Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(12): 28347–28373.
DOI: 10.3934/math.20251247
Received: 09 October 2025
Revised: 19 November 2025
Accepted: 26 November 2025
Published: 03 December 2025

Research article

A generalized framework for ς-neutrosophic fuzzy metric spaces and related fixed-point theorems

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Abstract: This paper introduces ς-neutrosophic fuzzy metric spaces (ς-NFMSs), a significant generalization of neutrosophic fuzzy metric spaces (NFMSs). By extending the parameter space from a single dimension $(0, \infty)$ to a multi-dimensional vector space $(0, \infty)^{\varsigma}$, this framework offers enhanced flexibility for modeling complex systems where uncertainty depends on multiple factors simultaneously. The study investigates the topological properties of ς-NFMSs, rigorously proving that their topology is first-countable and that the associated space is Hausdorff. Furthermore, a generalized fixed-point theorem is established within this new framework, extending previous results in NFMSs.

Keywords: neutrosophic fuzzy metric space; ς-NFMS; Hausdorff topology; fixed-point theorem; t-norm; t-conorm; multi-parameter generalization
Mathematics Subject Classification: 47H10, 54A40, 54E70

1. Introduction and background

A mathematical method for handling sets in which elements may have partial degrees of membership is provided by fuzzy set theory. Unlike classical set theory, where membership is binary–an element is either in or out–fuzzy sets provide a more adaptable framework, especially useful for modeling situations with unclear or uncertain boundaries. This innovative concept was first introduced by Zadeh in [1], which has profoundly influenced numerous fields of study. Building on Zadeh's pioneering ideas, further developments enriched the theory. For example, Smarandache [2] introduced neutrosophic sets, whereas Atanassov [3] proposed intuitionistic fuzzy sets, both of which broadened the theoretical and practical horizons of fuzzy sets. Although intuitionistic fuzzy sets are fuzzy sets, this is not always the case [4]. Indeed, intuitionistic fuzzy set theory is better suited to some circumstances [5]. These initial breakthroughs paved the way for subsequent advancements. Kramosil and Michálek [6] defined fuzzy metric spaces in 1975, and George and Veeramani [7] improved upon them in 1994. The usefulness and reach of fuzzy set theory were further expanded in 1997 by Çoker [8] with intuitionistic fuzzy topological spaces and in 2004 by Park [9] with intuitionistic fuzzy metric spaces (IFMSs). Vinoth and Jayalakshmi [10] explored new concept and examined some features of this concept in κ-IFMS.

The introduction of these concepts has had a significant influence on numerous subsequent research efforts by mathematicians. Kaleva and Seikkala [11] developed fuzzy metric spaces, for example, as the distances between two locations, expressed as positive fuzzy integers, in 1984. Neutosophic sets are an extension of intuitionistic fuzzy sets that were introduced by Smarandache [12] in 2006. In 2012, Salama and Alblowi [13] expanded the concepts of intuitionistic fuzzy topological spaces and fuzzy topological spaces to the neutrosophic set setting. The modal operator and normalization are two algebraic procedures that Ejegwa [14] introduced to intuitionistic fuzzy sets in 2014. The area was further advanced by Majumdar [15], who looked at the real-world uses of neutrosophic sets in decision-making.

The drive to generalize mathematical structures extends beyond the standard fuzzy framework. Various approaches have been developed to handle generalized norms, exemplified by the study of linear n-normed spaces and their completions using ideal convergence [16]. Furthermore, the analysis of uncertainty utilizes diverse mathematical tools; for instance, advanced convergence concepts, such as statistical convergence, have been successfully applied in spaces modeling uncertainty, like credibility spaces [17]. The analytical techniques underpinning the study of these generalized spaces often leverage deep results from functional analysis. Advanced techniques utilizing Banach algebras [18] and operator theory–including the analysis of spectral properties on specific spaces like the Wiener algebra [19] and the application of reproducing kernels [20]–continue to offer powerful analytical frameworks that often intersect with the study of generalized metric structures.

Probabilistic metric spaces (PMSs), introduced by K. Menger [21], extended usual metric spaces (MSs) by incorporating a probabilistic approach to distance. Instead of using a numerical value, Menger employed a distribution function $\upsilon_{\beta,\gamma}$ for each pair of elements β and γ. For each real number Θ, $\upsilon_{\beta,\gamma}(\Theta)$ represents the probability that the distance from β to γ is less than Θ. The distribution function υ is a left-continuous, non-decreasing function $\mathbb{R} \to [0, 1]$, with $\inf_{\Theta\in\mathbb{R}} \upsilon(\mathbb{R}) = 0$ and $\sup_{\Theta\in\mathbb{R}} \upsilon(\mathbb{R}) = 1$. An important characteristic of fuzzy metrics is the inclusion of a parameter Θ, which has found applications across various domains, including engineering, economics, marketing, and medicine.

Heilpern [22] introduced the fixed-point theorem for fuzzy contraction maps. Heilpern's analysis was expanded by Bose and Sahani [23]. Fixed-point theorems pertaining to IFMS are presented to Alaca et al. [24]. A major breakthrough in the field was made in 2020 when Kirişci and Şimşek [25] established the idea of neutrosophic metric spaces. Kirişçi and Şimşek [26] explored the notions of neutrosophic contractive and neutrosophic mappings.

In this research, we derived multiple findings regarding the fixed points of a neutrosophic mapping. Subsequently, several fixed-point results have been established within this framework [27, 28], along with further generalizations of neutrosophic fuzzy metric spaces, enhancing their theoretical and practical applications. Indeed, the field of neutrosophic metric spaces is expanding rapidly, with researchers exploring various generalizations in different directions. For instance, some studies focus on relaxing the classical metric axioms, leading to the development of neutrosophic metric-like spaces [29] and other generalized structures like neutrosophic E_β-metric spaces and neutrosophic quasi-S_β-metric spaces [30]. Other works focus on strengthening the foundational fixed-point theory within the standard neutrosophic fuzzy metric space (NFMS) itself, establishing key results like the Banach, Edelstein, and Kannan fixed-point theorems [31]. However, these valuable generalizations primarily focus on modifying the metric axioms or applying existing contractions. The challenge of modeling uncertainty that depends on multiple, independent parameters simultaneously (a multi-dimensional parameter vector) remains largely unaddressed. Our work aims to fill this specific gap by introducing the ς-neutrosophic fuzzy metric space (ς-NFMS) framework, a novel generalization focused on the parameter space $(0,\infty)^\varsigma$ rather than the metric axioms. This field remains highly active, with recent studies exploring novel fuzzy contractions for applications in engineering science [32] and convergence results for specific contractions like the graph-Reich type [33].

The theory of fuzzy metric spaces has advanced significantly since Das et al. [34] proposed the idea of neutrosophic fuzzy sets. In their study, Ghosh et al. [35] investigated the idea of neutrosophic fuzzy metric space.

Neutroposophic fuzzy metric spaces were introduced because they are better than typical fuzzy or crisp sets in modeling uncertainty and indeterminacy in real-world occurrences. Neutrosophic sets give a more thorough representation of uncertainty by representing items with three components: truth, indeterminacy, and falsity.

In the area of medical diagnosis, for example, NFMS can handle situations in which a patient's symptoms are not indicative of a particular illness. Conventional fuzzy sets might use a scale of 0 to 1 to reflect the probability of a diagnosis. This approach, therefore, might not adequately convey the ambiguity or contradictory character of symptoms. Conversely, neutrophilic sets take into consideration the degrees of truth, indeterminacy, and falsity connected to every symptom and possible diagnosis. Through the use of NFMS, a metric space can be created in which the distance between two diagnoses represents both their resemblance and the intrinsic uncertainty or indeterminacy of the diagnostic process.

A robust mathematical foundation that closely matches the intricacy and unpredictability of real-world issues is offered by the introduction of NFMS. Applications include identifying patterns in confusing data sets, making decisions under ambiguity, and diagnosing medical conditions. Our goal in creating NFMS theory is to enhance our capacity to model, evaluate, and make wise choices in unpredictable situations.

This study aims to develop a generalized framework by extending the concept of NFMS. We consider an NFMS where the fuzzy distance is characterized by degrees of truth, indeterminacy, and falsity relative to a parameter Θ. For example, Θ could represent the uncertainty in diagnosing a medical condition, allowing for varying degrees of similarity and dissimilarity between potential diagnoses based on conflicting or ambiguous symptoms.

Consider the uncertainty in diagnosing a medical condition as a way to measure the "closeness" between two potential diagnoses, β and γ. Incorporating factors such as symptom ambiguity, conflicting test results, and patient history as parameters adds complexity, reflecting the multifaceted nature of medical assessments influenced by numerous variables. Classical NFMSs successfully model uncertainty with a single parameter Θ, for instance, the "closeness" between two potential medical diagnoses [36]. However, this approach can be insufficient for complex systems where uncertainty arises from multiple, independent factors simultaneously. For example, a final medical diagnosis may depend not only on ambiguous symptoms but also on the reliability of lab test results, the contradictory nature of patient history, and the severity of the symptoms. Each of these factors represents an independent dimension of uncertainty that can be modeled by a component of our ς-parameter vector. The ς-NFMS framework developed here [2], where $\varsigma \in \{1, 2, 3, ...\}$, provides a more flexible and realistic mechanism to model this multi-dimensional uncertainty structure. This is the primary motivation for our generalization. The multi-parameter approach can also be seen as a contribution parallel to the motivation presented for κ-IFMS in [10], which motivates the concept of an ς-NFMS ($\varsigma \in \{1, 2, 3, ...\}$), where the distance between elements is defined in terms of truth, indeterminacy, and falsity components relative to a parameter Θ.

The following is a summary of this paper's structure: We outline some essential features and basic ideas of neutrosophic fuzzy sets and neutrosophic metric spaces in Section 2. After introducing the idea of ς-NFMS, Section 3 provides examples that show how to use it. The topological characteristics of the generalized metric space are also illustrated in this part, emphasizing significant findings such as nowhere denseness, the Hausdorff property, compactness, and completeness. We will now examine these findings in more depth as we proceed to the paper's primary findings. The paper's conclusion, Section 4, establishes a fixed-point theorem that expands and generalizes earlier findings on ς-NFMSs.

2. Preliminaries

The key terms that provide the basis for determining the primary findings are introduced in this section. The non-standard finite numbers are defined as follows: $1^+ = 1 + \tau$, where "1" is its standard part and τ is its non-standard part; and $0^- = 1 + \tau$, where "0" is its standard part and τ is its non-standard part. In this case, $]0^-, 1^+[$ denotes a non-standard unit interval. The non-standard unit interval $]0^-, 1^+[$ contains the non-standard integers 0 and 1, which are infinitesimally tiny but less than 0 and infinitesimally small but higher than 1, respectively.

Triangular norms (t-norms) were introduced by Menger [21] in the context of measuring distances between elements in a space. Menger suggested using probability distributions rather than numbers to express distances. t-norms extend the concept of the triangle inequality to probabilistic metric spaces. Their dual counterparts, triangular conorms (t-conorms), are similarly important. Both t-norms and t-conorms play a critical role in fuzzy operations, particularly in modeling intersections and unions.

Definition 2.1. *Consider an operation* $\oplus : [0, 1] \times [0, 1] \to [0, 1]$. *The operation* $\oplus$ *is referred to as a continuous t-norm if it fulfills the following properties for all* $s, t, u, v \in [0, 1]$*:*

i. $s \oplus 1 = s$.
ii. When $s \leq u$ *and* $t \leq v$, *then* $s \oplus t \leq u \oplus v$.
iii. The operation $\oplus$ *is continuous.*
iv. The operation $\oplus$ *satisfies both commutativity and associativity.*

Definition 2.2. *Consider an operation* $[0, 1] \times [0, 1] \to [0, 1]$. *The operation* $\circledast$ *is referred to as a continuous t-conorm if it fulfills the following properties for all* $s, t, u, v \in [0, 1]$*:*
(i) $s \circledast 0 = s$.
(ii) *When* $s \leq u$ *and* $t \leq v$, *then* $s \circledast t \leq u \circledast v$.
(iii) *The operation* $\circledast$ *is continuous.*
(iv) *The operation* $\circledast$ *satisfies both commutativity and associativity.*

Definition 2.3. *A 6-tuple* $(\Psi, \upsilon, \varpi, \lambda, \oplus, \circledast)$ *is called a neutrosophic metric space (NMS) if* Ψ *is a non-empty set,* $\oplus$ *and* $\circledast$ *are continuous t-norm and t-conorm operations, respectively, and* υ, ϖ, *and* λ *are fuzzy sets defined on* $\Psi^2 \times (0, \infty)$, *satisfying specific conditions for all* $u, v, w \in \Psi$ *and* $y, z > 0$,
i. $0 \leq \upsilon(u, v, z) \leq 1$, $0 \leq \varpi(u, v, z) \leq 1$, $0 \leq \lambda(u, v, z) \leq 1$,
ii. $\upsilon(u, v, z) + \varpi(u, v, z) + \lambda(u, v, z) \leq 3$,
iii. $\upsilon(u, v, z) = 1$ *if* $u = v$,
iv. $\upsilon(u, v, z) = \upsilon(v, u, z)$,
v. $\upsilon(u, v, z) \oplus \upsilon(v, w, y) \leq \upsilon(u, w, z + y)$,
vi. $\upsilon(u, v, .) : [0, \infty) \to [0, 1]$ *is continuous,*
vii. $\lim_{z\to\infty} \upsilon(u, v, z) = 1$,
viii. $\varpi(u, v, z) = 0$ *if* $u = v$,
ix. $\varpi(u, v, z) = \varpi(v, u, z)$,
x. $\varpi(u, v, z) \circledast \varpi(v, w, y) \geq \varpi(u, w, z + y)$,
xi. $\varpi(u, v, .) : [0, \infty) \to [0, 1]$ *is continuous,*
xii. $\lim_{z\to\infty} \varpi(u, v, z) = 0$,
xiii. $\lambda(u, v, z) = 0$ *if* $u = v$,
xiv. $\lambda(u, v, z) = \lambda(v, u, z)$,
xv. $\lambda(u, v, z) \circledast \lambda(v, w, y) \geq \lambda(u, w, z + y)$,
xvi. $\lambda(u, v, .) : [0, \infty) \to [0, 1]$ *is continuous,*
xvii. $\lim_{z\to\infty} \lambda(u, v, z) = 0$,
xviii. if $z \leq 0$, *then* $\upsilon(u, v, z) = 0$, $\varpi(u, v, z) = 1$ *and* $\lambda(u, v, z) = 1$.

The degrees of nearness, neutralness, and non-nearness between u and v with regard to z are represented by $\upsilon(u, v, z)$, $\varpi(u, v, z)$, *and* $\lambda(u, v, z)$, *respectively, in this context.*

Now, we transition to the section that covers the key results derived from this study, offering a deeper exploration of the significant properties of ς-NFMSs.

3. Main results

The creation of ς-NFMSs is the main topic of this section, which also examines other aspects that help the framework operate.

Definition 3.1. *ς-NFMS: Consider Ψ as a non-empty set equipped with a continuous t-norm $\oplus$ and a continuous t-conorm $\circledast$. Let υ, ϖ, λ, and τ represent fuzzy sets (FSs) defined on $\Psi^2 \times (0,\infty)^{\varsigma}$. An ordered 7-tuple $(\Psi, \upsilon, \varpi, \lambda, \tau, \oplus, \circledast)$ satisfies the conditions of a ς-NFMS if the following properties hold for all β, $\gamma \in \Psi$ and for all $\Theta_i \mid_{i=1}^{\varsigma} = \Theta_1, \Theta_2, \ldots, \Theta_{\varsigma} > 0$:*

(ς1)

$$0 \leq \upsilon(\beta, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\varsigma}) \leq 1,\ 0 \leq \varpi(\beta, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\varsigma}) \leq 1,$$
$$0 \leq \lambda(\beta, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\varsigma}) \leq 1,\ 0 \leq \tau(\beta, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\varsigma}) \leq 1,$$

(ς2)

$$\begin{aligned}&\upsilon(\beta, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\varsigma}) + \varpi(\beta, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\varsigma}\\ &+ \lambda(\beta, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\varsigma}) + \tau(\beta, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\varsigma} \leq 4,\end{aligned}$$

(ς3) $\upsilon(\beta, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\varsigma}) = \upsilon(\gamma, \beta, \Theta_1, \Theta_2, \ldots, \Theta_{\varsigma})$,
(ς4) $\upsilon(\beta, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\varsigma}) = 1 \Longleftrightarrow \beta = \gamma$,
(ς5) $\lim_{\varsigma \to \infty} \upsilon(\beta, \gamma, \Theta_1, \Theta_2, \cdots, \Theta_{\varsigma}) = 1$,
(ς6) *for any* $\mathfrak{h} \in \{1, 2, \cdots, \varsigma\}$, *we have*

$$\begin{aligned}&\upsilon(\beta, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\mathfrak{h}-1}, \Theta + \varrho, \Theta_{\mathfrak{h}+1}, ..., \Theta_{\varsigma}\\ &\geq \upsilon(\beta, \varsigma, \Theta_1, \Theta_2, \ldots, \Theta_{\mathfrak{h}-1}, \Theta, \Theta_{\mathfrak{h}+1}, ..., \Theta_{\varsigma}) \oplus \upsilon(\varsigma, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\mathfrak{h}-1}, \varrho, \Theta_{\mathfrak{h}+1}, ..., \Theta_{\varsigma}),\end{aligned} \tag{3.1}$$

(ς7) $\upsilon(\beta, \gamma, .) : (0,\infty)^{\varsigma} \to [0, 1]$ *is continuous,*
(ς8) $\varpi(\beta, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\varsigma}) = \varpi(\gamma, \beta, \Theta_1, \Theta_2, \ldots, \Theta_{\varsigma})$,
(ς9) $\varpi(\beta, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\varsigma}) = 1 \Longleftrightarrow \beta = \gamma$,
(ς10) $\lim_{\varsigma \to \infty} \varpi(\beta, \gamma, \Theta_1, \Theta_2, \cdots, \Theta_{\varsigma}) = 1$,
(ς11) *for any* $\mathfrak{h} \in \{1, 2, \cdots, \varsigma\}$, *we have*

$$\begin{aligned}&\varpi(\beta, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\mathfrak{h}-1}, \Theta + \varrho, \Theta_{\mathfrak{h}+1}, ..., \Theta_{\varsigma})\\ &\geq \varpi(\beta, \varsigma, \Theta_1, \Theta_2, \ldots, \Theta_{\mathfrak{h}-1}, \Theta, \Theta_{\mathfrak{h}+1}, ..., \Theta_{\varsigma}) \oplus \varpi(\varsigma, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\mathfrak{h}-1}, \varrho, \Theta_{\mathfrak{h}+1}, ..., \Theta_{\varsigma}),\end{aligned} \tag{3.2}$$

