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Research article

Artificial intelligence powered agricultural field robots selection problem in spatial planning: applications of $\mathcal{L}^p$-intuitionistic fuzzy sets

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Abstract: It is a well-known fact that $\mathcal{L}^p$-spaces provide a robust and flexible framework for analyzing functions with different types of behavior, uncertainty, and regularity. They are widely applicable in many areas of mathematics, science, and engineering. In this study, we introduced a novel generalization that combines interval intuitionistic fuzzy sets (IFSs), as proposed by Atanassov [8], with circular intuitionistic fuzzy sets (C-IFS), introduced by Atanassov [9], because these classical sets restrict us. This new concept is known as the $\mathcal{L}^p$-intuitionistic fuzzy set (value) ($\mathcal{L}^p$-$IFS(V)$). The degrees of membership and non-membership in a $\mathcal{L}^p$-IFS are depicted by a diamond shape, circle, star shape, and square with its center defined by non-negative real numbers "κ" and "$\mathscr{s}$", ensuring that $\kappa + \mathscr{s} \leq 1$. The structure of a $\mathcal{L}^p$-IFS facilitates the representation of information through points on different shapes with respect to pth-norm with a designated center and norm "$\aleph$", thereby enabling a more precise characterization of the fuzziness inherent in uncertain data. As a result, a $\mathcal{L}^p$-IFS empowers decision-makers to evaluate options within a broader and more flexible framework, leading to the possibility of making more nuanced decisions. After establishing the concept of $\mathcal{L}^p$-IFS, some fundamental operations involving $\mathcal{L}^p$-$IFSs$ were outlined. To establish a novel scoring function and an accuracy function that incorporates the decision-makers' attitude (λ), the set's optimistic and pessimistic points were defined. When the decision-maker's viewpoint (λ) approached 1, the defuzzification of $\mathcal{L}^p$-IFS occurred near its optimistic point, while it occurred near its pessimistic point as (λ) approached 0. Moreover, a technique for converting a collection of intuitionistic fuzzy values into a $\mathcal{L}^p$-intuitionistic fuzzy values ($\mathcal{L}^p$-$IFVs$) was formulated. Additionally, several algebraic operations between $\mathcal{L}^p$-IFV using general triangular t-norms and triangular t-conorms were proposed. To transform input values represented by $\mathcal{L}^p$-$IFVs$ into a single output value, specific weighted aggregation operators based on these algebraic methods were introduced. The proposed methodology was applied to a problem concerning the selection of the optimal artificial intelligence (AI) agricultural field robots multi-attribute decision-making ($MADM$) framework. Finally, a framework was also presented for addressing $MADM$ challenges within a $\mathcal{L}^p$-intuitionistic fuzzy context. It is interesting to note that the time complexity of the proposed method and a comparative analysis were evaluated.

Keywords: $\mathcal{L}^p$-intuitionistic fuzzy sets; $\mathcal{L}^p$-intuitionistic fuzzy score function; $\mathcal{L}^p$-intuitionistic fuzzy aggregation operators; $\mathcal{L}^p$-intuitionistic fuzzy multi-attribute decision making problem; artificial intelligence powered agricultural field robots' selection
Mathematics Subject Classification: 90B50, 91B06, 03E72, 47S40, 03B52

1. Introduction

There are clear and well-defined boundaries between members and non-members of a collection. However, many classification concepts that we commonly use in everyday conversations involve sets that lack this characteristic, such as groups of tall people, expensive cars, highly contagious diseases, short driving distances, modest benefits, numbers close to a specific value, or sunny days. In these cases, there are subtle distinctions that enable gradual transitions between membership and non-membership. Zadeh's fuzzy set theory [49] effectively captures such ambiguous concepts in natural language. Real-world situations often require the inclusion of negative information, which cannot be easily inferred from positive aspects alone. For instance, while antibiotics are effective in treating certain illnesses, they may also have adverse side effects on the body. The positive aspect of this information can be considered the membership degree, whereas the negative aspect represents the non-membership degree, which is separate from the membership. In particular, Atanassov [4] introduced the concept of incorporating both membership and non-membership degrees, known as the IFS. Figure 1 offers a geometrical representation of an IFS, depicted as an ordered pair within a triangular region. In Figure 1, the points $(1,0)$ and $(0,1)$ represent total agreement and complete disagreement, respectively, while $(0,0)$ signifies a lack of knowledge or uncertainty about the situation. Within the triangular region, the ordered pair $\big(\kappa_T(\omega), \mathscr{s}_T(\omega)\big)$, referred to as the intuitionistic fuzzy value IFV), reflects that an individual agrees with situation ω by κ_T and disagrees by $\mathscr{s}_T$.

Intuitionistic fuzzy sets (IFS) have been applied across fields due to their strong capacity to handle uncertainty. In decision-making (DM), two main approaches stand out. One involves multi-criteria decision-making ($MCDM$) techniques that rely on information measures such as distance, similarity/dissimilarity, divergence, knowledge, and entropy. For Pythagorean fuzzy set and related concepts, see [47,48] and the references therein.

The other approach uses aggregation operators (AOs), which combine multiple pieces of information into a single value. Xu [44] introduced average aggregation operators for IFS, while Liu et al. [35] expanded prioritized AOs for IFS applications. Several researchers subsequently focused on developing aggregation operators for IFSs [45]. Boran et al. [12] explored the TOPSIS method for IFSs and applied it to solve supplier selection problems. Khan et al. [28] examined the $VIKOR$ method for IFSs, applying it to the selection of renewable energy sources. Khan and colleagues [29] provided theoretical foundations for the empirically effective $VIKOR$ method. For more details on $MCDM$ techniques, refer to Alinezhad and Khalili [2]. Additionally, Akram et al. [1] introduced an intuitionistic fuzzy logic controller for a heater fan system.

Divergence measures were initially introduced to quantify the difference between two probability distributions in classical probability theory. Bhandari et al. [8] extended this concept to fuzzy sets, defining a formula to measure how distinct two fuzzy sets are from each other. Their approach proposed a non-negative, symmetric measure that satisfies the identity of indiscernible. Later, Montes et al. [39] developed an axiomatic framework for fuzzy divergence. These measures have become essential tools in various fields, such as figure skating scoring [27], decision-making [10], and image thresholding and processing. Divergence measures play a crucial role in various scientific domains, such as pattern recognition, decision-making, market forecasting, image processing, and machine learning. Mishra et al. [37] applied a divergence-based $MABAC$ method for smartphone selection. Luo and Wang [36] extended the $VIKOR$ approach to IFS. Zhou et al. [50] introduced differentiation measures for Pythagorean fuzzy sets using belief functions, applying them in medical diagnostics. Rani et al. [41] employed a divergence-based $VIKOR$ method to assess renewable energy systems in a Pythagorean fuzzy context. The axiomatically supported divergence measurements for the q-rung orthopair fuzzy environment were proposed by Khan et al. [26]. The correlation coefficients and their uses in pattern recognition and clustering analysis were covered by Riaz et al. [42]. Borujeni et al. [11] studied dynamic intuitionistic fuzzy group decision analysis for sustainability risk assessment in surface mining operation projects. Moreover, Gitinavard, et al. [20,21] presented a novel variation of interval-valued hesitant fuzzy group outranking approach and its application in green supplier evaluation in manufacturing systems. Mousavi [38] discussed evaluating construction projects by a new group decision-making model based on intuitionistic fuzzy logic concepts. For more related results related to fuzzy sets and their generalization, see [10,30,31], and the references therein.

On the other hand, in recent years, the study of uncertainty modeling and intelligent decision-making has witnessed substantial progress through the development of advanced soft computing and fuzzy-set–based frameworks. Dalkılıç has played a significant role in this evolution by introducing several innovative theoretical structures and decision-making tools. The Dalkılıç [14], a novel uncertainty framework, the VFPIFS-cluster model, was proposed to enhance clustering performance under vague and imprecise environments. This work established a foundation for extending fuzzy and intuitionistic structures with more flexible parameterization. Further advancements were presented in Dalkılıç [15], where hyperflexible sets with neutrosophic parameters were generalized to support complex decision-maker preferences in uncertain domains. Building on these contributions, Dalkılıç [16] introduced VFP-soft sets, offering a powerful comparative decision-making mechanism that improves the interpretability and reliability of multi-criteria evaluations. In another significant contribution, Dalkılıç [17] developed decision-making approaches focusing on the optimal parameter–object pair, providing an efficient computational paradigm for soft-set-based analysis. Additionally, the interaction between heterogeneous object sets was explored through the concept of inverse object interaction sets for binary soft sets [18], addressing the need to analyze relationships across universes. Collectively, these studies demonstrate a continuous effort to refine uncertainty modeling and strengthen decision-making methodologies, providing a robust theoretical basis for further advancements in soft computing and fuzzy systems, see [22,23,32].

Although IFS has been applied to numerous problems, uncertainties often complicate the accurate prediction of membership and non-membership degrees. To address this, Atanassov [6], and Garg and Rani [19] introduced the concept of interval-valued membership $u_{\mathcal{A}}(\omega)$ and non-membership $v_{\mathcal{A}}(\omega)$ degrees instead of assigning a specific value. This approach, referred to as an interval-valued intuitionistic fuzzy set ($IVIFS$), was defined by Atanassov in 1989. Unlike IFS, which uses a single value, $IVIFS$ is represented by a rectangular region R (as shown in Figure 2). In this figure, total agreement, total disagreement, and complete ignorance correspond to the points $([1,1],[0,0])$, $([0,0],[1,1])$, and $([0,0],[0,0])$, respectively. When an individual cannot assign exact membership and non-membership values for a situation ω, the intervals $u_{\mathcal{A}}(\omega)$) and $v_{\mathcal{A}}(\omega)$ form a region R. Deveci et al. [13] assessed the public bus transportation service utilizing $IVIFS$s. Subsequently, additional researchers explored $IVIFS$s and applied them in diverse contexts (Xu and Gou, [46]). Although $IVIFS$ offers the opportunity to give membership and non-membership degree intervals rather than exact values, handling their representation is challenging. Therefore, a different actual extension of IFSs is suggested, in which a circular region is straightforward representation instead of a rectangular one. This is referred to as C-IFS. Atanassov [5] proposed the concept of $IVFS$. In this framework, the circle with center $(\kappa(\omega), \mathcal{s}(\omega))$ and radius r replaces the rectangular region R depicted in Figure 1. The C-IFS reduces to a standard IFS when $r = 0$. Atanassov and Marinov [7] introduced distance metrics for C-IFSs. Boltürk and Kahraman [9] characterized interval-valued IFSs. Alkan and Kahraman [3] explored the application of C-IFSs in the selection of hospital placements during a pandemic. Kahraman and Alkan developed the TOPSIS method for C-IFSs and applied it to supplier selection scenarios [25]. Otay and Kahraman [24] tackled the multi-expert supplier evaluation issue by adapting the AHP and $VIKOR$ methodologies for C-IFSs. For triangular norm and conorm, see [33,34,43] and the references therein.

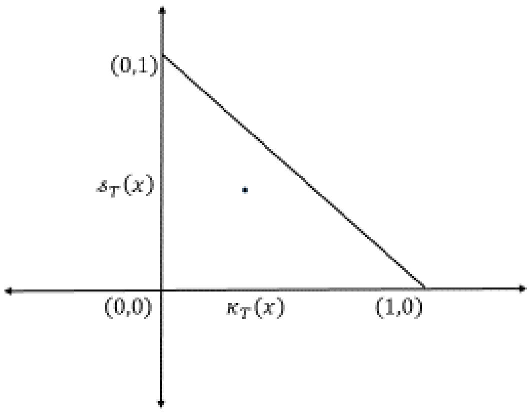

Figure 1. Geometric presentation of IFS.

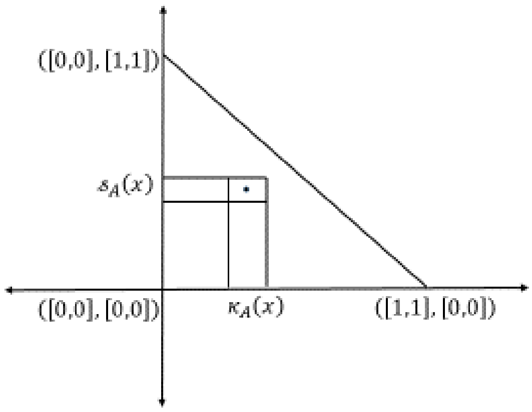

Figure 2. Geometric presentation of $IVIFS$.

In this research, we introduce the concept of a $\mathcal{L}^p$-IFSs, which extends the idea of representing membership and non-membership degrees as different shapes into the $\mathcal{L}^p$-intuitionistic fuzzy framework. Instead of representing an element's membership and non-membership degrees with precise values, this new fuzzy set model employs circles centered at $\langle\omega, \kappa(\omega), \mathcal{s}(\omega)\rangle$, governed by the more flexible condition $\kappa(\omega) + \mathcal{s}(\omega) \leq 1$. As shown in Figures 2 and 4, this extends the $IVIFS$

concept and the C-IFS model. Since decision-makers (DMs) can work with circles representing certain characteristics rather than precise numerical values, the decision-making process becomes more refined and responsive. The improvement brought by $\mathcal{L}^p$-IFS fuzzy sets is illustrated in Figure 4. The key contributions of the paper are outlined as follows:

- The ideas of $\mathcal{L}^p$-IFS and $\mathcal{L}^p$-IFV are introduced in this study.
- A technique for converting a set of IFV into a $\mathcal{L}^p$-IFS is obtained, and the multi-criteria group decision making $MCGDM$ can be resolved in this manner.
- $\mathcal{L}^p$-shapes indicate an element's membership or non-membership in a $\mathcal{L}^p$-IFS. Its structure enables more sensitive modeling in the continuous environment using multi-attribute decision-making ($MADM$) theory.
- To establish a novel scoring function and an accuracy function that incorporates the decision-makers' attitude (λ), the set's optimistic and pessimistic points are also defined. When the decision-maker's viewpoint (λ) approaches 1, the defuzzification of $\mathcal{L}^p$-IFS occurs near its optimistic point, while it occurs near its pessimistic point as (λ) approaches 0.
- For $\mathcal{L}^p$-IFS, certain algebraic operations are defined using t-norms and t-conorms.
- Some weighted arithmetic and geometric aggregation operators are supplied with the support of these operations. In $MADM$, these aggregation operators are employed.
- To support our proposed methodology, we include illustrated examples.

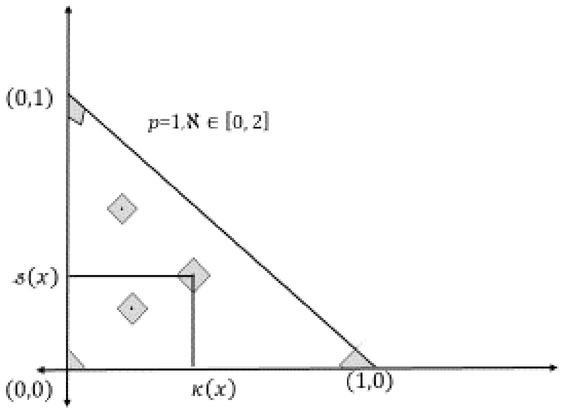

Figure 3. Geometric presentation of $\mathcal{L}_1$-IFS.

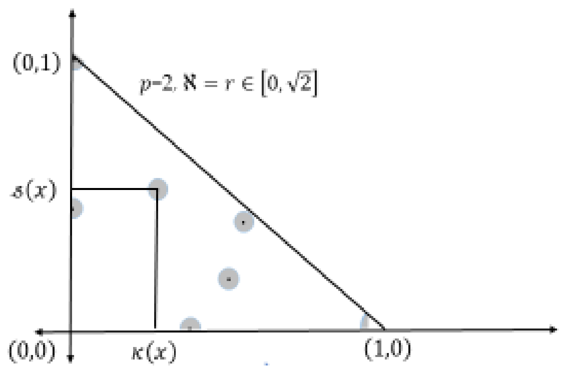

Figure 4. Geometric presentation of $\mathcal{L}_2$-IFS.

The structure of the paper is as follows: In Section 2, we cover a review of some fundamental concepts. In Section 3, we introduce the concept of $\mathcal{L}^p$-intuitionistic fuzzy sets ($\mathcal{L}^p$-IFSs) as a new extension of both intuitionistic fuzzy set and circular intuitionistic fuzzy set, along with defining basic set-theoretic properties for $\mathcal{L}^p$-IFS. Additionally, some novel score and accuracy functions are defined. Then, in Sections 4 and 5, we propose several algebraic procedures for $\mathcal{L}^p$-intuitionistic fuzzy values ($\mathcal{L}^p$-IFVs) using max-min rules and continuous Archimedean t-norms and t-conorms, respectively. In Section 6, based on these operations, we also introduce a few weighted aggregation operators for $\mathcal{L}^p$-IFVs via continuous Archimedean t-norms and t-conorms. In Section 7, steps of the algorithm for the $MADM$ technique is discussed. The proposed methodology is applied to a problem concerning the selection of the optimal agricultural field robots $MADM$ framework. An outline of the study and a discussion of the findings are provided in Section 8. The study is concluded in the final section by outlining the benefits and drawbacks of the suggested strategies and suggesting additional research for

$\mathcal{L}^p$-IFSs.

2. Preliminaries

In this section, we present several classical definitions, results, and concepts that will facilitate the discussion of the major findings.

Definition 1 ([5])**.** Let us have a fixed universe E and its sub-set T. The set

$$T = \{\langle \omega, \kappa_T(\omega), \mathscr{s}_T(\omega)\rangle : \text{for all } \omega \in E\},$$

where $0 \leq \kappa_T(\omega) + \mathscr{s}_T(\omega) \leq 1$ is called the intuitionistic fuzzy set (IFS) and functions $\kappa_T, \mathscr{s}_T : E \to [0, 1]$ indicate the degree of membership (validity, etc.) and non-membership (non-validity, etc.) of element $\omega \in E$ to a fixed set $T \subseteq E$. Now, we can define also function $\pi_T : E \to [0, 1]$ by means of

$$\pi_T(\omega) = 1 - \kappa_T(\omega) - \mathscr{s}_T(\omega),$$

and it corresponds to degree of indeterminacy (uncertainty, etc.). An intuitionistic fuzzy value (IFV) is the pair "$< \kappa_T(\omega), \mathscr{s}_T(\omega) >$" given an element ω of X. To make things easier to understand, we can write $\tilde{t} =< \kappa_{\tilde{t}}, \mathscr{s}_{\tilde{t}} >$, where $\kappa_{\tilde{t}} \in [0,1]$, $\mathscr{s}_{\tilde{t}} \in [0,1]$ and $0 \leq \kappa_{\tilde{t}} + \mathscr{s}_{\tilde{t}} \leq 1$. The degree of indeterminacy is represented by $\pi_{\tilde{t}}$, subject to the constraints that $\pi_{\tilde{t}} \in [0,1]$ and $\pi_{\tilde{t}} = 1 - \kappa_{\tilde{t}} - \mathscr{s}_{\tilde{t}}$.

The definition of the complement of an IFV $\tilde{t} =< \mathscr{s}_{\tilde{t}}, \kappa_{\tilde{t}}, \pi_{\tilde{t}} >$ is as follows:

$$\tilde{t}^C =< \mathscr{s}_{\tilde{t}}, \kappa_{\tilde{t}}, \pi_{\tilde{t}} >.$$

Definition 2. Let $D[0,1]$ denote the set of all closed subintervals of $[0,1]$. An interval-valued intuitionistic fuzzy set ($IVIFS$) $\mathcal{A}$ in X is defined as $\mathcal{A} = \{< \omega, u_{\mathcal{A}}(\omega), v_{\mathcal{A}}(\omega) > \omega \in X\}$ where $u_{\mathcal{A}}: X \to D[0,1]$ and "$v_{\mathcal{A}}: X \to D[0,1]$", with the condition "$0 \leq \sup u_{\mathcal{A}}(\omega) + \sup v_{\mathcal{A}}(\omega) \leq 1, \omega \in X$". The membership and non-membership degrees of X to $\mathcal{A}$ are represented by the intervals $u_{\mathcal{A}}(\omega)$ and $v_{\mathcal{A}}(\omega)$, respectively.

An interval-valued intuitionistic fuzzy number ($IVIFV$) is the pair $< u_{\mathcal{A}}(\omega), v_{\mathcal{A}}(\omega) >$ for any $\omega \in X$, see [6]. In this study, $\tilde{\mathcal{A}} = \left([u_{\tilde{\mathcal{A}}}^-, u_{\tilde{\mathcal{A}}}{}^+], [v_{\tilde{\mathcal{A}}}^-, v_{\tilde{\mathcal{A}}}^+]\right)$ is used to conveniently denote an $IVIF\aleph$. Here, $[u_{\tilde{\mathcal{A}}}^-, u_{\tilde{\mathcal{A}}}{}^+] \in D[0,1]$, $[v_{\tilde{\mathcal{A}}}^-, v_{\tilde{\mathcal{A}}}{}^+] \in D[0,1]$ and $u_{\tilde{\mathcal{A}}}{}^+ + v_{\tilde{\mathcal{A}}}{}^+ \leq 1$.

The concepts of t-norm and t-conorm are vital in statistics and decision-making. In algebra, binary operations defined on the closed unit interval are known as t-norms and t-conorms.

Definition 3 ([33,34,43])**.** A t-norm is a function $\mathcal{T}: [0,1] \times [0,1] \to [0,1]$ that that adheres to the following properties:
($\mathcal{T}1$) Border condition: $\mathcal{T}(\alpha, 1) = \alpha$ for all $\alpha \in [0,1]$.
($\mathcal{T}2$) Commutativity: $\mathcal{T}(\alpha, \beta) = \mathcal{T}(\beta, \alpha)$ for all $\alpha, \beta \in [0,1]$.
($\mathcal{T}3$) Associativity: $\mathcal{T}(\alpha, \mathcal{T}(\beta, \curlyvee)) = \mathcal{T}(\mathcal{T}(\alpha, \beta), \curlyvee)$ for all $\alpha, \beta, \curlyvee \in [0,1]$.
($\mathcal{T}4$) Monotonicity: $\mathcal{T}(\alpha, \beta) \leqslant \mathcal{T}(\alpha', \beta')$ whenever $\alpha \leqslant \alpha'$ and $\beta \leqslant \beta'$ for all $\alpha, \alpha', \beta, \beta' \in [0,1]$.

Definition 4 ([33,34,43])**.** A t-conorm is a function $\mathcal{S}: [0,1] \times [0,1] \to [0,1]$ that that adheres to the following properties:
($\mathcal{S}1$) Border condition: $\mathcal{S}(\alpha, 0) = \alpha$ for all $\alpha \in [0,1]$ (border condition).
($\mathcal{S}2$) Commutativity: $\mathcal{S}(\alpha, \beta) = \mathcal{S}(\beta, \alpha)$ for all $\alpha, \beta \in [0,1]$ (commutativity).
($\mathcal{S}3$) Associativity: $\mathcal{S}(\alpha, \mathcal{S}(\beta, \curlyvee)) = \mathcal{S}(\mathcal{S}(\alpha, \beta), \curlyvee)$ for all $\alpha, \beta, \curlyvee \in [0,1]$ (associativity).
($\mathcal{S}4$) Monotonicity: $\mathcal{S}(\alpha, \beta) \leqslant \mathcal{S}(\alpha', \beta'$) whenever $\alpha \leqslant \alpha'$ and $\beta \leqslant \beta'$ for all $\alpha, \alpha', \beta, \beta' \in [0,1]$ (monotonicity).

