Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(12): 28396–28406.
DOI:10.3934/math.20251249
Received: 07 October 2025
Revised: 26 November 2025
Accepted: 27 November 2025
Published: 03 December 2025

Research article

Cyclic codes over $F_2[u, v, w]/\langle u^2 = v^2, uv = 0, w^2 = w\rangle$ and its applications

Merve BULUT YILGÖR*

Department of Basic Sciences, Faculty of Engineering and Architecture, Altınbaş University, 34218, Mahmutbey, İstanbul, Türkiye

* **Correspondence:** Email: merve.yilgor@altinbas.edu.tr.

Abstract: We investigate linear and cyclic codes over the ring $F_2[u, v, w]/\langle u^2 = v^2,\ uv = 0,\ w^2 = w\rangle$. This is a commutative Frobenius non-chain ring, which, to the best of our knowledge, is studied here for the first time in the literature. We define a homogeneous weight on the ring and, with respect to a Gray map induced by this weight, obtain the optimal Reed-Muller code $RM\ (1, 7)$. We analyze the algebraic structure of the ring in detail, determine its ideals, and present code constructions together with their Gray images.

Keywords: algebraic coding theory; codes over rings; homogeneous weight
Mathematics Subject Classification: 11T71, 95B15

1. Introduction

Algebraic coding theory studies error-correcting codes built from algebraic structures such as finite fields and finite rings. Since the early 2000s, this perspective has remained a central theme of the field, with cyclic codes playing a distinguished role thanks to their rich algebraic structure and efficient description via ideals.

The seminal work of Hammons et al. [11] revealed connections between non-binary linear codes and non-linear binary codes, and motivated extensive research on codes over finite rings. Subsequent studies have investigated chains [1, 6] and non-chains of ideals [12, 14] (chain and non-chain rings), Gray maps, and homogeneous weights [9, 13]. There has also been sustained interest in Frobenius chain and non-chain rings and their applications, including DNA code constructions [3, 14].

In recent years, cyclic codes have been constructed over Frobenius rings of order 16 [12, 14]. Dougherty et al. [7] defined a Gray map on a local Frobenius non-chain ring of order 16 and described the corresponding binary images with respect to the Lee weight. Constantinescu and Heise [5] introduced homogeneous weights on rings, while Greferath and Schmidt [8] developed Gray isometries for finite chain rings. Gray maps based on homogeneous weights for non-chain rings were studied further in [12].

In this paper, we focus on the Frobenius non-chain ring

$$R = F_2[u, v, w]/\langle u^2 = v^2,\ uv = 0,\ w^2 = w\rangle,$$

with a commutative ring of characteristic 2 and order 256. We determine the structure of R, construct linear and cyclic codes over R, and define a Gray map associated with a homogeneous weight, yielding optimal binary images.

2. Linear codes over the Frobenius non-chain ring $\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + v^2\mathbb{F}_2 + w\mathbb{F}_2 + wu\mathbb{F}_2 + wv\mathbb{F}_2$ $+wv^2\mathbb{F}_2$

In this section, the basic definitions and concepts needed in this paper will be provided. Throughout, let

$$R = \mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + v^2\mathbb{F}_2 + w\mathbb{F}_2 + wu\mathbb{F}_2 + wv\mathbb{F}_2 + wv^2\mathbb{F}_2$$

be the quotient ring $F_2[u, v, w]/\langle u^2 = v^2, uv = 0, w^2 = w\rangle$, which is a commutative non-chain ring. Every element can be written uniquely as $a_0 + a_1u + a_2v + a_3v^2 + a_4w + a_5wu + a_6wv + a_7wv^2$, with $u^2 = v^2, uv = 0$, and $w^2 = w$, where $a_i \in \mathbb{F}_2, 0 \le i \le 7$.

R has 48 ideals. All ideals of R are given in Tables 1 and 2 located in the Appendix. It has two maximal ideals and many principal ideals. The set of units of R is $U_R = \{1, 1 + u^2, 1 + w, 1 + u^2 + w\}$. We observe that U_R is isomorphic to the Klein four group, $\mathbb{Z}_2 \times Z_2$.

The ideal hierarchy is illustrated in Figure 1. There are eight levels; the ideal with the ID number 31 lies at the eighth level. The arrows show which lower-level ideal is encompassed by a higher-level ideal.

R is a three-variable residue ring. It can be simplified as follows:

$$R = \mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + w\mathbb{F}_2 + uw\mathbb{F}_2 + vw\mathbb{F}_2 + u^2\mathbb{F}_2 + wu^2\mathbb{F}_2,$$
$$u^2 = v^2, uv = 0, w^2 = w.$$
$$R = (\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + v^2\mathbb{F}_2) + w(\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + v^2\mathbb{F}_2),$$
$$u^2 = v^2, uv = 0, w^2 = w.$$
$$R = \mathcal{R}+w\mathcal{R},\ w^2 = w.$$

Here $\mathcal{R}$ is a local Frobenius non-chain ring $\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + v^2\mathbb{F}_2$ with $u^2 = v^2$ and $uv = 0$.