(ς12) $\varpi(\beta, \gamma, .) : (0,\infty)^{\varsigma} \to (0, 1]$ *is continuous,*
(ς13) $\lambda(\beta, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\varsigma}) = \lambda(\gamma, \beta, \Theta_1, \Theta_2, \ldots, \Theta_{\varsigma})$,
(ς14) $\lambda(\beta, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\varsigma}) = 0 \Longleftrightarrow \beta = \gamma$,
(ς15) $\lim_{\varsigma \to \infty} \lambda(\beta, \gamma, \Theta_1, \Theta_2, \cdots, \Theta_{\varsigma}) = 0$,
(ς16) *for any* $\mathfrak{h} \in \{1, 2, \cdots, \varsigma\}$, *we have*

$$\begin{aligned}&\lambda(\beta, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\mathfrak{h}-1}, \Theta + \varrho, \Theta_{\mathfrak{h}+1}, ..., \Theta_{\varsigma})\\ &\leq \lambda(\beta, \varsigma, \Theta_1, \Theta_2, \ldots, \Theta_{\mathfrak{h}-1}, \Theta, \Theta_{\mathfrak{h}+1}, ..., \Theta_{\varsigma}) \circledast \lambda(\varsigma, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\mathfrak{h}-1}, \varrho, \Theta_{\mathfrak{h}+1}, ..., \Theta_{\varsigma}),\end{aligned}$$

(ς17) $\lambda(\beta, \gamma, .) : (0,\infty)^{\varsigma} \to (0, 1]$ *is continuous,*
(ς18) $\tau(\beta, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\varsigma}) = \tau(\gamma, \beta, \Theta_1, \Theta_2, \ldots, \Theta_{\varsigma})$,
(ς19) $\tau(\beta, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\varsigma}) = 0 \Longleftrightarrow \beta = \gamma$,
(ς20) $\lim_{\varsigma \to \infty} \tau(\beta, \gamma, \Theta_1, \Theta_2, \cdots, \Theta_{\varsigma}) = 0$,
(ς21) *for any* $\mathfrak{h} \in \{1, 2, \cdots, \varsigma\}$, *we have*

$$\begin{aligned}&\tau(\beta, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\mathfrak{h}-1}, \Theta + \varrho, \Theta_{\mathfrak{h}+1}, ..., \Theta_{\varsigma})\\ &\leq \tau(\beta, \varsigma, \Theta_1, \Theta_2, \ldots, \Theta_{\mathfrak{h}-1}, \Theta, \Theta_{\mathfrak{h}+1}, ..., \Theta_{\varsigma}) \circledast \tau(\varsigma, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\mathfrak{h}-1}, \varrho, \Theta_{\mathfrak{h}+1}, ..., \Theta_{\varsigma}),\end{aligned}$$

(ς22) $\tau(\beta, \gamma, .) : (0, \infty)^{\varsigma} \to (0, 1]$ *is continuous,*
(ς23) *for* $\Theta_i \leq 0$,

$$\upsilon(\beta, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\varsigma}) = 0,\ \varpi(\beta, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\varsigma}) = 0,$$
$$\lambda(\beta, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\varsigma}) = 1,\ \tau(\beta, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\varsigma}) = 1.$$

In this framework, $\upsilon(\beta, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\varsigma})$ *represents the certainty that the distance between* β *and* γ *is less than* Θ_i*. Similarly,* $\varpi(\beta, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\varsigma})$ *indicates the degree of nearness,* $\lambda(\beta, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\varsigma})$ *indicates the degree of neutralness, and* $\tau(\beta, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\varsigma})$ *stands for the degree of non-nearness between* β *and* γ *with regard to* Θ_i*, respectively.*

Remark 3.1. *The primary motivation for extending the single parameter* $\Theta \in (0, \infty)$ *of classical NFMSs to the multi-dimensional vector* $\Theta = (\Theta_1, \ldots, \Theta_{\varsigma}) \in (0, \infty)^{\varsigma}$ *is to model complex systems where uncertainty is influenced by multiple, independent factors. As discussed in the introduction, a single parameter* Θ *may not be sufficient to capture the distinct roles of, for example, symptom ambiguity, test result reliability, and patient history in a medical diagnosis. The* ς*-dimensional parameter space allows each of these factors to be represented by its own component* (Θ_i)*, providing a more flexible and granular framework for modeling multi-faceted uncertainty.*

When $\varsigma = 1$, the ς-NFMS reduces to the NFMS, introduced by Ghosh et al. [35].

Example 3.1. (Induced ς-NFMS) Let (Ψ, d) denote a metric space, where $\Psi = (-\infty, \infty)$ and $d(\beta, \gamma) = |\beta - \gamma|$. Define the t-norm and t-conorm, $\oplus$ and $\circledast$ as $\beta \oplus \gamma = \min\{\beta, \gamma\}$ and $\beta \circledast \gamma = \max\{\beta, \gamma\}$. Let the FS $\upsilon, \varpi, \lambda, \tau$ on $\Psi^2 \times (0, \infty)$ be defined as

$$\begin{aligned}
\upsilon(\beta, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\varsigma}) &= \tfrac{\Theta_i + d(\beta,\gamma)}{\Theta_i + 2d(\beta,\gamma)},\\
\varpi(\beta, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\varsigma}) &= \tfrac{\Theta_i}{\Theta_i + d(\beta,\gamma)},\\
\lambda(\beta, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\varsigma}) &= \tfrac{d(\beta,\gamma)}{\Theta_i + d(\beta,\gamma)},\\
\tau(\beta, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\varsigma}) &= \tfrac{d(\beta,\gamma)}{\Theta_i},
\end{aligned}$$

for all $\beta, \gamma \in \Psi$, and $\Theta_i \,|_{i=1}^{\varsigma} > 0$.

Notice that:
(ς1) $0 \leq \upsilon(\beta, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\varsigma}) \leq 1$,
(ς2) since $d(\beta, \gamma) = d(\gamma, \beta)$, we have $\upsilon(\beta, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\varsigma}) = \upsilon(\gamma, \beta, \Theta_1, \Theta_2, \ldots, \Theta_{\varsigma})$,
(ς3) $\upsilon(\beta, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\varsigma}) = 1$ if $\beta = \gamma$,
(ς4) $\lim_{i\to\infty} \frac{\Theta_i + d(\beta,\gamma)}{\Theta_i + 2d(\beta,\gamma)} = 1$, for all $\beta, \gamma \in \Psi$ and $\Theta_i \,|_{i=1}^{\varsigma} = (\Theta_1, \Theta_2, \ldots, \Theta_{\varsigma}) > 0$.
(ς5)

$$\begin{aligned}
&\upsilon(\beta, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\mathfrak{h}-1}, \Theta + \varrho, \Theta_{\mathfrak{h}+1}, ..., \Theta_{\varsigma})\\
&\geq \upsilon(\beta, \varsigma, \Theta_1, \Theta_2, \ldots, \Theta_{\mathfrak{h}-1}, \Theta, \Theta_{\mathfrak{h}+1}, ..., \Theta_{\varsigma}) \oplus \upsilon(\varsigma, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\mathfrak{h}-1}, \varrho, \Theta_{\mathfrak{h}+1}, ..., \Theta_{\varsigma}),
\end{aligned}$$

for all $\beta, \gamma, \varsigma \in \Psi$, and $\Theta, \varrho > 0$.

Similarly, all the conditions for ϖ, λ, τ are satisfied. Let us briefly verify why these functions satisfy the core conditions. For instance, consider condition (ς5). Since the standard metric d satisfies the triangle inequality $d(\beta, \gamma) \leq d(\beta, \varsigma) + d(\varsigma, \gamma)$, and the function $f(x) = \frac{\Theta + x}{\Theta + 2x}$ is monotonically decreasing with respect to x, the structural compatibility between the metric d and the fuzzy set υ ensures the triangle inequality in the fuzzy setting is preserved. Similarly, the boundary conditions are satisfied as $\lim_{\Theta\to\infty}\frac{\Theta+d}{\Theta+2d}=1$ and $\lim_{\Theta\to 0}\frac{\Theta+d}{\Theta+2d}=\frac{1}{2}$ (for $d\neq 0$), consistent with the properties of ς-NFMSs. Hence, $(\Psi,\upsilon,\varpi,\lambda,\tau,\oplus,\circledast)$ forms an ς-NFMS induced by a metric d, referred to as the standard ς-NFMS.

Remark 3.2. *The 7-tuple* $(\Psi,\upsilon,\varpi,\lambda,\tau,\oplus,\circledast)$ *defined in above Example 3.1 would not constitute as a NFMS if the t-norm* $\oplus$ *is defined as* $\beta\oplus\gamma=\max\{0,\beta+\gamma-1\}$ *and the t-conorm* $\circledast$ *is defined as* $\beta\circledast\gamma=\beta+\gamma-\beta\gamma$.

Example 3.2. Let Ψ be the set of natural numbers. Consider the operations $\oplus$ and $\circledast$ defined as follows: the t-norm $\beta\oplus\gamma=\max\{0,\beta+\gamma-1\}$ and t-conorm $\beta\circledast\gamma=\beta+\gamma-\beta\gamma$. For all $\beta,\gamma\in\Psi$, $\Theta_i\,|_{i=1}^{\varsigma}\in(0,\infty)$,

$$\upsilon(\beta,\gamma,\Theta_1,\Theta_2,\ldots,\Theta_\varsigma)=1-\frac{|\beta-\gamma|}{2\Theta_i},$$

$$\varpi(\beta,\gamma,\Theta_1,\Theta_2,\ldots,\Theta_\varsigma)=\frac{\Theta_i^3-|\beta-\gamma|}{\Theta_i^3},$$

$$\lambda(\beta,\gamma,\Theta_1,\Theta_2,\ldots,\Theta_\varsigma)=\begin{cases}\frac{\gamma-\beta}{\gamma+\Theta_i}, & \text{if } \beta\leq\gamma,\\ \frac{\beta-\gamma}{\beta+\Theta_i}, & \text{if } \gamma\leq\beta,\end{cases}$$

$$\tau(\beta,\gamma,\Theta_1,\Theta_2,\ldots,\Theta_\varsigma)=\begin{cases}\frac{\gamma^2-\beta^2}{\gamma^2+\Theta_i^2}, & \text{if } \beta\leq\gamma,\\ \frac{\beta^2-\gamma^2}{\beta^2+\Theta_i^2}, & \text{if } \gamma\leq\beta.\end{cases}$$

Note that:
(1) $0\leq\upsilon(\beta,\gamma,\Theta_1,\Theta_2,\ldots,\Theta_\varsigma)\leq 1$,
(2) since $|\beta-\gamma|=|\gamma-\beta|$, we have

$$\upsilon(\beta,\gamma,\Theta_1,\Theta_2,\ldots,\Theta_\varsigma)=\upsilon(\gamma,\beta,\Theta_1,\Theta_2,\ldots,\Theta_\varsigma),$$

(3) $\upsilon(\beta,\gamma,\Theta_1,\Theta_2,\ldots,\Theta_\varsigma)=1$,
(4) $\lim_{i\to\infty}\frac{\Theta_i^3-|\beta-\gamma|}{\Theta_i^3}=1$, for all $\beta,\gamma\in\Psi$, $\Theta_i\,|_{i=1}^{\varsigma}>0$;
(5)

$$\begin{aligned}&\upsilon(\beta,\gamma,\Theta_1,\Theta_2,\ldots,\Theta_{\mathfrak{h}-1},\Theta+\varrho,\Theta_{\mathfrak{h}+1},...,\Theta_\varsigma)\\ &\geq\upsilon(\beta,\varsigma,\Theta_1,\Theta_2,\ldots,\Theta_{\mathfrak{h}-1},\Theta,\Theta_{\mathfrak{h}+1},...,\Theta_\varsigma)\oplus\upsilon(\varsigma,\gamma,\Theta_1,\Theta_2,\ldots,\Theta_{\mathfrak{h}-1},\varrho,\Theta_{\mathfrak{h}+1},...,\Theta_\varsigma),\end{aligned}$$

for all $\beta,\gamma,\varsigma\in\Psi$, and $\Theta,\varrho>0$.

Similarly, all the conditions for ϖ,λ,τ are satisfied. Under these conditions, $(\Psi,\upsilon,\varpi,\tau,\oplus,\circledast)$ forms a ς-NFMS.

For simplicity, we use $\upsilon\left(\beta,\gamma,\Theta_j^\varsigma\right)$ in place of $\upsilon(\beta,\gamma,\Theta_1,\Theta_2,\ldots,\Theta_\varsigma)$, $\varpi\left(\beta,\gamma,\Theta_j^\varsigma\right)$ instead of $\varpi(\beta,\gamma,\Theta_1,\Theta_2,\ldots,\Theta_\varsigma)$, $\lambda\left(\beta,\gamma,\Theta_j^\varsigma\right)$ for $\lambda(\beta,\gamma,\Theta_1,\Theta_2,\ldots,\Theta_\varsigma)$, and $\tau\left(\beta,\gamma,\Theta_j^\varsigma\right)$ to represent $\tau(\beta,\gamma,\Theta_1,\Theta_2,\ldots,\Theta_\varsigma)$, where $j=1,2,...,\varsigma$.

Lemma 3.1. *Let* $(\Psi, \upsilon, \varpi, \lambda, \tau, \oplus, \circledast)$ *be a* ς*-NFMS with* $\Theta_1, \Theta_2, \cdots, \Theta_\varsigma > 0$*. If* $\Theta_\mathfrak{h} < \Theta$ *for some* $\mathfrak{h} \in \{1, 2, 3, \cdots, \varsigma\}$ *such that* $\Theta_\mathfrak{h} < \Theta$*, then the following inequalities supply:*

$$\begin{aligned}
&\upsilon\left(\beta, \gamma, \Theta_j^\varsigma\right) \leq \upsilon\left(\beta, \gamma, \Theta_1, \Theta_2, \cdots, \Theta_{\mathfrak{h}-1}, \Theta_\mathfrak{h}, \Theta_{\mathfrak{h}+1}, \cdots, \Theta_\varsigma\right), \\
&\varpi\left(\beta, \gamma, \Theta_j^\varsigma\right) \leq \varpi\left(\beta, \gamma, \Theta_1, \Theta_2, \cdots, \Theta_{\mathfrak{h}-1}, \Theta_\mathfrak{h}, \Theta_{\mathfrak{h}+1}, \cdots, \Theta_\varsigma\right), \\
&\lambda\left(\beta, \gamma, \Theta_j^\varsigma\right) \geq \lambda\left(\beta, \gamma, \Theta_1, \Theta_2, \cdots, \Theta_{\mathfrak{h}-1}, \Theta_\mathfrak{h}, \Theta_{\mathfrak{h}+1}, \cdots, \Theta_\varsigma\right), \text{ and} \\
&\tau\left(\beta, \gamma, \Theta_j^\varsigma\right) \geq \tau\left(\beta, \gamma, \Theta_1, \Theta_2, \cdots, \Theta_{\mathfrak{h}-1}, \Theta_\mathfrak{h}, \Theta_{\mathfrak{h}+1}, \cdots, \Theta_\varsigma\right).
\end{aligned}$$

Proof. By applying the properties of t-norms along with condition (ς6), we can establish that for any β and γ in the set Ψ, the following holds:

$$\begin{aligned}
\upsilon\left(\beta, \gamma, \Theta_j^\varsigma\right) &= \upsilon\left(\beta, \gamma, \Theta_j^\varsigma\right) \oplus 1 = \upsilon\left(\beta, \gamma, \Theta_j^\varsigma\right) \oplus \upsilon\left(\gamma, \gamma, \Theta_1, \Theta_2, \cdots, \Theta_{\mathfrak{h}-1}, \Theta_\mathfrak{h}, \Theta_{\mathfrak{h}+1}, \cdots, \Theta_\varsigma\right) \\
&\leq \upsilon\left(\beta, \gamma, \Theta_1, \Theta_2, \cdots, \Theta_{\mathfrak{h}-1}, \Theta_\mathfrak{h}, \Theta_{\mathfrak{h}+1}, \cdots, \Theta_\varsigma\right).
\end{aligned}$$

Utilizing the properties of t-norms and condition (ς11), we deduce that for every β, γ in Ψ, the following result holds:

$$\begin{aligned}
\varpi\left(\beta, \gamma, \Theta_j^\varsigma\right) &= \varpi\left(\beta, \gamma, \Theta_j^\varsigma\right) \oplus 1 = \varpi\left(\beta, \gamma, \Theta_j^\varsigma\right) \oplus \varpi\left(\gamma, \gamma, \Theta_1, \Theta_2, \cdots, \Theta_{\mathfrak{h}-1}, \Theta_\mathfrak{h}, \Theta_{\mathfrak{h}+1}, \cdots, \Theta_\varsigma\right) \\
&\leq \varpi\left(\beta, \gamma, \Theta_1, \Theta_2, \cdots, \Theta_{\mathfrak{h}-1}, \Theta_\mathfrak{h}, \Theta_{\mathfrak{h}+1}, \cdots, \Theta_\varsigma\right).
\end{aligned}$$

Using the properties of t-conorms and condition (ς16),

$$\begin{aligned}
\lambda\left(\beta, \gamma, \Theta_j^\varsigma\right) &= \lambda\left(\beta, \gamma, \Theta_j^\varsigma\right) \circledast 0 = \lambda\left(\beta, \gamma, \Theta_j^\varsigma\right) \circledast \lambda\left(\gamma, \gamma, \Theta_1, \Theta_2, \cdots, \Theta_{\mathfrak{h}-1}, \Theta_\mathfrak{h}, \Theta_{\mathfrak{h}+1}, \cdots, \Theta_\varsigma\right) \\
&\geq \lambda\left(\beta, \gamma, \Theta_1, \Theta_2, \cdots, \Theta_{\mathfrak{h}-1}, \Theta_\mathfrak{h}, \Theta_{\mathfrak{h}+1}, \cdots, \Theta_\varsigma\right).
\end{aligned}$$

Finally, by utilizing the properties of t-conorms and condition (ς23), we can derive the following:

$$\tau\left(\beta, \gamma, \Theta_j^\varsigma\right) \geq \tau\left(\beta, \gamma, \Theta_1, \Theta_2, \cdots, \Theta_{\mathfrak{h}-1}, \Theta_\mathfrak{h}, \Theta_{\mathfrak{h}+1}, \cdots, \Theta_\varsigma\right).$$

Remark 3.3. *Let* $(\Psi, \upsilon, \varpi, \lambda, \tau, \oplus, \circledast)$ *be a* ς*-NFMS. If* $\upsilon\left(\beta, \gamma, \Theta_j^\varsigma\right) > 1 - \vartheta$*,* $\lambda\left(\beta, \gamma, \Theta_j^\varsigma\right) > 1 - \vartheta$*,* $\varpi\left(\beta, \gamma, \Theta_j^\varsigma\right) < \vartheta$ *and* $\tau\left(\beta, \gamma, \Theta_j^\varsigma\right) < \vartheta$ *for all* $\beta, \gamma \in \Psi$*,* $\Theta_j \mid_{j=1}^{\varsigma} = \Theta_1, \Theta_2, \cdots, \Theta_\varsigma > 0$*, and* $\vartheta \in (0, 1)$*, then for each* $\mathfrak{h} \in \{1, 2, 3, \cdots, \varsigma\}$*, there exists* $\Theta \in (0, \Theta_\mathfrak{h})$ *such that the following conditions hold:*

$$\begin{aligned}
&\upsilon\left(\beta, \gamma, \Theta_1, \Theta_2, \cdots, \Theta_{\mathfrak{h}-1}, \Theta_\mathfrak{h}, \Theta_{\mathfrak{h}+1}, \cdots, \Theta_\varsigma\right) > 1 - \vartheta, \\
&\varpi\left(\beta, \gamma, \Theta_1, \Theta_2, \cdots, \Theta_{\mathfrak{h}-1}, \Theta_\mathfrak{h}, \Theta_{\mathfrak{h}+1}, \cdots, \Theta_\varsigma\right) > 1 - \vartheta, \\
&\lambda\left(\beta, \gamma, \Theta_1, \Theta_2, \cdots, \Theta_{\mathfrak{h}-1}, \Theta_\mathfrak{h}, \Theta_{\mathfrak{h}+1}, \cdots, \Theta_\varsigma\right) < \vartheta, \text{ and} \\
&\tau\left(\beta, \gamma, \Theta_1, \Theta_2, \cdots, \Theta_{\mathfrak{h}-1}, \Theta_\mathfrak{h}, \Theta_{\mathfrak{h}+1}, \cdots, \Theta_\varsigma\right) < \vartheta.
\end{aligned}$$

