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Research article

Stationary distribution and extinction of a stochastic SEI epidemic model with logistic growth and nonlinear perturbation

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Abstract: In this paper, we proposed and studied a stochastic SEI (Susceptible-Exposed-Infectious) epidemic model with logistic growth and nonlinear perturbation. By constructing a series of suitable stochastic Lyapunov functions, we derived sufficient conditions for the existence of a unique ergodic stationary distribution. Furthermore, we obtained criteria which ensured the disease approached extinction at an exponential rate. At the same time, under these criteria, the distribution of susceptible individuals converged weakly to a unique invariant probability measure. The numerical simulations supported the theoretical results.

Keywords: stochastic SEI epidemic model; logistic growth; nonlinear perturbation; ergodicity; extinction
Mathematics Subject Classification: 92D25, 92D30, 60H10

1. Introduction

Recently, mathematical models have become indispensable tools in the research and control of infectious diseases worldwide. In 1927, Kermack and McKendrick introduced the first compartmentalized representation of disease transmission, later known as the SIR (Susceptible-Infectious-Removed) epidemic model [1]. Since then, a large number of mathematicians and ecologists have extended this framework to develop various epidemic models, which has enhanced our understanding of disease transmission and control strategies [2–4].

The SIR model is a foundational compartmental model in epidemiology that simulates disease spread by dividing a population into Susceptible (S), Infected (I), and Removed (R) groups. However, many authors argue that susceptible individuals exhibit a non-negligible latent period before infection, which should be incorporated in epidemic models [5, 6]. The latent period refers to the time delay between the initial infection and the host becoming infectious, during which the host is not yet capable of transmitting the disease. Accounting for the latent period enhances the biological realism of models. For example, Cao et al. [7] proposed a deterministic SEI epidemic model with logistic growth which takes the following form

$$\begin{cases} \dfrac{dS(t)}{dt} = rS(t)\left(1 - \dfrac{S(t)}{K}\right) - \dfrac{\beta S(t)I(t)}{1 + mS(t)}, \\ \dfrac{dE(t)}{dt} = \dfrac{\beta S(t)I(t)}{1 + mS(t)} - (\alpha + \mu)E(t), \\ \dfrac{dI(t)}{dt} = \alpha E(t) - \mu I(t), \end{cases} \tag{1.1}$$

where the parameters r, K, β, μ, m and α are positive constants. In system (1.1), $S(t), E(t)$ and $I(t)$ denote the number of individuals who are susceptible, exposed (i.e., in the latent period) and infectious, respectively. The parameters are defined as follows: r, K are the intrinsic birth rate and carrying capacity of susceptible host individuals, β denotes the disease transmission coefficient between compartments S and I, μ is the natural death rate of the exposed and infected individuals, m denotes the reciprocal of the half saturation constant and α is the inverse of latent period. Moreover, they use saturation incidence $\frac{\beta SI}{1+mS}$, which is considered to be more reasonable by many authors [8, 9]. This is because saturation incidence captures the psychological effect that individuals tend to reduce their number of contacts per unit time when the number of infectives becomes large. Cao et al. defined the basic reproduction number $R_0 = \frac{\alpha\beta K}{\mu(1+mK)(\alpha+\mu)}$, which governs the epidemic threshold for system (1.1) [7]. The basic reproduction number, R_0, is the average number of secondary cases generated by a single typical infectious individual in a completely susceptible population—in the absence of any interventions. Specifically, if $R_0 > 1$, the disease becomes endemic, whereas if $R_0 < 1$, the infection chain cannot be sustained, which leads to disease extinction.

However, in natural environments, epidemic dynamics are inevitably subject to environmental noise within ecological systems. The environmental variations have a significant influence on the development of an epidemic [10, 11]. Thus, the stochastic differential equation (SDE) models frequently offer more accurate representations of disease transmission processes [12–14]. In recent years, various forms of stochastic epidemic models with logistic growth have been formulated and studied [15–17]. Multiple modeling approaches which incorporate stochastic perturbations can be employed, each exerting distinct effects on population dynamics: parameter perturbations [18, 19], perturbations around the positive endemic equilibrium [20, 21] and the approach used by Imhof and Walcher [22]. In more realistic circumstances, the spread of epidemics is not only influenced by linear factors, but also by the intensity of interactions within populations, which is often described by nonlinear perturbations. Many authors have suggested that random perturbations may be dependent on square of the variables in a biomathematical model, so the second-order perturbations should be taken into consideration [23–25].

In this paper, we incorporate the nonlinear perturbation into system (1.1), similar to the work of Liu and Jiang [26]. Then system (1.1) can be extended to a stochastic model

$$\begin{cases} dS(t) = \left[rS(t)\left(1 - \dfrac{S(t)}{K}\right) - \dfrac{\beta S(t)I(t)}{1 + mS(t)} \right] dt + S(t)(\sigma_{11} + \sigma_{12}S(t))dB_1(t), \\ dE(t) = \left[\dfrac{\beta S(t)I(t)}{1 + mS(t)} - (\alpha + \mu)E(t) \right] dt + E(t)(\sigma_{21} + \sigma_{22}E(t))dB_2(t), \\ dI(t) = [\alpha E(t) - \mu I(t)]\, dt + I(t)(\sigma_{31} + \sigma_{32}I(t))dB_3(t), \end{cases} \tag{1.2}$$

where $\dot{B}_i(t)$, $i = 1, 2, 3$, are the white noise, which are formally regarded as the derivative of the Brownian motions $B_i(t)$, i.e., $\dot{B}_i(t) = dB_i(t)/dt$, $\sigma_{ij} > 0$, $j = 1, 2$, are the intensities of the white noise.

As far as we know, the dynamical behavior of system (1.2) has not been studied yet. In this paper, we are devoted to studying the dynamics of system (1.2) and establish sufficient conditions for the disease to prevail and disappear. The organization of this paper is as follows: In Section 2, we introduce the preliminaries used throughout the paper. In Section 3, we obtain sufficient conditions for the unique ergodic stationary distribution of system (1.2), via construction of appropriate stochastic Lyapunov functions. In Section 4, we establish sufficient conditions under which the disease dies out exponentially while the distribution of susceptible individuals converges weakly to a unique invariant probability measure. In Section 5, we perform numerical simulations to validate the theoretical results. Finally, we present some concluding remarks and future directions.

2. Preliminaries

Throughout this paper, unless otherwise specified, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). We also let $B_i(t)$ be defined on the complete probability space, $i = 1, 2, 3$. Define

$$\mathbb{R}^d_+ = \{x = (x_1, \cdots, x_d) \in \mathbb{R}^d : x_i > 0, 1 \leq i \leq d\}.$$

We begin by considering a general d-dimensional stochastic differential equation

$$dX(t) = f(X(t), t)dt + g(X(t), t)dB(t) \text{ for } t \geq t_0, \tag{2.1}$$

with the initial value $X(0) = X_0 \in \mathbb{R}^d$. $B(t)$ denotes a d-dimensional standard Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Let $C^{2,1}(\mathbb{R}^d \times [t_0, \infty); \mathbb{R}_+)$ represent the class of all nonnegative functions $V(X, t)$ on $\mathbb{R}^d \times [t_0, \infty)$ that are twice continuously differentiable with respect to X and once with respect to t. The differential operator $\mathcal{L}$ of Eq (2.1) is defined by [27]

$$\mathcal{L} = \frac{\partial}{\partial t} + \sum_{i=1}^{d} f_i(X, t)\frac{\partial}{\partial X_i} + \frac{1}{2}\sum_{i,j=1}^{d} [g^T(X, t)g(X, t)]_{ij}\frac{\partial^2}{\partial X_i \partial X_j}.$$

When applied to a function $V \in C^{2,1}(\mathbb{R}^d \times [t_0, \infty); \mathbb{R}_+)$, this operator yields

$$\mathcal{L}V(X, t) = V_t(X, t) + V_X(X, t)f(X, t) + \frac{1}{2}\text{trace}[g^T(X, t)V_{XX}(X, t)g(X, t)],$$

where $V_t = \frac{\partial V}{\partial t}$, $V_X = (\frac{\partial V}{\partial X_1}, \ldots, \frac{\partial V}{\partial X_d})$, $V_{XX} = (\frac{\partial^2 V}{\partial X_i \partial X_j})_{d\times d}$. Applying Itô's formula [27] to $V(X(t), t)$ for $X(t) \in \mathbb{R}^d$ gives

$$dV(X(t), t) = \mathcal{L}V(X(t), t)dt + V_X(X(t), t)g(X(t), t)dB(t).$$

In order to study the dynamical behavior of an epidemic model, the global existence and positivity of solutions must first be established. Since the proof is standard and similar to the statement of Lemma 2.1 in Lu et al. [25, 27], we put it in the appendix.

Theorem 2.1. *For any given initial value* $(S(0), E(0), I(0)) \in \mathbb{R}^3_+$, *there exists a unique solution* $(S(t), E(t), I(t))$ *of system (1.2) on* $t \geq 0$ *and the solution will remain in* $\mathbb{R}^3_+$ *with probability one, namely,* $(S(t), E(t), I(t)) \in \mathbb{R}^3_+$ *for all* $t \geq 0$ *almost surely (a.s.).*

With the existence and uniqueness of the global positive solution established, we can explore properties like ergodicity and extinction for the stochastic model (1.2) in the following sections.

3. Ergodicity of stochastic system (1.2)

In studying the dynamical behavior of stochastic biological systems, ergodicity is a crucial property [28]. If a stochastic system is ergodic, it admits a unique stationary distribution, which implies that the sample paths of the stochastic process $X(t,\omega)$ will almost surely enter a small neighborhood of the equilibrium state of the corresponding deterministic system, thereby ensuring the weak stability of the stochastic system. In the following analysis, we first define a parameter

$$R_0^S = \frac{\alpha\beta K}{(1+mK)\left(\mu+\frac{\sigma_{31}^2}{2}\right)\left(\alpha+\mu+\frac{\sigma_{21}^2}{2}\right)}\left(1-\frac{\sigma_{11}^2}{2r}-\frac{\sigma_{12}}{2r}(2\sigma_{11}+K\sigma_{12})K\right)^3.$$

In this section, we will focus on the dynamical behavior of system (1.2) provided that $R_0^S > 1$. More specifically, we will establish sufficient conditions for the existence and uniqueness of an ergodic stationary distribution of the system (1.2) by constructing a series of suitable stochastic Lyapunov functions.

Let $X(t)$ be a regular time-homogeneous Markov process in $\mathbb{R}^d$ described by the following stochastic differential equation

$$dX(t) = b(X)dt + \sum_{r=1}^{k} g_r(X)dB_r(t).$$

The diffusion matrix is defined as follows

$$A(x) = (a_{ij}(x)), a_{ij}(x) = \sum_{r=1}^{k} g_r^i(x)g_r^j(x).$$

Lemma 3.1. *[29]. The Markov process* $X(t)$ *has a unique ergodic stationary distribution* $\pi(\cdot)$ *if there exists a bounded domain* $D \subset \mathbb{R}^d$ *with regular boundary* Γ, *having the following properties:*

A_1: There is a positive number M such that $\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq M|\xi|^2,\ x \in D,\ \xi \in \mathbb{R}^d$.