Definition 5 ([33,34])**.** A function $g:[0,1]\to[0,\infty]$ with $g(1)=0$ that is strictly decreasing and satisfies $g(1)=0$ is referred to as the **additive generator** of a t-norm $\mathcal{T}$ if the relationship $\mathcal{T}(\alpha,\beta)=g^{-1}(g(\alpha)+g(\beta))$ holds for all $(\alpha,\beta)\in[0,1]\times[0,1]$.

The concept of a fuzzy complement is required to determine the additive generator of a dual t-conorm defined on the interval $[0,1]$.

Definition 6 ([47,48])**.** A fuzzy complement is a function $N:[0,1]\to[0,1]$ that meets the following criteria:
(N1) $N(0)=1$ and $N(1)=0$ (boundary conditions).
(N2) $N(\alpha)\geqslant N(\beta)$ whenever $\alpha\leqslant\beta$ for all $\alpha,\beta\in[0,1]$ (monotonicity).
(N3) Continuity.
(N4) $N(N(\alpha))=\alpha$ for all $\alpha\in[0,1]$ (involution).

The function $N:[0,1]\to[0,1]$ given by $N(\alpha)=(1-\alpha^p)^{1/p}$, where $p\in(0,\infty)$ [47], represents a fuzzy complement. When $p=1, N$ simplifies to the intuitionistic fuzzy complement $N(\alpha)=1-\alpha$.

Definition 7 ([32,48])**.** Let $\mathcal{T}$ be a t-norm and $\mathcal{S}$ be a t-conorm on the interval [0, 1]. If $\mathcal{T}(\alpha,\beta)=N(\mathcal{S}(N(\alpha),N(\beta)))$ and $\mathcal{S}(\alpha,\beta)=N(\mathcal{T}(N(\alpha),N(\beta)))$, then $\mathcal{T}$ and $\mathcal{S}$ are referred to as dual with respect to the fuzzy complement N.

Remark 1. Let $\mathcal{T}$ represent a t-norm on the interval $[0,1]$. The corresponding dual t-conorm $\mathcal{S}$ with regard to the intuitionistic fuzzy complement N is defined as follows:

$$\mathcal{S}(\alpha,\beta)=1-\mathcal{T}(1-\alpha,1-\beta).$$

It is important to mention that $\mathcal{T}$ qualifies as an Archimedean t-norm if and only if $\mathcal{T}(\alpha,\alpha)<\alpha$ for all $\alpha\in(0,1)$, while $\mathcal{S}$ is classified as an Archimedean t-conorm if and only if $\mathcal{S}(\alpha,\alpha)>\alpha$ [33]. Klement et al. [34] demonstrated that continuous Archimedean t-norms can be represented through their additive generators, as established in the following theorem.

Theorem 1 ([34])**.** Let $\mathcal{T}$ represent a t-norm on [0, 1]. The following statements are equivalent:
(i) $\mathcal{T}$ is a continuous Archimedean t-norm.
(ii) $\mathcal{T}$ possesses a continuous additive generator, meaning there exists a continuous, strictly decreasing function $g:[0,1]\to[0,\infty]$ with $t(1)=0$, such that $\mathcal{T}(\alpha,\beta)=g^{-1}(g(\alpha)+g(\beta))$ for all $(\alpha,\beta)\in[0,1]\times[0,1]$.

This new fuzzy set is an extension of the IFS and $IVIFS$, distinguished by different $\mathcal{L}^p$-shape representations of the degrees of membership and nonmembership.

3. $\mathcal{L}^p$-intuitionistic fuzzy sets

We begin with the primary definition of an $\mathcal{L}^p$-intuitionistic fuzzy set, which is as follows:

Definition 8. Let us have a fixed universe E and its sub-set T. The set

$$\mathcal{L}_{\aleph}^{p}=\{\langle\omega,\kappa(\omega),\mathscr{s}(\omega);\aleph\rangle\mid\omega\in E\},$$

where $0\le\kappa(\omega)+\mathscr{s}(\omega)\le1$ and $\aleph\in\left[0,2^{\frac{1}{p}}\right]$ with $p\ge1$ are called $\mathcal{L}^p$-IFS and functions $\kappa,\mathscr{s}:E\to[0,1]$ indicate the degree of membership (validity, etc.) and non-membership (non-validity, etc.) of element $\omega\in E$ to a fixed set $T\subseteq E$. Now, we can define function $\pi:E\to[0,1]$ by means of

$$\pi(\omega) = 1 - \kappa(\omega) - \mathcal{s}(\omega),$$

and it corresponds to degree of indeterminacy (uncertainty, etc.) (see Figures 3–9).

On the other hand, $\mathcal{L}_{\aleph}^{p}$ can also be defined using the following approach, such that:

Let $\mathcal{L}_1 = \{\langle h, m\rangle : h, m \in [0,1], \text{and } h + m \leq 1\}$. Then,

$$\mathcal{L}_{\aleph}^{p} = \{\langle \omega, \aleph^1(\kappa(\omega), \mathcal{s}(\omega))\rangle : \omega \in E\},$$

where

$$\aleph^1(\kappa(\omega), \mathcal{s}(\omega)) = \left\{\langle h, m\rangle : h, m \in [0,1] \text{ and } (|\kappa(\omega) - h|^p + |\mathcal{s}(\omega) - m|^p)^{\frac{1}{p}} \leq \aleph\right\} \cap \mathcal{L}_1,$$

$$= \left\{\langle h, m\rangle : h, m \in [0,1], (|\kappa(\omega) - h|^p + |\mathcal{s}(\omega) - m|^p)^{\frac{1}{p}} \leq \aleph \text{ and } h + m \leq 1\right\}.$$

To simplify matters, we consider the convex part of the $\mathcal{L}^p$-IFS in Definition 8 for $p \geq 1$. However, the readers interested in exploring this further may consider the nonconvex part for $p > 0$.

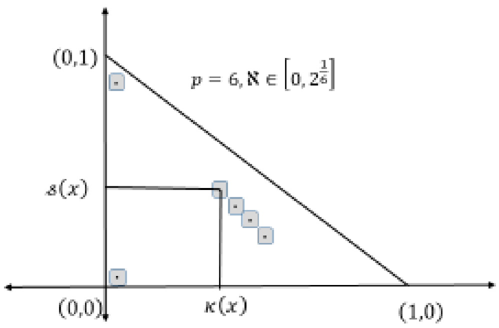

Figure 5. Geometric presentation of $\mathcal{L}_6$-IFS.

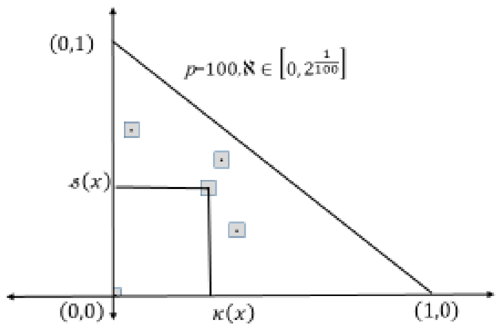

Figure 6. Geometric presentation of $\mathcal{L}_{100}$-IFS.

Note that if we want to cover the $\mathcal{L}^1$ and $\mathcal{L}^2$-intuitionistic fuzzy interpretation triangle, then $\aleph \in [0,2]$, and $\aleph = r \in [0, \sqrt{2}]$, respectively, see Figures 8 and 9.

Similarly, other shapes can be defined for $p \geq 1$.

Here is the restriction of Definition 8; The intuitionistic fuzzy interpretation triangle cannot be fully covered.

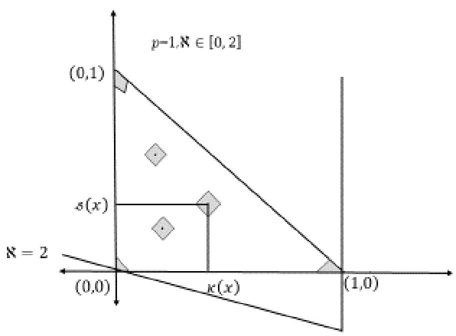

Figure 7. Triangular coverage of different $\aleph$ values of $\mathcal{L}_2$-IFS.

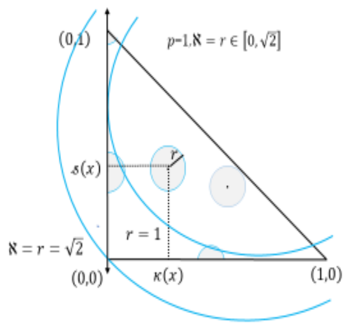

Figure 8. Triangular coverage of different $\aleph$ values of $\mathcal{L}_2$-IFS.

Definition 9. Let us have a fixed universe E and its sub-set T. The set

$$\mathcal{L}^p_{\aleph} = \{\langle \omega, \kappa(\omega), \mathcal{s}(\omega); \aleph \rangle | \omega \in E\},$$

where $0 \leq \kappa(\omega) + \mathcal{s}(\omega) \leq 1$ and $\aleph \in [0,1]$ with $p \geq 1$ is called $\mathcal{L}^p$-IFS and functions $\kappa, \mathcal{s}: E \to [0, 1]$ indicate the degree of membership (validity, etc.) and non-membership (non-validity, etc.) of element $\omega \in E$ to a fixed set $T \subseteq E$. Now, we can define function $\pi: E \to [0, 1]$ by means of

$$\pi(\omega) = 1 - \kappa(\omega) - \mathcal{s}(\omega).$$

This corresponds to degree of indeterminacy (uncertainty, etc.) (see Figures 3–6, 9, and 10).

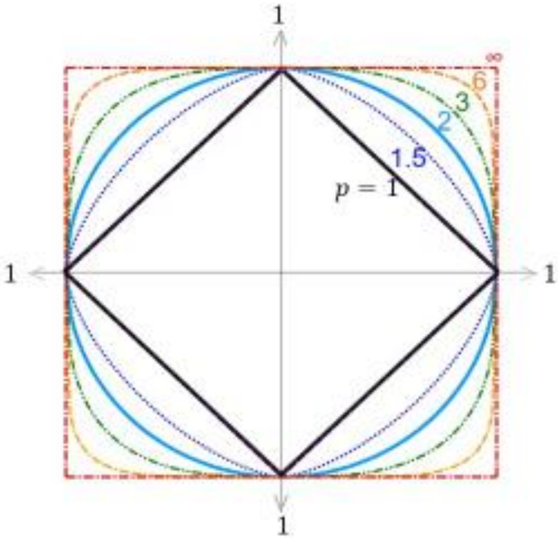

Figure 9. $\mathcal{L}^p$-intuitionistic fuzzy sets with $\aleph \in [0,1]$.

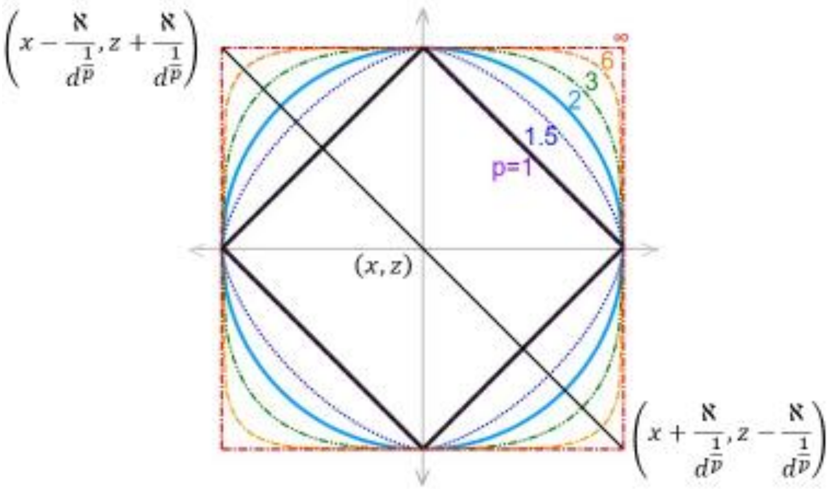

Figure 10. Optimistic and pessimistic points of $\mathcal{L}^p-IFS$ with $d = 1$.

Note that for $p \geq 1, \aleph = 2^{\frac{1}{p}} \geq 1$, Figure 9 will be reversed.

Development of $\mathcal{L}^p$-intuitionistic fuzzy sets

In this section, we discuss the procedure of calculating the $\mathcal{L}^p$-IFS in order to convert IFS to $\mathcal{L}^p$-IFS.

Assume that there are intuitionistic fuzzy pairs in an IFS $\mathcal{L}^p_{\aleph_i}$ with the following shapes: $\{(\kappa_{i,1}, s_{i,1}), (\kappa_{i,2}, s_{i,2}), (\kappa_{i,3}, s_{i,3}), \ldots\ldots\}$, where m is a numerical value of an IFS $\mathcal{L}^p_{\aleph_i}$ that contains n_i, the number of intuitionistic fuzzy pairs with $\mathcal{L}^p_{\aleph_i}$.

The following formula can be used to determine the "arithmetic average" of spherical fuzzy intuitionistic fuzzy pairs:

$$\left(\kappa\left(\mathcal{L}^p_{\aleph_i}\right), s\left(\mathcal{L}^p_{\aleph_i}\right)\right) = \left(\sum_{j=1}^{n_i} \frac{\kappa_{i,j}}{n_i}, \sum_{j=1}^{n_i} \frac{s_{i,j}}{n_i}\right).$$

The $\aleph$ of $\left(\kappa\left(\mathcal{L}^p_{\aleph_i}\right), s\left(\mathcal{L}^p_{\aleph_i}\right)\right)$ has the maximum Euclidean distance value.

$$\aleph_i = \max_{1\le j\le n_i}\left(\left(\left|\kappa\left(\mathcal{L}^p_{\aleph_i}\right) - \kappa_{i,j}\right|^p + \left|s\left(\mathcal{L}^p_{\aleph_i}\right) - s_{i,j}\right|^p\right)^{\frac{1}{p}}\right).$$

Thus, IFS is being changed into $\mathcal{L}^p$-IFS.

In order to facilitate collective decision-making, we create a mechanism for converting collections of IFSs into a $\mathcal{L}^p - IFVs$.

Proposition 1. Let a set of IFVs be denoted as

$$\left\{\mathcal{L}^p_{\aleph_1} = (\kappa_1, s_1; \aleph_1), \mathcal{L}^p_{\aleph_2} = (\kappa_2, s_2; \aleph_2), \ldots\ldots\ldots, \mathcal{L}^p_{\aleph_n} = (\kappa_n, s_n; \aleph_n)\right\}.$$

Then

$$\mathcal{L}^p_{\aleph} = \langle \kappa, s; \aleph\rangle,$$

is a $\mathcal{L}^p - IFV$ with

$$\kappa = \sum_{j=1}^{n_i} \frac{\kappa_{i,j}}{n_i} \text{ and } s = \sum_{j=1}^{n_i} \frac{s_{i,j}}{n_i},$$

$$\aleph = \max_{1\le j\le n_i}\left(\left(\left|\kappa - \kappa_{i,j}\right|^p + \left|s - s_{i,j}\right|^p\right)^{\frac{1}{p}}, 2^{\frac{1}{p}}\right).$$

Proof. Since $\kappa = \sum_{j=1}^{n_i} \frac{\kappa_{i,j}}{n_i}$ and $s = \sum_{j=1}^{n_i} \frac{s_{i,j}}{n_i}$, then we have

$$\kappa + s = \sum_{j=1}^{n_i} \frac{\kappa_{i,j}}{n_i} + \sum_{j=1}^{n_i} \frac{s_{i,j}}{n_i} = \frac{\sum_{j=1}^{n_i}(\kappa_{i,j}+s_{i,j})}{n_i} \le \frac{\sum_{j=1}^{n_i} 1}{n_i} = 1.$$

Furthermore, it is obvious that $\aleph \in \left[0, 2^{\frac{1}{p}}\right]$. Note that, for Definition 9, we have

$$\aleph = \min\left(\max_{1\le j\le n_i}\left(\left(\left|\kappa - \kappa_{i,j}\right|^p + \left|s - s_{i,j}\right|^p\right)^{\frac{1}{p}}\right), 1\right).$$

Example 1. The following sets of $IFSs$ are represented as:

$$\{(0.3, 0.7), (0.2, 0.7), (0.6, 0.2)\},$$

$$\{(0.2, 0.5), (0.3, 0.4), (0.9, 0.1)\},$$

and

$$\{(0.1, 0.6), (0.5, 0.5), (0.1, 0.8)\}.$$

With the help of Proposition 1, we find the corresponding $\mathcal{L}^p$-$IFSs$, we have:

- When $p = 1$, we have

$$(0.37, 0.53; 0.63), (0.47, 0.33; 0.93), (0.23, 0.63; 0.53);$$

- When $p = 2$, we have

$$(0.37, 0.53; 0.45), (0.47, 0.33; 0.73), (0.23, 0.63; 0.50);$$

- When $p = 5$, we have

$$(0.37, 0.53; 0.41), (0.47, 0.33; 0.71), (0.23, 0.63; 0.51);$$

- When $p = 20$, we have

$$(0.37, 0.53; 0.40), (0.47, 0.33; 0.70), (0.23, 0.63; 0.50),$$

and so on.

4. Basic operations and relations for the $\mathcal{L}^p$-intuitionistic fuzzy set

In this section, we propose some of the basic operations for $\mathcal{L}^p$-$IFSs$ like inclusion, union, intersection, complement, and some compositions. Some properties are also illustrated. For the sake of easy understanding, we take the following three $\mathcal{L}^p$-$IFSs$ over fixed universe E:

$$\mathcal{L}^p_{\aleph_1} = \{\langle \omega, \kappa_{\mathcal{L}_1}(\omega), \mathscr{s}_{\mathcal{L}_1}(\omega); \aleph_{\mathcal{L}_1}\rangle : \text{for all } \omega \in E\},$$

$$\mathcal{L}^p_{\aleph_2} = \{\langle \omega, \kappa_2(\omega), \mathscr{s}_2(\omega); \aleph_2\rangle : \text{for all } \omega \in E\},$$

$$\mathcal{L}^p_{\aleph_3} = \{\langle \omega, \kappa_3(\omega), \mathscr{s}_3(\omega); \aleph_3\rangle : \text{for all } \omega \in E\}.$$

Operations

Some basic operations between two $\mathcal{L}^p$-$IFSs$ $\mathcal{L}^p_{\aleph_1}$ and $\mathcal{L}^p_{\aleph_2}$ are as follows:

Definition 10. Let $\mathcal{L}^p_{\aleph_1}$ and $\mathcal{L}^p_{\aleph_2}$ be two $\mathcal{L}^p$-$IFSs$. Then,

i. $\neg\mathcal{L}^p_{\aleph_1} = \{\langle \omega, \mathscr{s}_1(\omega), \kappa_1(\omega); \aleph_1\rangle : \text{for all } \omega \in E\}$,
ii. $\mathcal{L}^p_{\aleph_1} \cup_{min} \mathcal{L}^p_{\aleph_2} = \{\langle \omega, max(\kappa_1(\omega), \kappa_2(\omega)), min(\mathscr{s}_1(\omega), \mathscr{s}_2(\omega)); min(\aleph_1, \aleph_2)\rangle :$ for all $\omega \in E\}$,
iii. $\mathcal{L}^p_{\aleph_1} \cup_{max} \mathcal{L}^p_{\aleph_2} = \{\langle \omega, max(\kappa_1(\omega), \kappa_2(\omega)), min(\mathscr{s}_1(\omega), \mathscr{s}_2(\omega)); max(\aleph_1, \aleph_2)\rangle :$ for all $\omega \in E\}$,

iv. $\mathcal{L}^p_{\aleph_1} \cap_{min} \mathcal{L}^p_{\aleph_2} = \{\langle \omega,\ min(\kappa_1(\omega), \kappa_2(\omega)),\ max(s_1(\omega), s_2(\omega)); min(\aleph_1, \aleph_2)\rangle :$ for all $\omega \in E\}\mathcal{L}^p_{\aleph_1} \cap_{max} \mathcal{L}^p_{\aleph_2} = \{\langle \omega,\ min(\kappa_1(\omega), \kappa_2(\omega)),$ $max(s_1(\omega), s_2(\omega)); max(\aleph_1, \aleph_2)\rangle :$ for all $\omega \in E\}$,

v. $\mathcal{L}^p_{\aleph_1} \otimes_{min} \mathcal{L}^p_{\aleph_2} = (\kappa_1(\omega) \cdot \kappa_2(\omega), s_1(\omega) + s_2(\omega) - s_1(\omega) \cdot s_2(\omega); min(\aleph_1, \aleph_2))$,

vi. $\mathcal{L}^p_{\aleph_1} \otimes_{max} \mathcal{L}^p_{\aleph_2} = (\kappa_1(\omega) \cdot \kappa_2(\omega), s_1(\omega) + s_2(\omega) - s_1(\omega) \cdot s_2(\omega); max(\aleph_1, \aleph_2))$,

vii. $\mathcal{L}^p_{\aleph_1} \oplus_{min} \mathcal{L}^p_{\aleph_2} = (\kappa_1(\omega) + \kappa_2(\omega) - \kappa_1(\omega) \cdot \kappa_2(\omega), s_1(\omega) \cdot s_2(\omega); min(\aleph_1, \aleph_2))$,

viii. $\mathcal{L}^p_{\aleph_1} \oplus_{max} \mathcal{L}^p_{\aleph_2} = (\kappa_1(\omega) + \kappa_2(\omega) - \kappa_1(\omega) \cdot \kappa_2(\omega), s_1(\omega) \cdot s_2(\omega); max(\aleph_1, \aleph_2))$,

ix. $\lambda \mathcal{L}^p_{\aleph_1} = \left(1 - (1 - \kappa_1(\omega))^{\lambda}, s_1(\omega)^{\lambda}; \aleph_1\right);\ \lambda > 0$,

x. ${\mathcal{L}^p_{\aleph_1}}^{\lambda} = ({\kappa_1}^{\lambda}, 1 - (1 - s_1)^{\lambda}; \aleph_1)$,

xi. $\mathcal{L}^p_{\aleph_1} @_{min} \mathcal{L}^p_{\aleph_2} = (\kappa_1(\omega) + \kappa_2(\omega), s_1(\omega) + s_2(\omega); min(\aleph_1, \aleph_2))$,

xii. $\mathcal{L}^p_{\aleph_1} @_{max} \mathcal{L}^p_{\aleph_2} = (\kappa_1(\omega) + \kappa_2(\omega), s_1(\omega) + s_2(\omega); max(\aleph_1, \aleph_2))$.