Definition 3.2. *Let* β *be a point in a* ς*-NFMS* $(\Psi, \upsilon, \varpi, \lambda, \tau, \oplus, \circledast)$*. For any real number* $\vartheta \in (0, 1)$*, the set*

$$\begin{aligned}
\mathfrak{B}\left(\beta, \vartheta, \Theta_1, \Theta_2, \ldots, \Theta_\varsigma\right) = \Big\{&\gamma \in \Psi : \upsilon\left(\beta, \gamma, \Theta_j^\varsigma\right) > 1 - \vartheta, \\
&\varpi\left(\beta, \gamma, \Theta_j^\varsigma\right) > 1 - \vartheta, \ \lambda\left(\beta, \gamma, \Theta_j^\varsigma\right) < \vartheta \text{ and } \tau\left(\beta, \gamma, \Theta_j^\varsigma\right) < \vartheta\Big\}
\end{aligned}$$

is called an open ball centered at $\beta \in \Psi$ *with radius* $\vartheta \in (0, 1)$*, and is defined with respect to the parameters* $\Theta_j \mid_{j=1}^{\varsigma} > 0$*.*

Definition 3.3. *Let $(\Psi, \upsilon, \varpi, \lambda, \tau, \oplus, \circledast)$ be a ς-NFMS. A set $\mathfrak{O}$ is said to be open in Ψ if and only if, for every open ball $\mathfrak{D}$, it holds that $\mathfrak{D} \subset \mathfrak{O}$.*

Definition 3.4. *Let $(\Psi, \upsilon, \varpi, \lambda, \tau, \oplus, \circledast)$ be a ς-NFMS. A set $\mathfrak{O}$ is considered open if and only if its complement, $\Psi \setminus \mathfrak{O}$, is a closed set.*

Theorem 3.1. *Let $(\Psi, \upsilon, \varpi, \lambda, \tau, \oplus, \circledast)$ be a ς-NFMS. Every open ball is an open set.*

Proof. Let $(\Psi, \upsilon, \varpi, \lambda, \tau, \oplus, \circledast)$ be a ς-NFMS, and consider $\beta \in \Psi$, $\Theta_1, \Theta_2, \cdots, \Theta_\varsigma > 0$ and $\vartheta \in (0,1)$. Assume that $\gamma \in \mathfrak{B}(\beta, \vartheta, \Theta_1, \Theta_2, \ldots, \Theta_\varsigma)$. This implies

$$\upsilon\left(\beta, \gamma, \Theta_j^\varsigma\right) > 1 - \vartheta,\ \varpi\left(\beta, \gamma, \Theta_j^\varsigma\right) > 1 - \vartheta,\ \lambda\left(\beta, \gamma, \Theta_j^\varsigma\right) < \vartheta \text{ and } \tau\left(\beta, \gamma, \Theta_j^\varsigma\right) < \vartheta.$$

Then, there exist $\mathfrak{h} \in \{1, 2, 3, \cdots, \varsigma\}$ and $\Theta \in (0, \Theta_m)$ such that

$$\vartheta_0 = \upsilon\left(\beta, \gamma, \Theta_1, \Theta_2, \cdots, \Theta_{\mathfrak{h}-1}, \Theta_{\mathfrak{h}}, \Theta_{\mathfrak{h}+1}, \cdots, \Theta_\varsigma\right).$$

Since $\vartheta_0 > 1 - \vartheta$, there exists $\vartheta' \in (0,1)$ such that $\vartheta_0 > 1 - \vartheta' > 1 - \vartheta$. Given ϑ_0 and ϑ' with $\vartheta_0 > 1 - \vartheta'$, there exist $\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4 \in (0,1)$, satisfying $\vartheta_0 \oplus \vartheta_1 > 1 - \vartheta'$, $\vartheta_0 \oplus \vartheta_2 > 1 - \vartheta'$, $(1 - \vartheta_0) \circledast (1 - \vartheta_3) \leq \vartheta'$, and $(1 - \vartheta_0) \circledast (1 - \vartheta_4) \leq \vartheta'$. Define $\vartheta_5 = \max\{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4\}$. Our goal is to show that the original ball $\mathfrak{B}(\beta, \ldots)$ contains a smaller open ball centered at γ. To do this, we construct the new ball using ϑ_5 as a radius and $(\Theta - \Theta_{\mathfrak{h}})$ as the parameter. We will now show that the new ball is fully contained within the original. Let

$$\mathfrak{B}\left(\gamma, 1 - \vartheta_5, \Theta_1, \Theta_2, \cdots, \Theta_{\mathfrak{h}-1}, \Theta - \Theta_{\mathfrak{h}}, \Theta_{\mathfrak{h}+1}, \cdots, \Theta_\varsigma\right)$$

be an open ball. We claim that

$$\mathfrak{B}\left(\gamma, 1 - \vartheta_5, \Theta_1, \Theta_2, \cdots, \Theta_{\mathfrak{h}-1}, \Theta - \Theta_{\mathfrak{h}}, \Theta_{\mathfrak{h}+1}, \cdots, \Theta_\varsigma\right) \subset \mathfrak{B}\left(\beta, \vartheta, \Theta_j^\varsigma\right).$$

To verify this, suppose $\omega \in \mathfrak{B}\left(\gamma, 1 - \vartheta_5, \Theta_1, \Theta_2, \cdots, \Theta_{\mathfrak{h}-1}, \Theta - \Theta_{\mathfrak{h}}, \Theta_{\mathfrak{h}+1}, \cdots, \Theta_\varsigma\right)$. Then,

$$\begin{aligned}
&\upsilon\left(\gamma, \omega, \Theta_1, \Theta_2, \cdots, \Theta_{\mathfrak{h}-1}, \Theta - \Theta_{\mathfrak{h}}, \Theta_{\mathfrak{h}+1}, \cdots, \Theta_\varsigma\right) > \vartheta_5,\\
&\varpi\left(\gamma, \omega, \Theta_1, \Theta_2, \cdots, \Theta_{\mathfrak{h}-1}, \Theta - \Theta_{\mathfrak{h}}, \Theta_{\mathfrak{h}+1}, \cdots, \Theta_\varsigma\right) > \vartheta_5,\\
&\lambda\left(\gamma, \omega, \Theta_1, \Theta_2, \cdots, \Theta_{\mathfrak{h}-1}, \Theta - \Theta_{\mathfrak{h}}, \Theta_{\mathfrak{h}+1}, \cdots, \Theta_\varsigma\right) < \vartheta_5,
\end{aligned}$$

and

$$\tau\left(\gamma, \omega, \Theta_1, \Theta_2, \cdots, \Theta_{\mathfrak{h}-1}, \Theta - \Theta_{\mathfrak{h}}, \Theta_{\mathfrak{h}+1}, \cdots, \Theta_\varsigma\right) < \vartheta_5.$$

Next, combining these inequalities with the properties of the ς-NFMS and the triangle inequality under υ, ϖ, and τ, it follows that:

$$\begin{aligned}
\upsilon\left(\beta, \omega, \Theta_j^\varsigma\right) &\geq \upsilon\left(\beta, \gamma, \Theta_1, \Theta_2, \cdots, \Theta_{\mathfrak{h}-1}, \Theta_{\mathfrak{h}}, \Theta_{\mathfrak{h}+1}, \cdots, \Theta_\varsigma\right)\\
&\quad \oplus \upsilon\left(\gamma, \omega, \Theta_1, \Theta_2, \cdots, \Theta_{\mathfrak{h}-1}, \Theta - \Theta_{\mathfrak{h}}, \Theta_{\mathfrak{h}+1}, \cdots, \Theta_\varsigma\right)\\
&\geq \vartheta_0 \oplus \vartheta_5 \geq \vartheta_0 \oplus \vartheta_1 \geq 1 - \vartheta' > 1 - \vartheta,
\end{aligned}$$

$$\begin{aligned}
\varpi\left(\beta, \omega, \Theta_j^\varsigma\right) &\geq \varpi\left(\beta, \gamma, \Theta_1, \Theta_2, \cdots, \Theta_{\mathfrak{h}-1}, \Theta_{\mathfrak{h}}, \Theta_{\mathfrak{h}+1}, \cdots, \Theta_\varsigma\right)\\
&\quad \oplus \varpi\left(\gamma, \omega, \Theta_1, \Theta_2, \cdots, \Theta_{\mathfrak{h}-1}, \Theta - \Theta_{\mathfrak{h}}, \Theta_{\mathfrak{h}+1}, \cdots, \Theta_\varsigma\right)\\
&\geq \vartheta_0 \oplus \vartheta_5 \geq \vartheta_0 \oplus \vartheta_2 \geq 1 - \vartheta' > 1 - \vartheta,
\end{aligned}$$

$$\begin{aligned}\lambda\left(\beta,\omega,\Theta_j^{\varsigma}\right) &\geq \lambda\left(\beta,\gamma,\Theta_1,\Theta_2,\cdots,\Theta_{\mathfrak{h}-1},\Theta_{\mathfrak{h}},\Theta_{\mathfrak{h}+1},\cdots,\Theta_{\varsigma}\right)\\ &\quad \circledast \lambda\left(\gamma,\omega,\Theta_1,\Theta_2,\cdots,\Theta_{\mathfrak{h}-1},\Theta-\Theta_{\mathfrak{h}},\Theta_{\mathfrak{h}+1},\cdots,\Theta_{\varsigma}\right)\\ &\leq (1-\vartheta_0)\circledast(1-\vartheta_5)\leq(1-\vartheta_0)\circledast(1-\vartheta_3)\geq\vartheta'<\vartheta,\end{aligned}$$

and

$$\begin{aligned}\tau\left(\beta,\omega,\Theta_j^{\varsigma}\right) &\leq \varpi\left(\beta,\gamma,\Theta_1,\Theta_2,\cdots,\Theta_{\mathfrak{h}-1},\Theta_{\mathfrak{h}},\Theta_{\mathfrak{h}+1},\cdots,\Theta_{\varsigma}\right)\\ &\quad \circledast \varpi\left(\gamma,\omega,\Theta_1,\Theta_2,\cdots,\Theta_{\mathfrak{h}-1},\Theta-\Theta_{\mathfrak{h}},\Theta_{\mathfrak{h}+1},\cdots,\Theta_{\varsigma}\right)\\ &\leq (1-\vartheta_0)\circledast(1-\vartheta_5)\leq(1-\vartheta_0)\circledast(1-\vartheta_4)\leq\vartheta'<\vartheta.\end{aligned}$$

Thus, $\omega\in\mathfrak{B}\left(\beta,\gamma,\Theta_j^{\varsigma}\right)$ confirms

$$\mathfrak{B}\left(\gamma,1-\vartheta_4,\Theta_j^{\varsigma}-\Theta_0\right)\subset\mathfrak{B}\left(\beta,\vartheta,\Theta_j^{\varsigma}\right).$$

The following consequence is derived from the previous theorem:

Corollary 3.1. *Let* $(\Psi,\upsilon,\varpi,\lambda,\tau,\oplus,\circledast)$ *be a* ς*-NFMS. Let*

$$\begin{aligned}\tau_{(\upsilon,\varpi,\lambda,\tau)}=\{&\mathcal{A}\subseteq\Psi:\forall\beta\in\Psi,\text{ there exist }\Theta_1,\Theta_2,\ldots,\Theta_{\varsigma}>0\text{ and }\vartheta\in(0,1)\\ &\text{such that }\mathfrak{B}\left(\beta,\vartheta:\Theta_1,\Theta_2,\ldots,\Theta_{\varsigma}\right)\subseteq\mathcal{A}\}.\end{aligned}$$

Then, $\tau_{(\upsilon,\varpi,\lambda,\tau)}$ *defines a topology on* Ψ.

From Theorem 3.1 and Corollary 3.1, for any ς-NFMS $(\upsilon,\varpi,\lambda,\tau)$ on Ψ, where $\tau_{(\upsilon,\varpi,\lambda,\tau)}$ is the induced topology on Ψ. This topology consists of the open sets

$$\{\mathfrak{B}\left(\beta,\vartheta:\Theta_1,\Theta_2,\ldots,\Theta_{\varsigma}\right):\beta\in\Psi,\vartheta\in(0,1),\Theta>0\}.$$

For $B_{\beta}=\left\{\mathfrak{B}\left(\beta,\frac{1}{\alpha}:\Theta_1,\Theta_2,\ldots,\Theta_{\varsigma}\right):\alpha\in\mathbb{N}\right\}$, where $\Theta_1=\Theta_2=\Theta_3=\cdots=\Theta_{\varsigma}=\frac{1}{\alpha}$, forms a local base at a point β. The topology $\tau_{(\upsilon,\varpi,\lambda,\tau)}$ is a first countable.

Theorem 3.2. *Every* ς*-NFMS is Hausdorff.*

Proof. Let $(\Psi,\upsilon,\varpi,\lambda,\tau,\oplus,\circledast)$ be a ς-NFMS. Let β and γ denote two different points in Ψ. For any given $\Theta_1,\Theta_2,\ldots,\Theta_{\varsigma}>0$, it follows that $0<\upsilon\left(\beta,\gamma,\Theta_j^{\varsigma}\right)<1$, $0<\varpi\left(\beta,\gamma,\Theta_j^{\varsigma}\right)<1$, $0<\lambda\left(\beta,\gamma,\Theta_j^{\varsigma}\right)<1$, and $0<\tau\left(\beta,\gamma,\Theta_j^{\varsigma}\right)<1$. Let $\vartheta_1=\upsilon\left(\beta,\gamma,\Theta_j^{\varsigma}\right)\in(0,1)$, $\vartheta_2=\varpi\left(\beta,\gamma,\Theta_j^{\varsigma}\right)\in(0,1)$, $\vartheta_3=\lambda\left(\beta,\gamma,\Theta_j^{\varsigma}\right)\in(0,1)$, $\vartheta_4=\tau\left(\beta,\gamma,\Theta_j^{\varsigma}\right)\in(0,1)$, and $\vartheta=\max\{\vartheta_1,\vartheta_2,1-\vartheta_3,1-\vartheta_4\}$. For each $\vartheta_0\in(\vartheta,1)$, there exist ϑ_5, ϑ_6, ϑ_7, and ϑ_8 such that $\vartheta_5\oplus\vartheta_5\geq\vartheta_0$, $\vartheta_6\oplus\vartheta_6\geq\vartheta_0$, $(1-\vartheta_7)\circledast(1-\vartheta_7)\leq1-\vartheta_0$, and $(1-\vartheta_8)\circledast(1-\vartheta_8)\leq1-\vartheta_0$. Put $\vartheta_9=\max\{\vartheta_5,\vartheta_6,\vartheta_7,\vartheta_8\}$ and consider the open balls,

$$\mathfrak{B}\left(\beta,1-\vartheta_9:\Theta_1,\Theta_2,\ldots,\Theta_{\frac{\mathfrak{h}}{2}},\cdots,\Theta_{\varsigma}\right)$$

and

$$\mathfrak{B}\left(\gamma,1-\vartheta_9:\Theta_1,\Theta_2,\ldots,\Theta_{\frac{\mathfrak{h}}{2}},\cdots,\Theta_{\varsigma}\right).$$

Then, clearly

$$\begin{aligned}B_{\beta,\gamma}&=\mathfrak{B}\left(\beta,1-\vartheta_9:\Theta_1,\Theta_2,\ldots,\Theta_{\frac{\mathfrak{h}}{2}},\cdots,\Theta_{\varsigma}\right)\\ &\quad\cap\mathfrak{B}\left(\gamma,1-\vartheta_9:\Theta_1,\Theta_2,\ldots,\Theta_{\frac{\mathfrak{h}}{2}},\cdots,\Theta_{\varsigma}\right)=\emptyset.\end{aligned}$$

Assume that $B_{\beta,\gamma} \neq \emptyset$, i.e., there exists $\omega \in B_{\beta,\gamma}$, then we have

$$\begin{aligned}\vartheta_1 &= \upsilon\left(\beta,\gamma,\Theta_j^{\varsigma}\right) \geq \upsilon\left(\beta,\omega,\Theta_1,\Theta_2,\ldots,\Theta_{\frac{\natural}{2}},\cdots,\Theta_{\varsigma}\right) \oplus \upsilon\left(\omega,\gamma,\Theta_1,\Theta_2,\ldots,\Theta_{\frac{\natural}{2}},\cdots,\Theta_{\varsigma}\right)\\ &\geq \vartheta_9 \oplus \vartheta_9 \geq \vartheta_5 \oplus \vartheta_5 \geq \vartheta_0 > \vartheta_1,\end{aligned}$$

$$\begin{aligned}\vartheta_2 &= \varpi\left(\beta,\gamma,\Theta_j^{\varsigma}\right) \geq \varpi\left(\beta,\omega,\Theta_1,\Theta_2,\ldots,\Theta_{\frac{\natural}{2}},\cdots,\Theta_{\varsigma}\right) \oplus \varpi\left(\omega,\gamma,\Theta_1,\Theta_2,\ldots,\Theta_{\frac{\natural}{2}},\cdots,\Theta_{\varsigma}\right)\\ &\geq \vartheta_9 \oplus \vartheta_9 \geq \vartheta_6 \oplus \vartheta_6 \geq \vartheta_0 > \vartheta_2,\end{aligned}$$

$$\begin{aligned}\vartheta_3 &= \lambda\left(\beta,\gamma,\Theta_j^{\varsigma}\right) \leq \lambda\left(\beta,\omega,\Theta_1,\Theta_2,\ldots,\Theta_{\frac{\natural}{2}},\cdots,\Theta_{\varsigma}\right) \circledast \lambda\left(\omega,\gamma,\Theta_1,\Theta_2,\ldots,\Theta_{\frac{\natural}{2}},\cdots,\Theta_{\varsigma}\right)\\ &\leq (1-\vartheta_9) \circledast (1-\vartheta_9) \leq (1-\vartheta_7) \circledast (1-\vartheta_7) \leq 1-\vartheta_0 < \vartheta_3,\end{aligned}$$

and

$$\begin{aligned}\vartheta_4 &= \tau\left(\beta,\gamma,\Theta_j^{\varsigma}\right) \leq \tau\left(\beta,\omega,\Theta_1,\Theta_2,\ldots,\Theta_{\frac{\natural}{2}},\cdots,\Theta_{\varsigma}\right) \circledast \tau\left(\omega,\gamma,\Theta_1,\Theta_2,\ldots,\Theta_{\frac{\natural}{2}},\cdots,\Theta_{\varsigma}\right)\\ &\leq (1-\vartheta_9) \circledast (1-\vartheta_9) \leq (1-\vartheta_8) \circledast (1-\vartheta_8) \leq 1-\vartheta_0 < \vartheta_4.\end{aligned}$$

Hence, it is contradiction. Therefore, $(\Psi,\upsilon,\varpi,\lambda,\tau,\oplus,\circledast)$ is a Hausdorff space.