A_2: There exists a nonnegative C^2-function V such that $\mathcal{L}V$ is negative for any $\mathbb{R}^d \setminus D$.

Lemma 3.2. *[30]. For any* $x > 0$, *the following inequality holds*

$$x + \frac{2}{3}x(1-x) \leq x^{\frac{1}{3}}.$$

Theorem 3.3. *Assume that* $R_0^S > 1$, *and* $\sigma_{11}^2 + 2K\sigma_{11}\sigma_{12} + K^2\sigma_{12}^2 < 2r$, *then for any initial value* $(S(0), E(0), I(0)) \in \mathbb{R}_+^3$, *the system (1.2) is ergodic and has a unique stationary distribution.*

Owing to the considerable length of the proof, it is included in the Appendix to enhance the readability of the main text. In the Appendix, we provide a detailed proof of the theorem following a three-step process for reference.

4. Extinction of the disease

In this section, we will derive sufficient conditions that guarantee the exponential extinction of the disease and the weak convergence of the distribution of susceptible individuals to a unique invariant probability measure. Firstly, we give a lemma.

Lemma 4.1. *[24] Consider the following one-dimensional stochastic differential equation*

$$dZ(t) = rZ(t)\left(1 - \frac{Z(t)}{K}\right)dt + Z(t)(\sigma_{11} + \sigma_{12}Z(t))dB_1(t), \tag{4.1}$$

with the initial value $Z(0) = z(0) > 0$. *Then system 4.1 has the ergodic property and the invariant density is given by*

$$\pi(z) = Qz^{-2+\frac{2r}{\sigma_{11}^2}}(\sigma_{11} + \sigma_{12}z)^{-2-\frac{2r}{\sigma_{11}^2}} e^{\frac{2r(\sigma_{11}+K\sigma_{12})}{K\sigma_{11}\sigma_{12}(\sigma_{11}+\sigma_{12}z)}},\ z \in (0, +\infty),$$

where Q *is a constant such that* $\int_0^{+\infty} \pi(z)dz = 1$.

In view of the stochastic comparison theorem for Itô's processes [31], we obtain

$$\mathbb{P}(S(t) \le Z(t), t \ge 0) = 1,\ S(0) = Z(0) = z. \tag{4.2}$$

Define

$$R_0^E = R_0\eta \int_0^{+\infty} |z - \frac{K}{1+mK}|\pi(z)dz + \frac{K}{1+mK}(R_0 - 1)\eta \mathrm{I}_{\{R_0>1\}} - \frac{\sigma_{22}^2 \wedge \sigma_{32}^2}{4},$$

where $\eta = \frac{\mu(1+mK)}{K}$, $\mathrm{I}_{\{R_0>1\}}$ denotes the indicator function, which takes the value 1 if $R_0 > 1$ and 0 otherwise, and $\wedge$ denotes the minimum operator.

Theorem 4.2. *Let* $(S(t), E(t), I(t))$ *be the solution of system (1.2) with any initial value* $(S(0), E(0), I(0)) \in \mathbb{R}_+^3$. *If* $R_0^E < 0$, *then the solution* $(S(t), E(t), I(t))$ *of the model (1.2) follows*

$$\limsup_{t\to\infty} \frac{1}{t} \ln(\rho_1 E(t) + \rho_2 I(t)) < R_0^E < 0 \ a.s.,$$

where

$$\rho_1 = \frac{\alpha K}{\mu(1+mK)(\alpha+\mu)},\ \rho_2 = \frac{\rho_1(\alpha+\mu)}{\alpha} = \frac{K}{\mu(1+mK)}.$$

It is equivalent to the following result

$$\lim_{t\to\infty} E(t) = \lim_{t\to\infty} I(t) = 0 \ a.s..$$

Moreover, the distribution of $S(t)$ *converges weakly to the unique invariant probability measure* μ^* *with density* $\pi(z)$.

Proof. Our proof is motivated by the works of Lu et al. [25]. Define a C^2-function $P(t)$ by

$$P(t) = \rho_1 E + \rho_2 I,$$

where the positive constants ρ_1, ρ_2 will be determined later. Making use of Itô's formula to $P(t)$ leads to

$$d(\ln P) = \mathcal{L}(\ln P)dt + \frac{1}{P}\left[\rho_1(\sigma_{21}E + \sigma_{22}E^2)dB_2(t) + \rho_2(\sigma_{31}I + \sigma_{32}I^2)dB_3(t)\right], \tag{4.3}$$

where

$$\begin{aligned}\mathcal{L}(\ln P) =& \frac{1}{P}\left[\frac{\rho_1\beta SI}{1+mS} - \rho_1(\alpha+\mu)E + \rho_2\alpha E - \rho_2\mu I\right] \\ & - \frac{\rho_1^2\left(\sigma_{21}E + \sigma_{22}E^2\right)^2}{2P^2} - \frac{\rho_2^2\left(\sigma_{31}I + \sigma_{32}I^2\right)^2}{2P^2} \\ =& \frac{1}{P}\left[\frac{\rho_1\beta SI}{1+mS} - \rho_2\mu I - (\rho_1(\alpha+\mu) - \rho_2\alpha)E\right] \\ & - \frac{\rho_1^2\left(\sigma_{21}E + \sigma_{22}E^2\right)^2}{2P^2} - \frac{\rho_2^2\left(\sigma_{31}I + \sigma_{32}I^2\right)^2}{2P^2}.\end{aligned}$$

Let

$$\rho_1 = \frac{\alpha K}{\mu(1+mK)(\alpha+\mu)},\ \rho_2 = \frac{\rho_1(\alpha+\mu)}{\alpha} = \frac{K}{\mu(1+mK)},$$

then we can get that

$$\begin{aligned}\mathcal{L}(\ln P) \leq& \frac{1}{P}\left[\rho_1\beta I\left(S - \frac{K}{1+mK}\right) + \frac{K}{1+mK}(R_0-1)I\right] \\ & - \frac{\rho_1^2\left(\sigma_{21}E + \sigma_{22}E^2\right)^2}{2P^2} - \frac{\rho_2^2\left(\sigma_{31}I + \sigma_{32}I^2\right)^2}{2P^2} \\ \leq& \left[R_0\eta\left(S - \frac{K}{1+mK}\right) + \frac{K}{1+mK}(R_0-1)\eta \mathrm{I}_{\{R_0>1\}}\right] \\ & - \frac{\rho_1^2\left(\sigma_{21}E + \sigma_{22}E^2\right)^2}{2P^2} - \frac{\rho_2^2\left(\sigma_{31}I + \sigma_{32}I^2\right)^2}{2P^2} \ a.s..\end{aligned} \tag{4.4}$$

Integrating from 0 to t and dividing by t on both sides of (4.3), and in view of (4.2) and (4.4), we have

$$\begin{aligned}\frac{\ln P(t)}{t} \leq& \frac{\ln P(0)}{t} + \frac{R_0\eta}{t}\int_0^t \left|Z - \frac{K}{1+mK}\right| d\tau + \frac{K}{1+mK}(R_0-1)\eta \mathrm{I}_{\{R_0>1\}} \\ & + \frac{1}{t}\sum_{i=1}^{2}(M_i(t) - N_i(t)) \ a.s.,\end{aligned}$$

where

$$M_1(t) = \int_0^t \frac{\rho_1(\sigma_{21}E(\tau) + \sigma_{22}E^2(\tau))}{P(\tau)}dB_2(\tau),\ N_1(t) = \int_0^t \frac{\rho_1^2\left(\sigma_{21}E(\tau) + \sigma_{22}E^2(\tau)\right)^2}{2P^2(\tau)}d\tau,$$

$$M_2(t)=\int_0^t \frac{\rho_2(\sigma_{31}I(\tau)+\sigma_{32}I^2(\tau))}{P(\tau)}dB_3(\tau),\ N_2(t)=\int_0^t \frac{\rho_2^2\left(\sigma_{31}I(\tau)+\sigma_{32}I^2(\tau)\right)^2}{2P^2(\tau)}d\tau.$$

The remainder of the proof follows the lines of proof of Theorem 3.1 in Lu et al. [25]. We have

$$\begin{aligned}\limsup_{t\to\infty}\frac{\ln P(t)}{t}&\le R_0\eta\int_0^{+\infty}|z-\frac{K}{1+mK}|\pi(z)dz+\frac{K}{1+mK}(R_0-1)\eta \mathrm{I}_{\{R_0>1\}}-\frac{\sigma_{22}^2\wedge\sigma_{32}^2}{4}\\&=R_0^E<0\ a.s.,\end{aligned}$$

which means $\lim_{t\to\infty}P(t)=0$ a.s., that is to say

$$\lim_{t\to\infty}E(t)=0\ a.s.,\ \ \lim_{t\to\infty}I(t)=0\ a.s..$$

As a result, for any small $\epsilon>0$ there exist t_0 and a set $\Omega_\epsilon\subset\Omega$ such that $P(\Omega_\epsilon)>1-\epsilon$ and $\frac{\beta SI}{1+mS}<\frac{\beta S\epsilon}{1+mS}$ for $t\ge t_0$ and $\omega\in\Omega_\epsilon$. Now from

$$\left[rS\left(1-\frac{S}{K}\right)-\frac{\beta S\epsilon}{1+mS}\right]+S(\sigma_{11}+\sigma_{12}S)dB_1(t)\le dS\le\left[rS\left(1-\frac{S}{K}\right)\right]+S(\sigma_{11}+\sigma_{12}S)dB_1(t),$$

it follows that the distribution of the process $S(t)$ converges weakly to the measure with the density $\pi(z)$. This completes the proof. □

5. Examples and numerical simulations

In this section, we introduce some numerical simulations to illustrate our theoretical results. We numerically simulate the solution of system (1.2) with the initial value $(S(0),E(0),I(0))=(0.6,0.6,0.8)$. For the numerical simulations, we use Milstein's Higher Order Method [32] to obtain the discretization equations of system (1.2):

$$\begin{cases}S_{j+1}=S_j+\left[rS_j\left(1-\frac{S_j}{K}\right)-\frac{\beta S_jI_j}{1+mS_j}\right]\Delta t+(\sigma_{11}S_j+\sigma_{12}S_j^2)\sqrt{\Delta t}\nu_{1,j}\\ \qquad+\frac{1}{2}(\sigma_{11}^2S_j+3\sigma_{11}\sigma_{12}S_j^2+2\sigma_{12}^2S_j^3)(\nu_{1,j}^2-1)\Delta t,\\ E_{j+1}=E_j+\left[\frac{\beta S_jI_j}{1+mS_j}-(\alpha+\mu)E_j\right]\Delta t+(\sigma_{21}E_j+\sigma_{22}E_j^2)\sqrt{\Delta t}\nu_{2,j}\\ \qquad+\frac{1}{2}(\sigma_{21}^2E_j+3\sigma_{21}\sigma_{22}E_j^2+2\sigma_{22}^2E_j^3)(\nu_{2,j}^2-1)\Delta t,\\ I_{j+1}=I_j+\left(\alpha E_j-\mu I_j\right)\Delta t+(\sigma_{31}I_j+\sigma_{32}I_j^2)\sqrt{\Delta t}\nu_{3,j}\\ \qquad+\frac{1}{2}(\sigma_{31}^2I_j+3\sigma_{31}\sigma_{32}I_j^2+2\sigma_{32}^2I_j^3)(\nu_{3,j}^2-1)\Delta t,\end{cases}$$

where the time increment $\Delta t>0$, $\nu_{i,j}$ $(i=1,2,3)$ denote independent Gaussian random variables which follow the distribution $N(0,1)$ for $j=1,2,\cdots,n$.