Relations

The relations over $\mathcal{L}^p$-$IFSs$ are first proposed as follows:

Definition 11. Let $\mathcal{L}^p_{\aleph_1}$ and $\mathcal{L}^p_{\aleph_2}$ be two $\mathcal{L}^p$-$IFSs$. Then, for all $\omega \in E$, we have

- $\mathcal{L}^p_{\aleph_1} \subset_{\upsilon} \mathcal{L}^p_{\aleph_2}$ iff $\left((\aleph_1 = \aleph_2)\ \&\ \begin{pmatrix} (\kappa_1(\omega) < \kappa_2(\omega)\ \&\ s_1(\omega) \geq s_2(\omega)) \\ \vee (\kappa_1(\omega) \leq \kappa_2(\omega)\ \&\ s_1(\omega) > s_2(\omega)) \\ \vee (\kappa_1(\omega) < \kappa_2(\omega)\ \&\ s_1(\omega) > s_2(\omega)) \end{pmatrix} \right)$;
- $\mathcal{L}^p_{\aleph_1} \subset_{\rho} \mathcal{L}^p_{\aleph_2}$ iff $((\aleph_1 < \aleph_2)\ \&\ \kappa_1(\omega) = \kappa_2(\omega)\ \&\ s_1(\omega) = s_2(\omega))$;
- $\mathcal{L}^p_{\aleph_1} \subset \mathcal{L}^p_{\aleph_2}$ iff $\left((\aleph_1 < \aleph_2)\ \&\ \begin{pmatrix} (\kappa_1(\omega) < \kappa_2(\omega)\ \&\ s_1(\omega) \geq s_2(\omega)) \\ \vee (\kappa_1(\omega) \leq \kappa_2(\omega)\ \&\ s_1(\omega) > s_2(\omega)) \\ \vee (\kappa_1(\omega) < \kappa_2(\omega)\ \&\ s_1(\omega) > s_2(\omega)) \end{pmatrix} \right)$;
- $\mathcal{L}^p_{\aleph_1} \subset_{\upsilon} \mathcal{L}^p_{\aleph_2}$ iff $\mathcal{L}^p_{\aleph_2} \supset_{\upsilon} \mathcal{L}^p_{\aleph_1}$;
- $\mathcal{L}^p_{\aleph_1} \subset_{\rho} \mathcal{L}^p_{\aleph_2}$ iff $\mathcal{L}^p_{\aleph_2} \supset_{\rho} \mathcal{L}^p_{\aleph_1}$;
- $\mathcal{L}^p_{\aleph_1} \subset \mathcal{L}^p_{\aleph_2}$ iff $\mathcal{L}^p_{\aleph_2} \supset \mathcal{L}^p_{\aleph_1}$;
- $\mathcal{L}^p_{\aleph_1} \subseteq_{\upsilon} \mathcal{L}^p_{\aleph_2}$ iff $((\aleph_1 = \aleph_2)\ \&\ \kappa_1(\omega) \leq \kappa_2(\omega)\ \&\ s_1(\omega) \geq s_2(\omega))$;
- $\mathcal{L}^p_{\aleph_1} \subseteq_{\rho} \mathcal{L}^p_{\aleph_2}$ iff $((\aleph_1 \leq \aleph_2)\ \&\ \kappa_1(\omega) = \kappa_2(\omega)\ \&\ s_1(\omega) = s_2(\omega))$;
- $\mathcal{L}^p_{\aleph_1} \subseteq \mathcal{L}^p_{\aleph_2}$ iff $((\aleph_1 \leq \aleph_2)\ \&\ \kappa_1(\omega) \leq \kappa_2(\omega)\ \&\ s_1(\omega) \geq s_2(\omega))$;
- $\mathcal{L}^p_{\aleph_1} \subseteq_{\upsilon} \mathcal{L}^p_{\aleph_2}$ iff $\mathcal{L}^p_{\aleph_2} \supseteq_{\upsilon} \mathcal{L}^p_{\aleph_1}$;
- $\mathcal{L}^p_{\aleph_1} \subseteq_{\rho} \mathcal{L}^p_{\aleph_2}$ iff $\mathcal{L}^p_{\aleph_2} \supseteq_{\rho} \mathcal{L}^p_{\aleph_1}$;
- $\mathcal{L}^p_{\aleph_1} \subseteq \mathcal{L}^p_{\aleph_2}$ iff $\mathcal{L}^p_{\aleph_2} \supseteq \mathcal{L}^p_{\aleph_1}$;
- $\mathcal{L}^p_{\aleph_1} =_{\upsilon} \mathcal{L}^p_{\aleph_2}$ iff $\kappa_1(\omega) = \kappa_2(\omega)\ \&\ s_1(\omega) = s_2(\omega)$;
- $\mathcal{L}^p_{\aleph_1} =_{\rho} \mathcal{L}^p_{\aleph_2}$ iff $\aleph_1 = \aleph_2$;
- $\mathcal{L}^p_{\aleph_1} = \mathcal{L}^p_{\aleph_2}$ iff $(\aleph_1 = \aleph_2)\&\kappa_1(\omega) = \kappa_2(\omega)\&s_1(\omega) = s_2(\omega)$.

From Definitions 10 and 11, we conclude the following results:

Proposition 2. Let $\mathcal{L}^p_{\aleph_1}$, and $\mathcal{L}^p_{\aleph_2}$ be two $\mathcal{L}^p$-$IFSs$. Then, the following properties hold, such that

- $\neg\left(\neg\mathcal{L}^p_{\aleph_1} \cup_{min} \neg\mathcal{L}^p_{\aleph_2}\right) = \mathcal{L}^p_{\aleph_1} \cap_{min} \mathcal{L}^p_{\aleph_2}$, and $\neg\left(\neg\mathcal{L}^p_{\aleph_1} \cup_{max} \neg\mathcal{L}^p_{\aleph_2}\right) = \mathcal{L}^p_{\aleph_1} \cap_{max} \mathcal{L}^p_{\aleph_2}$,
- $\neg\left(\neg\mathcal{L}^p_{\aleph_1} \cap_{min} \neg\mathcal{L}^p_{\aleph_2}\right) = \mathcal{L}^p_{\aleph_1} \cup_{min} \mathcal{L}^p_{\aleph_2}$, and $\neg\left(\neg\mathcal{L}^p_{\aleph_1} \cap_{max} \neg\mathcal{L}^p_{\aleph_2}\right) = \mathcal{L}^p_{\aleph_1} \cup_{max} \mathcal{L}^p_{\aleph_2}$,
- $\neg\left(\neg\mathcal{L}^p_{\aleph_1} \oplus_{min} \neg\mathcal{L}^p_{\aleph_2}\right) = \mathcal{L}^p_{\aleph_1} \otimes_{min} \mathcal{L}^p_{\aleph_2}$, and $\neg\left(\neg\mathcal{L}^p_{\aleph_1} \oplus_{max} \neg\mathcal{L}^p_{\aleph_2}\right) = \mathcal{L}^p_{\aleph_1} \otimes_{max} \mathcal{L}^p_{\aleph_2}$,
- $\neg\left(\neg\mathcal{L}^p_{\aleph_1} \otimes_{min} \neg\mathcal{L}^p_{\aleph_2}\right) = \mathcal{L}^p_{\aleph_1} \oplus_{min} \mathcal{L}^p_{\aleph_2}$, and $\neg\left(\neg\mathcal{L}^p_{\aleph_1} \otimes_{max} \neg\mathcal{L}^p_{\aleph_2}\right) = \mathcal{L}^p_{\aleph_1} \oplus_{max} \mathcal{L}^p_{\aleph_2}$,
- $\neg\left(\neg\mathcal{L}^p_{\aleph_1} @_{min} \neg\mathcal{L}^p_{\aleph_2}\right) = \mathcal{L}^p_{\aleph_1} @_{min} \mathcal{L}^p_{\aleph_2}$, and $\neg\left(\neg\mathcal{L}^p_{\aleph_1} @_{max} \neg\mathcal{L}^p_{\aleph_2}\right) = \mathcal{L}^p_{\aleph_1} @_{max} \mathcal{L}^p_{\aleph_2}$.

Proof. The proof follows similar steps as those used in the operations of $IFSs$, and as such, is excluded for brevity.

Proposition 3. Let $\mathcal{L}^p_{\aleph_1}$, $\mathcal{L}^p_{\aleph_2}$ and $\mathcal{L}^p_{\aleph_3}$ be three $\mathcal{L}^p$-$IFSs$. Then, the following properties hold, such that

1) $\mathcal{L}^p_{\aleph_1} \cup_{min} \mathcal{L}^p_{\aleph_1} = \mathcal{L}^p_{\aleph_1}$, and $\mathcal{L}^p_{\aleph_1} \cup_{max} \mathcal{L}^p_{\aleph_1} = \mathcal{L}^p_{\aleph_1}$,
2) $\mathcal{L}^p_{\aleph_1} \cap_{min} \mathcal{L}^p_{\aleph_1} = \mathcal{L}^p_{\aleph_1}$, and $\mathcal{L}^p_{\aleph_1} \cap_{max} \mathcal{L}^p_{\aleph_1} = \mathcal{L}^p_{\aleph_1}$,
3) $\mathcal{L}^p_{\aleph_1} @_{min} \mathcal{L}^p_{\aleph_1} = \mathcal{L}^p_{\aleph_1}$, and $\mathcal{L}^p_{\aleph_1} @_{max} \mathcal{L}^p_{\aleph_1} = \mathcal{L}^p_{\aleph_1}$,
4) $\mathcal{L}^p_{\aleph_1} \cup_{min} \mathcal{L}^p_{\aleph_2} = \mathcal{L}^p_{\aleph_2} \cup_{min} \mathcal{L}^p_{\aleph_1}$, and $\mathcal{L}^p_{\aleph_1} \cup_{max} \mathcal{L}^p_{\aleph_2} = \mathcal{L}^p_{\aleph_2} \cup_{max} \mathcal{L}^p_{\aleph_1}$,
5) $\mathcal{L}^p_{\aleph_1} \cap_{min} \mathcal{L}^p_{\aleph_2} = \mathcal{L}^p_{\aleph_2} \cap_{min} \mathcal{L}^p_{\aleph_1}$, and $\mathcal{L}^p_{\aleph_1} \cap_{max} \mathcal{L}^p_{\aleph_2} = \mathcal{L}^p_{\aleph_2} \cap_{max} \mathcal{L}^p_{\aleph_1}$,
6) $\mathcal{L}^p_{\aleph_1} \otimes_{min} \mathcal{L}^p_{\aleph_2} = \mathcal{L}^p_{\aleph_2} \otimes_{min} \mathcal{L}^p_{\aleph_1}$, and $\mathcal{L}^p_{\aleph_1} \otimes_{max} \mathcal{L}^p_{\aleph_2} = \mathcal{L}^p_{\aleph_2} \otimes_{max} \mathcal{L}^p_{\aleph_1}$,
7) $\mathcal{L}^p_{\aleph_1} \oplus_{min} \mathcal{L}^p_{\aleph_2} = \mathcal{L}^p_{\aleph_2} \oplus_{min} \mathcal{L}^p_{\aleph_1}$, and $\mathcal{L}^p_{\aleph_1} \oplus_{max} \mathcal{L}^p_{\aleph_2} = \mathcal{L}^p_{\aleph_2} \oplus_{max} \mathcal{L}^p_{\aleph_1}$,
8) $\mathcal{L}^p_{\aleph_1} @_{min} \mathcal{L}^p_{\aleph_1} = \mathcal{L}^p_{\aleph_2} @_{min} \mathcal{L}^p_{\aleph_1}$, and $\mathcal{L}^p_{\aleph_1} @_{max} \mathcal{L}^p_{\aleph_2} = \mathcal{L}^p_{\aleph_2} @_{max} \mathcal{L}^p_{\aleph_1}$,
9) $\mathcal{L}^p_{\aleph_1} \cup_{min} \left(\mathcal{L}^p_{\aleph_2} \cup_{min} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} \cup_{min} \mathcal{L}^p_{\aleph_1}\right) \cup_{min} \mathcal{L}^p_{\aleph_3}$ and $\mathcal{L}^p_{\aleph_1} \cup_{max} \left(\mathcal{L}^p_{\aleph_2} \cup_{max} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} \cup_{max} \mathcal{L}^p_{\aleph_2}\right) \cup_{max} \mathcal{L}^p_{\aleph_3}$,
10) $\mathcal{L}^p_{\aleph_1} \cap_{min} \left(\mathcal{L}^p_{\aleph_2} \cap_{min} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} \cap_{min} \mathcal{L}^p_{\aleph_2}\right) \cap_{min} \mathcal{L}^p_{\aleph_3}$ and $\mathcal{L}^p_{\aleph_1} \cap_{max} \left(\mathcal{L}^p_{\aleph_2} \cap_{max} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} \cap_{max} \mathcal{L}^p_{\aleph_2}\right) \cap_{max} \mathcal{L}^p_{\aleph_3}$,
11) $\mathcal{L}^p_{\aleph_1} \otimes_{min} \left(\mathcal{L}^p_{\aleph_2} \otimes_{min} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} \otimes_{min} \mathcal{L}^p_{\aleph_2}\right) \otimes_{min} \mathcal{L}^p_{\aleph_3}$ and $\mathcal{L}^p_{\aleph_1} \otimes_{max} \left(\mathcal{L}^p_{\aleph_2} \otimes_{max} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} \otimes_{max} \mathcal{L}^p_{\aleph_2}\right) \otimes_{max} \mathcal{L}^p_{\aleph_3}$,
12) $\mathcal{L}^p_{\aleph_1} \oplus_{min} \left(\mathcal{L}^p_{\aleph_2} \oplus_{min} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} \oplus_{min} \mathcal{L}^p_{\aleph_2}\right) \oplus_{min} \mathcal{L}^p_{\aleph_3}$ and $\mathcal{L}^p_{\aleph_1} \oplus_{max} \left(\mathcal{L}^p_{\aleph_2} \oplus_{max} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} \oplus_{max} \mathcal{L}^p_{\aleph_2}\right) \oplus_{max} \mathcal{L}^p_{\aleph_3}$,
13) $\mathcal{L}^p_{\aleph_1} \cap_{min} \left(\mathcal{L}^p_{\aleph_2} \cup_{min} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} \cap_{min} \mathcal{L}^p_{\aleph_2}\right) \cup_{min} \left(\mathcal{L}^p_{\aleph_1} \cap_{min} \mathcal{L}^p_{\aleph_3}\right)$ and $\mathcal{L}^p_{\aleph_1} \cap_{max} \left(\mathcal{L}^p_{\aleph_2} \cup_{min} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} \cap_{max} \mathcal{L}^p_{\aleph_2}\right) \cup_{min} \left(\mathcal{L}^p_{\aleph_1} \cap_{max} \mathcal{L}^p_{\aleph_3}\right)$,
14) $\mathcal{L}^p_{\aleph_1} \cap_{min} \left(\mathcal{L}^p_{\aleph_2} \cup_{max} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} \cap_{min} \mathcal{L}^p_{\aleph_2}\right) \cup_{max} \left(\mathcal{L}^p_{\aleph_1} \cap_{min} \mathcal{L}^p_{\aleph_3}\right)$ and $\mathcal{L}^p_{\aleph_1} \cap_{max} \left(\mathcal{L}^p_{\aleph_2} \cup_{max} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} \cap_{max} \mathcal{L}^p_{\aleph_2}\right) \cup_{max} \left(\mathcal{L}^p_{\aleph_1} \cap_{max} \mathcal{L}^p_{\aleph_3}\right)$,
15) $\mathcal{L}^p_{\aleph_1} \oplus_{min} \left(\mathcal{L}^p_{\aleph_2} \cup_{min} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} \oplus_{min} \mathcal{L}^p_{\aleph_2}\right) \cup_{min} \left(\mathcal{L}^p_{\aleph_1} \oplus_{min} \mathcal{L}^p_{\aleph_3}\right)$ and $\mathcal{L}^p_{\aleph_1} \oplus_{max} \left(\mathcal{L}^p_{\aleph_2} \cup_{min} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} \oplus_{max} \mathcal{L}^p_{\aleph_2}\right) \cup_{min} \left(\mathcal{L}^p_{\aleph_1} \oplus_{max} \mathcal{L}^p_{\aleph_3}\right)$,
16) $\mathcal{L}^p_{\aleph_1} \oplus_{min} \left(\mathcal{L}^p_{\aleph_2} \cup_{max} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} \oplus_{min} \mathcal{L}^p_{\aleph_2}\right) \cup_{max} \left(\mathcal{L}^p_{\aleph_1} \oplus_{min} \mathcal{L}^p_{\aleph_3}\right)$ and $\mathcal{L}^p_{\aleph_1} \oplus_{max} \left(\mathcal{L}^p_{\aleph_2} \cup_{max} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} \oplus_{max} \mathcal{L}^p_{\aleph_2}\right) \cup_{max} \left(\mathcal{L}^p_{\aleph_1} \oplus_{max} \mathcal{L}^p_{\aleph_3}\right)$,

17) $\mathcal{L}^p_{\aleph_1} \otimes_{min} \left(\mathcal{L}^p_{\aleph_2} \cup_{min} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} \otimes_{min} \mathcal{L}^p_{\aleph_2}\right) \cup_{min} \left(\mathcal{L}^p_{\aleph_1} \otimes_{min} \mathcal{L}^p_{\aleph_3}\right)$ and
$\mathcal{L}^p_{\aleph_1} \otimes_{max} \left(\mathcal{L}^p_{\aleph_2} \cup_{min} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} \otimes_{max} \mathcal{L}^p_{\aleph_2}\right) \cup_{min} \left(\mathcal{L}^p_{\aleph_1} \otimes_{max} \mathcal{L}^p_{\aleph_3}\right)$,
18) $\mathcal{L}^p_{\aleph_1} \otimes_{min} \left(\mathcal{L}^p_{\aleph_2} \cup_{max} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} \otimes_{min} \mathcal{L}^p_{\aleph_2}\right) \cup_{max} \left(\mathcal{L}^p_{\aleph_1} \otimes_{min} \mathcal{L}^p_{\aleph_3}\right)$ and
$\mathcal{L}^p_{\aleph_1} \otimes_{max} \left(\mathcal{L}^p_{\aleph_2} \cup_{max} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} \otimes_{max} \mathcal{L}^p_{\aleph_2}\right) \cup_{max} \left(\mathcal{L}^p_{\aleph_1} \otimes_{max} \mathcal{L}^p_{\aleph_3}\right)$,
19) $\mathcal{L}^p_{\aleph_1} @_{min} \left(\mathcal{L}^p_{\aleph_2} \cup_{min} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} @_{min} \mathcal{L}^p_{\aleph_2}\right) \cup_{min} \left(\mathcal{L}^p_{\aleph_1} @_{min} \mathcal{L}^p_{\aleph_3}\right)$ and
$\mathcal{L}^p_{\aleph_1} @_{max} \left(\mathcal{L}^p_{\aleph_2} \cup_{min} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} @_{max} \mathcal{L}^p_{\aleph_2}\right) \cup_{min} \left(\mathcal{L}^p_{\aleph_1} @_{max} \mathcal{L}^p_{\aleph_3}\right)$,
20) $\mathcal{L}^p_{\aleph_1} @_{min} \left(\mathcal{L}^p_{\aleph_2} \cup_{max} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} @_{min} \mathcal{L}^p_{\aleph_2}\right) \cup_{max} \left(\mathcal{L}^p_{\aleph_1} @_{min} \mathcal{L}^p_{\aleph_3}\right)$ and
$\mathcal{L}^p_{\aleph_1} @_{max} \left(\mathcal{L}^p_{\aleph_2} \cup_{max} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} @_{max} \mathcal{L}^p_{\aleph_2}\right) \cup_{max} \left(\mathcal{L}^p_{\aleph_1} @_{max} \mathcal{L}^p_{\aleph_3}\right)$,
21) $\mathcal{L}^p_{\aleph_1} \cup_{min} \left(\mathcal{L}^p_{\aleph_2} \cap_{min} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} \cup_{min} \mathcal{L}^p_{\aleph_2}\right) \cap_{min} \left(\mathcal{L}^p_{\aleph_1} \cup_{min} \mathcal{L}^p_{\aleph_3}\right)$ and
$\mathcal{L}^p_{\aleph_1} \cup_{max} \left(\mathcal{L}^p_{\aleph_2} \cap_{min} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} \cup_{max} \mathcal{L}^p_{\aleph_2}\right) \cap_{min} \left(\mathcal{L}^p_{\aleph_1} \cup_{max} \mathcal{L}^p_{\aleph_3}\right)$,
22) $\mathcal{L}^p_{\aleph_1} \cup_{min} \left(\mathcal{L}^p_{\aleph_2} \cap_{max} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} \cup_{min} \mathcal{L}^p_{\aleph_2}\right) \cap_{max} \left(\mathcal{L}^p_{\aleph_1} \cup_{min} \mathcal{L}^p_{\aleph_3}\right)$ and
$\mathcal{L}^p_{\aleph_1} \cup_{max} \left(\mathcal{L}^p_{\aleph_2} \cap_{max} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} \cup_{max} \mathcal{L}^p_{\aleph_2}\right) \cap_{max} \left(\mathcal{L}^p_{\aleph_1} \cup_{max} \mathcal{L}^p_{\aleph_3}\right)$,
23) $\mathcal{L}^p_{\aleph_1} \oplus_{min} \left(\mathcal{L}^p_{\aleph_2} \cap_{min} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} \oplus_{min} \mathcal{L}^p_{\aleph_2}\right) \cap_{min} \left(\mathcal{L}^p_{\aleph_1} \oplus_{min} \mathcal{L}^p_{\aleph_3}\right)$ and
$\mathcal{L}^p_{\aleph_1} \oplus_{max} \left(\mathcal{L}^p_{\aleph_2} \cap_{min} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} \oplus_{max} \mathcal{L}^p_{\aleph_2}\right) \cap_{min} \left(\mathcal{L}^p_{\aleph_1} \oplus_{max} \mathcal{L}^p_{\aleph_3}\right)$,
24) $\mathcal{L}^p_{\aleph_1} \oplus_{min} \left(\mathcal{L}^p_{\aleph_2} \cap_{max} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} \oplus_{min} \mathcal{L}^p_{\aleph_2}\right) \cap_{max} \left(\mathcal{L}^p_{\aleph_1} \oplus_{min} \mathcal{L}^p_{\aleph_3}\right)$ and
$\mathcal{L}^p_{\aleph_1} \oplus_{max} \left(\mathcal{L}^p_{\aleph_2} \cap_{max} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} \oplus_{max} \mathcal{L}^p_{\aleph_2}\right) \cap_{max} \left(\mathcal{L}^p_{\aleph_1} \oplus_{max} \mathcal{L}^p_{\aleph_3}\right)$,
25) $\mathcal{L}^p_{\aleph_1} \otimes_{min} \left(\mathcal{L}^p_{\aleph_2} \cap_{min} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} \otimes_{min} \mathcal{L}^p_{\aleph_2}\right) \cap_{min} \left(\mathcal{L}^p_{\aleph_1} \otimes_{min} \mathcal{L}^p_{\aleph_3}\right)$ and
$\mathcal{L}^p_{\aleph_1} \otimes_{max} \left(\mathcal{L}^p_{\aleph_2} \cap_{min} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} \otimes_{max} \mathcal{L}^p_{\aleph_2}\right) \cap_{min} \left(\mathcal{L}^p_{\aleph_1} \otimes_{max} \mathcal{L}^p_{\aleph_3}\right)$,
26) $\mathcal{L}^p_{\aleph_1} \otimes_{min} \left(\mathcal{L}^p_{\aleph_2} \cap_{max} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} \otimes_{min} \mathcal{L}^p_{\aleph_2}\right) \cap_{max} \left(\mathcal{L}^p_{\aleph_1} \otimes_{min} \mathcal{L}^p_{\aleph_3}\right)$ and
$\mathcal{L}^p_{\aleph_1} \otimes_{max} \left(\mathcal{L}^p_{\aleph_2} \cap_{max} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} \otimes_{max} \mathcal{L}^p_{\aleph_2}\right) \cap_{max} \left(\mathcal{L}^p_{\aleph_1} \otimes_{max} \mathcal{L}^p_{\aleph_3}\right)$,
27) $\mathcal{L}^p_{\aleph_1} @_{min} \left(\mathcal{L}^p_{\aleph_2} \cap_{min} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} @_{min} \mathcal{L}^p_{\aleph_2}\right) \cap_{min} \left(\mathcal{L}^p_{\aleph_1} @_{min} \mathcal{L}^p_{\aleph_3}\right)$ and
$\mathcal{L}^p_{\aleph_1} @_{max} \left(\mathcal{L}^p_{\aleph_2} \cap_{min} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} @_{max} \mathcal{L}^p_{\aleph_2}\right) \cap_{min} \left(\mathcal{L}^p_{\aleph_1} @_{max} \mathcal{L}^p_{\aleph_3}\right)$,
28) $\mathcal{L}^p_{\aleph_1} @_{min} \left(\mathcal{L}^p_{\aleph_2} \cap_{max} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} @_{min} \mathcal{L}^p_{\aleph_2}\right) \cap_{max} \left(\mathcal{L}^p_{\aleph_1} @_{min} \mathcal{L}^p_{\aleph_3}\right)$ and
$\mathcal{L}^p_{\aleph_1} @_{max} \left(\mathcal{L}^p_{\aleph_2} \cap_{max} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} @_{max} \mathcal{L}^p_{\aleph_2}\right) \cap_{max} \left(\mathcal{L}^p_{\aleph_1} @_{max} \mathcal{L}^p_{\aleph_3}\right)$.