Definition 3.5. *Let* $(\Psi,\upsilon,\varpi,\lambda,\tau,\oplus,\circledast)$ *be a* ς*-NFMS. Suppose that there exist* $\Theta_1,\Theta_2,\ldots,\Theta_\varsigma > 0$ *and* $0 < \vartheta < 1$ *such that for all* $\beta,\gamma \in \mathcal{A}$*, the following circumstances are true:*

$$\upsilon\left(\beta,\gamma,\Theta_j^{\varsigma}\right) > 1-\vartheta, \varpi\left(\beta,\gamma,\Theta_j^{\varsigma}\right) > 1-\vartheta, \lambda\left(\beta,\gamma,\Theta_j^{\varsigma}\right) < \vartheta \text{ and } \tau\left(\beta,\gamma,\Theta_j^{\varsigma}\right) < \vartheta,$$

where $\mathcal{A}$ *is a subset of* Ψ*. In this case,* $\mathcal{A}$ *is referred to as neutrosophic fuzzy bounded (NF-bounded).*

Remark 3.4. *Let a* ς*-NFMS* $(\Psi,\upsilon,\varpi,\lambda,\tau,\oplus,\circledast)$ *be induced by a metric space* d *on* Ψ*. The subset* $\mathcal{A} \subset \Psi$ *is neutrosophic fuzzy bounded (NF-bounded) if it is bounded.*

Theorem 3.3. *Let* $(\Psi,\upsilon,\varpi,\lambda,\tau,\oplus,\circledast)$ *be a* ς*-NFMS. Every compact subset* $\mathcal{A}$ *is NF-bounded.*

Proof. A compact subset of a ς-NFMS is denoted by $\mathcal{A}$. Let $\Theta_1,\Theta_2,\ldots,\Theta_\varsigma > 0$ and $\vartheta \in (0,1)$. Consider the open cover

$$\{\mathfrak{B}(\beta,\vartheta : \Theta_1,\Theta_2,\ldots,\Theta_{\natural},\cdots,\Theta_{\varsigma}) : \beta \in \mathcal{A}\}$$

of $\mathcal{A}$. Since $\mathcal{A}$ is compact, then there exist $\beta_1,\beta_2,\beta_3,\cdots,\beta_\lambda \in \mathcal{A}$ such that $\mathcal{A} \subseteq \bigcup_{i=1}^{\lambda} \mathfrak{B}(\beta_i,\vartheta,\Theta)$. For any $\beta,\gamma \in \mathcal{A}$, it follows that

$$\begin{aligned}\beta &\in \mathfrak{B}(\beta_i,\vartheta,\Theta_1,\Theta_2,\ldots,\Theta_{\natural},\cdots,\Theta_{\varsigma}),\\ \gamma &\in \mathfrak{B}(\beta_k,\vartheta,\Theta_1,\Theta_2,\ldots,\Theta_{\natural},\cdots,\Theta_{\varsigma})\end{aligned}$$

for some i, k. Then, we have $\upsilon\left(\beta,\beta_i,\Theta_j^{\varsigma}\right) > 1-\vartheta$, $\varpi\left(\beta,\beta_i,\Theta_j^{\varsigma}\right) > 1-\vartheta$, $\lambda\left(\beta,\beta_i,\Theta_j^{\varsigma}\right) < \vartheta$, $\tau\left(\beta,\beta_i,\Theta_j^{\varsigma}\right) < \vartheta$, $\upsilon\left(\gamma,\beta_k,\Theta_j^{\varsigma}\right) > 1-\vartheta$, $\varpi\left(\gamma,\beta_k,\Theta_j^{\varsigma}\right) > 1-\vartheta$, $\lambda\left(\beta,\beta_i,\Theta_j^{\varsigma}\right) < \vartheta$, and $\tau\left(\gamma,\beta_k,\Theta_j^{\varsigma}\right) < \vartheta$. Let

$$\begin{aligned}\alpha &= \min\left\{\upsilon\left(\beta_i,\beta_k,\Theta_j^{\varsigma}\right) : 1 \leq i,k \leq \gamma\right\},\\ \zeta &= \min\left\{\varpi\left(\beta_i,\beta_k,\Theta_j^{\varsigma}\right) : 1 \leq i,k \leq \gamma\right\},\\ \Phi &= \max\left\{\lambda\left(\beta_i,\beta_k,\Theta_j^{\varsigma}\right) : 1 \leq i,k \leq \gamma\right\},\\ \psi &= \max\left\{\tau\left(\beta_i,\beta_k,\Theta_j^{\varsigma}\right) : 1 \leq i,k \leq \gamma\right\}.\end{aligned}$$

Then $\alpha, \zeta, \Phi, \psi > 0$. Now we have

$$\begin{aligned}\upsilon\left(\beta,\gamma,3\Theta_j^{\varsigma}\right) &\geq \upsilon\left(\beta,\beta_i,\Theta_j^{\varsigma}\right)\oplus\upsilon\left(\beta_i,\beta_k,\Theta_j^{\varsigma}\right)\oplus\upsilon\left(\beta_k,\gamma,\Theta_j^{\varsigma}\right)\\ &\geq (1-\vartheta)\oplus(1-\vartheta)\oplus\alpha > 1-\vartheta_1', \text{ for some } 0<\vartheta_1'<1,\end{aligned}$$

$$\begin{aligned}\varpi\left(\beta,\gamma,3\Theta_j^{\varsigma}\right) &\geq \varpi\left(\beta,\beta_i,\Theta_j^{\varsigma}\right)\oplus\varpi\left(\beta_i,\beta_k,\Theta_j^{\varsigma}\right)\oplus\varpi\left(\beta_k,\gamma,\Theta_j^{\varsigma}\right)\\ &\geq (1-\vartheta)\oplus(1-\vartheta)\oplus\zeta > 1-\vartheta_2', \text{ for some } 0<\vartheta_2'<1,\end{aligned}$$

$$\begin{aligned}\lambda\left(\beta,\gamma,3\Theta_j^{\varsigma}\right) &\leq \lambda\left(\beta,\beta_i,\Theta_j^{\varsigma}\right)\circledast\lambda\left(\beta_i,\beta_k,\Theta_j^{\varsigma}\right)\circledast\lambda\left(\beta_k,\gamma,\Theta_j^{\varsigma}\right)\\ &\leq \vartheta\circledast\vartheta\circledast\Phi \leq \vartheta_3', \text{ for some } 0<\vartheta_3'<1,\end{aligned}$$

$$\begin{aligned}\tau\left(\beta,\gamma,3\Theta_j^{\varsigma}\right) &\leq \tau\left(\beta,\beta_i,\Theta_j^{\varsigma}\right)\circledast\tau\left(\beta_i,\beta_k,\Theta_j^{\varsigma}\right)\circledast\tau\left(\beta_k,\gamma,\Theta_j^{\varsigma}\right)\\ &\leq \vartheta\circledast\vartheta\circledast\psi \leq \vartheta_4', \text{ for some } 0<\vartheta_4'<1.\end{aligned}$$

If we take $\vartheta' = \max\left\{\vartheta_1', \vartheta_2', \vartheta_3', \vartheta_4'\right\}$ and $\Theta' = 3\Theta_j^{\varsigma}$, then we have $\upsilon(\beta,\gamma,\Theta') > 1-\vartheta'$, $\varpi(\beta,\gamma,\Theta') < \vartheta'$, and $\varsigma(\beta,\gamma,\Theta') < \vartheta'$, $\forall \beta,\gamma \in \mathscr{A}$. Hence, $\mathscr{A}$ is NF-bounded.

Remark 3.5. *According to the aforementioned Theorem 3.3 and Remark 3.4, any compact set in a ς-NFMS is closed and bounded.*

Theorem 3.4. *Let $(\Psi, \upsilon, \varpi, \lambda, \tau, \oplus, \circledast)$ be a ς-NFMS. Let $\tau_{(\upsilon,\varpi,\lambda,\tau)}$ be the topology on Ψ induced by the ς-NFMS. For a sequence $\left\{\beta_j\right\} \in \Psi$, we have*

$$\begin{aligned}\beta_j \to \beta \quad \Leftrightarrow \quad &\upsilon\left(\beta_j,\beta,\Theta_j^{\varsigma}\right)\to 1, \varpi\left(\beta_j,\beta,\Theta_j^{\varsigma}\right)\to 1,\\ &\lambda\left(\beta_j,\beta,\Theta_j^{\varsigma}\right)\to 0 \textit{ and } \tau\left(\beta_j,\beta,\Theta_j^{\varsigma}\right)\to 0 \textit{ as } j\to\infty.\end{aligned}$$

Proof. Let $\Theta_1, \Theta_2, \ldots, \Theta_{\mathfrak{h}}, \cdots, \Theta_{\varsigma} > 0$. Let $\left(\beta_j\right)$ be a real sequence. For any given $\vartheta \in (0,1)$, there exists $j_0 \in \mathbb{N}$ such that

$$\beta_j \in \mathfrak{B}\left(\beta, r : \Theta_1, \Theta_2, \ldots, \Theta_m, \cdots, \Theta_{\varsigma}\right)$$

for all $j \geq j_0$ with the following conditions satisfied:

$$1-\upsilon\left(\beta_j,\beta,\Theta_j^{\varsigma}\right) < \vartheta,\ 1-\varpi\left(\beta_j,\beta,\Theta_j^{\varsigma}\right) < \vartheta,\ \lambda\left(\beta_j,\beta,\Theta_j^{\varsigma}\right) < \vartheta \text{ and } \tau\left(\beta_j,\beta,\Theta_j^{\varsigma}\right) < \vartheta.$$

Then, we can express the following limits:

$$\upsilon\left(\beta_j,\beta,\Theta_j^{\varsigma}\right)\to 1,\ \varpi\left(\beta_j,\beta,\Theta_j^{\varsigma}\right)\to 1,\ \lambda\left(\beta_j,\beta,\Theta_j^{\varsigma}\right)\to 0 \text{ and } \tau\left(\beta_j,\beta,\Theta_j^{\varsigma}\right)\to 0 \text{ as } j\to\infty.$$

Conversely, if for every $\Theta_1, \Theta_2, \ldots, \Theta_{\mathfrak{h}}, \cdots, \Theta_{\varsigma} > 0$, $\upsilon\left(\beta_j,\beta,\Theta_j^{\varsigma}\right) \to 1$, $\varpi\left(\beta_j,\beta,\Theta_j^{\varsigma}\right) \to 1$, $\lambda\left(\beta_j,\beta,\Theta_j^{\varsigma}\right) \to 0$ and $\tau\left(\beta_j,\beta,\Theta_j^{\varsigma}\right) \to 0$ as $j \to \infty$. For any $\vartheta \in (0,1)$, there exists $j_0 \in \mathbb{N}$ such that $1-\upsilon\left(\beta_j,\beta,\Theta_j^{\varsigma}\right) < \vartheta$, $1-\varpi\left(\beta_j,\beta,\Theta_j^{\varsigma}\right) < \vartheta$, $\lambda\left(\beta_j,\beta,\Theta_j^{\varsigma}\right) < \vartheta$ and $\tau\left(\beta_j,\beta,\Theta_j^{\varsigma}\right) < \vartheta$ for all $j \geq j_0$. This implies that $\upsilon\left(\beta_j,\beta,\Theta_j^{\varsigma}\right) > 1-\vartheta$, $\varpi\left(\beta_j,\beta,\Theta_j^{\varsigma}\right) > 1-\vartheta$, $\lambda\left(\beta_j,\beta,\Theta_j^{\varsigma}\right) < \vartheta$ and $\tau\left(\beta_j,\beta,\Theta_j^{\varsigma}\right) < \vartheta$, for all $j \geq j_0$. Thus $\beta_j \in \mathfrak{B}\left(\beta, r : \Theta_1, \Theta_2, \ldots, \Theta_{\mathfrak{h}}, \cdots, \Theta_{\varsigma}\right)$, for all $j \geq j_0$, and $\beta_j \to \beta$.

Definition 3.6. *Consider $(\Psi, \upsilon, \varpi, \lambda, \tau, \oplus, \circledast)$ as a ς-NFMS. Let $\left(\beta_j\right) \in \Psi$ be said to converge to $\beta \in \Psi$, if, for every real number $\vartheta \in (0, 1)$, there exists a natural number j_0 such that for all $j > j_0$, the following conditions hold:*

$$\upsilon\left(\beta_j, \beta, \Theta_1^{\varsigma}\right) > 1 - \vartheta,\ \varpi\left(\beta_j, \beta, \Theta_1^{\varsigma}\right) > 1 - \vartheta,\ \lambda\left(\beta_j, \beta, \Theta_1^{\varsigma}\right) < \vartheta \text{ and } \tau\left(\beta_j, \beta, \Theta_1^{\varsigma}\right) < \vartheta,$$

where $\Theta_1, \Theta_2, \ldots, \Theta_{\varsigma} > 0$.

Lemma 3.2. *Let $(\Psi, \upsilon, \varpi, \lambda, \tau, \oplus, \circledast)$ be a ς-NFMS. A sequence $\beta_j \in \Psi$ is said to converge to $\beta \in \Psi$ if*

$$\lim_{j\to\infty} \upsilon\left(\beta_j, \gamma, \Theta_1^{\varsigma}\right) = 1,\ \lim_{j\to\infty} \varpi\left(\beta_j, \gamma, \Theta_1^{\varsigma}\right) = 1,\ \lim_{j\to\infty} \lambda\left(\beta_j, \gamma, \Theta_1^{\varsigma}\right) = 0 \text{ and } \lim_{j\to\infty} \tau\left(\beta_j, \gamma, \Theta_1^{\varsigma}\right) = 0,$$

for all $\Theta_1, \Theta_2, \ldots, \Theta_{\varsigma} > 0$, and $\beta, \gamma \in \Psi$.

Definition 3.7. *Let $(\Psi, \upsilon, \varpi, \lambda, \tau, \oplus, \circledast)$ be a ς-NFMS, then a sequence $\left(\beta_j\right) \in \Psi$ is said to be Cauchy if for $\vartheta > 0$ and each $\Theta_1, \Theta_2, \ldots, \Theta_{\varsigma} > 0$, and there exists a $j_0 \in \mathbb{N}$ such that $\upsilon\left(\beta_j, \beta_k, \Theta_1^{\varsigma}\right) > 1 - \vartheta$, $\varpi\left(\beta_j, \beta_k, \Theta_1^{\varsigma}\right) > 1 - \vartheta$, $\lambda\left(\beta_j, \beta_k, \Theta_1^{\varsigma}\right) < \vartheta$, and $\tau\left(\beta_j, \beta_k, \Theta_1^{\varsigma}\right) < \vartheta$, for all $j, k \geq j_0$.*

Definition 3.8. *Suppose that $(\Psi, \upsilon, \varpi, \lambda, \tau, \oplus, \circledast)$ is a ς-NFMS. Given the topology $\tau_{(\upsilon,\varpi,\tau)}$, if all Cauchy sequences are convergent, then $(\Psi, \upsilon, \varpi, \tau, \oplus, \circledast)$ is a complete ς-NFMS.*

Example 3.3. Let $\Psi = \left\{\frac{1}{j} : j \in \mathbb{N}\right\} \cup \{0\}$ and $\oplus$ be the continuous t-norm and $\circledast$ be the continuous t-conorm defined by $r \oplus s = rs$, $r \circledast s = \min\{1, r + s\}$, for all $r, s \in [0, 1]$, respectively. For any $\Theta_j^{\varsigma} \in (0, 1)^{\varsigma}$ and for any $\beta, \gamma \in \Psi$. Define FS $\upsilon, \varpi, \lambda, \tau$ on $\Psi^2 \times (0, \infty)^{\varsigma}$ by

$$\begin{aligned}
\upsilon\left(\beta, \gamma, \Theta_j^{\varsigma}\right) &= \left\{\frac{\Theta_j^{\varsigma}}{\Theta_j^{\varsigma} + d(\beta,\gamma)}, \Theta_j^{\varsigma} > 0\right\}, \varpi\left(\beta, \gamma, \Theta_j^{\varsigma}\right) = \left\{\frac{\Theta_j^{\varsigma} + d(\beta,\gamma)}{\Theta_j^{\varsigma}}, \Theta_j^{\varsigma} > 0\right\}, \\
\lambda\left(\beta, \gamma, \Theta_j^{\varsigma}\right) &= \left\{\frac{d(\beta,\gamma)}{\Theta_j^{\varsigma} + d(\beta,\gamma)}, \Theta_j^{\varsigma} > 0\right\}, \tau\left(\beta, \gamma, \Theta_j^{\varsigma}\right) = \left\{\frac{d(\beta,\gamma)}{\Theta_j^{\varsigma}}, \Theta_j^{\varsigma} > 0\right\},
\end{aligned}$$

then $(\Psi, \upsilon, \varpi, \lambda, \tau, \oplus, \circledast)$ is a complete ς-NFMS.