Example 5.1. *In order to check the existence of an ergodic stationary distribution, we choose the values of the system parameters as follows:* $r = 0.45$*,* $K = 1.4$*,* $\beta = 0.9$*,* $\mu = 0.2$*,* $\gamma = 0.2$*,* $\alpha = 0.3$*,* $m = 1.2$*,* $\sigma_{11} = 0.1$*,* $\sigma_{21} = 0.2$*,* $\sigma_{31} = 0.2$*,* $\sigma_{12} = 0.01$*,* $\sigma_{22} = 0.01$ *and* $\sigma_{32} = 0.05$*. Direct calculation leads to* $R_0^S = 1.1803 > 1$*. In other words, the condition of Theorem 3.3 holds. By Theorem 3.3, we can get that there is an ergodic stationary distribution of system (1.2). The simulation results presented in Figure 1 support our findings.*

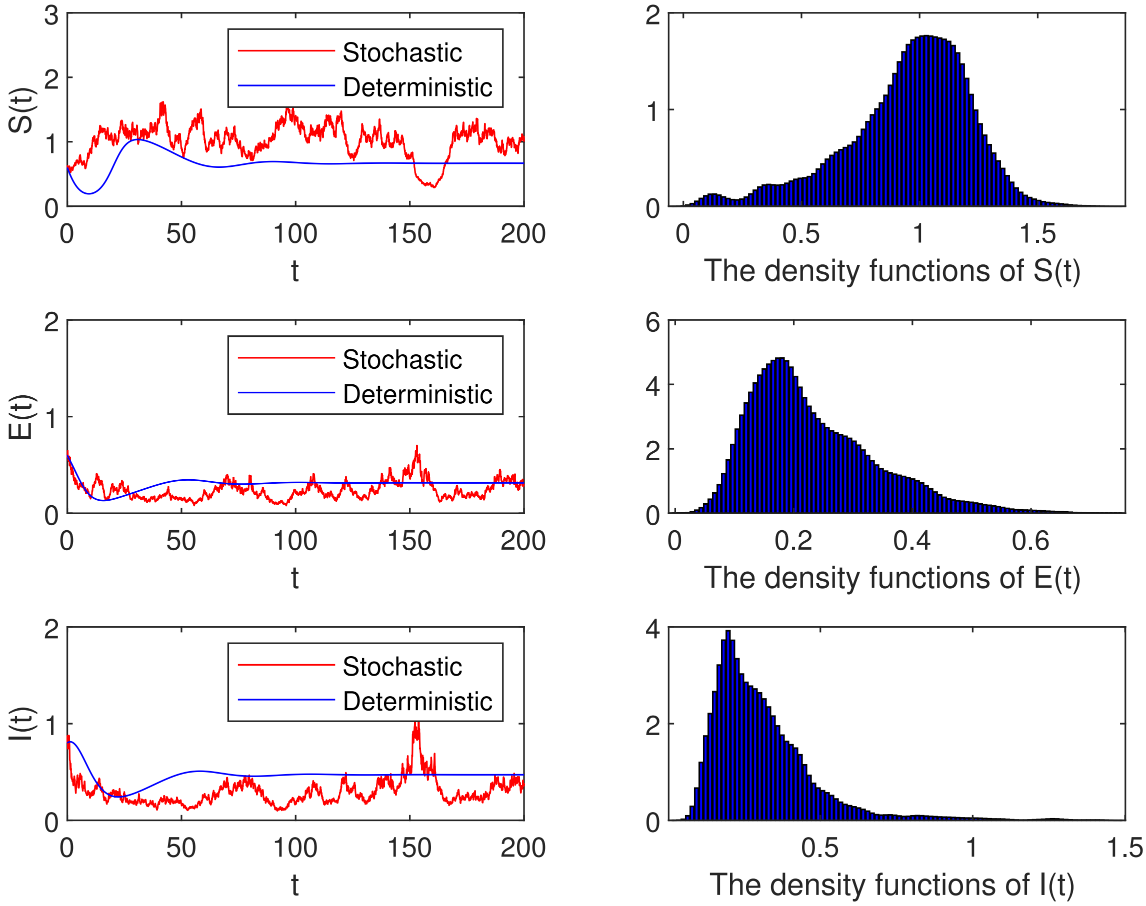

Figure 1. The left column shows the paths of $S(t)$, $E(t)$ and $I(t)$ of system (1.2) with the initial value $(S(0), E(0), I(0)) = (0.6, 0.6, 0.8)$. The right column displays the histogram of the probability density functions of $S(t)$, $E(t)$ and $I(t)$ with perturbations. Parameters are set as follows: $r = 0.45$, $K = 1.4$, $\beta = 0.9$, $\mu = 0.2$, $\gamma = 0.2$, $\alpha = 0.3$, $m = 1.2$, $\sigma_{11} = 0.1$, $\sigma_{21} = 0.2$, $\sigma_{31} = 0.2$, $\sigma_{12} = 0.01$, $\sigma_{22} = 0.01$ and $\sigma_{32} = 0.05$. The condition that $R_0^S > 1$ ensures the existence of an ergodic stationary distribution.

To validate the robustness of the results, we conducted 200 independent simulation runs. A step size of $h = 0.04$ was adopted in the simulations. The average trajectory of the stochastic model outputs is depicted in Figure 2. Given the sensitivity of the higher-order Milstein method to step size selection, We conducted simulations with different step sizes and compared the results, which are presented in Table 1. The presence of slight statistical noise is consistent with the model's stochastic nature and does not affect the overall conclusions. These results demonstrate robust stability in the data. Under such conditions, the disease is not expected to go extinct.

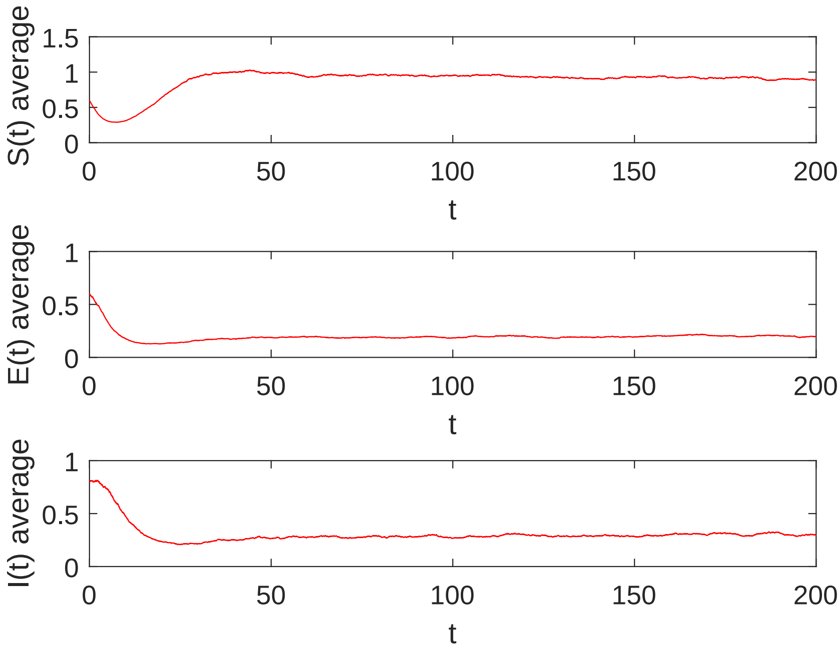

Figure 2. The column is the average trajectory of $S(t)$, $E(t)$ and $I(t)$ of system (1.2) with the initial value $(S(0), E(0), I(0)) = (0.6, 0.6, 0.8)$. We choose the values of the system parameters as follows: $r = 0.45$, $K = 1.4$, $\beta = 0.9$, $\mu = 0.2$, $\gamma = 0.2$, $\alpha = 0.3$, $m = 1.2$, $\sigma_{11} = 0.1$, $\sigma_{21} = 0.2$, $\sigma_{31} = 0.2$, $\sigma_{12} = 0.01$, $\sigma_{22} = 0.01$ and $\sigma_{32} = 0.05$. $R_0^S > 1$ ensures the existence of an ergodic stationary distribution.

Table 1. Comparison of results under different step sizes (Example 5.1).

Step size	95% confidence intervals for $S(t)$, $E(t)$ and $I(t)$	Standard deviations
$h = 0.02$	[0.8878, 0.9782], [0.1698, 0.2015], [0.2539, 0.3079]	0.0256, 0.0081, 0.0138
$h = 0.04$	[0.9048, 1.0078], [0.1644, 0.1959], [0.2412, 0.2988]	0.0263, 0.0080, 0.0147
$h = 0.08$	[0.9050, 1.0046], [0.1860, 0.2244], [0.2757, 0.3452]	0.0254, 0.0098, 0.0177
$h = 0.16$	[0.8992, 1.0033], [0.1620, 0.1950], [0.2326, 0.2907]	0.0266, 0.0084, 0.0148

Example 5.2. *In order to verify the extinction of the disease, the values of the system parameters are selected as follows:* $r = 0.5$, $K = 0.8$, $\beta = 0.85$, $\mu = 0.15$, $\gamma = 0.2$, $\alpha = 0.2$, $m = 1.2$, $\sigma_{11} = 0.6$, $\sigma_{21} = 0.2$, $\sigma_{31} = 0.2$, $\sigma_{12} = 0.02$, $\sigma_{22} = 1.0$, $\sigma_{32} = 1.0$. *By a simple calculation, we obtain* $R_0 = 1.3217 > 1$ *and* $R_0^E = -0.0651 < 0$. *That is to say, while the disease does not go extinct in the deterministic system, we establish that it will die out exponentially with probability one in its stochastic counterpart based on Theorem 4.2. We give the simulations displayed in Figure 3 to support our results. Following the approach in Example 5.1, the average trajectory of 200 simulations of the stochastic model is presented in Figure 4 after removing outliers.*

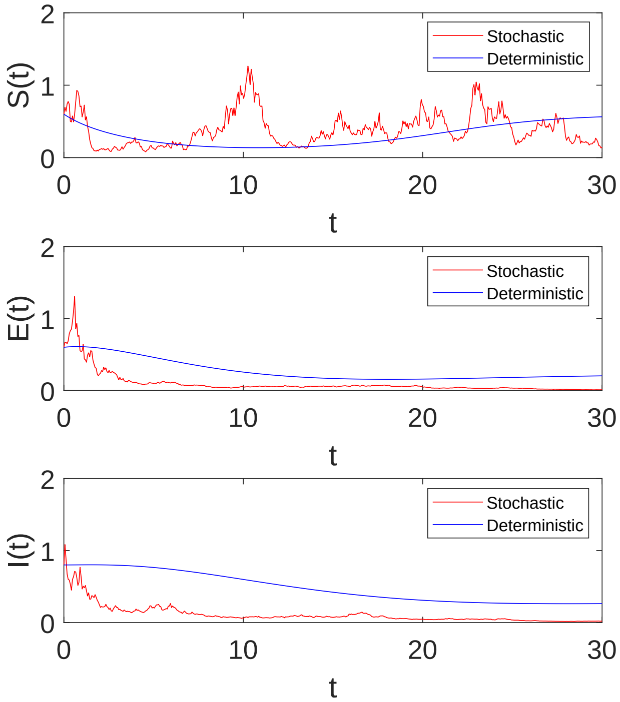

Figure 3. The column is the paths of $S(t)$, $E(t)$ and $I(t)$ of system (1.2) with the initial value $(S(0), E(0), I(0)) = (0.6, 0.6, 0.8)$. Parameter values are chosen as: $r = 0.5$, $K = 0.8$, $\beta = 0.85$, $\mu = 0.15$, $\gamma = 0.2$, $\alpha = 0.2$, $m = 1.2$, $\sigma_{11} = 0.6$, $\sigma_{21} = 0.2$, $\sigma_{31} = 0.2$, $\sigma_{12} = 0.02$, $\sigma_{22} = 1.0$, $\sigma_{32} = 1.0$. We calculate that $R_0 = 1.3217 > 1$ and $R_0^E = -0.0651 < 0$. In this scenario, the deterministic model does not go extinct, but the stochastic model will go extinct.