Proof. The proof follows a process similar to that used in the operations of $IFSs$ and is therefore omitted for the sake of brevity.

Proposition 4. Let $\mathcal{L}^p_{\aleph_1}$, $\mathcal{L}^p_{\aleph_2}$, and $\mathcal{L}^p_{\aleph_3}$ be three $\mathcal{L}^p$-$IFSs$. Then, the following properties hold, such that

a) $\mathcal{L}^p_{\aleph_1} \otimes_{min} \left(\mathcal{L}^p_{\aleph_2} \oplus_{min} \mathcal{L}^p_{\aleph_3}\right) \subset \left(\mathcal{L}^p_{\aleph_1} \otimes_{min} \mathcal{L}^p_{\aleph_2}\right) \oplus_{min} \left(\mathcal{L}^p_{\aleph_1} \otimes_{min} \mathcal{L}^p_{\aleph_3}\right)$ and
$\mathcal{L}^p_{\aleph_1} \otimes_{max} \left(\mathcal{L}^p_{\aleph_2} \oplus_{min} \mathcal{L}^p_{\aleph_3}\right) \subset \left(\mathcal{L}^p_{\aleph_1} \otimes_{max} \mathcal{L}^p_{\aleph_2}\right) \oplus_{min} \left(\mathcal{L}^p_{\aleph_1} \otimes_{max} \mathcal{L}^p_{\aleph_3}\right)$,

b) $\mathcal{L}^p_{\aleph_1} \otimes_{min} \left(\mathcal{L}^p_{\aleph_2} \oplus_{max} \mathcal{L}^p_{\aleph_3}\right) \subset \left(\mathcal{L}^p_{\aleph_1} \otimes_{min} \mathcal{L}^p_{\aleph_2}\right) \oplus_{max} \left(\mathcal{L}^p_{\aleph_1} \otimes_{min} \mathcal{L}^p_{\aleph_3}\right)$ and
$\mathcal{L}^p_{\aleph_1} \otimes_{max} \left(\mathcal{L}^p_{\aleph_2} \oplus_{max} \mathcal{L}^p_{\aleph_3}\right) \subset \left(\mathcal{L}^p_{\aleph_1} \otimes_{max} \mathcal{L}^p_{\aleph_2}\right) \oplus_{max} \left(\mathcal{L}^p_{\aleph_1} \otimes_{max} \mathcal{L}^p_{\aleph_3}\right)$,

c) $\mathcal{L}^p_{\aleph_1} @_{min} \left(\mathcal{L}^p_{\aleph_2} \oplus_{min} \mathcal{L}^p_{\aleph_3}\right) \subset \left(\mathcal{L}^p_{\aleph_1} @_{min} \mathcal{L}^p_{\aleph_2}\right) \oplus_{min} \left(\mathcal{L}^p_{\aleph_1} @_{min} \mathcal{L}^p_{\aleph_3}\right)$ and
$\mathcal{L}^p_{\aleph_1} @_{max} \left(\mathcal{L}^p_{\aleph_2} \oplus_{min} \mathcal{L}^p_{\aleph_3}\right) \subset \left(\mathcal{L}^p_{\aleph_1} @_{max} \mathcal{L}^p_{\aleph_2}\right) \oplus_{min} \left(\mathcal{L}^p_{\aleph_1} @_{max} \mathcal{L}^p_{\aleph_3}\right)$,

d) $\mathcal{L}^p_{\aleph_1} @_{min} \left(\mathcal{L}^p_{\aleph_2} \oplus_{max} \mathcal{L}^p_{\aleph_3}\right) \subset \left(\mathcal{L}^p_{\aleph_1} @_{min} \mathcal{L}^p_{\aleph_2}\right) \oplus_{max} \left(\mathcal{L}^p_{\aleph_1} @_{min} \mathcal{L}^p_{\aleph_3}\right)$ and
$\mathcal{L}^p_{\aleph_1} @_{max} \left(\mathcal{L}^p_{\aleph_2} \oplus_{max} \mathcal{L}^p_{\aleph_3}\right) \subset \left(\mathcal{L}^p_{\aleph_1} @_{max} \mathcal{L}^p_{\aleph_2}\right) \oplus_{max} \left(\mathcal{L}^p_{\aleph_1} @_{max} \mathcal{L}^p_{\aleph_3}\right)$.

e) $\mathcal{L}^p_{\aleph_1} \oplus_{min} \left(\mathcal{L}^p_{\aleph_2} \otimes_{min} \mathcal{L}^p_{\aleph_3}\right) \supset \left(\mathcal{L}^p_{\aleph_1} \oplus_{min} \mathcal{L}^p_{\aleph_2}\right) \otimes_{min} \left(\mathcal{L}^p_{\aleph_1} \oplus_{min} \mathcal{L}^p_{\aleph_3}\right)$ and
$\mathcal{L}^p_{\aleph_1} \oplus_{max} \left(\mathcal{L}^p_{\aleph_2} \otimes_{min} \mathcal{L}^p_{\aleph_3}\right) \supset \left(\mathcal{L}^p_{\aleph_1} \oplus_{max} \mathcal{L}^p_{\aleph_2}\right) \otimes_{min} \left(\mathcal{L}^p_{\aleph_1} \oplus_{max} \mathcal{L}^p_{\aleph_3}\right)$,

f) $\mathcal{L}^p_{\aleph_1} \oplus_{min} \left(\mathcal{L}^p_{\aleph_2} \otimes_{max} \mathcal{L}^p_{\aleph_3}\right) \supset \left(\mathcal{L}^p_{\aleph_1} \oplus_{min} \mathcal{L}^p_{\aleph_2}\right) \otimes_{max} \left(\mathcal{L}^p_{\aleph_1} \oplus_{min} \mathcal{L}^p_{\aleph_3}\right)$ and
$\mathcal{L}^p_{\aleph_1} \oplus_{max} \left(\mathcal{L}^p_{\aleph_2} \otimes_{max} \mathcal{L}^p_{\aleph_3}\right) \supset \left(\mathcal{L}^p_{\aleph_1} \oplus_{max} \mathcal{L}^p_{\aleph_2}\right) \otimes_{max} \left(\mathcal{L}^p_{\aleph_1} \oplus_{max} \mathcal{L}^p_{\aleph_3}\right)$,

g) $\mathcal{L}^p_{\aleph_1} @_{min} \left(\mathcal{L}^p_{\aleph_2} \otimes_{min} \mathcal{L}^p_{\aleph_3}\right) \supset \left(\mathcal{L}^p_{\aleph_1} @_{min} \mathcal{L}^p_{\aleph_2}\right) \otimes_{min} \left(\mathcal{L}^p_{\aleph_1} @_{min} \mathcal{L}^p_{\aleph_3}\right)$ and
$\mathcal{L}^p_{\aleph_1} @_{max} \left(\mathcal{L}^p_{\aleph_2} \otimes_{min} \mathcal{L}^p_{\aleph_3}\right) \supset \left(\mathcal{L}^p_{\aleph_1} @_{max} \mathcal{L}^p_{\aleph_2}\right) \otimes_{min} \left(\mathcal{L}^p_{\aleph_1} @_{max} \mathcal{L}^p_{\aleph_3}\right)$,

h) $\mathcal{L}^p_{\aleph_1} @_{min} \left(\mathcal{L}^p_{\aleph_2} \otimes_{max} \mathcal{L}^p_{\aleph_3}\right) \supset \left(\mathcal{L}^p_{\aleph_1} @_{min} \mathcal{L}^p_{\aleph_2}\right) \otimes_{max} \left(\mathcal{L}^p_{\aleph_1} @_{min} \mathcal{L}^p_{\aleph_3}\right)$ and
$\mathcal{L}^p_{\aleph_1} @_{max} \left(\mathcal{L}^p_{\aleph_2} \otimes_{max} \mathcal{L}^p_{\aleph_3}\right) \supset \left(\mathcal{L}^p_{\aleph_1} @_{max} \mathcal{L}^p_{\aleph_2}\right) \otimes_{max} \left(\mathcal{L}^p_{\aleph_1} @_{max} \mathcal{L}^p_{\aleph_3}\right)$.

i) $\mathcal{L}^p_{\aleph_1} \oplus_{min} \left(\mathcal{L}^p_{\aleph_2} @_{min} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} \oplus_{min} \mathcal{L}^p_{\aleph_2}\right) @_{min} \left(\mathcal{L}^p_{\aleph_1} \oplus_{min} \mathcal{L}^p_{\aleph_3}\right)$ and
$\mathcal{L}^p_{\aleph_1} \oplus_{max} \left(\mathcal{L}^p_{\aleph_2} @_{min} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} \oplus_{max} \mathcal{L}^p_{\aleph_2}\right) @_{min} \left(\mathcal{L}^p_{\aleph_1} \oplus_{max} \mathcal{L}^p_{\aleph_3}\right)$,

j) $\mathcal{L}^p_{\aleph_1} \oplus_{min} \left(\mathcal{L}^p_{\aleph_2} @_{max} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} \oplus_{min} \mathcal{L}^p_{\aleph_2}\right) @_{max} \left(\mathcal{L}^p_{\aleph_1} \oplus_{min} \mathcal{L}^p_{\aleph_3}\right)$ and
$\mathcal{L}^p_{\aleph_1} \oplus_{max} \left(\mathcal{L}^p_{\aleph_2} @_{max} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} \oplus_{max} \mathcal{L}^p_{\aleph_2}\right) @_{max} \left(\mathcal{L}^p_{\aleph_1} \oplus_{max} \mathcal{L}^p_{\aleph_3}\right)$,

k) $\mathcal{L}^p_{\aleph_1} \otimes_{min} \left(\mathcal{L}^p_{\aleph_2} @_{min} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} \otimes_{min} \mathcal{L}^p_{\aleph_2}\right) @_{min} \left(\mathcal{L}^p_{\aleph_1} \otimes_{min} \mathcal{L}^p_{\aleph_3}\right)$ and
$\mathcal{L}^p_{\aleph_1} \otimes_{max} \left(\mathcal{L}^p_{\aleph_2} @_{min} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} \otimes_{max} \mathcal{L}^p_{\aleph_2}\right) @_{min} \left(\mathcal{L}^p_{\aleph_1} \otimes_{max} \mathcal{L}^p_{\aleph_3}\right)$,

l) $\mathcal{L}^p_{\aleph_1} \otimes_{min} \left(\mathcal{L}^p_{\aleph_2} @_{max} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} \otimes_{min} \mathcal{L}^p_{\aleph_2}\right) @_{max} \left(\mathcal{L}^p_{\aleph_1} \otimes_{min} \mathcal{L}^p_{\aleph_3}\right)$ and
$\mathcal{L}^p_{\aleph_1} \otimes_{max} \left(\mathcal{L}^p_{\aleph_2} @_{max} \mathcal{L}^p_{\aleph_3}\right) = \left(\mathcal{L}^p_{\aleph_1} \otimes_{max} \mathcal{L}^p_{\aleph_2}\right) @_{max} \left(\mathcal{L}^p_{\aleph_1} \otimes_{max} \mathcal{L}^p_{\aleph_3}\right)$,

m) $\mathcal{L}^p_{\aleph_1} \otimes_{min} \mathcal{L}^p_{\aleph_2} \subseteq_v \mathcal{L}^p_{\aleph_1} \cap_{min} \mathcal{L}^p_{\aleph_2} \subseteq_v \mathcal{L}^p_{\aleph_1} @_{min} \mathcal{L}^p_{\aleph_2} \subseteq_v \mathcal{L}^p_{\aleph_1} \cup_{min} \mathcal{L}^p_{\aleph_2} \subseteq_v \mathcal{L}^p_{\aleph_1} \oplus_{min} \mathcal{L}^p_{\aleph_2}$,

n) $\mathcal{L}^p_{\aleph_1} \otimes_{max} \mathcal{L}^p_{\aleph_2} \subseteq_v \mathcal{L}^p_{\aleph_1} \cap_{max} \mathcal{L}^p_{\aleph_2} \subseteq_v \mathcal{L}^p_{\aleph_1} @_{max} \mathcal{L}^p_{\aleph_2} \subseteq_v \mathcal{L}^p_{\aleph_1} \cup_{max} \mathcal{L}^p_{\aleph_2} \subseteq_v \mathcal{L}^p_{\aleph_1} \oplus_{max} \mathcal{L}^p_{\aleph_2}$.

Proof. This proof also follows similar steps as those used in the operations of $IFSs$, and as such, is excluded for brevity.

Remark 2. If we take $\aleph_1 = 0 = \aleph_2 = \aleph_3$, then all operations and relations reduce for $\mathcal{L}^p$-$IFSs$ as well as $IFSs$.

$\mathcal{L}^p$-intuitionistic fuzzy model operators

In this subsection, some of the new model operators are introduced using the intuitionistic fuzzy approach and similar to logic operators' "necessity" and "possibility". Moreover, some extensions are obtained with the help of some parameters. First, we start with these two operators, such that:

Definition 12. Let $\mathcal{L}_{\aleph}^{p}$ be a $\mathcal{L}^{p}$-IFS. Then, we have

- $\mathcal{L}_{\aleph}^{p} = \{\langle \omega, \kappa(\omega), 1-\kappa(\omega); \aleph \rangle | \omega \in E\} = \{\langle \omega, \aleph^{1}(\kappa(\omega), 1-\kappa(\omega))\rangle | \omega \in E\}$.
- $\mathcal{L}_{\aleph}^{p} = \{\langle \omega, 1-s(\omega), s(\omega); \aleph \rangle | \omega \in E\} = \{\langle \omega, \aleph^{1}(1-s(\omega), s(\omega))\rangle | \omega \in E\}$.

Let $\omega, \gamma \in [0,1]$ be fixed numbers. Then, the following are the extensions of $\mathcal{L}^{p}$-intuitionistic fuzzy model operators:

$$
\begin{aligned}
D_{\omega}(\mathcal{L}_{\aleph}^{p}) &= \{\langle \omega, \kappa(\omega)+\omega\pi(\omega), s(\omega)+(1-\omega)\pi(\omega); \aleph \rangle | \omega \in E\}\\
&= \{\langle \omega, \aleph^{1}(\kappa(\omega)+\omega\pi(\omega), s(\omega)+(1-\omega)\pi(\omega))\rangle | \omega \in E\},\\
F_{\omega,\gamma}(\mathcal{L}_{\aleph}^{p}) &= \{\langle \omega, \kappa(\omega)+\omega\pi(\omega), s(\omega)+\beta\pi(\omega); \aleph \rangle | \omega \in E\}\\
&= \{\langle \omega, \aleph^{1}(\kappa(\omega)+\omega\pi(\omega), s(\omega)+\beta\pi(\omega))\rangle | \omega \in E\},\\
G_{\omega,\gamma}(\mathcal{L}_{\aleph}^{p}) &= \{\langle \omega, \omega\kappa(\omega), \beta s(\omega); \aleph \rangle | \omega \in E\}\\
&= \{\langle \omega, \aleph^{1}(\omega\kappa(\omega), \beta s(\omega))\rangle | \omega \in E\},\\
H_{\omega,\gamma}(\mathcal{L}_{\aleph}^{p}) &= \{\langle \omega, \omega\kappa(\omega), s(\omega)+\beta\pi(\omega); \aleph \rangle | \omega \in E\}\\
&= \{\langle \omega, \aleph^{1}(\omega\kappa(\omega), s(\omega)+\beta\pi(\omega))\rangle | \omega \in E\},\\
H^{*}{}_{\omega,\gamma}(\mathcal{L}_{\aleph}^{p}) &= \{\langle \omega, \omega\kappa(\omega), s(\omega)+\beta(1-\omega\kappa(\omega)-s(\omega)); \aleph \rangle | \omega \in E\}\\
&= \{\langle \omega, \aleph^{1}\left(\omega\kappa(\omega), s(\omega)+\beta(1-\omega\kappa(\omega)-s(\omega))\right)\rangle | \omega \in E\},\\
\overline{H}_{\omega,\gamma}(\mathcal{L}_{\aleph}^{p}) &= \{\langle \omega, \omega\kappa(\omega), s(\omega)+\beta-\beta s(\omega); \aleph \rangle | \omega \in E\}\\
&= \{\langle \omega, \aleph^{1}(\omega\kappa(\omega), s(\omega)+\beta-\beta s(\omega))\rangle | \omega \in E\},\\
J_{\omega,\gamma}(\mathcal{L}_{\aleph}^{p}) &= \{\langle \omega, \kappa(\omega)+\omega\pi(\omega), \beta s(\omega); \aleph \rangle | \omega \in E\}\\
&= \{\langle \omega, \aleph^{1}(\kappa(\omega)+\omega\pi(\omega), \beta s(\omega))\rangle | \omega \in E\},\\
J^{*}{}_{\omega,\gamma}(\mathcal{L}_{\aleph}^{p}) &= \{\langle \omega, \frac{1}{2}\left(\kappa(\omega)+\omega(1-\kappa(\omega)-\beta s(\omega))\right), \beta s(\omega); \aleph \rangle | \omega \in E\}\\
&= \{\langle \omega, \aleph^{1}\left(\frac{1}{2}\left(\kappa(\omega)+\omega(1-\kappa(\omega)-\beta s(\omega))\right), \beta s(\omega)\right)\rangle | \omega \in E\},\\
\bar{J}_{\omega,\gamma}(\mathcal{L}_{\aleph}^{p}) &= \{\langle \omega, \kappa(\omega)+\omega-\omega\kappa(\omega), \beta s(\omega); \aleph \rangle | \omega \in E\}\\
&= \{\langle \omega, \aleph^{1}(\kappa(\omega)+\omega-\omega\kappa(\omega), \beta s(\omega))\rangle | \omega \in E\}.
\end{aligned}
$$

The defuzzification function is essential for $\mathcal{L}^{p}$-IFS to enhance its applicability in various methods. The score and accuracy functions must be defined, similar to those in IFS and $IVIFS$. In this context, we propose novel accuracy and scoring functions for $\mathcal{L}^{p}$-IFS, derived from various perspectives. Initially, the classical defuzzification formulas for IFS and $IVIFS$ are presented below to clarify the source of these functions:

Definition 13. Let $T = (\kappa, s)$ be a IFV. Then, score function $(S_{IFV}(T))$ and accuracy functions

$\left(H_{FN}(T)\right)$ of T are denoted and defined as, such that

$$S_{IFV}(T) = \kappa - \mathcal{s},$$

where $-1 \leq S_{IFV}(T) \leq 1$.

$$H_{IFV}(T) = \kappa + \mathcal{s},$$

where $0 \leq H_{IFV}(T) \leq 1$, respectively.

Definition 14. Let $A = ([\kappa^-, \kappa^+], [\mathcal{s}^-, \mathcal{s}^+])$ be a $IVIFV$. Then, score function $\left(S_{IVIFV}(A)\right)$ and accuracy functions $\left(H_{IVFN}(A)\right)$ of A are denoted and defined as, such that

$$S_{IVIFV}(A) = \frac{S_{IFV}(\kappa^-, \mathcal{s}^-) + S_{IFV}(\kappa^+, \mathcal{s}^+)}{2} = \frac{\kappa^+ - \mathcal{s}^+ + \kappa^- - \mathcal{s}^-}{2},$$

where $-1 \leq S_{IVIFV}(A) \leq 1$.

$$H_{IVIFV}(A) = \frac{H_{IFV}(\kappa^-, \mathcal{s}^-) + H_{IFV}(\kappa^+, \mathcal{s}^+)}{2} = \frac{\kappa^+ + \mathcal{s}^+ + \kappa^- + \mathcal{s}^-}{2},$$

where $0 \leq H_{IVIFV}(A) \leq 1$, respectively.

Before offering any recommendations, let us begin by explaining how defuzzification functions are utilized in $\mathcal{L}^p$-IFS. Essentially, $\mathcal{L}^p$-IFS forms different shapes with a norm $\aleph$ around the central IFS point. In reality, every point within $\mathcal{L}^p$-IFS represents an IFS. Therefore, the points within the $\mathcal{L}^p$-IFS can be utilized to generate a score value.

We split the $\mathcal{L}^p$-IFS values into four equal parts for interpretation. As shown in Figure 10, increasing the κ and $\mathcal{s}$ values from the central IFS point $(\kappa, \mathcal{s})$ leads to values falling in the first quarter (Q1). Consequently, the membership and non-membership values will increase. By decreasing the value of κ and increasing the value of $\mathcal{s}$, the second quartile (Q2) values can be obtained from the IFS with the center at $(\kappa, \mathcal{s})$. In this case, the minimum membership value and the highest non-membership value of the $\mathcal{L}^p$-IFS are identified when the angle is set to 45^0 angle, and the point is at a distance of $\aleph$. This point, also an IFS, is referred to as the "pessimistic point" of the $\mathcal{L}^p$-IFS. Alternatively, the $\mathcal{L}^p$-IFS attains the point with the maximum membership and minimum non-membership values by moving to a position in the fourth quartile (Q4) at a distance of $\aleph$ and a 45^0 angle. This point, also an IFS, is termed the "optimistic point" of the $\mathcal{L}^p$-IFS. As a result, the two specified points within the $\mathcal{L}^p$-shapes can be utilized to calculate the score and accuracy values for the $\mathcal{L}^p$-IFS, similar to the use of endpoints in $IVIFS$.