Theorem 3.5. *Let $(\Psi, \upsilon, \varpi, \lambda, \tau, \oplus, \circledast)$ be a ς-NFMS. A convergent subsequence for each Cauchy sequence in Ψ indicates that the ς-NFMS $(\Psi, \upsilon, \varpi, \lambda, \tau, \oplus, \circledast)$ is complete.*

Proof. Given a Cauchy sequence $(\beta_k) \in \Psi$, let $\left(\beta_{j_k}\right)$ be a subsequence of (β_k) and let $\beta_k \to \beta$. Suppose that $\Theta_1, \Theta_2, \ldots, \Theta_{\varsigma} > 0$ and $\vartheta \in (0, 1)$. Choose $\vartheta_1 \in (0, 1)$ such that $(1-\vartheta_1) \oplus (1-\vartheta_1) \geq 1 - \vartheta$ and $\vartheta_1 \circledast \vartheta_1 \leq \vartheta$. Therefore, (β_k) is a Cauchy sequence, there is $k_0 \in \mathbb{N}$ such that

$$\begin{aligned}
&\upsilon\left(\beta_i, \beta_k, \Theta_1, \Theta_2, \ldots, \Theta_{\frac{\mathfrak{b}}{2}}, \cdots, \Theta_{\varsigma}\right) > 1 - \vartheta_1, \\
&\varpi\left(\beta_i, \beta_k, \Theta_1, \Theta_2, \ldots, \Theta_{\frac{\mathfrak{b}}{2}}, \cdots, \Theta_{\varsigma}\right) > 1 - \vartheta_1, \\
&\lambda\left(\beta_i, \beta_k, \Theta_1, \Theta_2, \ldots, \Theta_{\frac{\mathfrak{b}}{2}}, \cdots, \Theta_{\varsigma}\right) < \vartheta_1, \\
&\tau\left(\beta_i, \beta_k, \Theta_1, \Theta_2, \ldots, \Theta_{\frac{\mathfrak{b}}{2}}, \cdots, \Theta_{\varsigma}\right) < \vartheta_1,
\end{aligned}$$

for all $i, k \geq k_0$. Since $\beta_{k_j} \to \beta$, there is a positive integer j_u such that $j_u > k_0$,

$$\begin{aligned}
&\upsilon\left(\beta_{j_u}, \beta, \Theta_1, \Theta_2, \ldots, \Theta_{\frac{\mathfrak{b}}{2}}, \cdots, \Theta_{\varsigma}\right) > 1 - \vartheta_1, \\
&\varpi\left(\beta_{j_u}, \beta, \Theta_1, \Theta_2, \ldots, \Theta_{\frac{\mathfrak{b}}{2}}, \cdots, \Theta_{\varsigma}\right) > 1 - \vartheta_1, \\
&\lambda\left(\beta_{j_u}, \beta, \Theta_1, \Theta_2, \ldots, \Theta_{\frac{\mathfrak{b}}{2}}, \cdots, \Theta_{\varsigma}\right) < \vartheta_1, \\
&\tau\left(\beta_{j_u}, \beta, \Theta_1, \Theta_2, \ldots, \Theta_{\frac{\mathfrak{b}}{2}}, \cdots, \Theta_{\varsigma}\right) < \vartheta_1.
\end{aligned}$$

So, if $k > j_0$,

$$\begin{aligned}
\upsilon\left(\beta_i,\beta_k,\Theta_1^{\varsigma}\right) &\geq \upsilon\left(\beta_k,\beta_{j_u},\Theta_1,\Theta_2,\ldots,\Theta_{\frac{\mathfrak{b}}{2}},\cdots,\Theta_{\varsigma}\right) \oplus \upsilon\left(\beta_{j_u},\beta,\Theta_1,\Theta_2,\ldots,\Theta_{\frac{\mathfrak{b}}{2}},\cdots,\Theta_{\varsigma}\right) \\
&\geq (1-\vartheta_1)\oplus(1-\vartheta_1) \geq 1-\vartheta, \\
\varpi\left(\beta_i,\beta_k,\Theta_1^{\varsigma}\right) &\geq \varpi\left(\beta_k,\beta_{j_u},\Theta_1,\Theta_2,\ldots,\Theta_{\frac{\mathfrak{b}}{2}},\cdots,\Theta_{\varsigma}\right) \oplus \varpi\left(\beta_{j_u},\beta,\Theta_1,\Theta_2,\ldots,\Theta_{\frac{\mathfrak{b}}{2}},\cdots,\Theta_{\varsigma}\right) \\
&\geq (1-\vartheta_1)\oplus(1-\vartheta_1) \geq 1-\vartheta, \\
\lambda\left(\beta_i,\beta_k,\Theta_1^{\varsigma}\right) &\leq \lambda\left(\beta_k,\beta_{j_u},\Theta_1,\Theta_2,\ldots,\Theta_{\frac{\mathfrak{b}}{2}},\cdots,\Theta_{\varsigma}\right) \circledast \lambda\left(\beta_{j_u},\beta,\Theta_1,\Theta_2,\ldots,\Theta_{\frac{\mathfrak{b}}{2}},\cdots,\Theta_{\varsigma}\right) \\
&\leq \vartheta_1 \circledast \vartheta_1 < \vartheta, \\
\tau\left(\beta_i,\beta_k,\Theta_1^{\varsigma}\right) &\leq \tau\left(\beta_k,\beta_{j_u},\Theta_1,\Theta_2,\ldots,\Theta_{\frac{\mathfrak{b}}{2}},\cdots,\Theta_{\varsigma}\right) \circledast \tau\left(\beta_{j_u},\beta,\Theta_1,\Theta_2,\ldots,\Theta_{\frac{\mathfrak{b}}{2}},\cdots,\Theta_{\varsigma}\right) \\
&\leq \vartheta_1 \circledast \vartheta_1 < \vartheta.
\end{aligned}$$

As a result, we get $\beta_k \to \beta$. This represents the intended outcome.

Theorem 3.6. *Let $(\Psi,\upsilon,\varpi,\lambda,\tau,\oplus,\circledast)$ be a ς-NFMS, and let $\mathcal{U}$ be a subset of Ψ with subspace ς-NFMS*

$$(\upsilon_{\mathcal{U}},\varpi_{\mathcal{U}},\lambda_{\mathcal{U}},\tau_{\mathcal{U}}) = \left(\upsilon\,|_{\mathcal{U}\times(0,1)^{\varsigma}},\varpi\,|_{\mathcal{U}\times(0,1)^{\varsigma}},\lambda\,|_{\mathcal{U}\times(0,1)^{\varsigma}},\tau\,|_{\mathcal{U}\times(0,1)^{\varsigma}}\right).$$

Then, $(\mathcal{U},\upsilon_{\mathcal{U}},\varpi_{\mathcal{U}},\lambda_{\mathcal{U}},\tau_{\mathcal{U}},\oplus,\circledast)$ is complete$\Leftrightarrow$ $\mathcal{U}$ is closed subset of Ψ.

Proof. Let $\mathcal{U}$ be a closed subset of Ψ, and let (β_k) be a Cauchy sequence in $(\mathcal{U},\upsilon_{\mathcal{U}},\varpi_{\mathcal{U}},\lambda_{\mathcal{U}},\tau_{\mathcal{U}},\oplus,\circledast)$. Since (β_k) is also a Cauchy sequence in Ψ, it follows that $(\beta_k) \to \beta \in \Psi$. Given that $\mathcal{U}$ is closed, we have $\beta \in \mathcal{U}$. Thus, (β_k) converges in $\mathcal{U}$. Therefore, the completeness of $(\mathcal{U},\upsilon_{\mathcal{U}},\varpi_{\mathcal{U}},\lambda_{\mathcal{U}},\tau_{\mathcal{U}},\oplus,\circledast)$ is established.

Conversely, suppose that $(\mathcal{U},\upsilon_{\mathcal{U}},\varpi_{\mathcal{U}},\lambda_{\mathcal{U}},\tau_{\mathcal{U}},\oplus,\circledast)$ is complete, but $\mathcal{U}$ is not a closed subset of Ψ. Let $\beta \in \overline{\mathcal{U}}\setminus\mathcal{U}$. Then, there exists a sequence (β_k) in Ψ that converges to β, implying that (β_k) is a Cauchy sequence. Thus, for any $\vartheta \in (0,1)$ and all $\Theta > 0$, there exists a $j_0 \in \mathbb{N}$ such that $\upsilon\left(\beta_k,\beta_m,\Theta_j^{\varsigma}\right) > 1-\vartheta$, $\varpi\left(\beta_k,\beta_m,\Theta_j^{\varsigma}\right) > 1-\vartheta$, $\lambda\left(\beta_k,\beta_m,\Theta_j^{\varsigma}\right) < \vartheta$, and $\tau\left(\beta_k,\beta_m,\Theta_j^{\varsigma}\right) < \vartheta$, for all $k,m \geq j_0$. By leveraging the completeness of $(\mathcal{U},\upsilon_{\mathcal{U}},\varpi_{\mathcal{U}},\lambda_{\mathcal{U}},\tau_{\mathcal{U}},\oplus,\circledast)$, there exists $\gamma \in \mathcal{U}$ such that (β_k) converges to γ. This convergence is characterized by conditions similar to those for β, given by $\upsilon\left(\gamma,\beta_k,\Theta_j^{\varsigma}\right) > 1-\vartheta$, $\varpi\left(\gamma,\beta_k,\Theta_j^{\varsigma}\right) > 1-\vartheta$, $\lambda\left(\gamma,\beta_k,\Theta_j^{\varsigma}\right) < \vartheta$, and $\tau\left(\gamma,\beta_k,\Theta_j^{\varsigma}\right) < \vartheta$, for all $k \geq j_0$. Since (β_k) is a sequence in $\mathcal{U}$ and $\gamma \in \mathcal{U}$, it follows that $\upsilon\left(\gamma,\beta_k,\Theta_j^{\varsigma}\right) = \upsilon_{\mathcal{U}}\left(\gamma,\beta_k,\Theta_j^{\varsigma}\right)$, $\varpi\left(\gamma,\beta_k,\Theta_j^{\varsigma}\right) = \varpi_{\mathcal{U}}\left(\gamma,\beta_k,\Theta_j^{\varsigma}\right)$, $\lambda\left(\gamma,\beta_k,\Theta_j^{\varsigma}\right) = \lambda_{\mathcal{U}}\left(\gamma,\beta_k,\Theta_j^{\varsigma}\right)$, and $\tau\left(\gamma,\beta_k,\Theta_j^{\varsigma}\right) = \tau_{\mathcal{U}}\left(\gamma,\beta_k,\Theta_j^{\varsigma}\right)$. As a result, (β_k) converges to both β and γ in $(\mathcal{U},\upsilon_{\mathcal{U}},\varpi_{\mathcal{U}},\lambda_{\mathcal{U}},\tau_{\mathcal{U}},\oplus,\circledast)$. However, this leads to a contradiction since $\beta \notin \mathcal{U}$ and $\gamma \in \mathcal{U}$, with $\beta \neq \gamma$, violating the initial assumptions.

Lemma 3.3. *Let $(\Psi,\upsilon,\varpi,\tau,\oplus,\circledast)$ be a ς-NFMS. If $\Theta_1,\Theta_2,\ldots,\Theta_{\varsigma} > 0$ and $\vartheta_1,\vartheta_2 \in (0,1)$ such that $(1-\vartheta_2)\oplus(1-\vartheta_2) \geq 1-\vartheta_1$ and $\vartheta_2 \circledast \vartheta_2 \leq \vartheta_1$, then $\mathfrak{B}\left(\beta,\vartheta_2,\Theta_1,\Theta_2,\ldots,\Theta_{\frac{\mathfrak{b}}{2}},\cdots,\Theta_{\varsigma}\right) \subset \mathfrak{B}\left(\beta,\vartheta_1,\Theta_j^{\varsigma}\right)$.*

Theorem 3.7. *In a ς-NFMS $(\Psi,\upsilon,\varpi,\tau,\oplus,\circledast)$, a subset $\mathcal{Y}$ is said to be nowhere dense if each nonempty open set in Ψ contains an open ball whose closure does not intersect $\mathcal{Y}$.*

Proof. Suppose that $\mathcal{Z}$ is a nonempty open subset of Ψ. So, there exists a nonempty open subset $\mathcal{P} \subseteq \mathcal{Z}$ such that $\mathcal{P} \cap \overline{\mathcal{Y}} \neq \emptyset$. For any $\beta \in \mathcal{P}$, there exist $\vartheta_1 \in (0,1)$ and $\Theta_1, \Theta_2, \ldots, \Theta_\varsigma > 0$ such that $\mathfrak{B}(\beta, \vartheta_1, \Theta_1, \Theta_2, \ldots, \Theta_\varsigma) \subseteq \mathcal{P}$. Choose $\vartheta_2 \in (0,1)$ such that $(1-\vartheta_2) \oplus (1-\vartheta_2) \geq 1-\vartheta_1$ and $\vartheta_2 \circledast \vartheta_2 \leq \vartheta_1$. By Lemma 3.3,

$$\overline{\mathfrak{B}\left(\beta, \vartheta_2, \Theta_1, \Theta_2, \ldots, \Theta_{\frac{\mathfrak{h}}{2}}, \cdots, \Theta_\varsigma\right)} \subset \mathfrak{B}\left(\beta, \vartheta_1, \Theta_j^\varsigma\right).$$

Therefore,

$$\mathfrak{B}\left(\beta, \vartheta_2, \Theta_1, \Theta_2, \ldots, \Theta_{\frac{\mathfrak{h}}{2}}, \cdots, \Theta_\varsigma\right) \subseteq \mathcal{Z}$$

and

$$\overline{\mathfrak{B}\left(\beta, \vartheta_2, \Theta_1, \Theta_2, \ldots, \Theta_{\frac{\mathfrak{h}}{2}}, \cdots, \Theta_\varsigma\right)} \cap \mathcal{Y} = \emptyset.$$

Now, suppose $\mathcal{Y}$ is not nowhere dense. Hence, $\mathrm{int}(\mathcal{Y}) \neq \emptyset$, meaning there exists a nonempty open subset $\mathcal{Z} \subseteq \overline{\mathcal{Y}}$. Let $\mathfrak{B}(\beta, \vartheta_1, \Theta_1, \Theta_2, \ldots, \Theta_\varsigma)$ denote an open ball such that $\mathfrak{B}(\beta, \vartheta_1, \Theta_1, \Theta_2, \ldots, \Theta_\varsigma) \subseteq \mathcal{Z}$. This implies, $\overline{\mathfrak{B}(\beta, \vartheta_1, \Theta_1, \Theta_2, \ldots, \Theta_\varsigma)} \cap \mathcal{Y} \neq \emptyset$, leading to a contradiction.

4. Fixed-point theorems on ς-NFMSs

We present several fixed-point results within a ς-NFMS. For simplicity, in any ς-NFMS, where $\mathfrak{h} \in \{1, 2, .., \varsigma\}$, $\varrho > 0$, $\beta, \gamma \in \Psi$, and $\Theta_1, \Theta_2, \ldots, \Theta_\varsigma > 0$, the notations $\upsilon_{\mathfrak{h}}^{\varrho}\left(\beta, \gamma, \Theta_j^\varsigma\right)$, $\varpi_{\mathfrak{h}}^{\varrho}\left(\beta, \gamma, \Theta_j^\varsigma\right)$, $\lambda_{\mathfrak{h}}^{\varrho}\left(\beta, \gamma, \Theta_j^\varsigma\right)$ and $\tau_{\mathfrak{h}}^{\varrho}\left(\beta, \gamma, \Theta_j^\varsigma\right)$ are used as shorthand for the more detailed expressions

$$\upsilon\left(\beta, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\mathfrak{h}-1}, \Theta_{\mathfrak{h}/\varrho}, \Theta_{\mathfrak{h}+1}, \cdots, \Theta_\varsigma\right),$$
$$\varpi\left(\beta, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\mathfrak{h}-1}, \Theta_{\mathfrak{h}/\varrho}, \Theta_{\mathfrak{h}+1}, \cdots, \Theta_\varsigma\right),$$
$$\lambda\left(\beta, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\mathfrak{h}-1}, \Theta_{\mathfrak{h}/\varrho}, \Theta_{\mathfrak{h}+1}, \cdots, \Theta_\varsigma\right),$$

and

$$\tau\left(\beta, \gamma, \Theta_1, \Theta_2, \ldots, \Theta_{\mathfrak{h}-1}, \Theta_{\mathfrak{h}/\varrho}, \Theta_{\mathfrak{h}+1}, \cdots, \Theta_\varsigma\right),$$

respectively.

Theorem 4.1. *Consider a complete ς-NFMS, denoted by $(\Psi, \upsilon, \varpi, \tau, \oplus, \circledast)$, and a mapping $\Xi : \Psi \to \Psi$ that satisfies the following conditions:*

$$\upsilon_{\mathfrak{h}}^{1/\varkappa}\left(\Xi\beta, \Xi\gamma, \Theta_j^\varsigma\right) \geq \upsilon\left(\beta, \gamma, \Theta_j^\varsigma\right), \tag{4.1}$$

$$\varpi_{\mathfrak{h}}^{1/\varkappa}\left(\Xi\beta, \Xi\gamma, \Theta_j^\varsigma\right) \geq \varpi\left(\beta, \gamma, \Theta_j^\varsigma\right), \tag{4.2}$$

$$\lambda_{\mathfrak{h}}^{1/\varkappa}\left(\Xi\beta, \Xi\gamma, \Theta_j^\varsigma\right) \leq \lambda\left(\beta, \gamma, \Theta_j^\varsigma\right), \tag{4.3}$$

$$\tau_{\mathfrak{h}}^{1/\varkappa}\left(\Xi\beta, \Xi\gamma, \Theta_j^\varsigma\right) \leq \tau\left(\beta, \gamma, \Theta_j^\varsigma\right), \tag{4.4}$$

for all $\beta, \gamma \in \Psi$, $\Theta_1, \Theta_2, \ldots, \Theta_\varsigma > 0$, $\mathfrak{h} \in \{1, 2, .., \varsigma\}$ and $\varkappa \in (0,1)$. Assuming that $(\Psi, \upsilon, \varpi, \tau, \oplus, \circledast)$ is a $\mathfrak{h}$-natural ς-NFMS, it follows that the mapping Ξ possesses a unique fixed point.

Proof. Assume that there is just one fixed point in the mapping Ξ. Let $\mathfrak{r}$ and $\mathfrak{t}$ be fixed points of Ξ. By using the conditions provided in (4.1)–(4.4), the following holds:

$$\begin{aligned}\upsilon\left(\mathfrak{r},\mathfrak{t},\Theta_j^{\varsigma}\right) &= \upsilon\left(\Xi\mathfrak{r},\Xi\mathfrak{t},\Theta_j^{\varsigma}\right) \geq \upsilon\left(\mathfrak{r},\mathfrak{t},\Theta_1,\Theta_2,\ldots,\Theta_{\mathfrak{h}-1},\Theta_{\mathfrak{h}/\varkappa},\Theta_{\mathfrak{h}+1},\cdots,\Theta_{\varsigma}\right)\\ &= \upsilon_{\mathfrak{h}}^{\varkappa}\left(\mathfrak{r},\mathfrak{t},\Theta_j^{\varsigma}\right),\end{aligned}$$

$$\begin{aligned}\varpi\left(\mathfrak{r},\mathfrak{t},\Theta_j^{\varsigma}\right) &= \varpi\left(\Xi\mathfrak{r},\Xi\mathfrak{t},\Theta_j^{\varsigma}\right) \geq \varpi\left(\mathfrak{r},\mathfrak{t},\Theta_1,\Theta_2,\ldots,\Theta_{\mathfrak{h}-1},\Theta_{\mathfrak{h}/\varkappa},\Theta_{\mathfrak{h}+1},\cdots,\Theta_{\varsigma}\right)\\ &= \varpi_{\mathfrak{h}}^{\varkappa}\left(\mathfrak{r},\mathfrak{t},\Theta_j^{\varsigma}\right),\\ \lambda\left(\mathfrak{r},\mathfrak{t},\Theta_j^{\varsigma}\right) &= \lambda\left(\Xi\mathfrak{r},\Xi\mathfrak{t},\Theta_j^{\varsigma}\right) \leq \lambda\left(\mathfrak{r},\mathfrak{t},\Theta_1,\Theta_2,\ldots,\Theta_{\mathfrak{h}-1},\Theta_{\mathfrak{h}/\varkappa},\Theta_{\mathfrak{h}+1},\cdots,\Theta_{\varsigma}\right)\\ &= \lambda_{\mathfrak{h}}^{\varkappa}\left(\mathfrak{r},\mathfrak{t},\Theta_j^{\varsigma}\right),\end{aligned}$$

and

$$\begin{aligned}\tau\left(r,t,\Theta_j^{\varsigma}\right) &= \tau\left(\Xi r,\Xi t,\Theta_j^{\varsigma}\right) \leq \tau\left(r,t,\Theta_1,\Theta_2,\ldots,\Theta_{\mathfrak{h}-1},\Theta_{\mathfrak{h}/\varkappa},\Theta_{\mathfrak{h}+1},\cdots,\Theta_{\varsigma}\right)\\ &= \tau_{\mathfrak{h}}^{\varkappa}\left(\mathfrak{r},\mathfrak{t},\Theta_j^{\varsigma}\right).\end{aligned}$$

By iteratively applying the given inequalities (4.1)–(4.4), we get the following outcomes for all $\xi \in \mathbb{N}$:

$$\begin{gathered}\upsilon\left(\mathfrak{r},\mathfrak{t},\Theta_j^{\varsigma}\right) \geq \upsilon_{\mathfrak{h}}^{\varkappa^{\xi}}\left(\mathfrak{r},\mathfrak{t},\Theta_j^{\varsigma}\right),\ \varpi\left(\mathfrak{r},\mathfrak{t},\Theta_j^{\varsigma}\right) \geq \varpi_{\mathfrak{h}}^{\varkappa^{\xi}}\left(\mathfrak{r},\mathfrak{t},\Theta_j^{\varsigma}\right),\\ \tau\left(\mathfrak{r},\mathfrak{t},\Theta_j^{\varsigma}\right) \leq \tau_{\mathfrak{h}}^{\varkappa^{\xi}}\left(\mathfrak{r},\mathfrak{t},\Theta_j^{\varsigma}\right) \text{ and } \tau\left(\mathfrak{r},\mathfrak{t},\Theta_j^{\varsigma}\right) \leq \tau_{\mathfrak{h}}^{\varkappa^{\xi}}\left(\mathfrak{r},\mathfrak{t},\Theta_j^{\varsigma}\right).\end{gathered} \tag{4.5}$$

If $\left(q_{\xi}\right)$ is a sequence with $q_{\xi} > 0$ and $\lim_{\xi\to\infty} q_{\xi} = 0$, then $\mathfrak{h}$-natural property of $(\Psi,\upsilon,\varpi,\tau,\oplus,\circledast)$ ensures the following:

$$\begin{aligned}&\lim_{\xi\to\infty}\upsilon_{\mathfrak{h}}^{q_{\xi}}\left(\beta,\gamma,\Theta_j^{\varsigma}\right) = 1,\ \lim_{\xi\to\infty}\varpi_{\mathfrak{h}}^{q_{\xi}}\left(\beta,\gamma,\Theta_j^{\varsigma}\right) = 1,\ \lim_{\xi\to\infty}\lambda_{\mathfrak{h}}^{q_{\xi}}\left(\beta,\gamma,\Theta_j^{\varsigma}\right) = 0,\\ &\text{and } \lim_{\xi\to\infty}\tau_{\mathfrak{h}}^{q_{\xi}}\left(\beta,\gamma,\Theta_j^{\varsigma}\right) = 0,\end{aligned}$$

for all $\Theta_1,\Theta_2,\ldots,\Theta_{\varsigma} > 0$. Using this in (4.5), we have $\upsilon\left(\mathfrak{r},\mathfrak{t},\Theta_j^{\varsigma}\right) = 1$, $\varpi\left(\mathfrak{r},\mathfrak{t},\Theta_j^{\varsigma}\right) = 1$, $\lambda\left(\mathfrak{r},\mathfrak{t},\Theta_j^{\varsigma}\right) = 0$, and $\tau\left(\mathfrak{r},\mathfrak{t},\Theta_j^{\varsigma}\right) = 0$, for all $\Theta_1,\Theta_2,\ldots,\Theta_{\varsigma} > 0$. These conditions imply $\mathfrak{r} = \mathfrak{t}$, confirming the uniqueness of the fixed point of Ξ.