Let C be a linear code over R with length n, in which case C is an R-submodule of R^n. An element of a linear code is a codeword. A linear code C with length n is defined as a cyclic code if, for all $c = (c_0, c_1, \dots, c_{n-1})$ in C, its cyclic shift $(c_{n-1}, c_0, \dots, c_{n-2})$ is also a codeword in C. In a cyclic code, for each codeword $c = (c_0, c_1, \dots, c_{n-1}) \in C$, there is a corresponding polynomial $c(x) = c_0 + c_1x + \dots + c_{n-1}x^{n-1} \in R_n = R[x]/\langle x^n - 1\rangle$. In the polynomial representation of a cyclic code, there is a one-to-one correspondence with an ideal of R_n. In this paper, the quotient ring $R[x]/\langle x^n - 1\rangle$ will be denoted as R_n.

Yilgor et al. [14] constructed cyclic codes in the ring $F_2 + uF_2 + vF_2 + v^2F_2$. R is not isomorphic to this ring. However, the code construction performed here is used in constructing R.

Let R be a ring and $a \in R$. If a is different from 0 the Hamming weight of a is $w_H(a) = 1$; otherwise $w_H(a) = 0$. In the case of $a \in R^n$, any element will be a vector like $a = (a_1, a_2, ...a_n)$ and the Hamming weight of the vector a is the sum of the Hamming weights of its terms; that is, $w_H(a) = \sum_{i=1}^{n} w_H(a_i)$. The Hamming distance between two codewords a and b in R^n is given by $d(a, b) = w_H(a - b)$. It is important to note that d is a distance-preserving map.

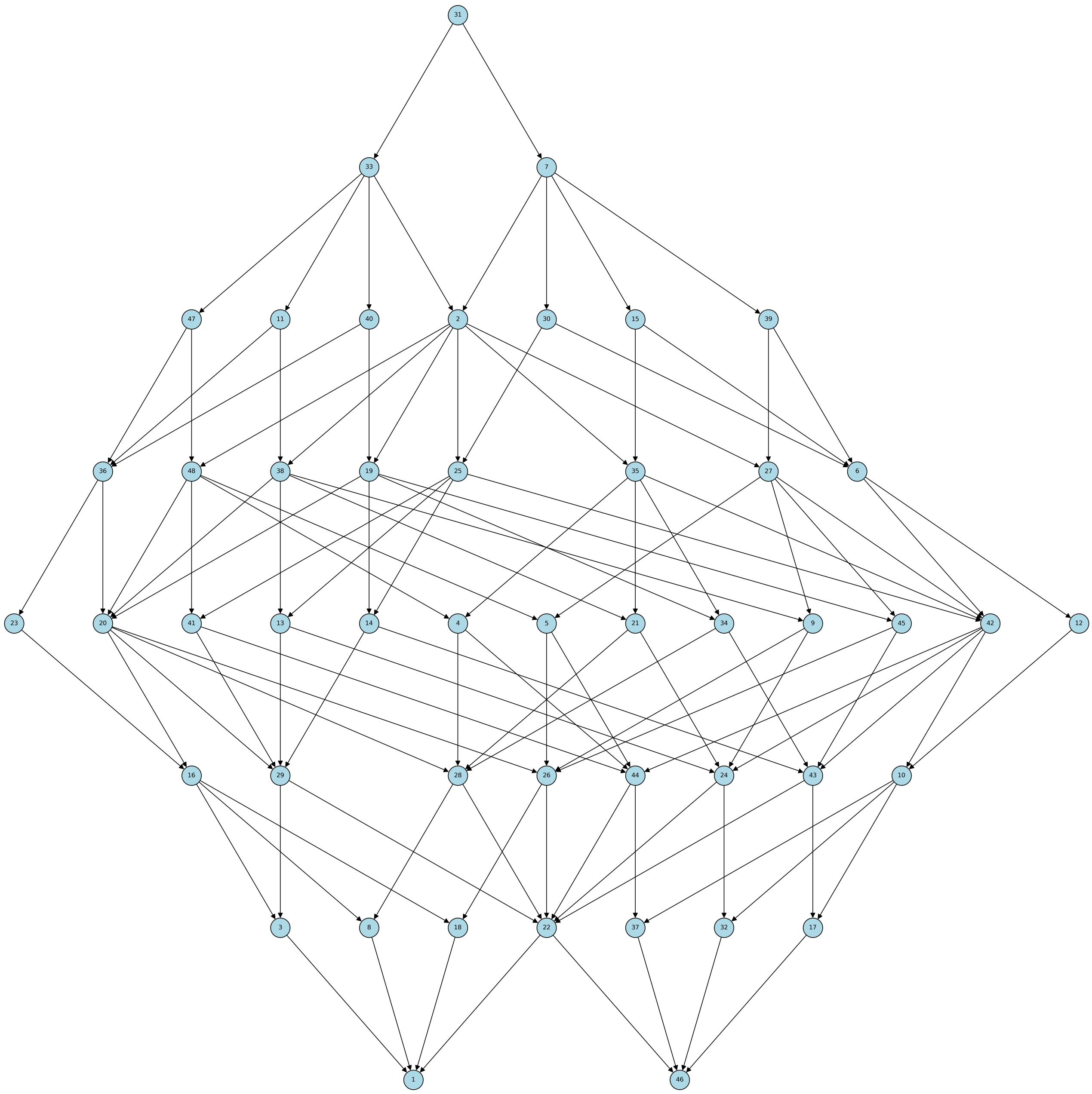

Figure 1. Hierarchy of ideals of $\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + u^2\mathbb{F}_2 + w\mathbb{F}_2 + wu\mathbb{F}_2 + wv\mathbb{F}_2 + wu^2\mathbb{F}_2$.

