type solutions emphasize coalition-wise stability and do not necessarily admit a potential representation. The weighted max-value does not need to belong to the (multi-choice) core, but it provides a potential-based and dividend-based allocation rule that is particularly suited for applications where the maximal attainable surplus and the heterogeneous exposure of participants must be considered simultaneously.

(5) From an applied perspective, the weighted max-value admits a tractable structure due to its closed-form representation based on max-dividends and proportional weights over relative coalitions. Once the maximal dividend vector is determined, the calculation of the weighted max-value reduces to a linear aggregation formed via relative weights. This suggests the feasibility of implementing efficient algorithms for large-scale instances, for example, by dynamic programming or sparsity-aware subset enumeration, especially when the underlying utility function possesses supermodular or decomposable properties. A systematic analysis of the computational complexity of such algorithms and the design of approximation schemes for very large games constitute an important topic for future research.

(6) Building on the maximal-utilities of multi-choice step vectors and weights related to constituents, this framework naturally lends itself to several extensions. First, dynamic adjustments of weights under evolving coalition structures and network-based interactions may be incorporated, leading to weighted max-values in dynamic or networked multi-choice TU games. Second, stochastic or fuzzy components in the utility functions and in weight distributions may be introduced to better capture uncertainty and ambiguity in real applications. Developing corresponding potential and dividend formulations in such uncertain environments, and deriving axiomatic characterizations for the resulting resolutions, are promising directions for further investigation.

Author contributions

Yan-An Hwang: Conceptualization, methodology, writing—original draft preparation, writing—review and editing; Yu-Hsien Liao: writing—review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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