Now, let $\beta_0 \in \Psi$ and define the iterative sequence $\left(\beta_{\xi}\right)$ by setting $\beta_{\xi} = \Xi\beta_{\xi-1}$, for all $\xi \in \mathbb{N}$. The unique fixed point of Ξ is reached by this sequence $\left(\beta_{\xi}\right)$. Assume that $\beta_{\xi} \neq \beta_{\xi-1}$, for all $\xi \in \mathbb{N}$. For any given $\xi \in \mathbb{N}$ and $\Theta_1,\Theta_2,\ldots,\Theta_{\varsigma} > 0$, we deduce that

$$\begin{aligned}\upsilon\left(\beta_{\xi},\beta_{\xi+1},\Theta_j^{\varsigma}\right) &= \upsilon\left(\Xi\beta_{\xi-1},\Xi\beta_{\xi},\Theta_j^{\varsigma}\right)\\ &\geq \upsilon\left(\beta_{\xi-1},\beta_{\xi},\Theta_{\mathfrak{h}},\ldots,\Theta_{\mathfrak{h}-1},\Theta_{\frac{\mathfrak{h}}{\varkappa}},\Theta_{\mathfrak{h}+1},\cdots,\Theta_{\varsigma}\right) = \upsilon_{\mathfrak{h}}^{\varkappa}\left(\beta_{\xi-1},\beta_{\xi},\Theta_j^{\varsigma}\right),\end{aligned}$$

$$\begin{aligned}\varpi\left(\beta_{\xi},\beta_{\xi+1},\Theta_j^{\varsigma}\right) &= \varpi\left(\Xi\beta_{\xi-1},\Xi\beta_{\xi},\Theta_j^{\varsigma}\right)\\ &\geq \varpi\left(\beta_{\xi-1},\beta_{\xi},\Theta_{\mathfrak{h}},\ldots,\Theta_{\mathfrak{h}-1},\Theta_{\frac{\mathfrak{h}}{\varkappa}},\Theta_{\mathfrak{h}+1},\cdots,\Theta_{\varsigma}\right) = \varpi_{\mathfrak{h}}^{\varkappa}\left(\beta_{\xi-1},\beta_{\xi},\Theta_j^{\varsigma}\right),\\ \lambda\left(\beta_{\xi},\beta_{\xi+1},\Theta_j^{\varsigma}\right) &= \lambda\left(\Xi\beta_{\xi-1},\Xi\beta_{\xi},\Theta_j^{\varsigma}\right)\\ &\leq \lambda\left(\beta_{\xi-1},\beta_{\xi},\Theta_{\mathfrak{h}},\ldots,\Theta_{\mathfrak{h}-1},\Theta_{\frac{\mathfrak{h}}{\varkappa}},\Theta_{\mathfrak{h}+1},\cdots,\Theta_{\varsigma}\right) = \lambda_{\mathfrak{h}}^{\varkappa}\left(\beta_{\xi-1},\beta_{\xi},\Theta_j^{\varsigma}\right),\end{aligned}$$

$$\begin{aligned}\tau\left(\beta_{\xi},\beta_{\xi+1},\Theta_{j}^{\varsigma}\right) &= \tau\left(\Xi\beta_{\xi-1},\Xi\beta_{\xi},\Theta_{j}^{\varsigma}\right)\\ &\leq \tau\left(\beta_{\xi-1},\beta_{\xi},\Theta_{\mathfrak{h}},\ldots,\Theta_{\mathfrak{h}-1},\Theta_{\frac{\mathfrak{h}}{\varkappa}},\Theta_{\mathfrak{h}+1},\cdots,\Theta_{\varsigma}\right)=\tau_{\mathfrak{h}}^{\varkappa}\left(\beta_{\xi-1},\beta_{\xi},\Theta_{j}^{\varsigma}\right).\end{aligned}$$

By applying the iterative technique repeatedly, we obtain the following inequality for all $\xi\in\mathbb{N}$:

$$\upsilon\left(\beta_{\xi},\beta_{\xi+1},\Theta_{j}^{\varsigma}\right)\geq \upsilon_{\mathfrak{h}}^{\varkappa^{\xi}}\left(\beta_{0},\beta_{1},\Theta_{j}^{\varsigma}\right).$$

For each $\xi\in\mathbb{N}$, $\Theta_1,\Theta_2,\ldots,\Theta_{\varsigma}>0$ and $u>0$, we have

$$\begin{aligned}\upsilon\left(\beta_{\xi},\beta_{\xi+u},\Theta_{j}^{\varsigma}\right) &\geq \upsilon\left(\beta_{\xi},\beta_{\xi+1},\Theta_{\mathfrak{h}},\ldots,\Theta_{\mathfrak{h}-1},\Theta_{\frac{\mathfrak{h}}{2}},\Theta_{\mathfrak{h}-1},\cdots,\Theta_{\varsigma}\right)\\ &\quad\oplus \upsilon\left(\beta_{\xi+1},\beta_{\xi+u},\Theta_{\mathfrak{h}},\ldots,\Theta_{\mathfrak{h}-1},\Theta_{\frac{\mathfrak{h}}{2}},\Theta_{\mathfrak{h}},\cdots,\Theta_{\varsigma}\right)\\ &\geq \upsilon_{m}^{2}\left(\beta_{\xi},\beta_{\xi+1},\Theta_{j}^{\varsigma}\right)\oplus\upsilon\left(\beta_{\xi+1},\beta_{\xi+2},\Theta_{\mathfrak{h}},\ldots,\Theta_{\mathfrak{h}-1},\Theta_{\frac{\mathfrak{h}}{2^2}},\Theta_{\mathfrak{h}-1},\cdots,\Theta_{\varsigma}\right)\\ &\quad\oplus \upsilon\left(\beta_{\xi+2},\beta_{\xi+u},\Theta_{\mathfrak{h}},\ldots,\Theta_{\mathfrak{h}-1},\Theta_{\frac{\mathfrak{h}}{2^2}},\Theta_{\mathfrak{h}},\cdots,\Theta_{\varsigma}\right)\\ &\geq \upsilon_{\mathfrak{h}}^{2}\left(\beta_{\xi},\beta_{\xi+1},\Theta_{j}^{\varsigma}\right)\oplus\upsilon_{\mathfrak{h}}^{2^2}\left(\beta_{\xi+1},\beta_{\xi+2},\Theta_{j}^{\varsigma}\right)\oplus,...,\\ &\quad\oplus\upsilon_{\mathfrak{h}}^{2^{u-1}}\left(\beta_{\xi+u-2},\beta_{\xi+u-1},\Theta_{j}^{\varsigma}\right)\oplus\upsilon_{\mathfrak{h}}^{2^{u-1}}\left(\beta_{\xi+u-1},\beta_{\xi+u},\Theta_{j}^{\varsigma}\right),\end{aligned}$$

$$\begin{aligned}\varpi\left(\beta_{\xi},\beta_{\xi+u},\Theta_{j}^{\varsigma}\right) &\geq \varpi\left(\beta_{\xi},\beta_{\xi+1},\Theta_{\mathfrak{h}},\ldots,\Theta_{\mathfrak{h}-1},\Theta_{\frac{\mathfrak{h}}{2}},\Theta_{\mathfrak{h}-1},\cdots,\Theta_{\varsigma}\right)\\ &\quad\oplus \varpi\left(\beta_{\xi+1},\beta_{\xi+u},\Theta_{\mathfrak{h}},\ldots,\Theta_{\mathfrak{h}-1},\Theta_{\frac{\mathfrak{h}}{2}},\Theta_{\mathfrak{h}},\cdots,\Theta_{\varsigma}\right)\\ &\geq \varpi_{m}^{2}\left(\beta_{\xi},\beta_{\xi+1},\Theta_{j}^{\varsigma}\right)\oplus\varpi\left(\beta_{\xi+1},\beta_{\xi+2},\Theta_{\mathfrak{h}},\ldots,\Theta_{\mathfrak{h}-1},\Theta_{\frac{\mathfrak{h}}{2^2}},\Theta_{\mathfrak{h}-1},\cdots,\Theta_{\varsigma}\right)\\ &\quad\oplus \varpi\left(\beta_{\xi+2},\beta_{\xi+u},\Theta_{\mathfrak{h}},\ldots,\Theta_{\mathfrak{h}-1},\Theta_{\frac{\mathfrak{h}}{2^2}},\Theta_{\mathfrak{h}},\cdots,\Theta_{\varsigma}\right)\\ &\geq \varpi_{\mathfrak{h}}^{2}\left(\beta_{\xi},\beta_{\xi+1},\Theta_{j}^{\varsigma}\right)\oplus\varpi_{\mathfrak{h}}^{2^2}\left(\beta_{\xi+1},\beta_{\xi+2},\Theta_{j}^{\varsigma}\right)\oplus,...,\\ &\quad\oplus\varpi_{\mathfrak{h}}^{2^{u-1}}\left(\beta_{\xi+u-2},\beta_{\xi+u-1},\Theta_{j}^{\varsigma}\right)\oplus\varpi_{\mathfrak{h}}^{2^{u-1}}\left(\beta_{\xi+u-1},\beta_{\xi+u},\Theta_{j}^{\varsigma}\right),\end{aligned}$$

$$\begin{aligned}\lambda\left(\beta_{\xi},\beta_{\xi+u},\Theta_{j}^{\varsigma}\right) &\leq \lambda\left(\beta_{\xi},\beta_{\xi+1},\Theta_{\mathfrak{h}},\ldots,\Theta_{\mathfrak{h}-1},\Theta_{\frac{\mathfrak{h}}{2}},\Theta_{\mathfrak{h}-1},\cdots,\Theta_{\varsigma}\right)\\ &\quad\circledast \lambda\left(\beta_{\xi+1},\beta_{\xi+u},\Theta_{\mathfrak{h}},\ldots,\Theta_{\mathfrak{h}-1},\Theta_{\frac{\mathfrak{h}}{2}},\Theta_{\mathfrak{h}},\cdots,\Theta_{\varsigma}\right)\\ &\leq \lambda_{m}^{2}\left(\beta_{\xi},\beta_{\xi+1},\Theta_{j}^{\varsigma}\right)\circledast\lambda\left(\beta_{\xi+1},\beta_{\xi+2},\Theta_{\mathfrak{h}},\ldots,\Theta_{\mathfrak{h}-1},\Theta_{\frac{\mathfrak{h}}{2^2}},\Theta_{\mathfrak{h}-1},\cdots,\Theta_{\varsigma}\right)\\ &\quad\circledast \lambda\left(\beta_{\xi+2},\beta_{\xi+u},\Theta_{\mathfrak{h}},\ldots,\Theta_{\mathfrak{h}-1},\Theta_{\frac{\mathfrak{h}}{2^2}},\Theta_{\mathfrak{h}},\cdots,\Theta_{\varsigma}\right)\\ &\leq \lambda_{\mathfrak{h}}^{2}\left(\beta_{\xi},\beta_{\xi+1},\Theta_{j}^{\varsigma}\right)\circledast\lambda_{\mathfrak{h}}^{2^2}\left(\beta_{\xi+1},\beta_{\xi+2},\Theta_{j}^{\varsigma}\right)\circledast,...,\\ &\quad\circledast\lambda_{\mathfrak{h}}^{2^{u-1}}\left(\beta_{\xi+u-2},\beta_{\xi+u-1},\Theta_{j}^{\varsigma}\right)\circledast\lambda_{\mathfrak{h}}^{2^{u-1}}\left(\beta_{\xi+u-1},\beta_{\xi+u},\Theta_{j}^{\varsigma}\right),\end{aligned}$$

$$\begin{aligned}\tau\left(\beta_{\xi},\beta_{\xi+u},\Theta_{j}^{\varsigma}\right) &\leq \tau\left(\beta_{\xi},\beta_{\xi+1},\Theta_{\mathfrak{h}},\ldots,\Theta_{\mathfrak{h}-1},\Theta_{\frac{\mathfrak{h}}{2}},\Theta_{\mathfrak{h}-1},\cdots,\Theta_{\varsigma}\right)\\ &\quad\circledast \tau\left(\beta_{\xi+1},\beta_{\xi+u},\Theta_{\mathfrak{h}},\ldots,\Theta_{\mathfrak{h}-1},\Theta_{\frac{\mathfrak{h}}{2}},\Theta_{\mathfrak{h}},\cdots,\Theta_{\varsigma}\right)\\ &\leq \tau_{\mathfrak{h}}^{2}\left(\beta_{\xi},\beta_{\xi+1},\Theta_{j}^{\varsigma}\right)\circledast\tau\left(\beta_{\xi+1},\beta_{\xi+2},\Theta_{\mathfrak{h}},\ldots,\Theta_{\mathfrak{h}-1},\Theta_{\frac{\mathfrak{h}}{2^2}},\Theta_{\mathfrak{h}-1},\cdots,\Theta_{\varsigma}\right)\\ &\quad\circledast \tau\left(\beta_{\xi+2},\beta_{\xi+u},\Theta_{\mathfrak{h}},\ldots,\Theta_{\mathfrak{h}-1},\Theta_{\frac{\mathfrak{h}}{2^2}},\Theta_{\mathfrak{h}},\cdots,\Theta_{\varsigma}\right)\\ &\leq \tau_{\mathfrak{h}}^{2}\left(\beta_{\xi},\beta_{\xi+1},\Theta_{j}^{\varsigma}\right)\circledast\varpi_{\mathfrak{h}}^{2^2}\left(\beta_{\xi+1},\beta_{\xi+2},\Theta_{j}^{\varsigma}\right)\circledast,...,\\ &\quad\circledast\tau_{\mathfrak{h}}^{2^{u-1}}\left(\beta_{\xi+u-2},\beta_{\xi+u-1},\Theta_{j}^{\varsigma}\right)\circledast\tau_{\mathfrak{h}}^{2^{u-1}}\left(\beta_{\xi+u-1},\beta_{\xi+u},\Theta_{j}^{\varsigma}\right).\end{aligned}$$

By applying the result from inequality (4.5) and the iterative process described, we get

$$\begin{aligned}
&\upsilon\left(\beta_{\xi},\beta_{\xi+u},\Theta_{j}^{\varsigma}\right) \geq \upsilon^{2\varkappa^{\xi}}\left(\beta_{0},\beta_{1},\Theta_{j}^{\varsigma}\right)\oplus,...,\oplus\upsilon^{2^{2}\varkappa^{\xi+1}}\left(\beta_{0},\beta_{1},\Theta_{j}^{\varsigma}\right)\oplus\upsilon^{2^{u+1}\varkappa^{\xi+u-1}}\left(\beta_{0},\beta_{1},\Theta_{j}^{\varsigma}\right),\\
&\varpi\left(\beta_{\xi},\beta_{\xi+u},\Theta_{j}^{\varsigma}\right) \geq \varpi^{2\varkappa^{\xi}}\left(\beta_{0},\beta_{1},\Theta_{j}^{\varsigma}\right)\oplus,...,\oplus\varpi^{2^{2}\varkappa^{\xi+1}}\left(\beta_{0},\beta_{1},\Theta_{j}^{\varsigma}\right)\oplus\varpi^{2^{u+1}\varkappa^{\xi+u-1}}\left(\beta_{0},\beta_{1},\Theta_{j}^{\varsigma}\right),\\
&\lambda\left(\beta_{\xi},\beta_{\xi+u},\Theta_{j}^{\varsigma}\right) \leq \lambda^{2\varkappa^{\xi}}\left(\beta_{0},\beta_{1},\Theta_{j}^{\varsigma}\right)\circledast,...,\circledast\lambda^{2^{2}\varkappa^{\xi+1}}\left(\beta_{0},\beta_{1},\Theta_{j}^{\varsigma}\right)\circledast\lambda^{2^{u+1}\varkappa^{\xi+u-1}}\left(\beta_{0},\beta_{1},\Theta_{j}^{\varsigma}\right)\\
&\tau\left(\beta_{\xi},\beta_{\xi+u},\Theta_{j}^{\varsigma}\right) \leq \tau^{2\varkappa^{\xi}}\left(\beta_{0},\beta_{1},\Theta_{j}^{\varsigma}\right)\circledast,...,\circledast\tau^{2^{2}\varkappa^{\xi+1}}\left(\beta_{0},\beta_{1},\Theta_{j}^{\varsigma}\right)\circledast\tau^{2^{u+1}\varkappa^{\xi+u-1}}\left(\beta_{0},\beta_{1},\Theta_{j}^{\varsigma}\right).
\end{aligned}$$