Define the Gray map

$$\Phi : R = \mathcal{R} + w\mathcal{R} \longrightarrow \mathcal{R}^2,$$
$$a + wb \longrightarrow (a, a + b),$$

where $R = \mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + v^2\mathbb{F}_2 + w\mathbb{F}_2 + wu\mathbb{F}_2 + wv\mathbb{F}_2 + wv^2\mathbb{F}_2$ and $\mathcal{R} = \mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + v^2\mathbb{F}_2$ and $a, b \in \mathcal{R}$.

The Lee weight on R is the Hamming weight of the Gray image:

$$w_L(a + wb) = w_H(\Phi((a + wb)),$$

where $a, b \in \mathcal{R}$.

We record structural facts [2, 4] that justify the lists of ideals displayed in Tables 1–2 in the Appendix.

Lemma 2.1. *Let $\mathcal{R} = \mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + v^2\mathbb{F}_2$ with $u^2 = v^2$ and $uv = 0$, and let $R = \mathcal{R} + w\mathcal{R}$ with $w^2 = w$. The map*

$$\Psi : R \longrightarrow \mathcal{R} \times \mathcal{R}, \qquad \Psi(a + wb) = (a,\ a + b)$$

is a ring isomorphism with the inverse $(x, y) \mapsto x + w(y - x)$. In particular, $R \cong \mathcal{R} \times \mathcal{R}$.

Proof. A straightforward check shows that Ψ is bijective and multiplicative; note that $w^2 = w$ and the characteristic 2 imply $(a + wb)(c + wd) = ac + w(ad + bc + bd)$, which matches multiplication in $\mathcal{R} \times \mathcal{R}$ under $(a,\ a + b)(c,\ c + d) = (ac,\ ac + ad + bc + bd)$. □

Corollary 2.1. *Under the isomorphism in Lemma 2.1, every ideal $I \subseteq R$ corresponds to a pair of ideals (I_1, I_2) of $\mathcal{R}$ via $I = \Psi^{-1}(I_1 \times I_2)$. In particular, I is principal if and only if both I_1 and I_2 are principal, generated by $r_1, r_2 \in \mathcal{R}$, in which case*

$$I = \langle\, r_1 + w(r_2 - r_1)\,\rangle.$$

Remark 2.1. *The maximal ideals of R correspond to pairs where exactly one component is maximal in $\mathcal{R}$ and the other is the whole ring:*

$$\mathrm{Max}(R) \ \cong\ \big(\mathrm{Max}(\mathcal{R}) \times \{\mathcal{R}\}\big) \ \cup\ (\{\mathcal{R}\} \times \mathrm{Max}(\mathcal{R})).$$

Concretely, if $\mathfrak{m} \subset \mathcal{R}$ is maximal then

$$\Psi^{-1}(\mathfrak{m} \times \mathcal{R}) = \{\, a + wb :\ a \in \mathfrak{m},\ b \in \mathcal{R}\,\}, \qquad \Psi^{-1}(\mathcal{R} \times \mathfrak{m}) = \{\, a + wb :\ a + b \in \mathfrak{m}\,\}$$

are maximal in R. This explains the two maximal ideals listed in the paper (obtained from the two maximal ideals of $\mathcal{R}$) and why their generators have the displayed form.

Remark 2.2. *Corollary 2.1 explains why so many ideals of R are principal: Each line in Tables 1–2 located in the Appendix corresponds to a pair of generators (r_1, r_2) of ideals of $\mathcal{R}$, encoded in R by $r_1 + w(r_2 - r_1)$.*

If A and B are codes, the tensor product of these two codes is defined as $A \bigotimes B = \{(a, b) | a \in A, b \in B\}$, and direct sum is defined as $A \bigoplus B = \{a + b | a \in A, b \in B)\}$. For a linear code C with a length n over $\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + w\mathbb{F}_2 + uw\mathbb{F}_2 + vw\mathbb{F}_2 + v^2\mathbb{F}_2 + wv^2\mathbb{F}_2$, we define

$$C_1 = \{a + b \in \mathcal{R} \mid w(a + b) + (w + 1)a \in C,\ a, b \in \mathcal{R}\},$$
$$C_2 = \{a \in \mathcal{R} \mid w(a + b) + (w + 1)a \in C,\ b \in \mathcal{R}\}.$$

Then C_1 and C_2 are linear codes over $\mathcal{R}$ and $C = wC_1 \bigoplus (w + 1)C_2$.