In this context, new functions for scoring $(S_{\mathcal{L}^p-IFV})$ and accuracy $(H_{\mathcal{L}^p-IFV})$ are introduced for $\mathcal{L}^p-IFV$, for $d = 2$.

Definition 15. A $\mathcal{L}^p$-intuitionistic fuzzy number is a collection of

$$\mathcal{L}_\aleph^p = (\kappa, \mathcal{s}; \aleph),$$

where $\mathcal{L}_\aleph^p$ represent the $\mathcal{L}^p$-intuitionistic fuzzy number with conditions:
(i) $0 \leq \kappa(x) + \mathcal{s}(x) \leq 1$.
(ii) $0 \leq \kappa(x), \mathcal{s}(x), \aleph \leq 1$.
(ii) $0 \leq \aleph \leq 2$.

For the sake of simplicity, the set of the $\mathcal{L}^p$-intuitionistic fuzzy number is $(\mathcal{L}^p - FN\mathcal{s})$.

Definition 16. Let $\mathcal{L}_\aleph^p = (\kappa, \mathcal{s}; \aleph)$ be a $\mathcal{L}^p - IFV$ with optimistic point $\left(\kappa + \frac{\aleph}{2^{\frac{1}{p}}}, \mathcal{s} - \frac{\aleph}{2^{\frac{1}{p}}}\right)$ and pessimistic point $\left(\kappa-\frac{\aleph}{2^{\frac{1}{p}}}, s+\frac{\aleph}{2^{\frac{1}{p}}}\right)$. Then, score function $\left(S_{\mathcal{L}^p-IFV}\left(\mathcal{L}_{\aleph}^p\right)\right)$ and accuracy functions $\left(H_{\mathcal{L}^p-IFV}\left(\mathcal{L}_{\aleph}^p\right)\right)$ of $\mathcal{L}_{\aleph}^p$ are denoted and defined with respect to the decision-maker's preference information $\lambda \in [0,1]$:

$$S_{\mathcal{L}^p-IFV}\left(\mathcal{L}_{\aleph}^p\right)=\frac{\lambda^* S_{IFV}\left(\kappa+\frac{\aleph}{2^{\frac{1}{p}}}, s-\frac{\aleph}{2^{\frac{1}{p}}}\right)+(1-\lambda)^* S_{IFV}\left(\kappa-\frac{\aleph}{2^{\frac{1}{p}}}, s+\frac{\aleph}{2^{\frac{1}{p}}}\right)}{3}=\frac{\kappa-s+2^{\frac{1}{p}}\aleph(2\lambda-1)}{3},$$

where $-1 \leq S_{\mathcal{L}^p-IFV}\left(\mathcal{L}_{\aleph}^p\right) \leq 1$.

$$H_{\mathcal{L}^p-IFV}\left(\mathcal{L}_{\aleph}^p\right)=\lambda^* H_{IFV}\left(\kappa+\frac{\aleph}{2^{\frac{1}{p}}}, s-\frac{\aleph}{2^{\frac{1}{p}}}\right)+(1-\lambda)^* H_{IFV}\left(\kappa-\frac{\aleph}{2^{\frac{1}{p}}}, s+\frac{\aleph}{2^{\frac{1}{p}}}\right)=\kappa+s,$$

where $0 \leq H_{\mathcal{L}^p-IFV}\left(\mathcal{L}_{\aleph}^p\right) \leq 1$, respectively. Note that parameter λ represents the decision-maker's viewpoint on the model. When λ is zero, it reflects a fully pessimistic view, while a value of one indicates a completely optimistic outlook. Common interpretations suggest that $\lambda \in [0, 0.5)$ reflects a pessimistic stance, and $\lambda \in (0.5, 1]$ reflects an optimistic stance. A neutral or indifferent attitude is indicated when $\lambda = 0.5$.

These rules define the comparison between two $\mathcal{L}^p-IFVs$ $\mathcal{L}_{\aleph_1}^p$ and $\mathcal{L}_{\aleph_2}^p$, such that

a) $\mathcal{L}_{\aleph_1}^p$ is higher ranked than $\mathcal{L}_{\aleph_2}^p$ if $S_{\mathcal{L}^p-IFV}\left(\mathcal{L}_{\aleph_1}^p\right) > S_{\mathcal{L}^p-IFV}\left(\mathcal{L}_{\aleph_2}^p\right)$,

b) $\mathcal{L}_{\aleph_1}^p$ is lower ranked than $\mathcal{L}_{\aleph_2}^p$ if $S_{\mathcal{L}^p-IFV}\left(\mathcal{L}_{\aleph_1}^p\right) < S_{\mathcal{L}^p-IFV}\left(\mathcal{L}_{\aleph_2}^p\right)$, when $S_{\mathcal{L}^p-IFV}\left(\mathcal{L}_{\aleph_1}^p\right) = S_{\mathcal{L}^p-IFV}\left(\mathcal{L}_{\aleph_2}^p\right)$ for two $\mathcal{L}^p-IFVs$, then,

c) $\mathcal{L}_{\aleph_1}^p$ is higher ranked than $\mathcal{L}_{\aleph_2}^p$ if $H_{\mathcal{L}^p-IFV}\left(\mathcal{L}_{\aleph_1}^p\right) > H_{\mathcal{L}^p-IFV}\left(\mathcal{L}_{\aleph_2}^p\right)$,

d) $\mathcal{L}_{\aleph_1}^p$ is lower ranked than $\mathcal{L}_{\aleph_2}^p$ if $H_{\mathcal{L}^p-IFV}\left(\mathcal{L}_{\aleph_1}^p\right) < H_{\mathcal{L}^p-IFV}\left(\mathcal{L}_{\aleph_2}^p\right)$,

e) $\mathcal{L}_{\aleph_1}^p$ is similar $\mathcal{L}_{\aleph_2}^p$ if $H_{\mathcal{L}^p-IFV}\left(\mathcal{L}_{\aleph_1}^p\right) = H_{\mathcal{L}^p-IFV}\left(\mathcal{L}_{\aleph_2}^p\right)$.

5. Operations on $\mathcal{L}^p$-intuitionistic fuzzy set via t-norms and t-conorms

By applying universal t-norms and t-conorms, algebraic operations among $\mathcal{L}^p$-IFS in Definition 10 (vii-x) can be expanded. Note that, for upcoming results, $\aleph \in [0,1]$.

Definition 17. Let $\mathcal{L}_{\aleph_1}^p = \langle \kappa_1, s_1; \aleph_1 \rangle$ and $\mathcal{L}_{\aleph_2}^p = \langle \kappa_2, s_2; \aleph_2 \rangle$ be two $\mathcal{L}^p$-$IFSs$. Assume that, in the context of $\mathcal{L}^p$-intuitionistic fuzzy complement, $N(\hbar) = 1 - \hbar$ with $\mathfrak{Q}$ as the norm or conorm, $\mathcal{T}$ and $\mathcal{S}$ as the dual t-norm and t-conorm, respectively. The general algebraic operations among $\mathcal{L}^p$-$IFSs$ are defined as follows:

1) $\mathcal{L}_{\aleph_1}^p \oplus_{\mathfrak{Q}} \mathcal{L}_{\aleph_2}^p = \left(\mathcal{S}(\kappa_1, \kappa_2), \mathcal{T}(s_1, s_2); \mathfrak{Q}(\aleph_1, \aleph_2)\right)$,

2) $\mathcal{L}_{\aleph_1}^p \otimes_{\mathfrak{Q}} \mathcal{L}_{\aleph_2}^p = \left(\mathcal{T}(\kappa_1, \kappa_2), \mathcal{S}(s_1, s_2); \mathfrak{Q}(\aleph_1, \aleph_2)\right)$.

It is clear that the operations presented in Definition 17 are based on those outlined in Definition 10, with particular selections for $\mathcal{T}$, $\mathcal{S}$, and $\mathfrak{Q}$.

We now show that the sum and product of two $\mathcal{L}^p$-$IFSs$ also result in $\mathcal{L}^p$-$IFSs$, as stated in the following proposition:

Proposition 5. Let $\mathcal{L}_{\aleph_1}^p = \langle \kappa_1, s_1; \aleph_1 \rangle$ and $\mathcal{L}_{\aleph_2}^p = \langle \kappa_2, s_2; \aleph_2 \rangle$ be two $\mathcal{L}^p$-$IFSs$. Assume that, in the context of intuitionistic fuzzy complement, $N(\hbar) = 1 - \hbar$ with $\mathfrak{Q}$ as the norm or conorm, $\mathcal{T}$ and $\mathcal{S}$ as the dual t-norm and t-conorm, respectively. Then $\mathcal{L}^{p}_{\aleph_1} \oplus_{\mathfrak{Q}} \mathcal{L}^{p}_{\aleph_2}$ and $\mathcal{L}^{p}_{\aleph_1} \otimes_{\mathfrak{Q}} \mathcal{L}^{p}_{\aleph_2}$ are also $\mathcal{L}^{p}$-IFS.

Proof. Sine $\mathcal{S}$ is a t-conorm corresponding to intuitionistic fuzzy complement N, then $\mathcal{S}(\hbar, ð) = 1 - \mathcal{T}(1-\hbar, 1-ð)$. We know that $\hbar \le 1 - ð$ and $\mathcal{T}$ is nondecreasing, then we have

$$\mathcal{T}(\hbar, ð) + \mathcal{S}(\hbar, ð) = \mathcal{T}(\hbar, ð) + 1 - \mathcal{T}(1-\hbar, 1-ð)$$
$$\le \mathcal{T}(1-\hbar, 1-ð) + 1 - \mathcal{T}(1-\hbar, 1-ð) = 1.$$

Furthermore, as the domain of $\mathfrak{Q}$ is the unit closed interval, we deduce that $\mathcal{L}^{p}_{\aleph_1} \otimes_{\mathfrak{Q}} \mathcal{L}^{p}_{\aleph_2}$ is a $\mathcal{L}^{p}$-IFS. It can also be demonstrated that $\mathcal{L}^{p}_{\aleph_1} \oplus_{\mathfrak{Q}} \mathcal{L}^{p}_{\aleph_2}$ is a $\mathcal{L}^{p} - IFV$.

Continuous Archimedean t-norms and t-conorm can be stated using their additive generators, as demonstrated by Klement et al. [46]. Consequently, one can define some algebraic operations among $\mathcal{L}^{p} - IFV$s by employing additive generators of strict Archimedean t-norms and t-conorms.

Definition 18. Let $\lambda > 0$, and assume that $\mathcal{L} = \langle \kappa_{\mathcal{L}}, s_{\mathcal{L}}; \aleph_{\mathcal{L}} \rangle$ and $\mathfrak{L} = \langle \kappa_{\mathfrak{L}}, s_{\mathfrak{L}}; \aleph_{\mathfrak{L}} \rangle$ are two $\mathcal{L}^{p} - IFV$s. Suppose that the additive generator of a continuous Archimedean t-norm is $\mathrm{g}: [0,1] \to [0, \infty]$, and the additive generator of a continuous Archimedean t-norm or t-conorm is $ɘ: [0,1] \to [0, \infty]$, with $\mathrm{ħ}(t) = \mathrm{g}(1-t)$. The following definitions describe algebraic operations for $\mathcal{L}^{p} - IFV$:

i. $\mathcal{L} \oplus_{ɘ} \mathfrak{L} = \langle \mathrm{ħ}^{-1}(\mathrm{ħ}(\kappa_{\mathcal{L}}) + \mathrm{ħ}(\kappa_{\mathfrak{L}})), \mathrm{g}^{-1}(\mathrm{g}(s_{\mathcal{L}}) + \mathrm{g}(s_{\mathfrak{L}})); ɘ^{-1}(ɘ(\aleph_{\mathcal{L}}) + ɘ(\aleph_{\mathfrak{L}})) \rangle$,
ii. $\mathcal{L} \otimes_{ɘ} \mathfrak{L} = \langle \mathrm{g}^{-1}(\mathrm{g}(\kappa_{\mathcal{L}}) + \mathrm{g}(\kappa_{\mathfrak{L}})), \mathrm{ħ}^{-1}(\mathrm{ħ}(s_{\mathcal{L}}) + \mathrm{ħ}(s_{\mathfrak{L}})); ɘ^{-1}(ɘ(\aleph_{\mathcal{L}}) + ɘ(\aleph_{\mathfrak{L}})) \rangle$,
iii. $\lambda_{ɘ}\mathcal{L} = \langle \mathrm{ħ}^{-1}(\lambda \mathrm{ħ}(\kappa_{\mathcal{L}})), \mathrm{g}^{-1}(\lambda \mathrm{g}(s_{\mathcal{L}})); ɘ^{-1}(\lambda ɘ(\aleph_{\mathcal{L}})) \rangle$,
iv. $\mathcal{L}^{\lambda_{ɘ}} = \langle \mathrm{g}^{-1}(\lambda \mathrm{g}(\kappa_{\mathcal{L}})), \mathrm{ħ}^{-1}(\lambda \mathrm{ħ}(s_{\mathcal{L}})); ɘ^{-1}(\lambda ɘ(\aleph_{\mathcal{L}})) \rangle$.

The following statement verifies that $\mathcal{L}^{p} - IFV$s is also multiplication by constant and power of $\mathcal{L}^{p} - IFV$s.

Proposition 6. Let $\lambda > 0$, and assume that $\mathcal{L} = \langle \kappa_{\mathcal{L}}, s_{\mathcal{L}}; \aleph_{\mathcal{L}} \rangle$ and $\mathfrak{L} = \langle \kappa_{\mathfrak{L}}, s_{\mathfrak{L}}; \aleph_{\mathfrak{L}} \rangle$ are two $\mathcal{L}^{p} - IFV$s. Suppose that the additive generator of a continuous Archimedean t-norm is $\mathrm{g}: [0,1] \to [0, \infty]$, and the additive generator of a continuous Archimedean t-norm or t-conorm is $ɘ: [0,1] \to [0, \infty]$, with $\mathrm{ħ}(t) = \mathrm{g}(1-t)$. Then $\mathcal{L} \oplus_{ɘ} \mathfrak{L}$, $\mathcal{L} \otimes_{ɘ} \mathfrak{L}$, $\lambda_{ɘ}\mathcal{L}$ and $\mathcal{L}^{\lambda_{ɘ}}$.

Proof. The Proposition 6 makes it obvious that $\mathcal{L} \oplus_{ɘ} \mathfrak{L}$ and $\mathcal{L} \otimes_{ɘ} \mathfrak{L}$ are $\mathcal{L}^{p} - IFV$s. It is well known that $\mathrm{ħ}^{-1}(t) = 1 - \mathrm{g}^{-1}(t)$ and $\mathrm{g}(t) = \mathrm{ħ}(1-t)$. Now, $\kappa \le 1 - s$ and $\mathrm{ħ}, \mathrm{ħ}^{-1}$ are non-decreasing, then

$$0 \le \mathrm{ħ}^{-1}(\lambda \mathrm{ħ}(\kappa_{\mathcal{L}})) + \mathrm{g}^{-1}(\lambda \mathrm{g}(s_{\mathcal{L}}))$$
$$\le \mathrm{ħ}^{-1}(\lambda \mathrm{ħ}(1 - s_{\mathcal{L}})) + \mathrm{g}^{-1}(\lambda \mathrm{g}(s_{\mathcal{L}}))$$
$$= 1 - \mathrm{g}^{-1}(\lambda \mathrm{ħ}(1 - s_{\mathcal{L}})) + \mathrm{g}^{-1}(\lambda \mathrm{g}(s_{\mathcal{L}}))$$
$$= 1 - \mathrm{g}^{-1}(\lambda \mathrm{g}(s_{\mathcal{L}})) + \mathrm{g}^{-1}(\lambda \mathrm{g}(s_{\mathcal{L}}))$$
$$= 1.$$

Furthermore, as the range of $ɘ^{-1}$ is the unit closed interval, we deduce that $\lambda_{ɘ}\mathcal{L}$ is a $\mathcal{L}^{p} - IFV$. Likewise, it can be demonstrated that $\mathcal{L}^{\lambda_{ɘ}}$ is a $\mathcal{L}^{p} - IFV$.

Example 2. Assume that $\mathrm{g}, \mathrm{ħ}, ɘ, \sigma: [0,1] \to [0, \infty]$ characterized by $\lambda > 0$, $\mathrm{g}(t) = -\log t, \mathrm{ħ}(t) = -\log(1-t), ɘ(t) = -\log t$ and $\sigma(t) = -\log(1-t)$. The algebraic operators are then obtained

a) $\mathcal{L} \oplus_{ɘ} \mathfrak{L} = \langle \kappa_{\mathcal{L}} + \kappa_{\mathfrak{L}} - \kappa_{\mathcal{L}}\kappa_{\mathfrak{L}}, s_{\mathcal{L}}s_{\mathfrak{L}}; \aleph_{\mathcal{L}}\aleph_{\mathfrak{L}} \rangle$,
b) $\mathcal{L} \oplus_{\sigma} \mathfrak{L} = \langle \kappa_{\mathcal{L}} + \kappa_{\mathfrak{L}} - \kappa_{\mathcal{L}}\kappa_{\mathfrak{L}}, s_{\mathcal{L}}s_{\mathfrak{L}}; \aleph_{\mathcal{L}} + \aleph_{\mathfrak{L}} - \aleph_{\mathcal{L}}\aleph_{\mathfrak{L}} \rangle$,
c) $\mathcal{L} \otimes_{ɘ} \mathfrak{L} = \langle \kappa_{\mathcal{L}}\kappa_{\mathfrak{L}}, s_{\mathcal{L}} + s_{\mathfrak{L}} - s_{\mathcal{L}}s_{\mathfrak{L}}; \aleph_{\mathcal{L}}\aleph_{\mathfrak{L}} \rangle$,
d) $\mathcal{L} \otimes_{\sigma} \mathfrak{L} = \langle \kappa_{\mathcal{L}}\kappa_{\mathfrak{L}}, s_{\mathcal{L}} + s_{\mathfrak{L}} - s_{\mathcal{L}}s_{\mathfrak{L}}; \aleph_{\mathcal{L}} + \aleph_{\mathfrak{L}} - \aleph_{\mathcal{L}}\aleph_{\mathfrak{L}} \rangle$,
e) $\lambda_{ɘ}\mathcal{L} = \left\langle 1 - (1-\kappa_{\mathcal{L}})^{\lambda}, s_{\mathcal{L}}^{\lambda}; \aleph_{\mathcal{L}}^{\lambda} \right\rangle$
f) $\lambda_{\sigma}\mathcal{L} = \left\langle 1 - (1-\kappa_{\mathcal{L}})^{\lambda}, s_{\mathcal{L}}^{\lambda}; 1 - (1-\aleph_{\mathcal{L}})^{\lambda} \right\rangle$,
g) $\mathcal{L}^{\lambda_{ɘ}} = \left\langle \kappa_{\mathcal{L}}^{\lambda}, 1 - (1-s_{\mathcal{L}})^{\lambda}; \aleph_{\mathcal{L}}^{\lambda} \right\rangle$,
h) $\mathcal{L}^{\lambda_{\sigma}} = \left\langle \kappa_{\mathcal{L}}^{\lambda}, 1 - (1-s_{\mathcal{L}})^{\lambda}; 1 - (1-\aleph_{\mathcal{L}})^{\lambda} \right\rangle$.

Some fundamental features of algebraic operations are provided by the following theorem.

Theorem 2. Let $\lambda, \gamma > 0$, and assume that $\mathcal{L} = \langle \kappa_{\mathcal{L}}, s_{\mathcal{L}}; \aleph_{\mathcal{L}} \rangle$, $\mathfrak{L} = \langle \kappa_{\mathfrak{L}}, s_{\mathfrak{L}}; \aleph_{\mathfrak{L}} \rangle$ and $\mathfrak{T} = \langle \kappa_{\mathfrak{T}}, s_{\mathfrak{T}}; \aleph_{\mathfrak{T}} \rangle$ are three $\mathcal{L}^p - IFV$s. Suppose that the additive generator of a continuous Archimedean t-norm is $g: [0,1] \to [0,\infty]$, and the additive generator of a continuous Archimedean t-norm or t-conorm is $ɘ: [0,1] \to [0,\infty]$, with $\hbar(t) = g(1-t)$. Then, followings hold such that

1) $\mathcal{L} \oplus_{ɘ} \mathfrak{L} = \mathfrak{L} \oplus_{ɘ} \mathcal{L}$,
2) $\mathcal{L} \otimes_{ɘ} \mathfrak{L} = \mathfrak{L} \otimes_{ɘ} \mathcal{L}$,
3) $(\mathcal{L} \oplus_{ɘ} \mathfrak{L}) \oplus_{ɘ} \mathfrak{T} = \mathcal{L} \oplus_{ɘ} (\mathfrak{L} \oplus_{ɘ} \mathfrak{T})$,
4) $(\mathcal{L} \otimes_{ɘ} \mathfrak{L}) \otimes_{ɘ} \mathfrak{T} = \mathcal{L} \otimes_{ɘ} (\mathfrak{L} \otimes_{ɘ} \mathfrak{T})$,
5) $\lambda_{ɘ}(\mathcal{L} \oplus_{ɘ} \mathfrak{L}) = \lambda_{ɘ}\mathcal{L} \oplus_{ɘ} \lambda_{ɘ}\mathfrak{L}$,
6) $(\lambda_{ɘ} + \gamma_{ɘ})\mathcal{L} = \lambda_{ɘ}\mathcal{L} \oplus_{ɘ} \gamma_{ɘ}\mathcal{L}$,
7) $(\mathcal{L} \otimes_{ɘ} \mathfrak{L})^{\lambda_{ɘ}} = \mathcal{L}^{\lambda_{ɘ}} \otimes_{ɘ} \mathfrak{L}^{\lambda_{ɘ}}$,
8) $\mathcal{L}^{\lambda_{ɘ}} \otimes_{ɘ} \mathcal{L}^{\gamma_{ɘ}} = \mathcal{L}^{\lambda_{ɘ}+\gamma_{ɘ}}$.