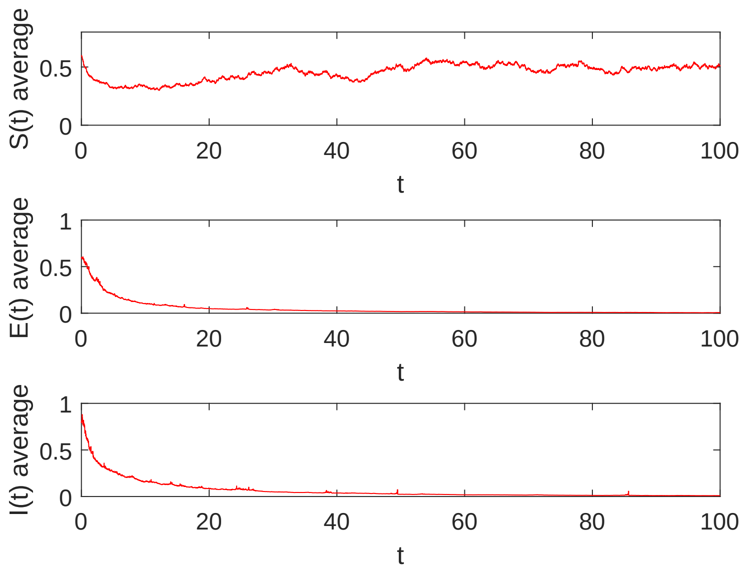

Figure 4. The column is the average trajectory of $S(t)$, $E(t)$ and $I(t)$ of system (1.2) with the initial value $(S(0), E(0), I(0)) = (0.6, 0.6, 0.8)$. Parameter values are chosen as: $r = 0.5$, $K = 0.8$, $\beta = 0.85$, $\mu = 0.15$, $\gamma = 0.2$, $\alpha = 0.2$, $m = 1.2$, $\sigma_{11} = 0.6$, $\sigma_{21} = 0.2$, $\sigma_{31} = 0.2$, $\sigma_{12} = 0.02$, $\sigma_{22} = 1.0$ and $\sigma_{32} = 1.0$. It follows that the stochastic model will go extinct.

Example 5.3. *To confirm the extinction of the disease, we adopt the same parameter values as those in Example 5.2, except with* $\mu = 0.2$*. A straightforward calculation shows that* $R_0 = 0.8673 < 1$ *and* $R_0^E = -0.1304 < 0$*. It follows directly that the condition of Theorem 4.2 is still satisfied. Then we derive the conclusion that the disease dies out exponentially with probability one in the stochastic system (1.2). Meanwhile, the disease is also driven to extinction in the deterministic system (1.1) due to the fact that* $R_0 < 1$*. We provide simulation results in Figure 5. Following the approach in Example 5.1, we show in Figure 6 the average trajectory of 200 simulations of the stochastic model after outlier removal.*

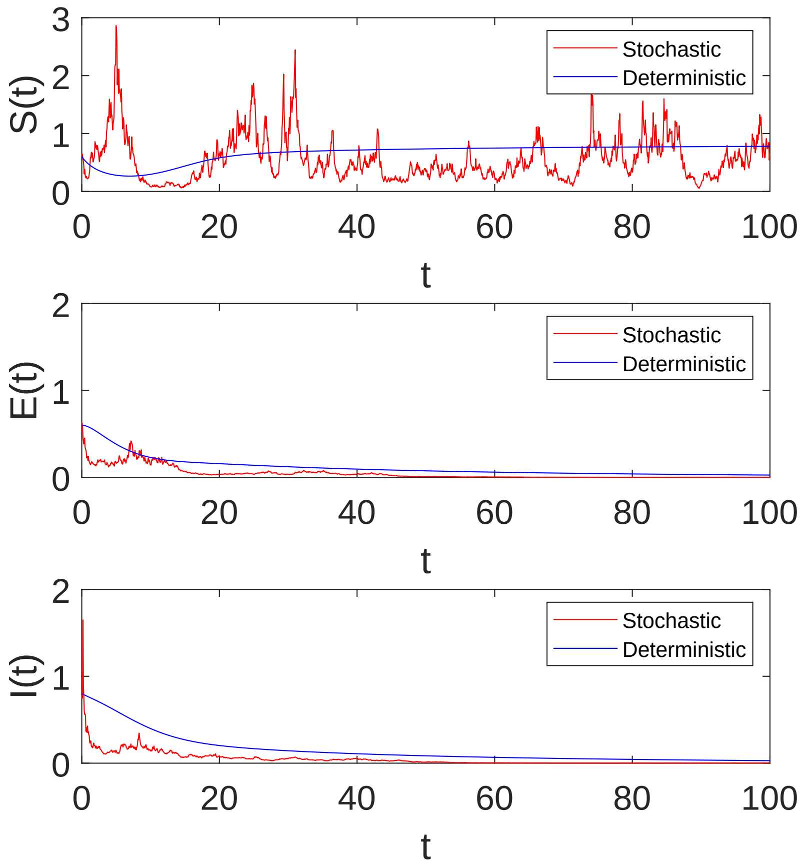

Figure 5. The column displays the paths of $S(t)$, $E(t)$ and $I(t)$ of system (1.2) with the initial value $(S(0), E(0), I(0)) = (0.6, 0.6, 0.8)$. We use the following parameter values: $r = 0.5$, $K = 0.8$, $\beta = 0.85$, $\mu = 0.2$, $\gamma = 0.2$, $\alpha = 0.2$, $m = 1.2$, $\sigma_{11} = 0.6$, $\sigma_{21} = 0.2$, $\sigma_{31} = 0.2$, $\sigma_{12} = 0.02$, $\sigma_{22} = 1.0$, $\sigma_{32} = 1.0$. We find that $R_0 = 0.8673 < 1$ and $R_0^E = -0.1304 < 0$. Under these conditions, both the deterministic and the stochastic models lead to extinction.

Remark 5.1. *For Examples 5.2 and 5.3, analogous simulations conducted with varying step sizes yielded consistent extinction outcomes, confirming the robustness of our results. The simulation results are provided in the appendix for completeness.*

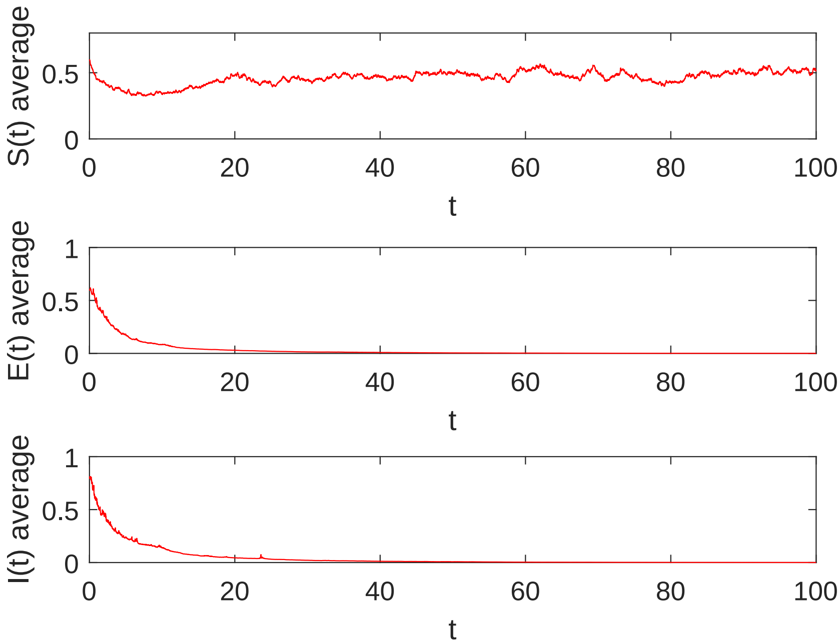

Figure 6. The column is the average trajectory of $S(t)$, $E(t)$ and $I(t)$ of system (1.2) with the initial value $(S(0), E(0), I(0)) = (0.6, 0.6, 0.8)$. Parameter values are chosen as: $r = 0.5$, $K = 0.8$, $\beta = 0.85$, $\mu = 0.2$, $\gamma = 0.2$, $\alpha = 0.2$, $m = 1.2$, $\sigma_{11} = 0.6$, $\sigma_{21} = 0.2$, $\sigma_{31} = 0.2$, $\sigma_{12} = 0.02$, $\sigma_{22} = 1.0$ and $\sigma_{32} = 1.0$. In this case, the stochastic system undergoes extinction.

6. Conclusions

In this paper, we considered a stochastic SEI epidemic model with logistic growth and nonlinear perturbation. It is evident that systems subjected to nonlinear stochastic noise pose substantially more analytical difficulties than their linear counterparts. By constructing stochastic Lyapunov functions, we studied the dynamics of the stochastic model (1.2) separately under the conditions $R_0^S > 1$ and $R_0^E < 0$.

Our analysis reveals that R_0^S is less than the basic reproduction number R_0 of the deterministic system (1.1), making it less likely for the disease to become endemic. Furthermore, from the expression of R_0^E, we can also see that an increase in the second-order noise intensity of $E(t)$ and $I(t)$ (i.e., σ_{22} and σ_{32}) may lead to the extinction of the disease. The findings of this paper indicate that the presence of perturbations has a suppressive effect on disease transmission.

Some interesting topics deserve further investigation. On the one hand, one can consider a model with reaction-diffusion [33–35] or explore the optimal control [36] in epidemic models. On the other hand, one can extend the results in this work by considering some discontinuous perturbations such as Lévy noise [37–39]. In addition, Numerous studies have shown that factors such as media coverage, awareness diffusion, and behavioral adaptations among susceptible populations significantly influence the transmission of infectious diseases [40, 41]. Therefore, future research could focus on developing relevant stochastic models. Furthermore, it is noteworthy that when an epidemic model incorporates elements such as nonlinearity, periodicity, and time delays, its dynamical behavior may evolve from simple stability or periodic oscillations into highly complex and long-term unpredictable chaos [42–44]. This evolution also represents a significant and promising direction for our future research. We leave these problems for our future work.