Theorem 2.1. [15] *Let C be a linear code of length n over $\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + w\mathbb{F}_2 + uw\mathbb{F}_2 + vw\mathbb{F}_2 + v^2\mathbb{F}_2 + wv^2\mathbb{F}_2, u^2 = v^2, uv = 0, w^2 = w$. Then $\Phi(C) = C_1 \bigotimes C_2$ and $|C| = |C_1||C_2|$.*

Lemma 2.2. [10] *If G_1 and G_2 are generator matrices of C_1 and C_2, respectively, then a generator matrix for C is*

$$\begin{pmatrix} wG_1 \\ (w+1)G_2 \end{pmatrix}.$$

Corollary 2.2. [15] *If $\Phi(C) = C_1 \bigotimes C_2$, then $C = wC_1 \bigoplus (w+1)C_2$.*

Proposition 2.1. *Let C be a linear code over R and let d_H and d_L denote the minimum Hamming and Lee distances of C, respectively. If $d(C_i)$ denotes the minimum distance of C_i, then $d_H = d_L = min\{d(C_1), d(C_2)\}$.*

Corollary 2.3. *Let $C = wC_1 \bigoplus (w+1)C_2$ be a linear code of length n over $\mathcal{R} + w\mathcal{R}$, where C_i is a linear code over $\mathcal{R}$ with dimension k_i and minimum Hamming distance $d(C_i)$. In this case, $\Phi(C)$ is a $[2n, k_1 + k_2, min(d(C_i))]$ linear code over $\mathcal{R}$.*

3. Cyclic codes over the ring $\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + w\mathbb{F}_2 + uw\mathbb{F}_2 + vw\mathbb{F}_2 + v^2\mathbb{F}_2 + wv^2\mathbb{F}_2$

Cyclic codes are an important subclass of algebraic codes, characterized by rich algebraic structures and numerous applications. In this section, we construct cyclic codes over R. Throughout the paper, we write $I_{u,v}f = \langle u, v\rangle f$ for $f \in R_n$ and abbreviate it as Zf.

Definition 3.1. *Let C be a linear code with a length n over R. C is called a cyclic code if it is invariant under the automorphism σ, which is $\sigma(c_0, c_1, \ldots, c_{n-1}) = (c_{n-1}, c_0, \ldots, c_{n-2})$.*

Theorem 3.1. [14] *Let n be a positive integer. For $i = \{1, 2, 3\}$ and $\gamma_1 = u$, $\gamma_2 = u + v$ and $\gamma_3 = v$, we define*

$$M_i = \langle f_1, Zf_2, \gamma_i f_3, v^2 f_4\rangle$$

as a cyclic code over $\mathcal{R}$, with $f_4|f_3|f_2|f_1|\,(x^n - 1)$.

Theorem 3.2. *Let M_1 and M_2 be cyclic codes of length n over $\mathcal{R}$. Then $C = (w)M_1 \bigoplus (w+1)M_2$ is a cyclic code over R.*

Proof. Let $x = (x_0, x_1, \ldots, x_{n-1})$ and $y = (y_0, y_1, \ldots, y_{n-1})$ are in M_1 and M_2, which are cyclic codes in $\mathcal{R}$. Suppose that $c = (c_0, c_1, \ldots, c_{n-1}) \in C$, where $c_i = wx_i + (1+w)y_i$ and $c_i = wx_i + y_i + wy_i = y_i + w(x_i + y_i)$ for all $i = 0, 1, , , , , n-1$. Then $\sigma(c) = (y_{n-1} + w(x_{n-1} + y_{n-1}), y_0 + w(x_0 + y_0), \ldots, y_{n-2} + w(x_{n-2} + y_{n-2})$ and $\sigma(c) = (y_{n-1}, y_0, \ldots, y_{n-2}) + w(x_{n-1} + y_{n-1}, x_0 + y_0, x_1 + y_1, \ldots, x_{n-2})$. $\sigma(c) = \sigma(y) + w(\sigma(x) + \sigma(y))$, and so M_1 and M_2 are cyclic. □

Theorem 3.3. [14] *Let n be a positive integer and $M_i = \langle f_1, Zf_2, \gamma_s f_3, v^2 f_4\rangle$ for $s = \{1, 2, 3\}$, and $\gamma_1 = u$, $\gamma_2 = u + v$, and $\gamma_3 = v$, be a cyclic code of length n over $\mathcal{R}$, where $f_4|f_3|f_2|f_1|\,(x^n - 1)$. Let $k_1 = n - \deg(f_1)$, $k_j = \deg(f_{j-1}) - \deg(f_j)$ for $j = \{2, 3, 4\}$ and*

$$S_1^i = \{x^t f_1 : 0 \le t \le k_1 - 1\},$$
$$S_2^i = \{x^t z f_2 : 0 \le t \le k_2 - 1, z \in Z\},$$

$$S_3^i = \{x^t\gamma_i f_3 : 0 \le t \le k_3 - 1\},$$
$$S_4^i = \{x^t v^2 f_4 : 0 \le t \le k_4 - 1\},$$

and $S^i = S_1^i \cup S_2^i \cup S_3^i \cup S_4^i$ *and* $|C| = 16^{k_1}8^{k_2}4^{k_3}2^{k_4}$.