Given that $(\Psi,\upsilon,\varpi,\lambda,\tau,\oplus,\circledast)$ is $\mathfrak{h}$-natural, the inequalities derived earlier imply the following limits:

$$\begin{aligned}
&\lim_{\xi\to\infty}\upsilon\left(\beta_{\xi},\beta_{\xi+u},\Theta_{j}^{\varsigma}\right)=1,\ \lim_{\xi\to\infty}\varpi\left(\beta_{\xi},\beta_{\xi+u},\Theta_{j}^{\varsigma}\right)=1,\\
&\lim_{\xi\to\infty}\lambda\left(\beta_{\xi},\beta_{\xi+u},\Theta_{j}^{\varsigma}\right)=0 \text{ and } \lim_{\xi\to\infty}\tau\left(\beta_{\xi},\beta_{\xi+u},\Theta_{j}^{\varsigma}\right)=0,
\end{aligned}$$

for all $\Theta_1,\Theta_2,\ldots,\Theta_\varsigma > 0$. Consequently, the sequence $\left(\beta_\xi\right)$ is Cauchy. Because $(\Psi,\upsilon,\varpi,\lambda,\oplus,\circledast)$ is complete, there exists $\mathfrak{z}\in\Psi$ such that

$$\begin{aligned}
&\lim_{\xi\to\infty}\upsilon\left(\beta_{\xi},\mathfrak{z},\Theta_{j}^{\varsigma}\right)=1,\ \lim_{\xi\to\infty}\varpi\left(\beta_{\xi},\mathfrak{z},\Theta_{j}^{\varsigma}\right)=1,\\
&\lim_{\xi\to\infty}\lambda\left(\beta_{\xi},\mathfrak{z},\Theta_{j}^{\varsigma}\right)=0,\text{ and } \lim_{\xi\to\infty}\tau\left(\beta_{\xi},\mathfrak{z},\Theta_{j}^{\varsigma}\right)=0.
\end{aligned}\tag{4.6}$$

Then, $\mathfrak{z}$ is a fixed point of Ξ. For each $\Theta_1,\Theta_2,\ldots,\Theta_\varsigma > 0$, we have

$$\begin{aligned}
\upsilon\left(\mathfrak{z},\Xi\mathfrak{z},\Theta_{j}^{\varsigma}\right) &\geq \upsilon_{\mathfrak{h}}^{2}\left(\mathfrak{z},\beta_{\xi},\Theta_{j}^{\varsigma}\right)\oplus\upsilon_{\mathfrak{h}}^{2}\left(\beta_{\xi},\Xi\mathfrak{z},\Theta_{j}^{\varsigma}\right)\\
&= \upsilon_{\mathfrak{h}}^{2}\left(\mathfrak{z},\beta_{\xi},\Theta_{j}^{\varsigma}\right)\oplus\upsilon_{\mathfrak{h}}^{2}\left(\Xi\beta_{\xi-1},\Xi\mathfrak{z},\Theta_{j}^{\varsigma}\right)\\
&\geq \upsilon_{\mathfrak{h}}^{2}\left(\mathfrak{z},\beta_{\xi},\Theta_{j}^{\varsigma}\right)\oplus\upsilon_{\mathfrak{h}}^{2\varkappa}\left(\beta_{\xi-1},\mathfrak{z},\Theta_{j}^{\varsigma}\right),
\end{aligned}$$

$$\begin{aligned}
\varpi\left(\mathfrak{z},\Xi\mathfrak{z},\Theta_{j}^{\varsigma}\right) &\geq \varpi_{\mathfrak{h}}^{2}\left(z,\beta_{\xi},\Theta_{j}^{\varsigma}\right)\oplus\varpi_{\mathfrak{h}}^{2}\left(\beta_{\xi},\Xi\mathfrak{z},\Theta_{j}^{\varsigma}\right)\\
&= \varpi_{\mathfrak{h}}^{2}\left(z,\beta_{\xi},\Theta_{j}^{\varsigma}\right)\oplus\varpi_{\mathfrak{h}}^{2}\left(\Xi\beta_{\xi-1},\Xi\mathfrak{z},\Theta_{j}^{\varsigma}\right)\\
&\geq \varpi_{\mathfrak{h}}^{2}\left(z,\beta_{\xi},\Theta_{j}^{\varsigma}\right)\oplus\varpi_{\mathfrak{h}}^{2\varkappa}\left(\beta_{\xi-1},\mathfrak{z},\Theta_{j}^{\varsigma}\right),\\
\lambda\left(\mathfrak{z},\Xi\mathfrak{z},\Theta_{j}^{\varsigma}\right) &\leq \lambda_{\mathfrak{h}}^{2}\left(\mathfrak{z},\beta_{\xi},\Theta_{j}^{\varsigma}\right)\circledast\lambda_{\mathfrak{h}}^{2}\left(\beta_{\xi},\Xi\mathfrak{z},\Theta_{j}^{\varsigma}\right)\\
&= \lambda_{\mathfrak{h}}^{2}\left(\mathfrak{z},\beta_{\xi},\Theta_{j}^{\varsigma}\right)\circledast\lambda_{\mathfrak{h}}^{2}\left(\Xi\beta_{\xi-1},\Xi\mathfrak{z},\Theta_{j}^{\varsigma}\right)\\
&\leq \lambda_{\mathfrak{h}}^{2}\left(\mathfrak{z},\beta_{\xi},\Theta_{j}^{\varsigma}\right)\circledast\lambda_{\mathfrak{h}}^{2\varkappa}\left(\beta_{\xi-1},\mathfrak{z},\Theta_{j}^{\varsigma}\right),\\
\tau\left(z,\Xi z,\Theta_{j}^{\varsigma}\right) &\leq \tau_{\mathfrak{h}}^{2}\left(\mathfrak{z},\beta_{\xi},\Theta_{j}^{\varsigma}\right)\circledast\tau_{\mathfrak{h}}^{2}\left(\beta_{\xi},\Xi\mathfrak{z},\Theta_{j}^{\varsigma}\right)\\
&= \tau_{\mathfrak{h}}^{2}\left(\mathfrak{z},\beta_{\xi},\Theta_{j}^{\varsigma}\right)\circledast\tau_{\mathfrak{h}}^{2}\left(\Xi\beta_{\xi-1},\Xi\mathfrak{z},\Theta_{j}^{\varsigma}\right)\\
&\leq \tau_{\mathfrak{h}}^{2}\left(\mathfrak{z},\beta_{\xi},\Theta_{j}^{\varsigma}\right)\circledast\tau_{\mathfrak{h}}^{2\varkappa}\left(\beta_{\xi-1},\mathfrak{z},\Theta_{j}^{\varsigma}\right).
\end{aligned}$$

Utilizing (4.6) in the preceding inequality, we derive

$$\upsilon\left(\mathfrak{z},\Xi\mathfrak{z},\Theta_{j}^{\varsigma}\right)=1,\ \varpi\left(\mathfrak{z},\Xi\mathfrak{z},\Theta_{j}^{\varsigma}\right)=1,\ \lambda\left(\mathfrak{z},\Xi\mathfrak{z},\Theta_{j}^{\varsigma}\right)=0,\ \text{and } \tau\left(\mathfrak{z},\Xi\mathfrak{z},\Theta_{j}^{\varsigma}\right)=0,$$

for all $\Theta_1,\Theta_2,\ldots,\Theta_\varsigma > 0$, that is, $\Xi\mathfrak{z}=\mathfrak{z}$, implying that Ξ possesses a fixed point of its own.

Example 4.1. Let (Ψ, d) be the complete metric space $\Psi = [0, 1]$ with the standard metric $d(\beta, \gamma) = |\beta - \gamma|$. Define the t-norm $\beta \oplus \gamma = \min\{\beta, \gamma\}$ and t-conorm $\beta \circledast \gamma = \max\{\beta, \gamma\}$.

Let $\varsigma = 1$ (the single-parameter case for simplicity) and define the ς-NFMS's functions for $\Theta > 0$ as

$$\upsilon(\beta, \gamma, \Theta) = \frac{\Theta}{\Theta + d(\beta, \gamma)}, \quad \varpi(\beta, \gamma, \Theta) = \frac{\Theta}{\Theta + d(\beta, \gamma)},$$
$$\lambda(\beta, \gamma, \Theta) = \frac{d(\beta, \gamma)}{\Theta + d(\beta, \gamma)}, \quad \tau(\beta, \gamma, \Theta) = \frac{d(\beta, \gamma)}{\Theta + d(\beta, \gamma)}.$$

This structure $(\Psi, \upsilon, \varpi, \lambda, \tau, \oplus, \circledast)$ forms a complete ς-NFMS and is $\mathfrak{h}$-natural (since $\lim_{\Theta\to\infty} \upsilon = 1$ and $\lim_{\Theta\to\infty} \lambda = 0$).

Now, define a mapping $\Xi : \Psi \to \Psi$ by $\Xi(\beta) = \frac{\beta}{4}$. Let us choose $\varkappa = 1/4 \in (0, 1)$. We must check the conditions of Theorem 4.1. For the υ condition (Eq (4.1)):

$$\begin{aligned} \upsilon_{\mathfrak{h}}^{1/\varkappa}(\Xi\beta, \Xi\gamma, \Theta) &= \upsilon\left(\frac{\beta}{4}, \frac{\gamma}{4}, \Theta/(1/4)\right) = \upsilon\left(\frac{\beta}{4}, \frac{\gamma}{4}, 4\Theta\right) \\ &= \frac{4\Theta}{4\Theta + d(\frac{\beta}{4}, \frac{\gamma}{4})} = \frac{4\Theta}{4\Theta + \frac{1}{4}|\beta - \gamma|} \\ &= \frac{16\Theta}{16\Theta + d(\beta, \gamma)}. \end{aligned}$$

We must check if $\frac{16\Theta}{16\Theta + d(\beta,\gamma)} \geq \upsilon(\beta, \gamma, \Theta) = \frac{\Theta}{\Theta + d(\beta,\gamma)}$. Let $d = d(\beta, \gamma)$. The inequality $\frac{16\Theta}{16\Theta + d} \geq \frac{\Theta}{\Theta + d}$ implies $16\Theta(\Theta + d) \geq \Theta(16\Theta + d)$, which simplifies to $15\Theta d \geq 0$. This is true for all $\Theta > 0$ and $d \geq 0$.

Similarly, for the λ condition (Eq (4.3)):

$$\begin{aligned} \lambda_{\mathfrak{h}}^{1/\varkappa}(\Xi\beta, \Xi\gamma, \Theta) &= \lambda\left(\frac{\beta}{4}, \frac{\gamma}{4}, 4\Theta\right) = \frac{d(\frac{\beta}{4}, \frac{\gamma}{4})}{4\Theta + d(\frac{\beta}{4}, \frac{\gamma}{4})} \\ &= \frac{\frac{1}{4}d}{4\Theta + \frac{1}{4}d} = \frac{d}{16\Theta + d}. \end{aligned}$$

We must check if $\frac{d}{16\Theta + d} \leq \lambda(\beta, \gamma, \Theta) = \frac{d}{\Theta + d}$. This inequality $d(\Theta + d) \leq d(16\Theta + d)$ simplifies to $0 \leq 15\Theta d$, which is also true. The conditions for ϖ and τ (Eqs (4.2) and (4.4)) are satisfied by a similar calculation. Since all conditions of Theorem 4.1 are met, Ξ must have a unique fixed point. Indeed, the mapping $\Xi(\beta) = \beta/4$ has a unique fixed point at $\mathfrak{z} = 0$ in $\Psi = [0, 1]$.

Remark 4.1. *In Theorem 4.1, we assume that the ς-NFMS is $\mathfrak{h}$-natural. It is important to note that this condition is essential for the uniqueness of the fixed point. The condition of $\mathfrak{h}$-naturalness cannot be replaced by $\mathfrak{m}$-naturalness, where $\mathfrak{m} \neq \mathfrak{h}$, nor can it be removed. The following example demonstrates that without this specific condition, the uniqueness of the fixed point is not guaranteed.*

Example 4.2. Let $\Psi = [0, 1]$. Define the t-norm $a \oplus b = ab$ and t-conorm $a \circledast b = \max\{a, b\}$. Let $\varsigma = 2$. Define the functions $\upsilon, \varpi, \lambda, \tau$ on $\Psi^2 \times (0, \infty)^2$ as follows:

$$\upsilon(\beta, \gamma, \Theta_1, \Theta_2) = \frac{\Theta_2}{\Theta_2 + |\beta - \gamma|}, \quad \varpi(\beta, \gamma, \Theta_1, \Theta_2) = \frac{\Theta_2}{\Theta_2 + |\beta - \gamma|},$$

$$\lambda(\beta,\gamma,\Theta_1,\Theta_2)=\frac{|\beta-\gamma|}{\Theta_2+|\beta-\gamma|},\quad \tau(\beta,\gamma,\Theta_1,\Theta_2)=\frac{|\beta-\gamma|}{\Theta_2}.$$

This space is a complete 2-NFMS. Notice that $\lim_{\Theta_2\to\infty}\upsilon = 1$, so the space is 2-natural. However, it is not 1-natural because the functions are independent of Θ_1, so $\lim_{\Theta_1\to\infty}\upsilon \neq 1$ (unless $\beta=\gamma$).

Now, define the mapping $\Xi:\Psi\to\Psi$ by $\Xi(\beta)=\beta$ (the identity map). Let us test the contraction conditions of Theorem 4.1 for $\mathfrak{h}=1$ (where the space is NOT natural). For any $\varkappa\in(0,1)$:

$$\upsilon_{\mathfrak{h}}^{1/\varkappa}(\Xi\beta,\Xi\gamma,\Theta_1,\Theta_2)=\upsilon(\beta,\gamma,\Theta_1/\varkappa,\Theta_2)=\frac{\Theta_2}{\Theta_2+|\beta-\gamma|}=\upsilon(\beta,\gamma,\Theta_1,\Theta_2).$$

Thus, the condition $\upsilon_{\mathfrak{h}}^{1/\varkappa}(\Xi\beta,\Xi\gamma,\Theta)\geq\upsilon(\beta,\gamma,\Theta)$ is satisfied (with equality). Similarly, all other conditions (4.2)–(4.4) are satisfied for $\mathfrak{h}=1$. However, despite satisfying the contraction inequalities, Ξ does not have a unique fixed point; in fact, every point in $[0,1]$ is a fixed point. This failure occurs because the space is not 1-natural. This proves that the $\mathfrak{h}$-naturalness condition assumed in Theorem 4.1 cannot be weakened.

Corollary 4.1. *Assume that $(\Psi,\upsilon,\varpi,\lambda,\tau,\oplus,\circledast)$ is a complete ς-NFMS. A function $\Xi:\Psi\to\Psi$ is defined if there exists $\xi\in(0,1)$ such that*

$$\begin{aligned}&\upsilon\left(\Xi\beta,\Xi\gamma,\Theta\right)\geq\upsilon(\beta,\gamma,\Theta),\ \varpi\left(\Xi\beta,\Xi\gamma,\Theta\right)\geq\varpi(\beta,\gamma,\Theta),\\&\lambda\left(\Xi\beta,\Xi\gamma,\Theta\right)\leq\lambda(\beta,\gamma,\Theta)\text{ and }\tau\left(\Xi\beta,\Xi\gamma,\Theta\right)\leq\tau(\beta,\gamma,\Theta),\ \forall\beta,\gamma\in\Psi.\end{aligned}\tag{4.7}$$

In this case, Ξ has a unique solution.

Lemma 4.1. *For any $m\in\{1,2,\cdots,\varsigma\}$, for all $\Theta_1,\Theta_2,\ldots,\Theta_\varsigma>0$ and $\phi\in(0,\Theta)$, if $\lim_{\xi\to\infty}\beta_\xi=\beta$ and $\lim_{\xi\to\infty}\gamma_\xi=\gamma$, then*

$$\begin{aligned}
&\upsilon\left(\beta,\gamma,\Theta_1,\Theta_2,\cdots,\Theta_{\mathfrak{h}-1},\Theta+\phi,\Theta_{\mathfrak{h}+1},\cdots,\Theta_\varsigma\right)\\
&\leq\lim_{\xi\to\infty}\inf\upsilon\left(\beta_\xi,\gamma_\xi,\Theta_1,\Theta_2,\cdots,\Theta_{\mathfrak{h}-1},\Theta+\phi,\Theta_{\mathfrak{h}+1},\cdots,\Theta_\varsigma\right),\\
&\varpi\left(\beta,\gamma,\Theta_1,\Theta_2,\cdots,\Theta_{\mathfrak{h}-1},\Theta+\phi,\Theta_{\mathfrak{h}+1},\cdots,\Theta_\varsigma\right)\\
&\leq\lim_{\xi\to\infty}\inf\varpi\left(\beta_\xi,\gamma_\xi,\Theta_1,\Theta_2,\cdots,\Theta_{\mathfrak{h}-1},\Theta+\phi,\Theta_{\mathfrak{h}+1},\cdots,\Theta_\varsigma\right),\\
&\lambda\left(\beta,\gamma,\Theta_1,\Theta_2,\cdots,\Theta_{\mathfrak{h}-1},\Theta+\phi,\Theta_{\mathfrak{h}+1},\cdots,\Theta_\varsigma\right)\\
&\geq\lim_{\xi\to\infty}\sup\lambda\left(\beta_\xi,\gamma_\xi,\Theta_1,\Theta_2,\cdots,\Theta_{\mathfrak{h}-1},\Theta+\phi,\Theta_{\mathfrak{h}+1},\cdots,\Theta_\varsigma\right),\\
&\tau\left(\beta,\gamma,\Theta_1,\Theta_2,\cdots,\Theta_{\mathfrak{h}-1},\Theta+\phi,\Theta_{\mathfrak{h}+1},\cdots,\Theta_\varsigma\right)\\
&\geq\lim_{\xi\to\infty}\sup\tau\left(\beta_\xi,\gamma_\xi,\Theta_1,\Theta_2,\cdots,\Theta_{\mathfrak{h}-1},\Theta+\phi,\Theta_{\mathfrak{h}+1},\cdots,\Theta_\varsigma\right),\\
&\upsilon\left(\beta,\gamma,\Theta_1,\Theta_2,\cdots,\Theta_{\mathfrak{h}-1},\Theta+\phi,\Theta_{\mathfrak{h}+1},\cdots,\Theta_\varsigma\right)\\
&\leq\lim_{\xi\to\infty}\sup\upsilon\left(\beta_\xi,\gamma_\xi,\Theta_1,\Theta_2,\cdots,\Theta_{\mathfrak{h}-1},\Theta+\phi,\Theta_{\mathfrak{h}+1},\cdots,\Theta_\varsigma\right),\\
&\varpi\left(\beta,\gamma,\Theta_1,\Theta_2,\cdots,\Theta_{\mathfrak{h}-1},\Theta+\phi,\Theta_{\mathfrak{h}+1},\cdots,\Theta_\varsigma\right)\\
&\leq\lim_{\xi\to\infty}\sup\varpi\left(\beta_\xi,\gamma_\xi,\Theta_1,\Theta_2,\cdots,\Theta_{\mathfrak{h}-1},\Theta+\phi,\Theta_{\mathfrak{h}+1},\cdots,\Theta_\varsigma\right),\\
&\lambda\left(\beta,\gamma,\Theta_1,\Theta_2,\cdots,\Theta_{\mathfrak{h}-1},\Theta+\phi,\Theta_{\mathfrak{h}+1},\cdots,\Theta_\varsigma\right)\\
&\geq\lim_{\xi\to\infty}\inf\lambda\left(\beta_\xi,\gamma_\xi,\Theta_1,\Theta_2,\cdots,\Theta_{\mathfrak{h}-1},\Theta+\phi,\Theta_{\mathfrak{h}+1},\cdots,\Theta_\varsigma\right),\\
&\tau\left(\beta,\gamma,\Theta_1,\Theta_2,\cdots,\Theta_{\mathfrak{h}-1},\Theta+\phi,\Theta_{\mathfrak{h}+1},\cdots,\Theta_\varsigma\right)\\
&\geq\lim_{\xi\to\infty}\inf\tau\left(\beta_\xi,\gamma_\xi,\Theta_1,\Theta_2,\cdots,\Theta_{\mathfrak{h}-1},\Theta+\phi,\Theta_{\mathfrak{h}+1},\cdots,\Theta_\varsigma\right).
\end{aligned}$$