Author contributions

Zeyu Xu: Software, writing-original draft; Liang Wang: Methodology, supervision, funding acquisition, writing-original draft, writing-review & editing, formal analysis. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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Appendices

A. Proof of Theorems 2.1 and 3.3

We now prove Theorem 2.1 on the existence and uniqueness of the global positive solution to the stochastic model (1.2).

Proof. Since the coefficients of system (1.2) satisfy the local Lipschitz condition, then for any given initial value $(S(0), E(0), I(0)) \in \mathbb{R}_+^3$, there is a unique local solution $(S(t), E(t), I(t)) \in \mathbb{R}_+^3$ on $t \in [0, \tau_e)$ a.s., where τ_e denotes the explosion time [27]. To show this solution is global, we only need to prove that $\tau_e = \infty$ a.s.. To this end, let $n_0 \geq 1$ be sufficiently large such that $S(0)$, $E(0)$, $I(0)$ all lie within the interval $\left[\frac{1}{n_0}, n_0\right]$. For each integer $n \geq n_0$, define the stopping time as [27]

$$\tau_n = \inf\left\{t \in [0, \tau_e) : \min\{S(t), E(t), I(t)\} \leq \frac{1}{n} \text{ or } \max\{S(t), E(t), I(t)\} \geq n\right\},$$

where we set $\inf \emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). Obviously, τ_n is increasing as $n \to \infty$. Set $\tau_\infty = \lim_{n\to\infty} \tau_n$, whence $\tau_\infty \leq \tau_e$ a.s. If $\tau_\infty = \infty$ a.s. is true, then $\tau_e = \infty$ a.s. and $(S(t), E(t), I(t)) \in \mathbb{R}_+^3$ a.s. for all $t \geq 0$. That is to say, to complete the proof we only need to prove $\tau_\infty = \infty$ a.s.. If this assertion is false, then there is a pair of constants $T > 0$ and $\epsilon \in (0, 1)$ such that

$$\mathbb{P}\{\tau_\infty \leq T\} > \epsilon.$$

Thus, there is an integer $n_1 \geq n_0$ such that

$$\mathbb{P}\{\tau_n \leq T\} \geq \epsilon, \ \forall n \geq n_1. \tag{A.1}$$

Define a C^2-function $W : \mathbb{R}_+^3 \to \mathbb{R}_+ \cup \{0\}$, for $0 < p < 1$, by

$$W(S, E, I) = \left(\frac{S^p}{p} - 1 - \ln S\right) + \left(\frac{E^p}{p} - 1 - \ln E\right) + \left(\frac{I^p}{p} - 1 - \ln I\right).$$

Applying Itô's formula [27], we have

$$\mathcal{L}\left(\frac{S^p}{p}-1-\ln S\right)=S^{p-1}\left[rS\left(1-\frac{S}{K}\right)-\frac{\beta SI}{1+mS}\right]-\frac{1}{2}(1-p)S^p(\sigma_{11}+\sigma_{12}S)^2 - \frac{1}{S}\left[rS\left(1-\frac{S}{K}\right)-\frac{\beta SI}{1+mS}\right]+\frac{1}{2}(\sigma_{11}+\sigma_{12}S)^2, \tag{A.2}$$

$$\mathcal{L}\left(\frac{E^p}{p}-1-\ln E\right)=E^{p-1}\left[\frac{\beta SI}{1+mS}-(\alpha+\mu)E\right]-\frac{1}{2}(1-p)E^p(\sigma_{21}+\sigma_{22}E)^2 - \frac{1}{E}\left[\frac{\beta SI}{1+mS}-(\alpha+\mu)E\right]+\frac{1}{2}(\sigma_{21}+\sigma_{22}E)^2, \tag{A.3}$$

$$\mathcal{L}\left(\frac{I^p}{p}-1-\ln I\right)=I^{p-1}\left[\alpha E-\mu I\right]-\frac{1}{2}(1-p)I^p(\sigma_{31}+\sigma_{32}I)^2-\frac{1}{I}\left[\alpha E-\mu I\right]+\frac{1}{2}(\sigma_{31}+\sigma_{32}I)^2. \tag{A.4}$$

In view of (A.2)–(A.4), we obtain

$$\begin{aligned}
\mathcal{L}W(S,E,I)=&S^{p-1}\left[rS\left(1-\frac{S}{K}\right)-\frac{\beta SI}{1+mS}\right]-\frac{1}{2}(1-p)S^p(\sigma_{11}+\sigma_{12}S)^2-\frac{1}{S}\left[rS\left(1-\frac{S}{K}\right)-\frac{\beta SI}{1+mS}\right]\\
&+\frac{1}{2}(\sigma_{11}+\sigma_{12}S)^2+E^{p-1}\left[\frac{\beta SI}{1+mS}-(\alpha+\mu)E\right]-\frac{1}{2}(1-p)E^p(\sigma_{21}+\sigma_{22}E)^2\\
&-\frac{1}{E}\left[\frac{\beta SI}{1+mS}-(\alpha+\mu)E\right]+\frac{1}{2}(\sigma_{21}+\sigma_{22}E)^2+I^{p-1}\left[\alpha E-\mu I\right]-\frac{1}{2}(1-p)I^p(\sigma_{31}+\sigma_{32}I)^2\\
&-\frac{1}{I}\left[\alpha E-\mu I\right]+\frac{1}{2}(\sigma_{31}+\sigma_{32}I)^2\\
\le& rS^p+\frac{r}{K}S+\frac{\beta I}{1+mS}+\frac{1}{2}(\sigma_{11}^2+2\sigma_{11}\sigma_{12}S+\sigma_{12}^2S^2)+\left(\frac{1}{E^{1-p}}-\frac{1}{E}\right)\frac{\beta SI}{1+mS}\\
&+(\alpha+\mu)+\frac{1}{2}(\sigma_{21}^2+2\sigma_{21}\sigma_{22}E+\sigma_{22}^2E^2)+\left(\frac{1}{I^{1-p}}-\frac{1}{I}\right)\alpha E+\mu\\
&+\frac{1}{2}(\sigma_{31}^2+2\sigma_{31}\sigma_{32}I+\sigma_{32}^2I^2)-\frac{1}{2}(1-p)\sigma_{12}^2S^{p+2}-\frac{1}{2}(1-p)\sigma_{22}^2E^{p+2}-\frac{1}{2}(1-p)\sigma_{32}^2I^{p+2}\\
\le& \sup_{(S,E,I)\in\mathbb{R}_+^3}\left\{rS^p+\frac{r}{K}S+\frac{\beta I}{1+mS}+\frac{1}{2}(2\sigma_{11}\sigma_{12}S+\sigma_{12}^2S^2)+\left(\frac{1}{E^{1-p}}-\frac{1}{E}\right)\frac{\beta SI}{1+mS}\right.\\
&+\frac{1}{2}(2\sigma_{21}\sigma_{22}E+\sigma_{22}^2E^2)+\left(\frac{1}{I^{1-p}}-\frac{1}{I}\right)\alpha E+\frac{1}{2}(2\sigma_{31}\sigma_{32}I+\sigma_{32}^2I^2)-\frac{1}{2}(1-p)\sigma_{12}^2S^{p+2}\\
&\left.-\frac{1}{2}(1-p)\sigma_{22}^2E^{p+2}-\frac{1}{2}(1-p)\sigma_{32}^2I^{p+2}\right\}+\frac{1}{2}(\sigma_{11}^2+\sigma_{21}^2+\sigma_{31}^2)+2\mu+\alpha:=\bar{K},
\end{aligned}$$

where $\bar{K}$ is a positive constant. Thus

$$dW(S,E,I)\le\bar{K}dt+(S^p-1)(\sigma_{11}+\sigma_{12}S)dB_1(t)+(E^p-1)(\sigma_{21}+\sigma_{22}E)dB_2(t) + (I^p-1)(\sigma_{31}+\sigma_{32}I)dB_3(t). \tag{A.5}$$

Integrating (A.5) from 0 to $\tau_k\wedge T=\min\{\tau_k,T\}$ and then taking the expectation on both sides, we have

$$\mathbb{E}W(S(\tau_k\wedge T),E(\tau_k\wedge T),I(\tau_k\wedge T))\le W(S(0),E(0),I(0))+\bar{K}\mathbb{E}(\tau_k\wedge T).$$

Hence

$$\mathbb{E}W(S(\tau_k \wedge T), E(\tau_k \wedge T), I(\tau_k \wedge T)) \le V(S(0), E(0), I(0)) + \bar{K}T. \tag{A.6}$$

Set $\Omega_k = \{\tau_k \le T\}$ for $k \ge k_1$ and by (A.1), we have $\mathbb{P}(\Omega_k) \ge \epsilon$. Note that for every $\omega \in \Omega_k$, there is $S(\tau_k, \omega)$, $E(\tau_k, \omega)$ or $I(\tau_k, \omega)$ equals either k or $\frac{1}{k}$. So $V(S(\tau_k, \omega), E(\tau_k, \omega), I(\tau_k, \omega))$ is no less than either

$$\frac{k^p}{p} - 1 - \ln p \text{ or } \frac{(\frac{1}{k})^p}{p} - 1 - \ln\frac{1}{k} = \frac{1}{pk^p} - 1 + \ln k.$$

Consequently,

$$W(S(\tau_k, \omega), E(\tau_k, \omega), I(\tau_k, \omega)) \ge \left(\frac{k^p}{p} - 1 - \ln p\right) \wedge \left(\frac{1}{pk^p} - 1 + \ln k\right).$$

In view of (A.6), we obtain

$$\begin{aligned} W(S(0), E(0), I(0)) + \bar{K}T \ge& \mathbb{E}[\mathrm{I}_{\Omega_k(\omega)} W(S(\tau_k, \omega), E(\tau_k, \omega), I(\tau_k, \omega))] \\ \ge& \epsilon\left[\left(\frac{k^p}{p} - 1 - \ln p\right) \wedge \left(\frac{1}{pk^p} - 1 + \ln k\right)\right], \end{aligned}$$

where I_{Ω_k} denotes the indicator function of Ω_k. Letting $k \to \infty$, then we obtain

$$\infty > W(S(0), E(0), I(0)) + \bar{K}T = \infty,$$

which leads to the contradiction. So we must have $\tau_\infty = \infty$ a.s.. This completes the proof. □

We now proceed with the proof of Theorem 3.3 in the following three steps.