Proof. (1)–(4) hold. For (5), we have

$$\begin{aligned}
\lambda_{ɘ}(\mathcal{L} \oplus_{ɘ} \mathfrak{L}) &= \lambda_{ɘ}\langle \hbar^{-1}\big(\hbar(\kappa_{\mathcal{L}}) + \hbar(\kappa_{\mathfrak{L}})\big), g^{-1}\big(g(s_{\mathcal{L}}) + g(s_{\mathfrak{L}})\big); ɘ^{-1}\big(ɘ(\aleph_{\mathcal{L}}) + ɘ(\aleph_{\mathfrak{L}})\big)\rangle \\
&= \langle \hbar^{-1}\Big(\lambda\hbar\Big(\hbar^{-1}\big(\hbar(\kappa_{\mathcal{L}}) + \hbar(\kappa_{\mathfrak{L}})\big)\Big)\Big), g^{-1}\Big(\lambda g\Big(g^{-1}\big(g(s_{\mathcal{L}}) \\
&\qquad + g(s_{\mathfrak{L}})\big)\Big)\Big); ɘ^{-1}\Big(\lambda ɘ\Big(ɘ^{-1}\big(ɘ(\aleph_{\mathcal{L}}) + ɘ(\aleph_{\mathfrak{L}})\big)\Big)\Big)\rangle \\
&= \langle \hbar^{-1}\big(\lambda\hbar(\kappa_{\mathcal{L}}) + \lambda\hbar(\kappa_{\mathfrak{L}})\big), g^{-1}\big(\lambda g(s_{\mathcal{L}}) + \lambda g(s_{\mathfrak{L}})\big); ɘ^{-1}\big(\lambda ɘ(\aleph_{\mathcal{L}}) + \lambda ɘ(\aleph_{\mathfrak{L}})\big)\rangle \\
&= \left\langle \begin{array}{c} \hbar^{-1}\Big(\hbar\Big(\hbar^{-1}\big(\lambda\hbar(\kappa_{\mathcal{L}})\big)\Big) + \hbar\Big(\hbar^{-1}\big(\lambda\hbar(\kappa_{\mathfrak{L}})\big)\Big)\Big), g^{-1}\Big(g\Big(g^{-1}\big(\lambda g(s_{\mathcal{L}})\big) \\ +g\Big(g^{-1}\big(\lambda g(s_{\mathfrak{L}})\big)\Big)\Big); ɘ^{-1}\Big(ɘ\Big(ɘ^{-1}\big(\lambda ɘ(\aleph_{\mathcal{L}})\big)\Big) + ɘ\Big(ɘ^{-1}\big(\lambda ɘ(\aleph_{\mathfrak{L}})\big)\Big)\Big) \end{array} \right\rangle \\
&= \langle \hbar^{-1}\big(\hbar(\kappa_{\lambda\mathcal{L}}) + \hbar(\kappa_{\lambda\mathfrak{L}})\big), g^{-1}\big(g(s_{\lambda\mathcal{L}}) + g(s_{\lambda\mathfrak{L}})\big); ɘ^{-1}\big(ɘ(\aleph_{\lambda\mathcal{L}}) + \lambda ɘ(s_{\lambda\mathfrak{L}})\big)\rangle \\
&= \lambda_{ɘ}\mathcal{L} \oplus_{ɘ} \lambda_{ɘ}\mathfrak{L}.
\end{aligned}$$

For (6), we have

$$\begin{aligned}
(\lambda_{ɘ} + \gamma_{ɘ})\mathcal{L} &= \langle \hbar^{-1}\big((\lambda+\gamma)\hbar(\kappa_{\mathcal{L}})\big), g^{-1}\big((\lambda+\gamma)g(s_{\mathcal{L}})\big); ɘ^{-1}\big((\lambda+\gamma)ɘ(\aleph_{\mathcal{L}})\big)\rangle \\
&= \langle \hbar^{-1}\big(\lambda\hbar(\kappa_{\mathcal{L}}) + \gamma\hbar(\kappa_{\mathcal{L}})\big), g^{-1}\big(\lambda g(s_{\mathcal{L}}) + \gamma g(s_{\mathcal{L}})\big); ɘ^{-1}\big(\lambda ɘ(\aleph_{\mathcal{L}}) + \gamma ɘ(\aleph_{\mathcal{L}})\big)\rangle \\
&= \left\langle \begin{array}{c} \hbar^{-1}\Big(\hbar\Big(\hbar^{-1}\big(\lambda\hbar(\kappa_{\mathcal{L}})\big)\Big) + \hbar\Big(\hbar^{-1}\big(\gamma\hbar(\kappa_{\mathcal{L}})\big)\Big)\Big), \\ g^{-1}\Big(g\Big(g^{-1}\big(\lambda g(s_{\mathcal{L}})\big)\Big) + g\Big(g^{-1}\big(\gamma g(s_{\mathcal{L}})\big)\Big)\Big); ɘ^{-1}\Big(ɘ\Big(ɘ^{-1}\big(\lambda ɘ(\aleph_{\mathcal{L}})\big)\Big) + ɘ\Big(ɘ^{-1}\big(\gamma ɘ(\aleph_{\mathcal{L}})\big)\Big)\Big) \end{array} \right\rangle
\end{aligned}$$

$$= \langle ħ^{-1}\left(ħ(\kappa_{\lambda_ə\mathcal{L}}) + ħ(\kappa_{\gamma_ə\mathcal{L}})\right), g^{-1}\left(g(s_{\lambda_ə\mathcal{L}}) + g(s_{\gamma_ə\mathcal{L}})\right); ə^{-1}\left(ə(\aleph_{\lambda_ə\mathcal{L}}) + ə(\aleph_{\gamma_ə\mathcal{L}})\right)\rangle$$

$$= \lambda_ə\mathcal{L} \oplus_ə \gamma_ə\mathcal{L}.$$

For (7), we have

$$(\mathcal{L} \otimes_ə \mathfrak{L})^{\lambda_ə} = \left\langle g^{-1}\left(\lambda g(\kappa_{\mathcal{L}\otimes_ə\mathfrak{L}})\right), ħ^{-1}\left(\lambda ħ(s_{\mathcal{L}\otimes_ə\mathfrak{L}})\right); ə^{-1}\left(\lambda ə(\aleph_{\mathcal{L}\otimes_ə\mathfrak{L}})\right)\right\rangle$$

$$= \left\langle g^{-1}\left(\lambda g\left(g^{-1}(g(\kappa_{\mathcal{L}}) + g(\kappa_{\mathfrak{L}}))\right)\right), ħ^{-1}\left(\lambda ħ\left(ħ^{-1}(ħ(s_{\mathcal{L}}) + ħ(s_{\mathfrak{L}}))\right)\right);\right.$$

$$\left. ə^{-1}\left(\lambda ə\left(ə^{-1}(ə(\aleph_{\mathcal{L}}) + ə(\aleph_{\mathfrak{L}}))\right)\right)\right\rangle$$

$$= \left\langle g^{-1}(\lambda g(\kappa_{\mathcal{L}}) + \lambda g(\kappa_{\mathfrak{L}})), ħ^{-1}(\lambda ħ(s_{\mathcal{L}}) + \lambda ħ(s_{\mathfrak{L}})); ə^{-1}(\lambda ə(\aleph_{\mathcal{L}}) + \lambda ə(\aleph_{\mathfrak{L}}))\right\rangle$$

$$= \left\langle g^{-1}\left(g\left(g^{-1}(\lambda g(\kappa_{\mathcal{L}}))\right) + g\left(g^{-1}(\lambda g(\kappa_{\mathfrak{L}}))\right)\right), ħ^{-1}\left(ħ\left(ħ^{-1}(\lambda ħ(s_{\mathcal{L}}))\right)\right.\right.$$

$$\left.\left. + ħ\left(ħ^{-1}(\lambda ħ(s_{\mathfrak{L}}))\right)\right); ə^{-1}\left(ə\left(ə^{-1}(\lambda ə(\aleph_{\mathcal{L}}))\right) + ə\left(ə^{-1}(\lambda ə(\aleph_{\mathfrak{L}}))\right)\right)\right\rangle$$

$$= \left\langle g^{-1}\left(g(\kappa_{\mathcal{L}^{\lambda_ə}}) + g(\kappa_{\mathfrak{L}^{\lambda_ə}})\right), ħ^{-1}\left(ħ(s_{\mathcal{L}^{\lambda_ə}}) + ħ(s_{\mathfrak{L}^{\lambda_ə}})\right);\right.$$

$$\left. ə^{-1}\left(ə(\aleph_{\mathcal{L}^{\lambda_ə}}) + ə(\aleph_{\mathfrak{L}^{\lambda_ə}})\right)\right\rangle = \mathcal{L}^{\lambda_ə} \otimes_ə \mathfrak{L}^{\lambda_ə}.$$

For (8), we have

$$\mathcal{L}^{\lambda_ə+\gamma_ə} = \left\langle g^{-1}((\lambda+\gamma)g(\kappa_{\mathcal{L}})), ħ^{-1}((\lambda+\gamma)ħ(s_{\mathcal{L}})); ə^{-1}((\lambda+\gamma)ə(\aleph_{\mathcal{L}}))\right\rangle$$

$$= \left\langle g^{-1}(\lambda g(\kappa_{\mathcal{L}}) + \gamma g(\kappa_{\mathcal{L}})), ħ^{-1}(\lambda ħ(s_{\mathcal{L}}) + \gamma ħ(s_{\mathcal{L}})); ə^{-1}(\lambda ə(\aleph_{\mathcal{L}}) + \gamma ə(\aleph_{\mathcal{L}}))\right\rangle$$

$$= \left\langle g^{-1}\left(g\left(g^{-1}(\lambda g(\kappa_{\mathcal{L}}))\right) + g\left(g^{-1}(\gamma g(\kappa_{\mathcal{L}}))\right)\right), ħ^{-1}\left(ħ\left(ħ^{-1}(\lambda ħ(s_{\mathcal{L}}))\right)\right.\right.$$

$$\left.\left. + ħ\left(ħ^{-1}(\gamma ħ(s_{\mathcal{L}}))\right)\right); ə^{-1}\left(ə\left(ə^{-1}(\lambda ə(\aleph_{\mathcal{L}}))\right) + ə\left(ə^{-1}(\gamma ə(\aleph_{\mathcal{L}}))\right)\right)\right\rangle$$

$$= \left\langle g^{-1}\left(g(\kappa_{\mathcal{L}^{\lambda}}) + g(\kappa_{\mathcal{L}^{\gamma}})\right), ħ^{-1}\left(ħ(s_{\mathcal{L}^{\lambda}}) + ħ(s_{\mathcal{L}\gamma})\right); ə^{-1}\left(ə(\aleph_{\mathcal{L}^{\lambda}}) + ə(\aleph_{\mathcal{L}^{\gamma}})\right)\right\rangle$$

$$= \mathcal{L}^{\lambda_ə} \otimes_ə \mathcal{L}^{\gamma_ə}.$$

6. Aggregation operators via $\mathcal{L}^p$-IFVs

Aggregation operators are crucial in converting input values expressed as fuzzy values into a single output value. In this section, we present a weighted arithmetic aggregation operator and a weighted geometric aggregation operator for $\mathcal{L}^p - IFV$s, utilizing the algebraic operations outlined in Section 3. Note that $\mathcal{L}^p$-intuitionistic fuzzy numbers ($\mathcal{L}^p - IFVs$) on E is denoted by $\mathcal{L}^p - IFV(E)$.

6.1. $\mathcal{L}^p$-intuitionistic fuzzy weighted averaging aggregation operators

Definition 19. Let $\{\mathcal{L}_i = \langle \kappa_{\mathcal{L}_i}, s_{\mathcal{L}_i}, ; \aleph_{\mathcal{L}_i}\rangle : i = 1, \ldots, n\}$ be the set of $\mathcal{L}^p - IFV$s. Suppose that the additive generator of a continuous Archimedean t-norm is $g:[0,1]\to[0,\infty]$, and the additive generator of a continuous Archimedean t-norm or t-conorm is $ɘ:[0,1]\to[0,\infty]$, with $ħ(t)=g(1-t)$. Then, $\mathcal{L}^p$-intuitionistic fuzzy weighted averaging aggregation ($\mathcal{L}^p-IFWAA$) operator with mapping $\mathcal{L}^p-IFWAA:\mathcal{L}^p-IFV(E)\to\mathcal{L}^p-IFV(E)$ is computed as follows:

$$\mathcal{L}^p-IFWAA_ɘ\,(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\dots\dots\dots,\mathcal{L}_n)=(ɘ)\oplus_{i=1}^n\varpi_{iɘ}\mathcal{L}_i,$$

with weight vector $\varpi=(\varpi_1,\varpi_2,\varpi_3,\dots,\varpi_n)^T$ with $0\le\varpi_j\le1$ and $\sum_{j=1}^n\varpi_j=1$.

Theorem 3. Let $\left\{\mathcal{L}_i=\langle\kappa_{\mathcal{L}_i},s_{\mathcal{L}_i};\aleph_{\mathcal{L}_i}\rangle: i=1,\dots,n\right\}$ be the set of $\mathcal{L}^p-IFV$s. Suppose that the additive generator of a continuous Archimedean t-norm is $g:[0,1]\to[0,\infty]$, and the additive generator of a continuous Archimedean t-norm or t-conorm is $ɘ:[0,1]\to[0,\infty]$, with $ħ(t)=g(1-t)$. If $\mathcal{L}^p$-intuitionistic fuzzy weighted averaging aggregation ($\mathcal{L}^p-IFWAA$) operator is defined with the help of this transformation $\mathcal{L}^p-IFWAA:\mathcal{L}^p-IFV(E)\to\mathcal{L}^p-IFV(E)$, then $\mathcal{L}^p-IFWAA_ɘ\,(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\dots\dots\dots,\mathcal{L}_i)$ is $\mathcal{L}^p-IFV$, and we have

$$\mathcal{L}^p-IFWAA_ɘ\,(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\dots\dots\dots,\mathcal{L}_n)$$
$$=\left\langle ħ^{-1}\left(\sum_{i=1}^n\varpi_iħ(\kappa_{\mathcal{L}_i})\right),g^{-1}\left(\sum_{i=1}^n\varpi_ig(s_{\mathcal{L}_i})\right);ɘ^{-1}\left(\sum_{i=1}^n\varpi_iɘ(\aleph_{\mathcal{L}_i})\right)\right\rangle,$$

with weight vector $\varpi=(\varpi_1,\varpi_2,\varpi_3,\dots,\varpi_n)^T$ with $0\le\varpi_j\le1$ and $\sum_{j=1}^n\varpi_j=1$.

Proof. As evident from Proposition 7, $\mathcal{L}^p-IFWAA_ɘ\,(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\dots\dots\dots,\mathcal{L}_n)$ is a $\mathcal{L}^p-IFV$. Using mathematical induction, it can be demonstrated that the second part also holds true. If $n=2$, then we have

$$\mathcal{L}^p-IFWAA_ɘ\,(\mathcal{L}_1,\mathcal{L}_2)=\varpi_{1ɘ}\mathcal{L}_1\oplus_ɘ\varpi_{2ɘ}\mathcal{L}_2$$
$$=\left\langle ħ^{-1}\left(ħ(\kappa_{\varpi_{1ɘ}\mathcal{L}_1})+ħ(\kappa_{\varpi_{2ɘ}\mathcal{L}_2})\right),g^{-1}\left(g(s_{\varpi_{1ɘ}\mathcal{L}_1})+g(s_{\varpi_{2ɘ}\mathcal{L}_2})\right)ɘ^{-1}\left(ɘ(\aleph_{\varpi_{1ɘ}\mathcal{L}_1})+ɘ(\aleph_{\varpi_{2ɘ}\mathcal{L}_2})\right)\right\rangle$$
$$=\left\langle ħ^{-1}\left(ħ\left(ħ^{-1}\left(\varpi_1ħ(\kappa_{\mathcal{L}_1})\right)\right)+ħ\left(ħ^{-1}\left(\varpi_1ħ(\kappa_{\mathcal{L}_1})\right)\right)\right);\right.$$
$$g^{-1}\left(g\left(g^{-1}\left(\varpi_1g(s_{\mathcal{L}_1})\right)\right)+g\left(g^{-1}\left(\varpi_2g(s_{\mathcal{L}_2})\right)\right)\right);$$
$$\left.ɘ^{-1}\left(ɘ\left(ɘ^{-1}\left(\varpi_1ɘ(\aleph_{\mathcal{L}_1})\right)\right)+ɘ\left(ɘ^{-1}\left(\varpi_2ɘ(\aleph_{\mathcal{L}_2})\right)\right)\right)\right\rangle$$
$$=\left\langle ħ^{-1}\left(\varpi_1ħ(\kappa_{\mathcal{L}_1})+\varpi_2ħ(\kappa_{\mathcal{L}_2})\right),g^{-1}\left(\varpi_1g(s_{\mathcal{L}_1})+\varpi_2g(s_{\mathcal{L}_2})\right);\right.$$
$$\left.ɘ^{-1}\left(\varpi_1ɘ(\aleph_{\mathcal{L}_1})+\varpi_2g(\aleph_{\mathcal{L}_2})\right)\right\rangle$$
$$=\left\langle ħ^{-1}\left(\sum_{j=1}^2\varpi_jħ\left(\kappa_{\mathcal{L}_j}\right)\right),g^{-1}\left(\sum_{j=1}^2\varpi_jg\left(s_{\mathcal{L}_j}\right)\right);ɘ^{-1}\left(\sum_{j=1}^2\varpi_jɘ\left(\aleph_{\mathcal{L}_j}\right)\right)\right\rangle.$$

Let us temporarily assume that the following expression hold, such that

$$A_{n-1}=\mathcal{L}^p-IFWAA_ɘ\,(\mathcal{L}_1,\dots,\mathcal{L}_{n-1})$$
$$=\left\langle ħ^{-1}\left(\sum_{j=1}^{n-1}\varpi_jħ\left(\kappa_{\mathcal{L}_j}\right)\right),g^{-1}\left(\sum_{j=1}^{n-1}\varpi_jg\left(s_{\mathcal{L}_j}\right)\right);ɘ^{-1}\left(\sum_{j=1}^{n-1}\varpi_jɘ\left(\aleph_{\mathcal{L}_j}\right)\right)\right\rangle.$$

We now have

$$\mathcal{L}^p - IFWAA_{ɘ}(\mathcal{L}_1, \dots, \mathcal{L}_n) = A_{n-1} \oplus_{ɘ} \varpi_{n_{ɘ}} \mathcal{L}_n$$

$$= \left\langle \hbar^{-1}\left(\sum_{j=1}^{n-1} \varpi_j \hbar\left(\kappa_{\mathcal{L}_j}\right)\right), g^{-1}\left(\sum_{j=1}^{n-1} \varpi_j g\left(s_{\mathcal{L}_j}\right)\right); ɘ^{-1}\left(\sum_{j=1}^{n-1} \varpi_j ɘ\left(\aleph_{\mathcal{L}_j}\right)\right)\right\rangle$$

$$\oplus_{ɘ} \left\langle \hbar^{-1}\left(\varpi_n\left(\kappa_{\mathcal{L}_n}\right)\right), g^{-1}\left(\varpi_n g\left(s_{\mathcal{L}_n}\right)\right); ɘ^{-1}\left(\varpi_n ɘ\left(\aleph_{\mathcal{L}_n}\right)\right)\right\rangle$$

$$= \left\langle \hbar^{-1}\left(\hbar\left(\hbar^{-1}\left(\sum_{j=1}^{n-1} \varpi_j \hbar\left(\kappa_{\mathcal{L}_j}\right)\right)\right) + \hbar\left(\hbar^{-1}\left(\varpi_n \hbar\left(\kappa_{\mathcal{L}_n}\right)\right)\right)\right),\right.$$

$$g^{-1}\left(g\left(g^{-1}\left(\sum_{j=1}^{n-1} \varpi_j g\left(s_{\mathcal{L}_j}\right)\right)\right) + g^{-1}\left(\varpi_n g\left(s_{\mathcal{L}_n}\right)\right)\right); ɘ^{-1}\left(ɘ\left(ɘ^{-1}\left(\sum_{j=1}^{n-1} \varpi_j ɘ\left(s_{\mathcal{L}_j}\right)\right)\right) + \right.$$

$$\left.\left. ɘ\left(ɘ^{-1}\left(\varpi_n ɘ\left(s_{\mathcal{L}_n}\right)\right)\right)\right)\right\rangle$$

$$= \left\langle \hbar^{-1}\left(\sum_{j=1}^{n-1} \varpi_j \hbar\left(\kappa_{\mathcal{L}_j}\right) + \varpi_n \hbar\left(\kappa_{\mathcal{L}_n}\right)\right), g^{-1}\left(\sum_{j=1}^{n-1} \varpi_j g\left(s_{\mathcal{L}_j}\right) + \varpi_n g\left(s_{\mathcal{L}_n}\right)\right);\right.$$

$$\left. ɘ^{-1}\left(\sum_{j=1}^{n-1} \varpi_j ɘ\left(\aleph_{\mathcal{L}_j}\right) + \varpi_n ɘ\left(\aleph_{\mathcal{L}_n}\right)\right)\right\rangle = \left\langle \hbar^{-1}\left(\sum_{i=1}^{n} \varpi_i \hbar\left(\kappa_{\mathcal{L}_i}\right)\right), g^{-1}\left(\sum_{i=1}^{n} \varpi_i g\left(s_{\mathcal{L}_i}\right)\right); ɘ^{-1}\left(\sum_{i=1}^{n} \varpi_i ɘ\left(\aleph_{\mathcal{L}_i}\right)\right)\right\rangle.$$

That concludes the proof.

Corollary 1. Assume that $g, \hbar, ɘ, \sigma: [0,1] \to [0, \infty]$ characterized by, $g(t) = -\log t, \hbar(t) = -\log(1-t), ɘ(t) = -\log t$ and $\sigma(t) = -\log(1-t)$. The algebraic $\mathcal{L}^p$-intuitionistic fuzzy weighted averaging aggregation operators are then obtained such that

$$\mathcal{L}^p - IFWAA_{ɘ}(\mathcal{L}_1, \dots, \mathcal{L}_n) = \left\langle 1 - \prod_{i=1}^{n} \left(1 - \kappa_{\mathcal{L}_i}\right)^{\varpi_i}, \prod_{i=1}^{n} s_{\mathcal{L}_i}^{\varpi_i}; \prod_{i=1}^{n} \aleph_{\mathcal{L}_i}^{\varpi_i}\right\rangle,$$

and

$$\mathcal{L}^p - IFWAA_{\sigma}(\mathcal{L}_1, \dots, \mathcal{L}_n) = \left\langle 1 - \prod_{i=1}^{n} \left(1 - \kappa_{\mathcal{L}_i}\right)^{\varpi_i}, \prod_{i=1}^{n} s_{\mathcal{L}_i}^{\varpi_i}; 1 - \prod_{i=1}^{n} \left(1 - \aleph_{\mathcal{L}_i}\right)^{\varpi_i}\right\rangle.$$

6.2. $\mathcal{L}^p$-intuitionistic fuzzy weighted geometric aggregation operators

Definition 20. Let $\left\{\mathcal{L}_i = \left\langle \kappa_{\mathcal{L}_i}, s_{\mathcal{L}_i}, ; \aleph_{\mathcal{L}_i}\right\rangle : i = 1, \dots, n\right\}$ be the set of $\mathcal{L}^p - IFV$s. Suppose that the additive generator of a continuous Archimedean t-norm is $g: [0,1] \to [0, \infty]$, and the additive generator of a continuous Archimedean t-norm or t-conorm is $ɘ: [0,1] \to [0, \infty]$, with $\hbar(t) = g(1-t)$. Then, $\mathcal{L}^p$-intuitionistic fuzzy weighted geometric aggregation ($\mathcal{L}^p - IFWAA_{ɘ}$) operator with mapping $\mathcal{L}^p - IFWAA: \mathcal{L}^p - IFV(E) \to \mathcal{L}^p - IFV(E)$ is computed as follows

$$\mathcal{L}^p - IFWAA_{\text{ɘ}}\,(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots, \mathcal{L}_n) = (\text{ɘ}) \bigoplus_{i=1}^{n} \mathcal{L}_i^{\varpi_{i\text{ɘ}}},$$

with weight vector $\varpi = (\varpi_1, \varpi_2, \varpi_3, \dots, \varpi_n)^T$ with $0 \le \varpi_j \le 1$ and $\sum_{j=1}^{n} \varpi_j = 1$.