Theorem 3.4. *Let* M_i *and* M_j *be cyclic codes with a length* n *over* $\mathcal{R}$. *Then the minimal spanning set for* $C = (w)M_i \bigoplus (w+1)M_j$ *is* $S = wS^i \cup (w+1)S^j$ *and* $|C| = 16^{k_1^i+k_1^j}\, 8^{k_2^i+k_2^j}\, 4^{k_3^i+k_3^j}\, 2^{k_4^i+k_4^j}$, *where* $k_1^i = n - \deg(f_1^i)$, $k_t^i = \deg(f_{t-1}^i) - \deg(f_t^i)$ $t = \{2,3,4\}$, *and similarly for* k_*^j.

Proof. The generator matrix for C over R is constructed using the method described in Lemma 2.2 and the minimal spanning set presented in Theorem 3.3. The proof is complete with the vectors forming the minimal spanning set. □

Example 3.1. *We know that* $x^3 - 1 = (1+x)(1+x+x^2)$ *and* $(1+x)|x^3 - 1$ *over* $\mathcal{R}$. *Let* $C_1 = g_1(x) =< (x+1) >$ *and* $C_2 = g_2(x) =< (x+1) >$. $C = wg_1(x) + (w+1)g_2(x)$ *is a cyclic code over* R. *In addition, the generator matrix of* C *is*

$$G = \begin{pmatrix} wG_1 \\ (1+w)G_2 \end{pmatrix} = \begin{pmatrix} w & w & 0 \\ 0 & w & w \\ 1+w & 1+w & 0 \\ 0 & 1+w & 1+w \end{pmatrix}.$$

Row-reducing yields $G' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, *which has two free rows, so* $|C| = 16^4$. *Therefore, the Gray image* $\Phi(C)$ *of* C *generates a* $[6,4,2]$ *linear code over* $\mathcal{R}$.

Let $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1})$, $\mathbf{y} = (y_0, y_1, \ldots, y_{n-1}) \in R^n$. In a Euclidean vector space, the inner product of vectors x and y is defined as $[\mathbf{x}, \mathbf{y}] = \sum_{i=0}^{n-1} x_i y_i$, where the calculations are performed in R.

Let C be a linear code of length n over R, the *dual code* of C is $C^\perp = \{\mathbf{w} \in R^n : [\mathbf{x}, \mathbf{y}] = 0\}$.

If $p(x) \in R[x]$, the reciprocal polynomial is $x^{\deg(p(x))}p(x^{-1})$.

Lemma 3.1. *Let* $C^\perp$ *be the dual code of* C. *Then* $\Phi(C^\perp) = \Phi(C)^\perp$. *In addition, if* C *is a self-dual code, so is* $\Phi(C)$.

Proof. Let $c_1 = a_1 + wb_1$ and $c_2 = a_2 + wb_2$, with $c_1, c_2 \in (\mathcal{R} + w\mathcal{R})^n$ and $a_i, b_i \in \mathcal{R}^n$, $i = 1, 2$. If $[c_1, c_2] = 0$ in R,

$$[c_1, c_2] = (a_1 + wb_1).(a_2 + wb_2) = a_1a_2 + w(a_1b_2 + a_2b_1 + b_1b_2) = 0,$$

and $a_1a_2 = 0$, and $a_1b_2 + a_2b_1 + b_1b_2 = 0$. On the other hand, $\Phi(c_1) = (a_1, a_1 + b_1)$ and $\Phi(c_2) = (a_2, a_2 + b_2)$. Then $[\Phi(c_1), \Phi(c_2))] = a_1a_2 + a_1a_2 + a_1b_2 + a_2b_1 + b_1b_2 = 0$. Therefore, $\Phi(C^\perp) \subseteq \Phi(C)^\perp$.

Now let C be a linear code with a length n and $|C| = 16^{k_1^i+k_1^j}8^{k_2^i+k_2^j}4^{k_3^i+k_3^j}2^{k_4^i+k_4^j} = 2^{4(k_1^i+k_1^j)+3(k_2^i+k_2^j)+2(k_3^i+k_3^j)+(k_4^i+k_4^j)}$. Since Φ is bijective, $|\Phi(C)| = |C|$, and therefore $\Phi(C)$ has the parameters $[2n,\ 4(k_1^i + k_1^j) + 3(k_2^i + k_2^j) + 2(k_3^i + k_3^j) + (k_4^i + k_4^j)]$. Thus $|\Phi(C)^\perp| = 2^{2n-(\,4(k_1^i+k_1^j)+3(k_2^i+k_2^j)+2(k_3^i+k_3^j)+(k_4^i+k_4^j)\,)}$.

On the other hand, using $|C^\perp| = 2^{2n}/|C|$ and $|\Phi(C^\perp)| = |C^\perp|$ again, we obtain $|\Phi(C^\perp)| = |C^\perp| = 2^{2n-(\,4(k_1^i+k_1^j)+3(k_2^i+k_2^j)+2(k_3^i+k_3^j)+(k_4^i+k_4^j)\,)}$. Hence, $|\Phi(C^\perp)| = |\Phi(C)^\perp|$.