Definition 4.1. *Let* $(\Psi, \upsilon, \varpi, \tau, \oplus, \circledast)$ *be a* ς*-NFMS. A mapping* $\Xi : \Psi \to \Psi$ *is called a* ς*-neutrosphic fuzzy contraction mapping (*ς*-NFCM) if* $0 \leq \varkappa < 1$ *such that, for every* $\beta, \gamma \in \Psi$ *and* $\Theta_1, \Theta_2, \ldots, \Theta_\varsigma > 0$*, the following circumstances are true:*

$$\frac{1}{\upsilon\left(\Xi\beta, \Xi\gamma, \Theta_j^\varsigma\right)} - 1 \leq \varkappa\left[\frac{1}{\upsilon\left(\beta, \gamma, \Theta_j^\varsigma\right)} - 1\right], \ \frac{1}{\varpi\left(\Xi\beta, \Xi\gamma, \Theta_j^\varsigma\right)} - 1 \leq \varkappa\left[\frac{1}{\varpi\left(\beta, \gamma, \Theta_j^\varsigma\right)} - 1\right],$$
$$\lambda\left(\Xi\beta, \Xi\gamma, \Theta_j^\varsigma\right) \leq \varkappa\lambda\left(\beta, \gamma, \Theta_j^\varsigma\right) \quad \textit{and} \quad \tau\left(\Xi\beta, \Xi\gamma, \Gamma_n^\kappa\right) \leq \varkappa\tau\left(\beta, \gamma, \Theta_j^\varsigma\right),$$

where $\varkappa$ *is the contractive factor of* Ξ*.*

Theorem 4.2. *Let* $(\Psi, \upsilon, \varpi, \lambda, \tau, \oplus, \circledast)$ *be a* ς*-NFMS and* $\Xi : \Psi \to \Psi$ *be a* ς*-NFCM. Then,* Ξ *has a unique fixed point.*

Proof. Let $\beta_0 \in \Psi$, and let $\left\{\beta_\xi\right\}$ be defined as $\beta_\xi = \Xi\beta_{\xi-1}$, for all $\xi \in \mathbb{N}$. For every $\xi \in \mathbb{N}$,

$$\begin{aligned}
\frac{1}{\upsilon\left(\beta_\xi, \beta_{\xi+1}, \Theta_j^\varsigma\right)} - 1 \quad &= \frac{1}{\upsilon\left(\Xi\beta_{\xi-1}, \Xi\beta_\xi, \Theta_j^\varsigma\right)} - 1 \leq \varkappa\left[\frac{1}{\upsilon\left(\beta_{\xi-1}, \beta_\xi, \Theta_j^\varsigma\right)} - 1\right] \\
&= \varkappa\left[\frac{1}{\upsilon\left(\Xi\beta_{\xi-2}, \Xi\beta_{\xi-1}, \Theta_j^\varsigma\right)} - 1\right] \leq \varkappa^2\left[\frac{1}{\upsilon\left(\beta_{\xi-2}, \beta_{\xi-1}, \Theta_j^\varsigma\right)} - 1\right] \\
&\vdots \\
&\leq \varkappa^\xi\left[\frac{1}{\upsilon\left(\beta_0, \beta_1, \Theta_j^\varsigma\right)} - 1\right].
\end{aligned}$$

Then, we obtain

$$\frac{1}{\upsilon\left(\beta_\xi, \beta_{\xi+1}, \Theta_j^\varsigma\right)} - 1 \leq \varkappa^\xi\left[\frac{1}{\upsilon\left(\beta_0, \beta_1, \Theta_j^\varsigma\right)} - 1\right]. \tag{4.8}$$

For each $\xi \in \mathbb{N}$, and for all $0 \leq \varkappa < 1$, we can deduce from Eq (4.8) that

$$\lim_{\xi\to\infty}\left[\frac{1}{\upsilon\left(\beta_\xi, \beta_{\xi+1}, \Theta_j^\varsigma\right)} - 1\right] \leq 0,$$

$$\text{i.e. } \lim_{\xi\to\infty} \frac{1}{\upsilon\left(\beta_\xi, \beta_{\xi+1}, \Theta_j^\varsigma\right)} = 1, \tag{4.9}$$

for all $\Theta_j^\varsigma > 0$.

For each $\xi \in \mathbb{N}$, $u > 0$ and $\Theta_j^\varsigma > 0$, we have

$$\begin{aligned}
\upsilon\left(\beta_\xi, \beta_{\xi+u}, \Theta_j^\varsigma\right) \quad &\geq \upsilon_\mathfrak{h}^2\left(\beta_\xi, \beta_{\xi+1}, \Theta_j^\varsigma\right) \oplus \upsilon_\mathfrak{h}^2\left(\beta_{\xi+1}, \beta_{\xi+u}, \Theta_j^\varsigma\right) \\
&\geq \upsilon_\mathfrak{h}^2\left(\beta_\xi, \beta_{\xi+1}, \Theta_j^\varsigma\right) \oplus \upsilon_\mathfrak{h}^{2^2}\left(\beta_{\xi+1}, \beta_{\xi+2}, \Theta_j^\varsigma\right) \oplus \cdots \\
&\quad \oplus \upsilon_\mathfrak{h}^{2^{u-1}}\left(\beta_{\xi+u-2}, \beta_{\xi+u-1}, \Theta_j^\varsigma\right) \oplus \upsilon_\mathfrak{h}^{2^{u-1}}\left(\beta_{\xi+u-1}, \beta_{\xi+u}, \Theta_j^\varsigma\right).
\end{aligned} \tag{4.10}$$

From Eq (4.9), we have $\lim_{\xi\to\infty} \upsilon_\mathfrak{h}^\mathfrak{z}\left(\beta_\xi, \beta_{\xi+1}, \Theta_j^\varsigma\right) = 1$, for every $\Theta_j^\varsigma > 0$ and $\mathfrak{z} > 0$. This suggests that

$$\lim_{\xi\to\infty} \upsilon\left(\beta_\xi, \beta_{\xi+1}, \Theta_j^\varsigma\right) \geq 1 \oplus 1 \oplus \cdots \oplus 1 = 1,$$

as does inequality (4.10). Given any positive real numbers $\Theta_1, \Theta_2, \ldots, \Theta_\varsigma$ and u, let $\left\{\beta_\xi\right\}$ be a Cauchy sequence in Ψ. The sequence is considered complete if it converges to a limit, that is, there exists $\mathfrak{q} \in \Psi$ such that $\left\{\beta_\xi\right\} \to \mathfrak{q}$.

To put it differently,

$$\lim_{\xi\to\infty} \upsilon\left(\beta_\xi, \mathfrak{q}, \Theta_j^\varsigma\right) = 1, \quad \text{for every } \Theta_j^\varsigma > 0. \tag{4.11}$$

Also, we get

$$\lim_{\xi\to\infty} \varpi\left(\beta_\xi, \mathfrak{q}, \Theta_j^\varsigma\right) = 1, \quad \text{for all } \Theta_j^\varsigma > 0.$$

This definition similarly implies that

$$\begin{aligned} \lambda\left(\beta_\xi, \beta_{\xi+1}, \Theta_j^\varsigma\right) &= \lambda\left(\Xi\beta_{\xi-1}, \Xi\beta_\xi, \Theta_j^\varsigma\right) \leq \varkappa\lambda\left(\beta_{\xi-1}, \beta_\xi, \Theta_j^\varsigma\right) \\ &= \varkappa\lambda\left(\Xi\beta_{\xi-2}, \Xi\beta_{\xi-1}, \Theta_j^\varsigma\right) \leq \varkappa^2\lambda\left(\beta_{\xi-2}, \beta_{\xi-1}, \Theta_j^\varsigma\right) \\ &\vdots \\ &= \varkappa^{\xi-1}\lambda\left(\Xi\beta_1, \Xi\beta_2, \Theta_j^\varsigma\right) \leq \varkappa^\xi\lambda\left(\beta_0, \beta_1, \Theta_j^\varsigma\right). \end{aligned} \tag{4.12}$$

Since $0 \leq \varkappa < 1$, we conclude from Eq (4.12) that

$$\lim_{\xi\to\infty} \lambda\left(\beta_\xi, \beta_{\xi+1}, \Theta_j^\varsigma\right) = 0, \quad \text{for all } \Theta_j^\varsigma > 0. \tag{4.13}$$

For $\xi \in \mathbb{N}$, $u > 0$ and $\Theta_j^\varsigma > 0$, we have

$$\begin{aligned} \lambda\left(\beta_\xi, \beta_{\xi+u}, \Theta_j^\varsigma\right) &\leq \lambda_{\mathfrak{h}}^2\left(\beta_\xi, \beta_{\xi+1}, \Theta_j^\varsigma\right) \circledast \lambda_{\mathfrak{h}}^2\left(\beta_{\xi+1}, \beta_{\xi+u}, \Theta_j^\varsigma\right) \\ &\leq \lambda_{\mathfrak{h}}^2\left(\beta_\xi, \beta_{\xi+1}, \Theta_j^\varsigma\right) \circledast \lambda_{\mathfrak{h}}^{2^2}\left(\beta_{\xi+1}, \beta_{\xi+2}, \Theta_j^\varsigma\right) \circledast \cdots \\ &\quad \circledast \lambda_{\mathfrak{h}}^{2^{u-1}}\left(\beta_{\xi+u-2}, \beta_{\xi+u-1}, \Theta_j^\varsigma\right) \circledast \lambda_{\mathfrak{h}}^{2^{u-1}}\left(\beta_{\xi+u-1}, \beta_{\xi+u}, \Theta_j^\varsigma\right). \end{aligned}$$

From Eq (4.13), we have $\lim_{\xi\to\infty} \lambda_{\mathfrak{h}}^{\mathfrak{z}}\left(\beta_\xi, \beta_{\xi+1}, \Theta_j^\varsigma\right) = 0$, for all $\Theta_j^\varsigma > 0$ and $\mathfrak{z} > 0$. This, combined with inequality (4.12), implies

$$\lim_{\xi\to\infty} \lambda\left(\beta_\xi, \beta_{\xi+u}, \Theta_j^\varsigma\right) = 0.$$

For any positive real numbers $\Theta_1, \Theta_2, \ldots, \Theta_\varsigma$ and u, let $\left\{\beta_\xi\right\}$ be a Cauchy sequence in Ψ. The sequence is said to be complete if it converges to itself, i.e., there exists $\mathfrak{q} \in \Psi$ such that $\left\{\beta_\xi\right\} \to \mathfrak{q}$.

Alternatively,

$$\lim_{\xi\to\infty} \lambda\left(\beta_\xi, \mathfrak{q}, \Theta_j^\varsigma\right) = 0, \quad \text{for all } \Theta_j^\varsigma > 0. \tag{4.14}$$

Similar operations also hold for another FS τ.

We will show that $\mathfrak{q}$ is a fixed point for Ψ. For all $\xi \in \mathbb{N}$ and $\Theta_j^\varsigma > 0$, we write that

$$\frac{1}{\upsilon\left(\beta_{\xi+1}, \Xi\mathfrak{q}, \Theta_j^\varsigma\right)} - 1 = \frac{1}{\upsilon\left(\Xi\beta_\xi, \Xi\mathfrak{q}, \Theta_j^\varsigma\right)} - 1 \leq \varkappa\left[\frac{1}{\upsilon\left(\beta_\xi, \mathfrak{q}, \Theta_j^\varsigma\right)} - 1\right].$$

$$\lim_{\xi\to\infty}\left[\frac{1}{\upsilon\left(\beta_{\xi+1},\Xi\mathfrak{q},\Theta_j^{\varsigma}\right)}-1\right]=0,\quad \text{using (4.10)},$$
$$\text{i.e., }\lim_{\xi\to\infty}\upsilon\left(\beta_{\xi+1},\Xi\mathfrak{q},\Theta_j^{\varsigma}\right)=1. \tag{4.15}$$

For any $\xi\in\mathbb{N}$ and for all $\Theta_j^{\varsigma}>0$, we have

$$\upsilon\left(\mathfrak{q},\Xi\mathfrak{q},\Theta_j^{\varsigma}\right)\geq\upsilon_{\mathfrak{h}}^2\left(\mathfrak{q},\beta_{\xi+1},\Theta_j^{\varsigma}\right)\oplus\upsilon_{\mathfrak{h}}^2\left(\beta_{\xi+1},\Xi\mathfrak{q},\Theta_j^{\varsigma}\right).$$

Taking the limit as $\xi\to\infty$ in the above inequality and using Eqs (4.11) and (4.15), leads to

$$\upsilon\left(\mathfrak{q},\Xi\mathfrak{q},\Theta_j^{\varsigma}\right)=1,\ \text{for all }\Theta_j^{\varsigma}>0.$$

In a comparable way, we have

$$\varpi\left(\mathfrak{q},\Xi\mathfrak{q},\Theta_j^{\varsigma}\right)=1,\ \text{for all }\Theta_j^{\varsigma}>0.$$

Now, we can write

$$\lambda\left(\beta_{\xi},\Xi\mathfrak{q},\Theta_j^{\varsigma}\right)=\lambda\left(\Xi\beta_{\xi},\Xi\mathfrak{q},\Theta_j^{\varsigma}\right)\leq\lambda\left(\beta_{\xi},\mathfrak{q},\Theta_j^{\varsigma}\right),$$
$$\lim_{\eta\to\infty}\lambda\left(\beta_{\xi+1},\Xi\mathfrak{q},\Theta_j^{\varsigma}\right)=0,\quad\text{by using (4.13)}. \tag{4.16}$$

For any $\xi\in\mathbb{N}$ and for all $\Theta_1,\Theta_2,\ldots,\Theta_{\varsigma}>0$, we have

$$\lambda\left(\mathfrak{q},\Xi\mathfrak{q},\Theta_j^{\varsigma}\right)\leq\lambda_{\mathfrak{h}}^2\left(\mathfrak{q},\beta_{\xi+1},\Theta_j^{\varsigma}\right)\circledast\lambda_{\mathfrak{h}}^2\left(\beta_{\xi+1},\Xi\mathfrak{q},\Theta_j^{\varsigma}\right),$$

which, together with (4.13) and (4.16), yields

$$\lambda\left(\mathfrak{q},\Xi\mathfrak{q},\Theta_j^{\varsigma}\right)=0,\ \text{for all }\Theta_j^{\varsigma}>0.$$

Additionally, in a similar manner, we have

$$\tau\left(\mathfrak{a},\Xi\mathfrak{q},\Theta_j^{\varsigma}\right)=0,\ \text{for all }\Theta_j^{\varsigma}>0.$$

This signifies that $\mathfrak{q}$ serves as a fixed point for Ψ, $\mathfrak{p}$ is another fixed point of Ψ, distinct from $\mathfrak{a}$. Therefore, there exist positive values $t_1,t_2,\cdots,t_{\varsigma}$ such that $\upsilon\left(\mathfrak{q},\mathfrak{p},t_1^{\varsigma}\right)<1$, $\varpi\left(\mathfrak{q},\mathfrak{p},t_1^{\varsigma}\right)<1$, $\lambda\left(\mathfrak{q},\mathfrak{p},t_1^{\varsigma}\right)>0$, $\tau\left(\mathfrak{q},\mathfrak{p},t_1^{\varsigma}\right)>0$. Now, we have

$$\begin{aligned}
&\tfrac{1}{\upsilon\left(\mathfrak{q},\mathfrak{p},t_1^{\varsigma}\right)}-1=\tfrac{1}{\upsilon\left(\Xi\mathfrak{q},\Xi\mathfrak{p},t_1^{\varsigma}\right)}-1\leq\varkappa\left[\tfrac{1}{\upsilon\left(\mathfrak{q},\mathfrak{p},t_1^{\varsigma}\right)}-1\right],\\
&\tfrac{1}{\varpi\left(\mathfrak{q},\mathfrak{p},t_1^{\varsigma}\right)}-1=\tfrac{1}{\varpi\left(\Xi\mathfrak{q},\Xi\mathfrak{p},t_1^{\varsigma}\right)}-1\leq\varpi\left[\tfrac{1}{\upsilon\left(\mathfrak{q},\mathfrak{p},t_1^{\varsigma}\right)}-1\right],\\
&\lambda\left(\mathfrak{q},\mathfrak{p},t_1^{\varsigma}\right)=\lambda\left(\Xi\mathfrak{q},\Xi\mathfrak{p},t_1^{\varsigma}\right)\leq\varkappa\lambda\left(\mathfrak{q},\mathfrak{p},t_1^{\varsigma}\right)<\lambda\left(\mathfrak{q},\mathfrak{p},t_1^{\varsigma}\right),\\
&\tau\left(\mathfrak{q},\mathfrak{p},t_1^{\varsigma}\right)=\tau\left(\Xi\mathfrak{q},\Xi\mathfrak{p},t_1^{\varsigma}\right)\leq\varkappa\tau\left(\mathfrak{q},\mathfrak{p},t_1^{\varsigma}\right)<\tau\left(\mathfrak{q},\mathfrak{p},t_1^{\varsigma}\right).
\end{aligned}$$

Since $\varkappa$ is less than 1, the aforementioned inequality leads to a contradiction. Therefore, it must be the case that $\mathfrak{q}=\mathfrak{p}$. As a result, a unique fixed point of Ψ is established.

5. Conclusions

This study introduced the concept of ς-NFMSs, a significant generalization of NFMSs. By incorporating multiple parameters ς into neutrosophic fuzzy sets, the proposed framework offers greater flexibility and applicability in analyzing mathematical structures. The fundamental properties of ς-NFMSs were explored, demonstrating that their topology is first-countable and that the corresponding metric space satisfies the Hausdorff condition. Additionally, a fixed-point theorem was established, which expanded and improved on previous findings in the setting of NFMSs. New research directions and real-world applications in mathematical analysis and related domains are made possible by these results. As an avenue for future research, it would be valuable to investigate whether the fixed-point theorems established in this work can be extended to proximal point results within the ς-NFMS framework, potentially drawing parallels to recent findings in related spaces [37].

Author contributions

Xiu-Liang Qiu and Qing-Bo Cai: Conceptualization, Methodology, Formal analysis, Writing–original draft, Writing–review & editing, Funding acquisition; Ömer Kişi and Mehmet Gürdal: Conceptualization, Methodology, Formal analysis, Writing–original draft, Writing–review & editing. All authors contributed equally to this work and have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work is supported by Fujian Provincial Natural Science Foundation of China (Grant No. 2024J01792) and Xiamen Natural Science Foundation Project (Grant No. 3502Z202573307).

Conflict of interest

The authors declare no conflicts of interest.

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