Theorem 4. Let $\{\mathcal{L}_i = \langle \kappa_{\mathcal{L}_i}, s_{\mathcal{L}_i}, ; \aleph_{\mathcal{L}_i} \rangle : i = 1, \dots, n\}$ be the set of $\mathcal{L}^p - IFV$s. Suppose that the additive generator of a continuous Archimedean t-norm is $\text{g}: [0,1] \to [0, \infty]$, and the additive generator of a continuous Archimedean t-norm or t-conorm is $\text{ɘ}: [0,1] \to [0, \infty]$, with $\hbar(t) = \text{g}(1 - t)$. If $\mathcal{L}^p$-intuitionistic fuzzy weighted geometric aggregation ($\mathcal{L}^p - IFWAA_{\text{ɘ}}$) operator is defined with the help of this transformation $\mathcal{L}^p - IFWAA_{\text{ɘ}}: \mathcal{L}^p - IFV(E) \to \mathcal{L}^p - IFV(E)$, then $\mathcal{L}^p - IFWAA_{\text{ɘ}}\,(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots, \mathcal{L}_n)$ is $\mathcal{L}^p - IFV$ and we have

$$\mathcal{L}^p - IFWGA_{\text{ɘ}}\,(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots \dots \dots, \mathcal{L}_n)$$

$$= \left\langle \text{g}^{-1}\left(\sum_{i=1}^{n} \varpi_i \text{g}(\kappa_{\mathcal{L}_i})\right), \hbar^{-1}\left(\sum_{i=1}^{n} \varpi_i \hbar(s_{\mathcal{L}_i})\right); \text{ɘ}^{-1}\left(\sum_{i=1}^{n} \varpi_i \text{ɘ}(\aleph_{\mathcal{L}_i})\right)\right\rangle,$$

with weight vector $\varpi = (\varpi_1, \varpi_2, \varpi_3, \dots, \varpi_n)^T$ with $0 \le \varpi_j \le 1$ and $\sum_{j=1}^{n} \varpi_j = 1$.

Proof. By using the same arguments like Theorem 4, it can be proven.

Corollary 2. Assume that $\text{g}, \hbar, \text{ɘ}, \sigma: [0,1] \to [0, \infty]$ characterized by, $\text{g}(t) = -\log t, \hbar(t) = -\log(1 - t), \text{ɘ}(t) = -\log t$ and $\sigma(t) = -\log(1 - t)$. The algebraic $\mathcal{L}^p$-intuitionistic fuzzy weighted geometric aggregation operators are then obtained, such that

$$\mathcal{L}^p - IFWGA_{\text{ɘ}}\,(\mathcal{L}_1, \dots, \mathcal{L}_n) = \left\langle \prod_{i=1}^{n} \kappa_{\mathcal{L}_i}^{\varpi_i}, 1 - \prod_{i=1}^{n} \left(1 - s_{\mathcal{L}_i}\right)^{\varpi_i}; \prod_{i=1}^{n} \aleph_{\mathcal{L}_i}^{\varpi_i} \right\rangle,$$

and

$$\mathcal{L}^p - IFWGA_{\sigma}\,(\mathcal{L}_1, \dots, \mathcal{L}_n) = \left\langle \prod_{i=1}^{n} \kappa_{\mathcal{L}_i}^{\varpi_i}, 1 - \prod_{i=1}^{n} \left(1 - s_{\mathcal{L}_i}\right)^{\varpi_i}; 1 - \prod_{i=1}^{n} \left(1 - \aleph_{\mathcal{L}_i}\right)^{\varpi_i} \right\rangle.$$

7. The $MADM$ framework based on the proposed techniques

The $MADM$ technique is particularly effective for selecting the most suitable alternative from a finite set of options due to its structured framework. To further improve the performance and reliability of existing methods, we present a procedure for the $MADM$ technique that incorporates four specialized operators: the $\mathcal{L}^p - IFWAA_{\text{ɘ}}$ operator, $\mathcal{L}^p - IFWAA_{\sigma}$ operator, $\mathcal{L}^p - IFWGA_{\text{ɘ}}$ operator, and $\mathcal{L}^p - IFWGA_{\sigma}$ operator. Our aim is to apply this procedure to real-world problems and thereby enhance the decision-making process.

We consider a finite set of alternatives denoted by $\omega = \{\omega_1, \omega_2, \dots, \omega_m\}$. Similarly, we define a finite set of attributes as $\tilde{\text{O}} = \{\tilde{\text{O}}_1, \tilde{\text{O}}_2, \dots, \tilde{\text{O}}_n\}$, associated with a weight vector $\varpi = (\varpi_1, \varpi_2, \dots, \varpi_n)^T$, where $\varpi_j > 0$ with $\sum_{j=1}^{n} \varpi_j = 1$. To construct the decision matrix for evaluating the optimal alternative, $\mathcal{L}^p - IFV$ values are assigned to each alternative–attribute pair. Here, $\kappa_{\mathcal{L}_i}$ and $s_{\mathcal{L}_i}$ represent the positive and negative membership degrees, respectively, while α_j and β_j denote reference parameters corresponding to an alternative $(\omega)_j$ under attribute $(\tilde{\text{O}}_j)$, as specified by the decision makers. These values satisfy the conditions $0 \le \kappa_{\mathcal{L}_i} + s_{\mathcal{L}_i} \le 1$ and $0 \le \aleph_{\mathcal{L}_i} \le 1$. Furthermore, the degree of refusal is given by $\pi_j = 1 - \kappa_{\mathcal{L}_i} - s_{\mathcal{L}_i}$. To validate the proposed approach, we also consider several real-world applications and demonstrate the evaluation process through the developed theoretical framework.

7.1. The proposed algorithm

Our main objective of this subsection is to present a process that illustrates the problem to be addressed in the subsequent section. The fundamental steps of the decision-making approach are as follows:

Step 1. Construct the team decision matrix by representing their evaluations in the $\mathcal{L}^p - IFV$ form.

Step 2. When assigning values, two types of criteria are considered: profit and cost. For cost-type criteria, normalization is performed as a first priority; for profit-type criteria, normalization is not required.

$$\mathcal{L}_i = \begin{cases} (\langle \kappa_{\mathcal{L}_i}, \mathscr{s}_{\mathcal{L}_i} \rangle), & \text{same type input data} \\ (\langle \mathscr{s}_{\mathcal{L}_i}, \kappa_{\mathcal{L}_i} \rangle), & \text{different type input data.} \end{cases}$$

In this case, as the input data for all attributes is uniform, normalization is unnecessary. All alternatives and criteria in the given problem share the same characteristics.

Step 3. Using the four various types of operators" $\mathcal{L}^p - IFWAA_{\vartheta}$ operator, $\mathcal{L}^p - IFWAA_{\sigma}$ operator, $\mathcal{L}^p - IFWGA_{\vartheta}$ operator, and $\mathcal{L}^p - IFWGA_{\sigma}$ operator," merge the dataset into a single representative set such that

$$\mathcal{L}^p - IFWAA_{\vartheta}(\mathcal{L}_1, \dots, \mathcal{L}_n) = \left\langle 1 - \prod_{i=1}^{n} \left(1 - \kappa_{\mathcal{L}_i}\right)^{\varpi_i}, \prod_{i=1}^{n} \mathscr{s}_{\mathcal{L}_i}^{\varpi_i}; \prod_{i=1}^{n} \aleph_{\mathcal{L}_i}^{\varpi_i} \right\rangle,$$

$$\mathcal{L}^p - IFWAA_{\sigma}(\mathcal{L}_1, \dots, \mathcal{L}_n) = \left\langle 1 - \prod_{i=1}^{n} \left(1 - \kappa_{\mathcal{L}_i}\right)^{\varpi_i}, \prod_{i=1}^{n} \mathscr{s}_{\mathcal{L}_i}^{\varpi_i}; 1 - \prod_{i=1}^{n} \left(1 - \aleph_{\mathcal{L}_i}\right)^{\varpi_i} \right\rangle,$$

$$\mathcal{L}^p - IFWGA_{\vartheta}(\mathcal{L}_1, \dots, \mathcal{L}_n) = \left\langle \prod_{i=1}^{n} \kappa_{\mathcal{L}_i}^{\varpi_i}, 1 - \prod_{i=1}^{n} \left(1 - \mathscr{s}_{\mathcal{L}_i}\right)^{\varpi_i}; \prod_{i=1}^{n} \aleph_{\mathcal{L}_i}^{\varpi_i} \right\rangle,$$

$$\mathcal{L}^p - IFWGA_{\sigma}(\mathcal{L}_1, \dots, \mathcal{L}_n) = \left\langle \prod_{i=1}^{n} \kappa_{\mathcal{L}_i}^{\varpi_i}, 1 - \prod_{i=1}^{n} \left(1 - \mathscr{s}_{\mathcal{L}_i}\right)^{\varpi_i}; 1 - \prod_{i=1}^{n} \left(1 - \aleph_{\mathcal{L}_i}\right)^{\varpi_i} \right\rangle.$$

Step 4. Determine the aggregated theories with respect to different score values, such as

$$S_{\mathcal{L}^p - IFV}(\mathcal{L}) = \frac{\kappa - \mathscr{s} + 2^{\frac{1}{p}} \aleph (2\lambda - 1)}{3},$$

where $-1 \leq S_{\mathcal{L}^p - IFV}(\mathcal{L}) \leq 1$.

If the score function fails to provide a satisfactory result, the accuracy function is then applied as follows:

$$H_{\mathcal{L}^p - IFV}(\mathcal{L}) = \kappa + \mathscr{s},$$

where $0 \leq H_{\mathcal{L}^p - IFV}(\mathcal{L}) \leq 1$.

Step 5. Our aim is to identify the most suitable alternative by analyzing the ranking results derived from the score values. To enhance the reliability of the proposed techniques and to demonstrate their practical applicability, several numerical examples are considered, highlighting the effectiveness and validity of the developed operators. Furthermore, the geometric interpretation of the proposed algorithm is illustrated in Figure 11.

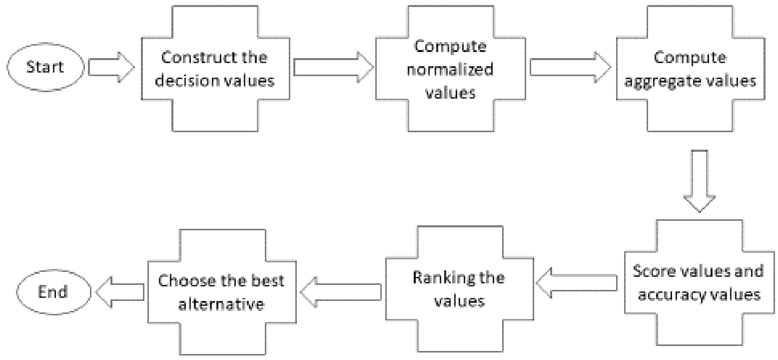

Figure 11. Flow chart of the proposed algorithm.

7.2. Types of agricultural field robots

Agricultural field robots, often referred to as agribots, are specialized autonomous machines designed to perform a variety of tasks in agricultural fields. They are increasingly being used in modern farming to improve efficiency, reduce labor costs, and address the growing challenges of food production, such as labor shortages, sustainability, and climate change. These robots can handle tasks such as planting, watering, weeding, monitoring crop health, and even harvesting. Here are some key types and applications of agricultural field robots:

ω_1: Harvesting robots are advanced machines designed to automate the picking and harvesting of crops in agriculture. These robots use technologies like artificial intelligence (AI), computer vision, and robotic arms to identify, select, and harvest ripe produce with precision. By reducing reliance on manual labor, they enhance efficiency, minimize crop damage, and enable around-the-clock operations, making them an essential innovation in modern farming. Harvesting robots are transforming modern agriculture by boosting productivity, minimizing waste, and addressing labor shortages.

ω_2: Spraying and irrigation robots are autonomous machines used in agriculture to optimize the application of water, fertilizers, and pesticides. These robots are equipped with sensors, GPS, and AI technologies to accurately deliver water and chemicals based on the specific needs of crops. They can move through fields autonomously, reducing waste, preventing overuse, and ensuring even distribution. By improving efficiency and precision, spraying and irrigation robots help conserve resources, protect the environment, and support sustainable farming practices. Spraying and irrigation robots are a crucial part of precision agriculture, ensuring optimal resource use while supporting crop health and sustainability.

ω_3: Crop monitoring robots are autonomous machines designed to track the health, growth, and conditions of crops in real time. These robots are equipped with sensors, cameras, and AI technologies to gather data on soil moisture, plant health, pest infestations, and nutrient levels. By providing precise, up-to-date information, crop monitoring robots help farmers make informed decisions, optimize resource use, and improve crop yields. They are an essential tool in precision agriculture, enabling early detection of issues and enhancing overall farm productivity. By offering real-time, accurate data, crop monitoring robots are transforming modern agriculture, helping farmers optimize their practices and improve sustainability while maximizing yields.

ω_4: Soil analysis robots are autonomous machines designed to assess soil conditions in agricultural fields. Equipped with sensors and sampling tools, these robots measure factors like soil moisture, pH levels, nutrient content, and temperature. By providing precise, real-time data, soil analysis robots help farmers understand the health of their soil and make informed decisions about irrigation, fertilization, and crop management. They are crucial in precision agriculture, enhancing resource efficiency, improving crop yields, and supporting sustainable farming practices. Soil analysis robots are revolutionizing how farmers manage their soil, providing detailed insights that lead to improved productivity, resource conservation, and sustainable agricultural practices.
ω_5: Swarming robots are groups of autonomous machines that work together collaboratively, often mimicking the behavior of natural swarms like ants or bees. In agriculture, these robots communicate and coordinate to perform tasks such as planting, weeding, harvesting, and monitoring crops. Swarming robots rely on decentralized control, AI, and sensor networks to efficiently cover large areas, making them ideal for precision farming. Their ability to work in groups enhances productivity, reduces labor costs, and increases the overall efficiency of farming operations.
Selecting the right agricultural robot involves considering a variety of factors to ensure it meets the needs of your farm and integrates well into your existing operations so followings are the attributes. Here are some key factors to take into account:
$\tilde{O}_1$: Compatibility for an agriculture robot refers to its ability to seamlessly integrate with existing farming systems, technologies, and operations. It ensures that the robot can work alongside current equipment, software, and infrastructure without issues. Key aspects include: Robot must connect with other farm machinery, like tractors or irrigation systems. It should work with farm management software, GPS systems, and data platforms for smooth operation and data sharing. The robot must be suited to the farm’s specific conditions, such as crop type, soil, and terrain. It should easily adapt to the farm's size and operations, whether for small-scale or large-scale farming. Ensuring compatibility enhances efficiency, reduces costs, and maximizes the robot’s effectiveness in agricultural tasks.
$\tilde{O}_2$: Technology and Features refer to the advanced tools and capabilities integrated into a robot to enhance its performance and efficiency. In the context of robotics, these include: Devices for detecting environmental conditions, monitoring crop health, and navigating autonomously. Enables the robot to analyze data, adapt to its environment, and improve decision-making over time. GPS, LiDAR, and other technologies that enable robots to move and operate independently in fields. Robotic arms, sprayers, or harvesting tools that enable accurate and efficient task execution. IoT, Wi-Fi, or Bluetooth capabilities that enable communication with other systems and data sharing. These technologies and features ensure that robots are capable, efficient, and suited for complex tasks, particularly in sectors like agriculture.
$\tilde{O}_3$: Maintenance and support refer to the ongoing care and services required to keep a robot functioning optimally throughout its lifespan. For agricultural robots, this includes: Routine checks to ensure the robot's hardware, such as sensors and moving parts, are in good working condition. Keeping the robot's software and AI algorithms up to date for improved functionality and bug fixes. Timely repairs to address any mechanical or electrical issues that may arise. Access to experts or service teams to help troubleshoot problems or provide guidance on usage. Providing operators with manuals, tutorials, and support for efficient use and maintenance of the robot. Proper maintenance and support ensure the robot’s reliability, extend its lifespan, and prevent costly breakdowns, making it essential for sustained operation.
$\tilde{O}_4$: Regulatory compliance refers to the adherence of agricultural robots to laws, guidelines, and standards set by governing bodies. It ensures that the robot operates safely and ethically within legal frameworks. Key aspects include: Ensuring the robot meets safety requirements to protect users,

workers, and the environment. Complying with laws regarding pesticide use, emissions, and sustainable farming practices. Adhering to rules on data collection and storage, especially when robots gather information from fields or connected devices. Obtaining necessary approvals and certifications from regulatory agencies before the robot is deployed. Meeting regulatory compliance ensures the robot is legally permitted for use and operates safely and responsibly in agricultural settings.

Step 1. Develop the team matrix by embedding their assessments within the $\mathcal{L}^p - IFV$ representation, see Table 1.

Table 1. Decision matrix of $\mathcal{L}^p - IF$ information.

	$\tilde{O}_1$	$\tilde{O}_2$	$\tilde{O}_3$	$\tilde{O}_4$	$\tilde{O}_5$
ω_1	$\langle 0.5,0.4; 0.8\rangle$	$\langle 0.7,0.2; 0.3\rangle$	$\langle 0.6,0.2; 0.6\rangle$	$\langle 0.3,0.7; 0.9\rangle$	$\langle 0.6,0.1; 1\rangle$
ω_2	$\langle 0.8,0.2; 0.7\rangle$	$\langle 0.3,0.7; 0.5\rangle$	$\langle 0.5,0.3; 0.8\rangle$	$\langle 0.9,0.1; 0.3\rangle$	$\langle 0.3,0.6; 0.9\rangle$
ω_3	$\langle 0.7,0.1; 0.4\rangle$	$\langle 0.6,0.3; 1\rangle$	$\langle 0.5,0.2; 0.6\rangle$	$\langle 0.3,0.6; 0.8\rangle$	$\langle 0.4,0.4; 0.4\rangle$
ω_4	$\langle 0.8,0.2; 0.4\rangle$	$\langle 0.5,0.2; 0.3\rangle$	$\langle 0.3,0.3; 0.7\rangle$	$\langle 0.7,0.3; 0.9\rangle$	$\langle 0.8,0.1; 0.9\rangle$

Step 2. When assigning values, two cases are considered: Data of the same type and data of different types. If the data are of different types, normalization is applied as the first priority, such that:

$$\mathcal{L}_j = \begin{cases} \left(\langle \kappa_{\mathcal{L}_i}, \mathcal{s}_{\mathcal{L}_i}; \aleph_{\mathcal{L}_i}\rangle\right), & \text{same type input data} \\ \left(\langle \mathcal{s}_{\mathcal{L}_i}, \kappa_{\mathcal{L}_i}; \aleph_{\mathcal{L}_i}\rangle\right), & \text{different type input data.} \end{cases}$$

In this case, as the input data for all attributes is of the same type, normalization is unnecessary. All alternatives and criteria in the given problem share a uniform nature.

Step 3. For $p = 3$, the data are aggregated into a singleton set using four operators $\mathcal{L}^3 - IFWAA_{\vartheta}$ operator, $\mathcal{L}^3 - IFWAA_{\sigma}$ operator, $\mathcal{L}^3 - IFWGA_{\vartheta}$ operator, and $\mathcal{L}^3 - IFWGA_{\sigma}$ along with the weight vector $\varpi = (0.27, 0.24,0.22,0.17,0.1)^T$. The results of this aggregation are presented in Tables 2 and 3.

Table 2. $\mathcal{L}^3 - IFWAA_{\vartheta}$ and $\mathcal{L}^3 - IFWAA_{\sigma}$ operators.

	$\mathcal{L}^3 - IFWAA_{\vartheta}$	$\mathcal{L}^3 - IFWAA_{\sigma}$
ω_1	$\langle 0.56,0.28; 1\rangle$	$\langle 0.56,0.28; 0.62\rangle$
ω_2	$\langle 0.67,0.29; 0.68\rangle$	$\langle 0.67,0.29; 0.59\rangle$
ω_3	$\langle 0.55,0.24; 1\rangle$	$\langle 0.55,0.24; 0.61\rangle$
ω_4	$\langle 0.65,0.22; 0.67\rangle$	$\langle 0.65,0.22; 0.53\rangle$

Table 3. $\mathcal{L}^3 - IFWGA_{\vartheta}$ and $\mathcal{L}^3 - IFWGA_{\sigma}$ operators.

	$\mathcal{L}^3 - IFWGA_{\vartheta}$	$\mathcal{L}^3 - IFWGA_{\sigma}$
ω_1	$\langle 0.53,0.37; 0.62\rangle$	$\langle 0.53,0.37; 1\rangle$
ω_2	$\langle 0.53,0.42; 0.59\rangle$	$\langle 0.53,0.42; 0.69\rangle$
ω_3	$\langle 0.51,0.31; 0.61\rangle$	$\langle 0.51,0.31; 1\rangle$
ω_4	$\langle 0.56,0.23; 0.53\rangle$	$\langle 0.56,0.23; 0.70\rangle$

Step 3. Refer to Tables 4 and 5 to find the aggregated theory's score values for decision maker's attitude $\lambda = 0.3$, such that:

Table 4. $\mathcal{L}^3 - IFWAA$ score values.

	$S_{\mathcal{L}^3-IFV}$			
	$S_{\mathcal{L}^3-IFV}(\omega_1)$	$S_{\mathcal{L}^3-IFV}(\omega_2)$	$S_{\mathcal{L}^3-IFV}(\omega_3)$	$S_{\mathcal{L}^3-IFV}(\omega_4)$
$\boldsymbol{\mathcal{L}^3 - IFWAA_q}$	0.01	0.07	0.02	0.09
$\boldsymbol{\mathcal{L}^3 - IFWAA_\sigma}$	0.04	0.08	0.05	0.10

Table 5. $\mathcal{L}^3 - IFWGA$ score values.

	$S_{\mathcal{L}^3-IFV}$			
	$S_{\mathcal{L}^3-IFV}(\omega_1)$	$S_{\mathcal{L}^3-IFV}(\omega_2)$	$S_{\mathcal{L}^3-IFV}(\omega_3)$	$S_{\mathcal{L}^3-IFV}(\omega_4)$
$\boldsymbol{\mathcal{L}^3 - IFWGA_q}$	0.01	−0.01	0.02	0.07
$\boldsymbol{\mathcal{L}^3 - IFWGA_\sigma}$	−0.01	−0.02	−0.03	0.05

Step 4. The ranking results are analyzed based on the computed score values to identify the most prominent alternative among the four. The detailed results are provided in Tables 6 and 7.