Since $\Phi(C^\perp) \subseteq \Phi(C)^\perp$ and both sets have the same finite cardinality, we conclude that $\Phi(C^\perp) = \Phi(C)^\perp$. □

Theorem 3.5. [14] *Let n be a positive integer. Let* $M_i = \langle f_1, Zf_2, \gamma_s f_3, v^2 f_4\rangle$ *for* $s = \{1,2,3\}$, *and let* $\gamma_1 = u$, $\gamma_2 = u+v$, *and* $\gamma_3 = v$, *be a cyclic code of length n over* $\mathcal{R}$, *where* $f_4|f_3|f_2|f_1|\,(x^n-1)$. *Then the dual code of* M_i *is*

$$M_i^{\perp} = \left\langle \left(\frac{x^n-1}{f_4}\right)^{*}, Z\left(\frac{x^n-1}{f_3}\right)^{*}, \gamma_i^{\perp}\left(\frac{x^n-1}{f_2}\right)^{*}, v^2\left(\frac{x^n-1}{f_1}\right)^{*} \right\rangle,$$

where $\gamma_1^{\perp} = v$, $\gamma_2^{\perp} = u+v$, *and* $\gamma_3^{\perp} = u$.

Using Lemma 3.1 and Theorem 3.5, the following theorem is proven.

Theorem 3.6. *Let C be a linear code of length n over R and* $\phi(C) = M_1 \bigotimes M_2$, *and* $C = wM_1 \bigoplus (w+1)M_2$. *Then* $\phi(C^{\perp}) = M_1^{\perp} \bigotimes M_2^{\perp}$ *and* $C^{\perp} = wM_1^{\perp} \bigoplus (w+1)M_2^{\perp}$.

Proof. By Lemma 3.1, $\phi(C)^{\perp} = (M_1 \bigotimes M_2)^{\perp}$. Therefore, we need to prove that $M_1^{\perp} \bigotimes M_2^{\perp} = (M_1 \bigotimes M_2)^{\perp}$. $M_1^{\perp} \bigotimes M_2^{\perp} \subseteq (M_1 \bigotimes M_2)^{\perp}$. On the other hand, let M_1 and M_2 be $[n, 4k_1+3k_2+2k_3+k_4]$ and $[n, 4k_1'+3k_2'+2k_3'+k_4']$ codes, respectively. Then $M_1^{\perp}, M_2^{\perp}$, and $M_1 \bigotimes M_2$ are $[n, n-(4k_1+3k_2+2k_3+k_4)]$, $[n-(4k_1'+3k_2'+2k_3'+k_4')]$, and $[2n-4(k_1+k_1')+3(k_2+k_2')+2(k_3+k_3')+(k_4+k_4')]$ binary linear codes, respectively. Then $|M_1^{\perp} \bigotimes M_2^{\perp}| = |M_1^{\perp}|.|M_2^{\perp}| = 2^{2n}-(4(k_1+k_1')+3(k_2+k_2')+2(k_3+k_3')+(k_4+k_4'))$. Hence, $M_1^{\perp} \bigotimes M_2^{\perp} = (M_1 \bigotimes M_2)^{\perp}$. In light of Corollary 2.2, we obtain the last statement. □

4. Homogeneous weight and binary image of linear codes over R

The homogeneous weights of two-variable non-chain rings have been presented in [12, 14]. This study determines the homogeneous weights of the three-variable non-chain ring R and obtains binary codes using the Gray map.

Definition 4.1. [9] *A real-valued function w on the finite ring S is called a (left) homogeneous weight if* $w(0) = 0$ *and the following are true.*

(i) For all $x, y \in S$, $Sx = Sy$ *implies* $w(x) = w(y)$.
(ii) A real number γ *exists such that*

$$\sum_{y \in Sx} w(y) = \gamma\,|Sx|\,.$$

The number γ is the average value of w on S, and from Condition (*ii*), we conclude that the average value of w is constant on all non-zero principal ideals of S.

We define the homogeneous weight for R in the same sense as [9]. For any $x \in R$, we define

$$w_{\text{hom}}(x) = \begin{cases} 0 & if\ x = 0, \\ 128 & if\ x = u^2 w, \\ 64 & otherwise. \end{cases}$$

In this case, a distance-preserving Gray map from (R, w_{hom}) to $(\mathbb{F}_2^{128}, w_{\text{H}})$, where w_{hom} and w_{H} denote the homogeneous weight and Hamming weight, respectively, can be determined as follows. Let

$$\varphi(u^2 w) = (1_{128}),$$

$$\begin{aligned}
&\varphi(u) = (1_{64}, 0_{64}),\\
&\varphi(v) = (1_{32}, 0_{32}, 1_{32}, 0_{32}),\\
&\varphi(w) = (1_{16}, 0_{16}, 1_{16}, 0_{16}, 1_{16}, 0_{16}, 1_{16}, 0_{16}),\\
&\varphi(uw) = (1_8, 0_8, 1_8, 0_8, 1_8, 0_8, 1_8, 0_8, 1_8, 0_8, 1_8, 0_8, 1_8, 0_8, 1_8, 0_8),\\
&\varphi(vw) = (1, 1, 1, 1, 0, 0, 0, 0, \dots, 1, 1, 1, 1, 0, 0, 0, 0),\\
&\varphi(u^2) = (1, 1, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0 \dots, 1, 1, 0, 0),\\
&\varphi(1) = (1, 0, 1, 0, \dots, 1, 0).
\end{aligned}$$

Here, 1_n and 0_n are vectors of length n that are composed entirely of 1s and 0s, respectively. The vectors corresponding to the Gray images of the basis elements that generate R have a length 128.