Proof. In order to prove Theorem 3.3, we only need to validate conditions A_1 and A_2 in Lemma 3.1.
Step 1. We first prove the condition A_1. The diffusion matrix of system (1.2) is given by

$$A = \begin{pmatrix} \left(\sigma_{11}S + \sigma_{12}S^2\right)^2 & 0 & 0 \\ 0 & \left(\sigma_{21}E + \sigma_{22}E^2\right)^2 & 0 \\ 0 & 0 & \left(\sigma_{31}I + \sigma_{32}I^2\right)^2 \end{pmatrix}.$$

Choose $\tilde{M} = \min_{(S,E,I)\in\bar{D}_k\subset\mathbb{R}^3_+} \{(\sigma_{11}S + \sigma_{12}S^2)^2, (\sigma_{21}E + \sigma_{22}E^2)^2, (\sigma_{31}I + \sigma_{32}I^2)^2\}$. We have

$$\begin{aligned} \sum_{i,j=1}^{3} a_{ij}(S, E, I)\xi_i\xi_j &= ((\sigma_{11}S + \sigma_{12}S^2)\xi_1 \quad (\sigma_{21}E + \sigma_{22}E^2)\xi_2 \quad (\sigma_{31}I + \sigma_{32}I^2)\xi_3) \begin{pmatrix} (\sigma_{11}S + \sigma_{12}S^2)\xi_1 \\ (\sigma_{21}E + \sigma_{22}E^2)\xi_2 \\ (\sigma_{31}I + \sigma_{32}I^2)\xi_3 \end{pmatrix} \\ &= (\sigma_{11}S + \sigma_{12}S^2)^2\xi_1^2 + (\sigma_{21}E + \sigma_{22}E^2)^2\xi_2^2 + (\sigma_{31}I + \sigma_{32}I^2)^2\xi_3^2 \\ &\ge \tilde{M}\,\|\xi\|^2 \quad \text{for any } (S, E, I) \in \bar{D}_k, \quad \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3, \end{aligned}$$

where $\bar{D}_k = [\frac{1}{k}, k] \times [\frac{1}{k}, k] \times [\frac{1}{k}, k]$, so the condition A_1 in Lemma 3.1 holds.

Step 2. Now we prove the condition A_2. The construction of a suitable Lyapunov function will be carried out step by step in this phase of the proof. In view of system (1.2), we have

$$\mathcal{L}S = rS\left(1-\frac{S}{k}\right)-\frac{\beta SI}{1+mS} \le rS\left(1-\frac{S}{k}\right) \le -\frac{r}{m}(1+mS)+\frac{r(1+mK)}{m}, \tag{A.7}$$

$$\mathcal{L}(-\ln E) = -\frac{\beta SI}{E(1+mS)}+(\alpha+\mu)+\left(\frac{\sigma_{21}^2}{2}+\sigma_{21}\sigma_{22}E+\frac{\sigma_{22}^2}{2}E^2\right), \tag{A.8}$$

and

$$\mathcal{L}(-\ln I) = -\frac{\alpha E}{I}+\mu+\left(\frac{\sigma_{31}^2}{2}+\sigma_{31}\sigma_{32}I+\frac{\sigma_{32}^2}{2}I^2\right).$$

Define

$$V_1(E) = -\ln E+\frac{d_1}{p}(\sigma_{21}+\sigma_{22}E)^p,\ V_2(I) = -c_1\ln I+\frac{c_1 d_2(\sigma_{31}+\sigma_{32}I)^p}{p}+\frac{c_1\sigma_{31}\sigma_{32}}{\mu}I,$$

where $p \in (0,1)$ and c_1, d_1 and d_2 are positive constants which will be determined later. Applying Itô's formula to $V_1(E)$ [27], we obtain

$$\begin{aligned}
\mathcal{L}V_1 =& -\frac{\beta SI}{E(1+mS)}+(\alpha+\mu)+\left(\frac{\sigma_{21}^2}{2}+\sigma_{21}\sigma_{22}E+\frac{\sigma_{22}^2}{2}E^2\right)\\
&+d_1\sigma_{22}(\sigma_{21}+\sigma_{22}E)^{p-1}\left[\frac{\beta SI}{1+mS}-(\alpha+\mu)E\right]-\frac{1}{2}d_1\sigma_{22}^2(1-p)(\sigma_{21}+\sigma_{22}E)^pE^2\\
\le& -\frac{\beta SI}{E(1+mS)}+(\alpha+\mu)+\left(\frac{\sigma_{21}^2}{2}+\sigma_{21}\sigma_{22}E+\frac{\sigma_{22}^2}{2}E^2\right)\\
&+d_1\sigma_{22}\sigma_{21}^{p-1}\frac{\beta SI}{1+mS}-\frac{1}{2}d_1\sigma_{22}^2(1-p)\sigma_{21}^pE^2.
\end{aligned}$$

Choose $d_1 = \frac{1}{(1-p)\sigma_{21}^p}$, we have

$$\begin{aligned}
\mathcal{L}V_1 \le& -\frac{\beta SI}{E(1+mS)}+(\alpha+\mu)+\left(\frac{\sigma_{21}^2}{2}+\sigma_{21}\sigma_{22}E+\frac{\sigma_{22}^2}{2}E^2\right)+\frac{\sigma_{22}}{(1-p)\sigma_{21}}\frac{\beta SI}{1+mS}-\frac{\sigma_{22}^2}{2}E^2\\
\le& -\frac{\beta SI}{E(1+mS)}+\left(\alpha+\mu+\frac{\sigma_{21}^2}{2}\right)+\sigma_{21}\sigma_{22}E+\frac{\sigma_{22}\beta}{(1-p)\sigma_{21}m}I.
\end{aligned} \tag{A.9}$$

Next, making use of Itô's formula to $V_2(I)$, we get

$$\begin{aligned}
\mathcal{L}V_2 =& -\frac{c_1\alpha E}{I}+c_1\mu+c_1\left(\frac{\sigma_{31}^2}{2}+\sigma_{31}\sigma_{32}I+\frac{\sigma_{32}^2}{2}I^2\right)+c_1d_2\sigma_{32}(\sigma_{31}+\sigma_{32}I)^{p-1}(\alpha E-\mu I)\\
&-\frac{1}{2}c_1d_2\sigma_{32}^2(1-p)(\sigma_{31}+\sigma_{32}I)^pI^2+\frac{c_1\sigma_{31}\sigma_{32}}{\mu}(\alpha E-\mu I)\\
\le& -\frac{c_1\alpha E}{I}+c_1\mu+c_1\left(\frac{\sigma_{31}^2}{2}+\sigma_{31}\sigma_{32}I+\frac{\sigma_{32}^2}{2}I^2\right)+c_1d_2\sigma_{32}\sigma_{31}^{p-1}(\alpha E-\mu I)\\
&-\frac{1}{2}c_1d_2\sigma_{32}^2(1-p)\sigma_{31}^pI^2+\frac{c_1\sigma_{31}\sigma_{32}}{\mu}(\alpha E-\mu I).
\end{aligned}$$

Choose $d_2 = \frac{1}{(1-p)\sigma_{31}^p}$, then

$$\mathcal{L}V_2 \leq -\frac{c_1\alpha E}{I} + c_1\left(\mu + \frac{\sigma_{31}^2}{2}\right) + \left(\frac{c_1\sigma_{32}\alpha}{(1-p)\sigma_{31}} + \frac{c_1\sigma_{31}\sigma_{32}\alpha}{\mu}\right)E. \tag{A.10}$$

Define

$$V_3(S, E, I) = V_1(E) + V_2(I) + c_2 S,$$

where c_2 is a positive constant to be determined later. By (A.7)–(A.10), we obtain

$$\begin{aligned}
\mathcal{L}V_3 \leq &-\frac{\beta SI}{E(1+mS)} - \frac{c_1\alpha E}{I} - \frac{c_2 r}{m}(1+mS) + \left(\alpha + \mu + \frac{\sigma_{21}^2}{2}\right) + c_1\left(\mu + \frac{\sigma_{31}^2}{2}\right) \\
&+ \frac{c_2 r(1+mK)}{m} + \frac{\sigma_{22}\beta}{(1-p)\sigma_{21}m}I + \left(\frac{c_1\sigma_{31}\sigma_{32}\alpha}{\mu} + \frac{c_1\sigma_{32}\alpha}{(1-p)\sigma_{31}} + \sigma_{21}\sigma_{22}\right)E \\
\leq &-3\sqrt[3]{\frac{c_1c_2\alpha\beta r}{m}}\sqrt[3]{S} + \left(\alpha + \mu + \frac{\sigma_{21}^2}{2}\right) + c_1\left(\mu + \frac{\sigma_{31}^2}{2}\right) + \frac{c_2 r(1+mK)}{m} \\
&+ \frac{\sigma_{22}\beta}{(1-p)\sigma_{21}m}I + \left(\frac{c_1\sigma_{31}\sigma_{32}\alpha}{\mu} + \frac{c_1\sigma_{32}\alpha}{(1-p)\sigma_{31}} + \sigma_{21}\sigma_{22}\right)E \\
= &-3\sqrt[3]{\frac{c_1c_2\alpha\beta r}{m}}\sqrt[3]{S} + \left(\alpha + \mu + \frac{\sigma_{21}^2}{2}\right) + c_1\left(\mu + \frac{\sigma_{31}^2}{2}\right) + \frac{c_2 r(1+mK)}{m} + K_1 I + K_2 E,
\end{aligned} \tag{A.11}$$

where

$$K_1 = \frac{\sigma_{22}\beta}{(1-p)\sigma_{21}m} \text{ and } K_2 = \frac{c_1\sigma_{31}\sigma_{32}\alpha}{\mu} + \frac{c_1\sigma_{32}\alpha}{(1-p)\sigma_{31}} + \sigma_{21}\sigma_{22}.$$

Additionally, we have

$$\mathcal{L}\left(\frac{S}{rK}\right) = \frac{S}{K}\left(1 - \frac{S}{K}\right) - \frac{\beta SI}{rK(1+mS)} \leq \frac{S}{K}\left(1 - \frac{S}{K}\right) \tag{A.12}$$

and

$$\begin{aligned}
\mathcal{L}\left(-\frac{\ln S}{r}\right) &= -\left(1 - \frac{S}{K}\right) + \frac{\beta I}{r(1+mS)} + \frac{1}{r}\left(\frac{\sigma_{11}^2}{2} + \sigma_{11}\sigma_{12}S + \frac{\sigma_{12}^2}{2}S^2\right) \\
&\leq \frac{S}{K} - 1 + \frac{\beta I}{r} + \frac{\sigma_{11}^2}{2r} + \frac{\sigma_{11}\sigma_{12}}{r}S + \frac{\sigma_{12}^2}{2r}S^2,
\end{aligned}$$

then we have

$$\begin{aligned}
\mathcal{L}\left(-\frac{\ln S}{r} + \frac{K\sigma_{12}^2}{2r^2}S\right) &\leq \frac{S}{K} - 1 + \frac{\beta I}{r} + \frac{\sigma_{11}^2}{2r} + \frac{\sigma_{11}\sigma_{12}}{r}S + \frac{\sigma_{12}^2}{2r}S^2 + \frac{K\sigma_{12}^2 S}{2r} - \frac{\sigma_{12}^2}{2r}S^2 \\
&= \frac{S}{K} - 1 + \frac{\beta I}{r} + \frac{\sigma_{11}^2}{2r} + \frac{\sigma_{12}}{2r}(2\sigma_{11} + K\sigma_{12})S.
\end{aligned}$$

Noting that

$$rS(1-\frac{S}{K})\le r(K-S),$$

we obtain

$$\mathcal{L}\left(\frac{1}{r}\cdot\frac{\sigma_{12}}{2r}(2\sigma_{11}+K\sigma_{12})S\right)\le\frac{\sigma_{12}}{2r}(2\sigma_{11}+K\sigma_{12})K-\frac{\sigma_{12}}{2r}(2\sigma_{11}+K\sigma_{12})S.$$