For any element of R, we obtain the Gray images via the following map:

$$\begin{aligned}
&\varphi(a_0 + a_1u + a_2v + a_3w + a_4uw + a_5vw + a_6u^2 + a_7wu^2)\\
=&a_0\varphi(1) + a_1\varphi(u) + a_2\varphi(v) + a_3\varphi(w) + a_4\varphi(uw) + a_5\varphi(vw) + a_6\varphi(v^2) + a_7\varphi(wu^2),
\end{aligned}$$

for all $a_i \in \mathbb{F}_2$.

The Gray images of elements of R form a binary linear code $[128, 8, 64]$. It is also optimal and equals $RM\ (1, 7)$. In addition, C is a self-orthogonal code, since $GG^T = 0$.

The Gray image extended to n-coordinates is obtained by setting

$$\phi(c) = (\varphi(c_0), \varphi(c_1), \dots, \varphi(c_{n-1})),$$

where $c = (c_0, c_1, \dots, c_{n-1})$. It is clear that if C is a linear code of length n over R, then $\phi(C)$ is a binary linear code of length $128n$.

Theorem 4.1. *If C is a linear code over $R = \mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + w\mathbb{F}_2 + uw\mathbb{F}_2 + vw\mathbb{F}_2 + u^2\mathbb{F}_2 + wu^2\mathbb{F}_2$, the Gray image of C is a binary self-orthogonal code.*

Proof. $\varphi(R)$ is a self-orthogonal code; therefore, the inner product of the Gray images of any two elements of R is zero. Let a and b be codewords in C of length n over R. We obtain $[\phi(a), \phi(b)] = \sum_{i=0}^{n-1} \varphi(a_i)\varphi(b_i) = 0$. Hence $\phi(C)$ is a self-orthogonal code. □

Theorem 4.2. *Let $C = R^n$. Then $\phi(C)$ is a binary linear code with the parameters* $[128n, 8n, 64]$.

Proof. First, by definition of φ, ϕ maps each coordinate of R to a binary vector of length 128, and hence the block length is $128n$.

Since $|R| = 256 = 2^8$, R is an eight-dimensional vector space over $\mathbb{F}_2$ via the Gray image. Let $e^{(j)} \in R^n$ be the vector whose j-th coordinate is $1 \in R$ and other coordinates are 0. Then $\phi(e^{(j)})$ occupies the j-th 128-block and spans an eight-dimensional subspace there. The n blocks are disjoint in support, so the total binary dimension is $8n$.

Finally, we show that the minimum distance is 64. Because φ is distance-preserving with respect to w_{hom} and w_H, the weight of $\phi(c)$ equals $\sum_{i=0}^{n-1} w_{\text{hom}}(c_i)$. By the definition of w_{hom}, every nonzero coordinate contributes either 64 or 128. Hence, any nonzero codeword has a weight of at least 64. Moreover, taking $c = e^{(j)}$ with the entry $1 \in R$ (which has $w_{\text{hom}}(1) = 64$) gives a codeword of weight 64. Therefore, $d_{\min} = 64$. Consequently, we have the result. □

5. Conclusions

We investigated the Frobenius non-chain ring $R = F_2[u, v, w]/\langle u^2 = v^2, uv = 0, w^2 = w\rangle$, determined its ideal structure, and developed linear and cyclic code constructions over R. We defined a homogeneous weight and showed that the associated Gray image yields optimal binary codes, including the Reed-Muller code $RM(1, 7)$. Since, to the best of our knowledge, this ring is investigated here for the first time, we expect further developments on codes over R, including applications to DNA codes and self-orthogonal code families.

Use of Generative-AI tools declaration

The author declares she has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflict of interest.