Thus, define

$$V_4(S)=-\frac{\ln S}{r}+\frac{K\sigma_{12}^2}{2r^2}S+\frac{\sigma_{12}}{2r^2}(2\sigma_{11}+K\sigma_{12})S.$$

We have

$$\mathcal{L}V_4\le\frac{S}{K}-1+\frac{\beta I}{r}+\frac{\sigma_{11}^2}{2r}+\frac{\sigma_{12}}{2r}(2\sigma_{11}+K\sigma_{12})K. \tag{A.13}$$

Define

$$V_5(S)=V_4(S)+\frac{2}{3}\cdot\frac{S}{rK}.$$

In view of (A.12) and (A.13), it follows from Lemma 3.2 that

$$\begin{aligned}\mathcal{L}V_5&\le\frac{S}{K}+\frac{2}{3}\frac{S}{K}\left(1-\frac{S}{K}\right)-1+\frac{\beta I}{r}+\frac{\sigma_{11}^2}{2r}+\frac{\sigma_{12}}{2r}(2\sigma_{11}+K\sigma_{12})K\\&\le\frac{\sqrt[3]{S}}{\sqrt[3]{K}}-1+\frac{\beta I}{r}+\frac{\sigma_{11}^2}{2r}+\frac{\sigma_{12}}{2r}(2\sigma_{11}+K\sigma_{12})K.\end{aligned} \tag{A.14}$$

Define

$$V_6(S,E,I)=V_3(S,E,I)+3\sqrt[3]{\frac{c_1c_2\alpha\beta rK}{m}}V_5(S),$$

then by (A.11) and (A.14), we get

$$\begin{aligned}\mathcal{L}V_6\le&\left(\alpha+\mu+\frac{\sigma_{21}^2}{2}\right)+c_1\left(\mu+\frac{\sigma_{31}^2}{2}\right)+\frac{c_2r(1+mK)}{m}\\&-3\sqrt[3]{\frac{c_1c_2\alpha\beta rK}{m}}\left(1-\frac{\sigma_{11}^2}{2r}-\frac{\sigma_{12}}{2r}(2\sigma_{11}+K\sigma_{12})K\right)+\left(K_1+\frac{3\beta}{r}\sqrt[3]{\frac{c_1c_2\alpha\beta rK}{m}}\right)I+K_2E.\end{aligned} \tag{A.15}$$

Let

$$c_1=\frac{\alpha\beta K\left(1-\frac{\sigma_{11}^2}{2r}-\frac{\sigma_{12}}{2r}(2\sigma_{11}+K\sigma_{12})K\right)^3}{(1+mK)\left(\mu+\frac{\sigma_{31}^2}{2}\right)^2},\ c_2=\frac{\alpha\beta Km\left(1-\frac{\sigma_{11}^2}{2r}-\frac{\sigma_{12}}{2r}(2\sigma_{11}+K\sigma_{12})K\right)^3}{r(1+mK)^2\left(\mu+\frac{\sigma_{31}^2}{2}\right)}.$$

By (A.15), we have

$$
\begin{aligned}
\mathcal{L}V_6 \leq & \left(\alpha + \mu + \frac{\sigma_{21}^2}{2}\right) - \frac{\alpha\beta K}{(1+mK)\left(\mu + \frac{\sigma_{31}^2}{2}\right)}\left(1 - \frac{\sigma_{11}^2}{2r} - \frac{\sigma_{12}}{2r}(2\sigma_{11} + K\sigma_{12})K\right)^3 \\
& + \left(K_1 + \frac{3\beta}{r}\sqrt[3]{\frac{c_1c_2\alpha\beta rK}{m}}\right)I + K_2E \\
= & -\left(\alpha + \mu + \frac{\sigma_{21}^2}{2}\right)\left(R_0^S - 1\right) + \left(K_1 + \frac{3\beta}{r}\sqrt[3]{\frac{c_1c_2\alpha\beta rK}{m}}\right)I + K_2E,
\end{aligned}
$$

where

$$
R_0^S = \frac{\alpha\beta K}{(1+mK)\left(\mu + \frac{\sigma_{31}^2}{2}\right)\left(\alpha + \mu + \frac{\sigma_{21}^2}{2}\right)}\left(1 - \frac{\sigma_{11}^2}{2r} - \frac{\sigma_{12}}{2r}(2\sigma_{11} + K\sigma_{12})K\right)^3.
$$

From the inequality $\sigma_{11}^2 + 2K\sigma_{11}\sigma_{12} + K^2\sigma_{12}^2 < 2r$, it follows that R_0^S is positive definite in the above equation. Then define

$$
V_7(S, E, I) = V_6(S, E, I) - \frac{K_2}{\alpha}I,
$$

then we have

$$
\begin{aligned}
\mathcal{L}V_7 \leq & -\left(\alpha + \mu + \frac{\sigma_{21}^2}{2}\right)\left(R_0^S - 1\right) + \left(K_1 + \frac{3\beta}{r}\sqrt[3]{\frac{c_1c_2\alpha\beta rK}{m}} + \frac{\mu K_2}{\alpha}\right)I \\
= & -\left(\alpha + \mu + \frac{\sigma_{21}^2}{2}\right)\left(R_0^S - 1\right) + K_3I,
\end{aligned}
$$

where

$$
K_3 = K_1 + \frac{3\beta}{r}\sqrt[3]{\frac{c_1c_2\alpha\beta rK}{m}} + \frac{\mu K_2}{\alpha}.
$$

Noting that

$$
\mathcal{L}\left(\frac{\alpha}{\alpha + \mu}E + I\right) = \frac{\alpha\beta SI}{(\alpha + \mu)(1 + mS)} - \mu I,
$$

then define

$$
V_8(S, E, I) = \frac{K_3}{\mu}\left(\frac{\alpha}{\alpha + \mu}E + I\right) + V_7(S, E, I).
$$

We obtain

$$
\mathcal{L}V_8 \leq -\left(\alpha + \mu + \frac{\sigma_{21}^2}{2}\right)\left(R_0^S - 1\right) + \frac{\alpha\beta K_3 SI}{\mu(\alpha + \mu)(1 + mS)}
$$

$$\leq -\left(\alpha + \mu + \frac{\sigma_{21}^2}{2}\right)\left(R_0^S - 1\right) + \frac{\alpha\beta K_3}{\mu(\alpha+\mu)}SI. \tag{A.16}$$

Define

$$V_9(S,E,I) = \frac{(\sigma_{11}+\sigma_{12}S)^p}{p} + \frac{(\sigma_{21}+\sigma_{22}E)^p}{p} + \frac{(\sigma_{31}+\sigma_{32}I)^p}{p},$$

where $\sigma_{ij} > 0$, $i = 1,2,3$, $j = 1,2$. Then we obtain

$$\begin{aligned}
\mathcal{L}V_9 =& \sigma_{12}(\sigma_{11}+\sigma_{12}S)^{p-1}\left[rS\left(1-\frac{S}{K}\right) - \frac{\beta SI}{1+mS}\right] - \frac{\sigma_{12}^2}{2}(1-p)(\sigma_{11}+\sigma_{12}S)^p S^2\\
&+ \sigma_{22}(\sigma_{21}+\sigma_{22}E)^{p-1}\left[\frac{\beta SI}{1+mS} - (\alpha+\mu)E\right] - \frac{\sigma_{22}^2}{2}(1-p)(\sigma_{21}+\sigma_{22}E)^p E^2\\
&+ \sigma_{32}(\sigma_{31}+\sigma_{32}I)^{p-1}\left[\alpha E - \mu I\right] - \frac{\sigma_{32}^2}{2}(1-p)(\sigma_{31}+\sigma_{32}I)^p I^2\\
\leq& \sigma_{12}\sigma_{11}^{p-1}\frac{rK}{4} - \frac{1-p}{2}\sigma_{12}^{p+2}S^{p+2} + \sigma_{22}\sigma_{21}^{p-1}\beta SI - \frac{1-p}{2}\sigma_{22}^{p+2}E^{p+2}\\
&+ \sigma_{32}\sigma_{31}^{p-1}\alpha E - \frac{1-p}{2}\sigma_{32}^{p+2}I^{p+2}.
\end{aligned} \tag{A.17}$$

Then we define a C^2-function $\tilde{V} : \mathbb{R}_+^3 \to \mathbb{R}_+$ as follows

$$\tilde{V}(S,E,I) = MV_8(S,E,I) - \ln E + V_9(S,E,I),$$

where $M > 0$ is a sufficiently large number satisfying the following condition

$$-M\left(\alpha + \mu + \frac{\sigma_{21}^2}{2}\right)\left(R_0^S - 1\right) + B \leq -2, \tag{A.18}$$

where B will be determined later.

Since the logarithmic terms $\ln S$, $\ln E$, and $\ln I$ are assigned negative coefficients, while the linear terms S, E, and I carry positive ones, $\tilde{V}(S,E,I)$ is not only continuous, but also tends to ∞ as (S,E,I) approaches the boundary of $\mathbb{R}_+^3$ and as $\|(S,E,I)\| \to \infty$, where $\|\cdot\|$ denotes the Euclidean norm of a point in $\mathbb{R}_+^3$. Thus, it must be lower bounded and achieve this lower bound at a point (S_0, E_0, I_0) in the interior of $\mathbb{R}_+^3$. Then we define a C^2-function $\bar{V} : \mathbb{R}_+^3 \to \mathbb{R}_+ \cup \{0\}$ in the following form

$$\bar{V}(S,E,I) = \tilde{V}(S,E,I) - \tilde{V}(S_0,E_0,I_0),$$

where $(S,E,I) \in (\frac{1}{n}, n) \times (\frac{1}{n}, n) \times (\frac{1}{n}, n)$ and $n > 1$ is a sufficiently large integer.