References

1. T. Abualrub, I. Siap, Cyclic codes over the rings $Z_2 + uZ_2$ and $Z_2 + uZ_2 + u^2Z_2$, *Des. Codes Crypt.*, **42** (2007), 273–287. https://doi.org/10.1007/s10623-006-9034-5
2. M. Badie, A. Aliabad, F. Obeidavi, On ideals of product of commutative rings and their applications, arXiv: 2506.08537. https://doi.org/10.48550/arXiv.2506.08537
3. T. Alsuraiheed, E. Oztas, S. Ali, M. Yilgor, Reversible codes and applications to DNA codes over $F_4^{2t}[u]/(u^2 - 1)$, *AIMS Mathematics*, **8** (2023), 27762–27774. https://doi.org/10.3934/math.20231421
4. I. Chajda, G. Eigenthaler, H. Länger, Ideals of direct products of rings, *Asian-Eur. J. Math.*, **11** (2018), 1850094. https://doi.org/10.1142/S1793557118500948
5. I. Constantinescu, W. Heise, A metric for codes over residue class rings of integers, *Probl. Peredachi Inf.*, **33** (1997), 22–28.
6. H. Dinh, S. López-Permouth, Cyclic and negacyclic codes over finite chain rings, *IEEE Trans. Inform. Theory*, **50** (2004), 1728–1744. https://doi.org/10.1109/TIT.2004.831789
7. S. Dougherty, A. Kaya, E. Salturk, Cyclic codes over local Frobenius rings of order 16, *Adv. Math. Commun.*, **11** (2017), 99–114. https://doi.org/10.3934/amc.2017005
8. M. Greferath, S. Schmidt, Gray isometries for finite chain rings and a nonlinear ternary $(36, 3^{12}, 15)$ code, *IEEE Trans. Inform. Theory*, **45** (1999), 2522–2524. https://doi.org/10.1109/18.796395
9. M. Greferath, M. O'Sullivan, On bounds for codes over Frobenius rings under homogeneous weights, *Discrete Math.*, **289** (2004), 11–24. https://doi.org/10.1016/j.disc.2004.10.002
10. F. Gursoy, I. Siap, B. Yildiz, Construction of skew cyclic codes over $\mathbb{F}_q + v\mathbb{F}_q$, *Adv. Math. Commun.*, **8** (2014), 313–322. https://doi.org/10.3934/amc.2014.8.313

11. A. Hammons, P. Kumar, A. Calderbank, N. Sloane, P. Sole, The Z_4-linearity of Kerdock, Preparate, Goethals, and related codes, *IEEE Trans. Inform. Theory*, **40** (1994), 301–319. https://doi.org/10.1109/18.312154

12. B. Yildiz, S. Karadeniz, Cyclic codes over $\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + uv\mathbb{F}_2$, *Des. Codes Crypt.*, **58** (2011), 221–234. https://doi.org/10.1007/s10623-010-9399-3

13. B. Yildiz, I. Kelebek, The homogeneous weight for R_k, related Gray map a new binary quasi-cyclic codes, *Filomat*, **31** (2017), 885–897. https://doi.org/10.2298/FIL1704885Y

14. M. Yilgor, F. Gursoy, E. Oztas, F. Demirkale, Cyclic codes over $\mathbb{F}_2 + u\mathbb{F}_2 + v\mathbb{F}_2 + v^2\mathbb{F}_2$ with respect to the homogeneous weight and their applications to DNA codes, *AAECC*, **32** (2021), 621–636. https://doi.org/10.1007/s00200-020-00416-0

15. S. Zhu, Y. Wang, M. Shi, Some results on cyclic codes over $F_2 + vF_2$, *IEEE Trans. Inform. Theory*, **56** (2010), 1680–1684. https://doi.org/10.1109/TIT.2010.2040896

Appendix

Table 1. Generators of ideals of R.

No.	Size	Ideal
1	2	$< u^2w + u^2 >$
46	2	$< u^2w >$
3	4	$< vw + v + uw + u >$
8	4	$< vw + v >$
17	4	$< uw >$
18	4	$< uw + u >$
22	4	$< u^2 >$
32	4	$< vw + uw >$
37	4	$< vw >$
10	8	$< vw, uw >$
16	8	$< vw + v, uw + u >$
24	8	$< vw + u^2 + uw >$
26	8	$< u^2w + uw + u >$
28	8	$< vw + v + u^2w >$
29	8	$< vw + v + u^2w + uw + u >$
43	8	$< u^2 + uw >$
44	8	$< vw + u^2 >$
4	16	$< v >$
5	16	$< vw + uw + u >$
9	16	$< vw + u >$
12	16	$< w >$
13	16	$< v + u >$
14	16	$< vw + v + u >$
20	16	$< vw + v, u^2 + uw + u >$

Table 2. Generators of ideals of R (continued).

No.	Size	Ideal
21	16	$< v + uw >$
23	16	$< w + 1 >$
34	16	$< vw + v + uw >$
41	16	$< v + uw + u >$
42	16	$< vw, u^2 + uw >$
45	16	$< u >$
6	32	$< u^2 + w >$
19	32	$< vw + v, u >$
25	32	$< vw, v + u >$
27	32	$< vw, u >$
35	32	$< vw, v + uw >$
36	32	$< u^2w + w + 1 >$
38	32	$< vw + v, vw + u >$
48	32	$< v, uw + u >$
2	64	$< v, u >$
11	64	$< vw + uw + w + 1 >$
15	64	$< v + w >$
30	64	$< v + u + w >$
39	64	$< u + w >$
40	64	$< uw + w + 1 >$
47	64	$< vw + w + 1 >$
7	128	$< v, u + w >$
33	128	$< vw, uw + w + 1 >$
31	256	$< w, w + 1 >$

AIMS Press

©2025 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (https://creativecommons.org/licenses/by/4.0)