By (A.8), (A.16), and (A.17), we obtain

$$\begin{aligned}
\mathcal{L}\bar{V} \leq& -M\left(\alpha + \mu + \frac{\sigma_{21}^2}{2}\right)\left(R_0^S - 1\right) + \frac{M\alpha\beta K_3}{\mu(\alpha+\mu)}SI - \frac{\beta SI}{E(1+mS)}\\
&+ (\alpha+\mu) + \left(\frac{\sigma_{21}^2}{2} + \sigma_{21}\sigma_{22}E + \frac{\sigma_{22}^2}{2}E^2\right) + \sigma_{12}\sigma_{11}^{p-1}\frac{rK}{4} - \frac{1-p}{2}\sigma_{12}^{p+2}S^{p+2}\\
&+ \sigma_{22}\sigma_{21}^{p-1}\beta SI - \frac{1-p}{2}\sigma_{22}^{p+2}E^{p+2} + \sigma_{32}\sigma_{31}^{p-1}\alpha E - \frac{1-p}{2}\sigma_{32}^{p+2}I^{p+2}.
\end{aligned}$$

Step 3. Now we are in the position to construct a compact subset D_ϵ such that the condition A_2 in Lemma 3.1 holds. Define the following bounded closed set

$$D_\epsilon = \left\{(S, E, I) \in \mathbb{R}^3_+ : \epsilon \le S \le \frac{1}{\epsilon}, \epsilon^3 \le E \le \frac{1}{\epsilon^3}, \epsilon \le I \le \frac{1}{\epsilon}\right\},$$

where $0 < \epsilon < 1$ is a sufficiently small constant. In the set $\mathbb{R}^3_+ \setminus D_\epsilon$, we can choose ϵ sufficiently small such that the following conditions hold

$$\epsilon < \frac{\mu(\alpha+\mu)(p+2)}{M\alpha\beta K_3(p+1)}, \tag{A.19}$$

$$\epsilon < \frac{\mu(\alpha+\mu)(1-p)(p+2)\left(\sigma_{12}^{p+2} \wedge \sigma_{32}^{p+2}\right)}{4M\alpha\beta K_3}, \tag{A.20}$$

$$-\frac{\beta}{\epsilon(1+m\epsilon)} + C \le -1, \tag{A.21}$$

$$-\frac{1-p}{4\epsilon^{p+2}}\left(\sigma_{12}^{p+2} \wedge \sigma_{32}^{p+2}\right) + C \le -1, \tag{A.22}$$

$$-\frac{1-p}{4\epsilon^{3(p+2)}}\sigma_{22}^{p+2} + C \le -1, \tag{A.23}$$

where C is a positive constant which will be given explicitly in expression (A.29), and the symbol $\wedge$ denotes the minimum operator (e.g., $\sigma_{12}^{p+2} \wedge \sigma_{32}^{p+2} = \min(\sigma_{12}^{p+2}, \sigma_{32}^{p+2})$). In view of (A.19) and (A.20), we have

$$\frac{M\alpha\beta K_3}{\mu(\alpha+\mu)}\frac{p+1}{p+2}\epsilon < 1, \tag{A.24}$$

$$\frac{M\alpha\beta K_3}{\mu(\alpha+\mu)}\frac{\epsilon}{p+2} - \frac{1-p}{4}\left(\sigma_{12}^{p+2} \wedge \sigma_{32}^{p+2}\right) < 0. \tag{A.25}$$

For convenience we can divide $D^c_\epsilon = \mathbb{R}^3_+ \setminus D_\epsilon$ into six domains,

$$D^1_\epsilon = \{(S, E, I) \in \mathbb{R}^3_+ : S < \epsilon\}, D^2_\epsilon = \{(S, E, I) \in \mathbb{R}^3_+ : I < \epsilon\},$$
$$D^3_\epsilon = \{(S, E, I) \in \mathbb{R}^3_+ : S \ge \epsilon, I \ge \epsilon, 0 < E < \epsilon^3\}, D^4_\epsilon = \{(S, E, I) \in \mathbb{R}^3_+ : S > \frac{1}{\epsilon}\},$$
$$D^5_\epsilon = \{(S, E, I) \in \mathbb{R}^3_+ : I > \frac{1}{\epsilon}\}, D^6_\epsilon = \{(S, E, I) \in \mathbb{R}^3_+ : E > \frac{1}{\epsilon^3}\},$$

It is easy to see that $D^c_\epsilon = D^1_\epsilon \cup D^2_\epsilon \cup D^3_\epsilon \cup D^4_\epsilon \cup D^5_\epsilon \cup D^6_\epsilon$. Next, we will prove that $L\bar{V}(S, E, I) \le -1$ for any $(S, E, I) \in D^c_\epsilon$, which is equivalent to prove it on the above six domains, respectively.

Case 1. If $(S, E, I) \in D^1_\epsilon$, due to $SI < \epsilon I \le \epsilon\frac{p+1+I^{p+2}}{p+2} = \frac{p+1}{p+2}\epsilon + \frac{\epsilon}{p+2}I^{p+2}$, we obtain

$$\begin{aligned}
\mathcal{L}\bar{V} &\le -M\left(\alpha+\mu+\frac{\sigma_{21}^2}{2}\right)\left(R_0^S - 1\right) + \frac{M\alpha\beta K_3}{\mu(\alpha+\mu)}\frac{p+1}{p+2}\epsilon \\
&\quad + \left(\frac{M\alpha\beta K_3}{\mu(\alpha+\mu)}\frac{\epsilon}{p+2} - \frac{1-p}{4}\sigma_{32}^{p+2}\right)I^{p+2} + B \\
&\le -2+1
\end{aligned}$$

$$= \quad -1, \tag{A.26}$$

where

$$B = \sup_{(S,E,I)\in\mathbb{R}_+^3} \Big\{ -\frac{\beta SI}{E(1+mS)} + (\alpha+\mu) + \left(\frac{\sigma_{21}^2}{2} + \sigma_{21}\sigma_{22}E + \frac{\sigma_{22}^2}{2}E^2\right) + \sigma_{12}\sigma_{11}^{p-1}\frac{rK}{4} - \frac{1-p}{4}\sigma_{12}^{p+2}S^{p+2} + \sigma_{22}\sigma_{21}^{p-1}\beta SI - \frac{1-p}{2}\sigma_{22}^{p+2}E^{p+2} + \sigma_{32}\sigma_{31}^{p-1}\alpha E - \frac{1-p}{4}\sigma_{32}^{p+2}I^{p+2}\Big\},$$

which follows from (A.18), (A.24), and (A.25). Thus

$$\mathcal{L}\bar{V} \le -1 \text{ for any } (S,E,I) \in D_\epsilon^1.$$

Case 2. If $(S,E,I) \in D_\epsilon^2$, due to $SI < \epsilon S \le \epsilon\frac{p+1+S^{p+2}}{p+2} = \frac{p+1}{p+2}\epsilon + \frac{\epsilon}{p+2}S^{p+2}$, we get

$$\begin{aligned} \mathcal{L}\bar{V} &\le -M\left(\alpha+\mu+\frac{\sigma_{21}^2}{2}\right)\left(R_0^S - 1\right) + \frac{M\alpha\beta K_3}{\mu(\alpha+\mu)}\frac{p+1}{p+2}\epsilon \\ &\quad + \left(\frac{M\alpha\beta K_3}{\mu(\alpha+\mu)}\frac{\epsilon}{p+2} - \frac{1-p}{4}\sigma_{12}^{p+2}\right)S^{\theta+2} + B \\ &\le -2+1 \\ &= -1, \end{aligned} \tag{A.27}$$

which follows from (A.18), (A.24), and (A.25). Therefore

$$\mathcal{L}\bar{V} \le -1 \text{ for any } (S,E,I) \in D_\epsilon^2.$$

Case 3. If $(S,E,I) \in D_\epsilon^3$, we have

$$\mathcal{L}\bar{V} \le -\frac{\beta SI}{E(1+mS)} + C \le -\frac{\beta\epsilon^2}{\epsilon^3(1+m\epsilon)} + C = -\frac{\beta}{\epsilon(1+m\epsilon)} + C \le -1, \tag{A.28}$$

which follows from (A.21) and

$$C = \sup_{(S,E,I)\in\mathbb{R}_+^3} \Big\{ \frac{M\alpha\beta K_3}{\mu(\alpha+\mu)}SI + (\alpha+\mu) + \left(\frac{\sigma_{21}^2}{2} + \sigma_{21}\sigma_{22}E + \frac{\sigma_{22}^2}{2}E^2\right) + \sigma_{12}\sigma_{11}^{p-1}\frac{rK}{4} - \frac{1-p}{4}\sigma_{12}^{p+2}S^{p+2} + \sigma_{22}\sigma_{21}^{p-1}\beta SI - \frac{1-p}{4}\sigma_{22}^{p+2}E^{p+2} + \sigma_{32}\sigma_{31}^{p-1}\alpha E - \frac{1-p}{4}\sigma_{32}^{p+2}I^{p+2}\Big\}. \tag{A.29}$$

So

$$\mathcal{L}\bar{V} \le -1 \text{ for any } (S,E,I) \in D_\epsilon^3.$$

Case 4. If $(S,E,I) \in D_\epsilon^4$, we get

$$\mathcal{L}\bar{V} \le -\frac{1-p}{4}\sigma_{12}^{p+2}S^{p+2} + C \le -\frac{1-p}{4\epsilon^{p+2}}\sigma_{12}^{p+2} + C \le -1, \tag{A.30}$$

which follows from (A.22). Hence

$$\mathcal{L}\bar{V} \leq -1 \text{ for any } (S, E, I) \in D_{\epsilon}^{4}.$$

Case 5. If $(S, E, I) \in D_{\epsilon}^{5}$, we have

$$\mathcal{L}\bar{V} \leq -\frac{1-p}{4}\sigma_{32}^{p+2}I^{p+2} + C \leq -\frac{1-p}{4\epsilon^{p+2}}\sigma_{32}^{p+2} + C \leq -1, \tag{A.31}$$

which follows from (A.22). Thus

$$\mathcal{L}\bar{V} \leq -1 \text{ for any } (S, E, I) \in D_{\epsilon}^{5}.$$

Case 6. If $(S, E, I) \in D_{\epsilon}^{6}$, we have

$$\mathcal{L}\bar{V} \leq -\frac{1-p}{4}\sigma_{22}^{p+2}E^{p+2} + C \leq -\frac{1-p}{4\epsilon^{3(p+2)}}\sigma_{22}^{p+2} + C \leq -1, \tag{A.32}$$

which follows from (A.23). Therefore

$$\mathcal{L}\bar{V} \leq -1 \text{ for any } (S, E, I) \in D_{\epsilon}^{6}.$$

Obviously, from (A.26)–(A.28), (A.30)–(A.32), we can obtain that for a sufficiently small ϵ,

$$\mathcal{L}\bar{V}(S, E, I) \leq -1 \text{ for any } (S, E, I) \in \mathbb{R}_{+}^{3} \setminus D_{\epsilon}.$$

So the condition A_2 in Lemma 3.1 holds. In view of Lemma 3.1, we can obtain that system (1.2) admits a unique ergodic stationary distribution. This completes the proof. □

B. Table of simulation results

Table 2. Comparison of results under different step sizes (Example 5.2).

Step size	95% confidence intervals for $S(t)$, $E(t)$ and $I(t)$	Standard deviations
$h = 0.02$	[0.4116, 0.5053], [−0.0065, 0.0049], [−0.0056, 0.0080]	0.0239, 0.0029, 0.0035
$h = 0.04$	[0.4166, 0.5189], [−0.0131, 0.0377], [−0.0139, 0.0073]	0.0261, 0.0043, 0.0054
$h = 0.08$	[0.4385, 0.5353], [−0.0165, 0.0020], [−0.0172, 0.0025]	0.0247, 0.0047, 0.0050

Table 3. Comparison of results under different step sizes (Example 5.3).

Step size	95% confidence intervals for $S(t)$, $E(t)$ and $I(t)$	Standard deviations
$h = 0.02$	[0.4324, 0.5332], [0.0001, 0.0003], [0.0002, 0.0004]	0.0257, 4.7019×10^{-5}, 5.3571×10^{-5}
$h = 0.04$	[0.4281, 0.5201], [0.0001, 0.0003], [0.0001, 0.0003]	0.0235, 4.5449×10^{-5}, 5.6935×10^{-5}
$h = 0.08$	[0.4150, 0.5035], [−0.0002, 0.0003], [−0.0003, 0.0004]	0.0226, 1.1538×10^{-4}, 1.6617×10^{-4}

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