Table 6. Ranking of $\mathcal{L}^3 - IFWAA$ operators w.r.t. $S_{\mathcal{L}^3-IFV}$.

	$S_{\mathcal{L}^3-IFV}$
$\boldsymbol{\mathcal{L}^3 - IFWAA_q}$	$\omega_4 > \omega_2 > \omega_3 > \omega_1$
$\boldsymbol{\mathcal{L}^3 - IFWAA_\sigma}$	$\omega_4 > \omega_2 > \omega_3 > \omega_1$

Table 7. Ranking of $\mathcal{L}^3 - IFWAA$ operators w.r.t. $S_{\mathcal{L}^3-IFV}$.

	$S_{\mathcal{L}^3-IFV}$
$\boldsymbol{\mathcal{L}^3 - IFWGA_q}$	$\omega_4 > \omega_3 > \omega_1 > \omega_2$
$\boldsymbol{\mathcal{L}^3 - IFWGA_\sigma}$	$\omega_4 > \omega_1 > \omega_2 > \omega_3$

The geometric representation of Table 6, in relation to Table 4, is given as follows (see Figure 12):

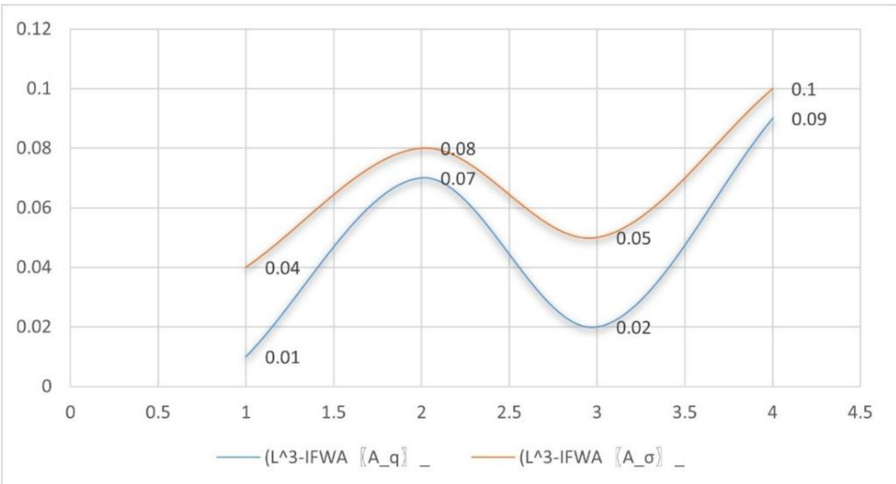

Figure 12. Scores of alternatives based on the two $\mathcal{L}^3 - IFWAA$.

The geometric representation of Table 7, in relation to Table 5, is given as follows (see Figure 13):

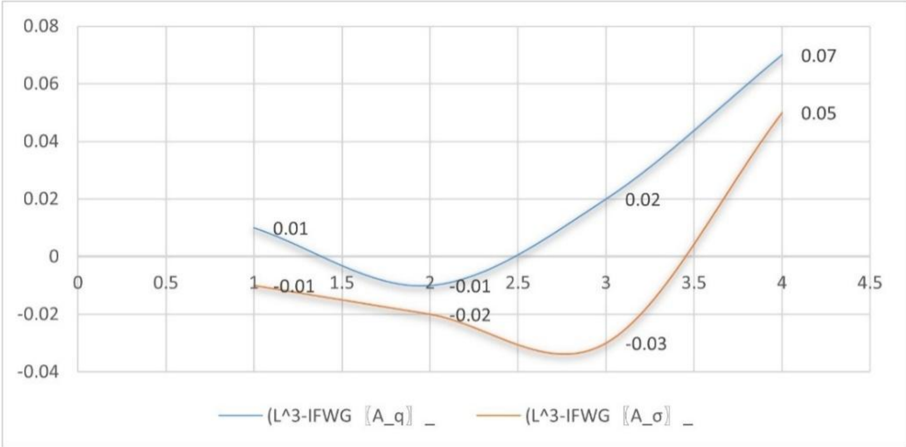

Figure 13. Scores of alternatives based on the two $\mathcal{L}^3 - IFWGA$.

It is evident that the most desirable decision is ω_4. Moreover, since each operator yields identical ranking results, the operators can be considered stable.

7.3. Advantages of agricultural robots

- Labor Efficiency: Robots can work around the clock, reducing the need for human labor, which is becoming increasingly scarce in rural areas.
- Precision Agriculture: Robots can perform tasks with high precision, reducing waste of resources like seeds, water, and pesticides.
- Sustainability: By optimizing resource use and reducing chemical inputs, robots contribute to more environmentally friendly farming practices.
- Yield Improvement: Early detection of pests, diseases, or nutrient deficiencies enables timely intervention, improving crop yield and quality.

7.4. Challenges and considerations

- High Initial Costs: The cost of acquiring and maintaining agricultural robots can be prohibitive for small farms.
- Field Variability: Different crops and field conditions may require tailored robotic solutions, limiting the general applicability of a single robot model.
- Technology Integration: Successful integration of robots into farming requires compatible software systems and trained personnel to manage them.
- Power Supply: Energy efficiency and battery life are limitations for many agricultural robots, especially in large-scale operations.

In the future, AI-powered agribots are expected to play an even bigger role in precision agriculture, leveraging machine learning and data analysis to enhance food security and optimize farming in the face of climate change and population growth.

8. Results and discussion

The individual overseeing the process evaluates the alternatives, considering the assigned weights. The project manager has also determined that all experts are of equal standing, making it suitable to apply weights aligned with the proposed model. In the next phase, decision-makers use predefined $\mathcal{L}^p - IF$ language concepts to assess the situation. This section compares the four proposed $\mathcal{L}^p - IF$ aggregation operators with the existing operators introduced using IF. The comparison highlights the effectiveness of these operators in addressing uncertain real-world decision-making problems (DMPs). A key advantage of this concept is that, through the $\mathcal{L}^p$-space, it gives freedom to take different membership functions by varying the value of p like $\mathcal{L}^1$, $\mathcal{L}^2$, …, $\mathcal{L}^\infty$. Tables 6 and 7 provide the rating results of four options using the proposed approach. Individuals should follow agriculture experts to select best one robot. The proposed method and the existing approach produced slightly different ranking outcomes, though both identified the same top choice. The comparison results are shown in Tables 6 and 7. By comparing the results with those found in the literature, it is clear that the $\mathcal{L}^p - IFS$ $MADM$ strategy proposed in this study aligns with the outcomes obtained from the IFS $MADM$ approach, which has been demonstrated and applied in various contexts. The new model introduces variations in the results due to the lambda (λ) value, representing the decision-maker's attitude, and the norm (ℵ) value, reflecting the uncertainty in the decisions. However, as shown in Tables 8 and 9, these factors enable the development of a case-specific structure, differentiating it from traditional IFS $MADM$ methods. This flexibility makes the use of $\mathcal{L}^p - IFS$ numbers in $MADM$ models highly relevant.

The ranking values obtained from the score function is examined to determine the most prominent alternative among the four, as presented in Table 8.

Table 8. $\mathcal{L}^3 - IFWAA$ and $IFWGA$ score values.

	$\mathcal{S}_{\mathcal{L}^3-IFV}$			
	$\mathcal{S}_{\mathcal{L}^3-IFV}(\omega_1)$	$\mathcal{S}_{\mathcal{L}^3-IFV}(\omega_2)$	$\mathcal{S}_{\mathcal{L}^3-IFV}(\omega_3)$	$\mathcal{S}_{\mathcal{L}^3-IFV}(\omega_4)$
$IFWAA$	0.29	0.37	0.32	0.43
$IFWGA$	0.16	0.11	0.20	0.33

Table 9. Ranking of $IFWAA$ and $IFWGA$ operators w.r.t. $\mathcal{S}_{IFV}$.

	$\mathcal{S}_{IFV}$
$IFWAA$	$\omega_4 > \omega_3 > \omega_2 > \omega_1$
$IFWGA$	$\omega_4 > \omega_3 > \omega_1 > \omega_2$

Next, we explore the applicability and flexibility of the developed method for handling diverse inputs and outputs.

Authenticity and ease of use of the suggested approach: We devised a system capable of handling any type of input data, and the proposed model effectively addresses uncertainty. This approach incorporates $IF\mathcal{s}$, $IVIFS$ and $\mathcal{L}^2 - IF$ or $C - IFS$ through the addition of the pth value of norm ℵ. By introducing the pth value, the interpretation of these parameters' changes, expanding both membership and non-membership spaces. While our method can be applied in various contexts, we focus on its application to the selection of agriculture robots. The proposed $\mathcal{L}^p - IF$ model is clear, easy to understand, and can be adapted to different outcomes.

Score Function's Impact: We begin by generalizing the associated accuracy functions and the types score functions that are introduced. Since each score function has its own distinct ordering and evaluation procedures due to pth value of norm $\aleph$ and decision maker's attitude λ, slight variations in results are expected. As shown in Table 10, there are minor differences in the rankings produced by new SF. However, it is important to note that the overall outcome remains largely consistent across all score functions.

Flexibility in aggregation with variable inputs and outputs: This approach is significantly more versatile than others, as it can adjust to different conditions in $MADM$ methods and the pth value of norm $\aleph$, which allows to get membership and non-membership values indifferent shapes. Moreover, it can be readily applied to a variety of input and output scenarios.

Sensitivity analysis: The results of the sensitivity analysis for the proposed models are presented in Tables 10–13. Both algorithms yield identical outcomes when λ variate, then the rank of attributes is changes. Although variations in score functions lead to changes in the ranking of alternatives, the optimal result remains unchanged. This indicates that both methods are influenced solely by the score functions used. The geometrically representation of Table 10 can be seen in Figure 14, we have

Table 10. Sensitivity analysis for λ value $\left(\mathcal{L}^3 - IFWAA_\vartheta\right)$.

	ω_1	ω_2	ω_3	ω_4	**Ranking**
$\boldsymbol{\lambda = 0}$	-0.32	-0.16	-0.31	-0.14	$\omega_4 > \omega_2 > \omega_3 > \omega_1$
$\boldsymbol{\lambda = 0.1}$	-0.24	-0.10	-0.23	-0.08	$\omega_4 > \omega_2 > \omega_3 > \omega_1$
$\boldsymbol{\lambda = 0.2}$	-0.16	-0.05	-0.15	-0.03	$\omega_4 > \omega_2 > \omega_3 > \omega_1$
$\boldsymbol{\lambda = 0.3}$	-0.07	0.01	-0.06	-0.03	$\omega_4 > \omega_2 > \omega_3 > \omega_1$
$\boldsymbol{\lambda = 0.4}$	0.1	0.7	0.2	0.9	$\omega_4 > \omega_2 > \omega_3 > \omega_1$
$\boldsymbol{\lambda = 0.5}$	0.10	0.12	0.11	0.14	$\omega_4 > \omega_2 > \omega_3 > \omega_1$
$\boldsymbol{\lambda = 0.6}$	0.17	0.18	0.19	0.20	$\omega_4 > \omega_2 > \omega_3 > \omega_1$
$\boldsymbol{\lambda = 0.7}$	0.26	0.24	0.27	0.25	$\omega_3 > \omega_1 > \omega_4 > \omega_2$
$\boldsymbol{\lambda = 0.8}$	0.35	0.30	0.36	0.31	$\omega_3 > \omega_1 > \omega_4 > \omega_2$
$\boldsymbol{\lambda = 0.9}$	0.43	0.35	0.44	0.37	$\omega_3 > \omega_1 > \omega_4 > \omega_2$
$\boldsymbol{\lambda = 1}$	0.52	0.41	0.53	0.42	$\omega_3 > \omega_1 > \omega_4 > \omega_2$

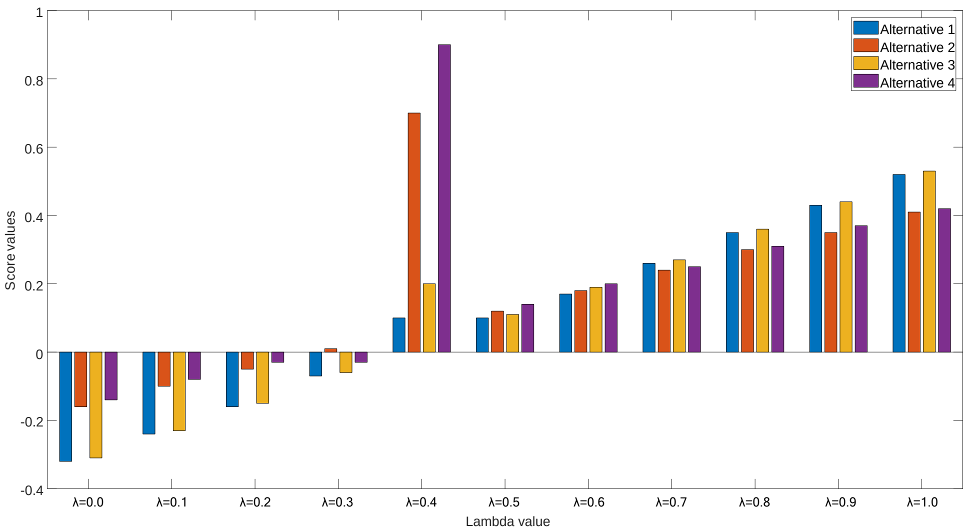

Figure 14. Scores of alternatives based on the $\mathcal{L}^3 - IFWAA_\vartheta$ w.r.t. λ.

Through the geometrically representation of Table 11 can be seen in Figure 15, we have

Table 11. Sensitivity analysis for λ value $\left(\mathcal{L}^3 - IFWAA_\sigma\right)$.

	ω_1	ω_2	ω_3	ω_4	**Ranking**
$\boldsymbol{\lambda = 0}$	−0.16	−0.12	−0.15	−0.08	$\omega_4 > \omega_2 > \omega_3 > \omega_1$
$\boldsymbol{\lambda = 0.1}$	−0.11	−0.07	−0.10	−0.03	$\omega_4 > \omega_2 > \omega_3 > \omega_1$
$\boldsymbol{\lambda = 0.2}$	−0.06	−0.02	−0.05	0.01	$\omega_4 > \omega_2 > \omega_3 > \omega_1$
$\boldsymbol{\lambda = 0.3}$	−0.01	0.03	0.01	0.05	$\omega_4 > \omega_2 > \omega_3 > \omega_1$
$\boldsymbol{\lambda = 0.4}$	0.04	0.08	0.05	0.10	$\omega_4 > \omega_2 > \omega_3 > \omega_1$
$\boldsymbol{\lambda = 0.5}$	0.10	0.12	0.11	0.14	$\omega_4 > \omega_2 > \omega_3 > \omega_1$
$\boldsymbol{\lambda = 0.6}$	0.15	0.17	0.16	0.19	$\omega_4 > \omega_2 > \omega_3 > \omega_1$
$\boldsymbol{\lambda = 0.7}$	0.20	0.22	0.21	0.23	$\omega_4 > \omega_2 > \omega_3 > \omega_1$
$\boldsymbol{\lambda = 0.8}$	0.25	0.27	0.26	0.28	$\omega_4 > \omega_2 > \omega_3 > \omega_1$
$\boldsymbol{\lambda = 0.9}$	0.29	0.32	0.30	0.31	$\omega_2 > \omega_4 > \omega_3 > \omega_1$
$\boldsymbol{\lambda = 1}$	0.34	0.37	0.35	0.36	$\omega_2 > \omega_4 > \omega_3 > \omega_1$

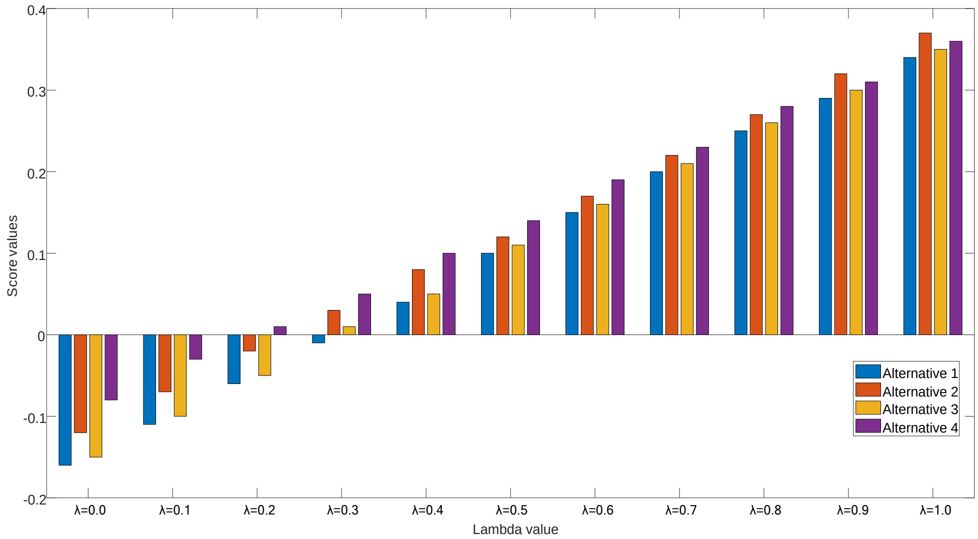

Figure 15. Scores of alternatives based on the $\mathcal{L}^3 - IFWAA\sigma$ w.r.t. λ.

Through the geometrically representation of Table 12 can be seen in Figure 16, we have

Table 12. Sensitivity analysis for λ value $\left(\boldsymbol{\mathcal{L}^3 - IFWGA_{\sigma}}\right)$.

	ω_1	ω_2	ω_3	ω_4	**Ranking**
$\boldsymbol{\lambda = 0}$	−0.20	−0.21	−0.19	−0.11	$\omega_4 > \omega_3 > \omega_1 > \omega_2$
$\boldsymbol{\lambda = 0.1}$	−0.15	−0.16	−0.14	−0.07	$\omega_4 > \omega_3 > \omega_1 > \omega_2$
$\boldsymbol{\lambda = 0.2}$	−0.10	−0.11	−0.09	−0.02	$\omega_4 > \omega_3 > \omega_1 > \omega_2$
$\boldsymbol{\lambda = 0.3}$	−0.05	−0.06	−0.04	0.02	$\omega_4 > \omega_3 > \omega_1 > \omega_2$
$\boldsymbol{\lambda = 0.4}$	0.01	−0.01	0.02	0.07	$\omega_4 > \omega_3 > \omega_1 > \omega_2$
$\boldsymbol{\lambda = 0.5}$	0.05	0.04	0.07	0.11	$\omega_4 > \omega_3 > \omega_1 > \omega_2$
$\boldsymbol{\lambda = 0.6}$	0.11	0.09	0.12	0.15	$\omega_4 > \omega_3 > \omega_1 > \omega_2$
$\boldsymbol{\lambda = 0.7}$	0.16	0.14	0.17	0.20	$\omega_4 > \omega_3 > \omega_1 > \omega_2$
$\boldsymbol{\lambda = 0.8}$	0.21	0.19	0.22	0.24	$\omega_4 > \omega_3 > \omega_1 > \omega_2$
$\boldsymbol{\lambda = 0.9}$	0.26	0.24	0.27	0.29	$\omega_4 > \omega_3 > \omega_1 > \omega_2$
$\boldsymbol{\lambda = 1}$	0.31	0.29	0.32	0.33	$\omega_4 > \omega_3 > \omega_1 > \omega_2$

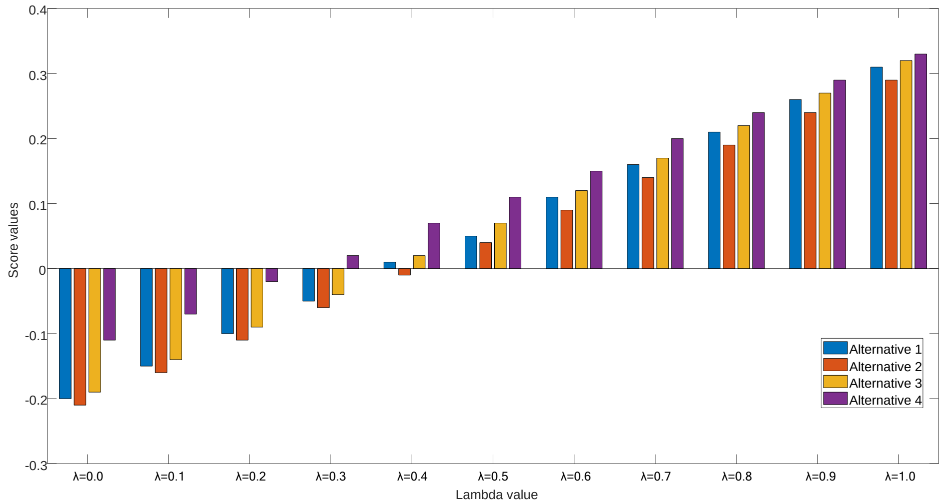

Figure 16. Scores of alternatives based on the $\mathcal{L}^3 - IFWGA_{\vartheta}$ w.r.t. λ.

Through the geometrically representation of Table 13 can be seen in Figure 17, we have

Table 13. Sensitivity analysis for λ value $\left(\mathcal{L}^3 - IFWGA_{\sigma}\right)$.

	ω_1	ω_2	ω_3	ω_4	**Ranking**
$\boldsymbol{\lambda = 0}$	-0.37	-0.25	-0.35	-0.18	$\omega_4 > \omega_2 > \omega_3 > \omega_1$
$\boldsymbol{\lambda = 0.1}$	-0.28	-0.20	-0.27	-0.12	$\omega_4 > \omega_2 > \omega_3 > \omega_1$
$\boldsymbol{\lambda = 0.2}$	-0.20	-0.14	-0.18	-0.07	$\omega_4 > \omega_2 > \omega_3 > \omega_1$
$\boldsymbol{\lambda = 0.3}$	-0.11	-0.08	-0.10	-0.01	$\omega_4 > \omega_2 > \omega_3 > \omega_1$
$\boldsymbol{\lambda = 0.4}$	-0.03	-0.02	-0.02	-0.5	$\omega_4 > \omega_2 > \omega_3 > \omega_1$
$\boldsymbol{\lambda = 0.5}$	0.05	0.04	0.07	0.11	$\omega_4 > \omega_2 > \omega_3 > \omega_1$
$\boldsymbol{\lambda = 0.6}$	0.14	0.10	0.15	0.17	$\omega_4 > \omega_3 > \omega_1 > \omega_2$
$\boldsymbol{\lambda = 0.7}$	0.22	0.15	0.24	0.23	$\omega_3 > \omega_4 > \omega_1 > \omega_2$
$\boldsymbol{\lambda = 0.8}$	0.31	0.21	0.32	0.29	$\omega_3 > \omega_1 > \omega_4 > \omega_2$
$\boldsymbol{\lambda = 0.9}$	0.39	0.27	0.40	0.34	$\omega_3 > \omega_1 > \omega_4 > \omega_2$
$\boldsymbol{\lambda = 1}$	0.47	0.33	0.49	0.40	$\omega_3 > \omega_1 > \omega_4 > \omega_2$

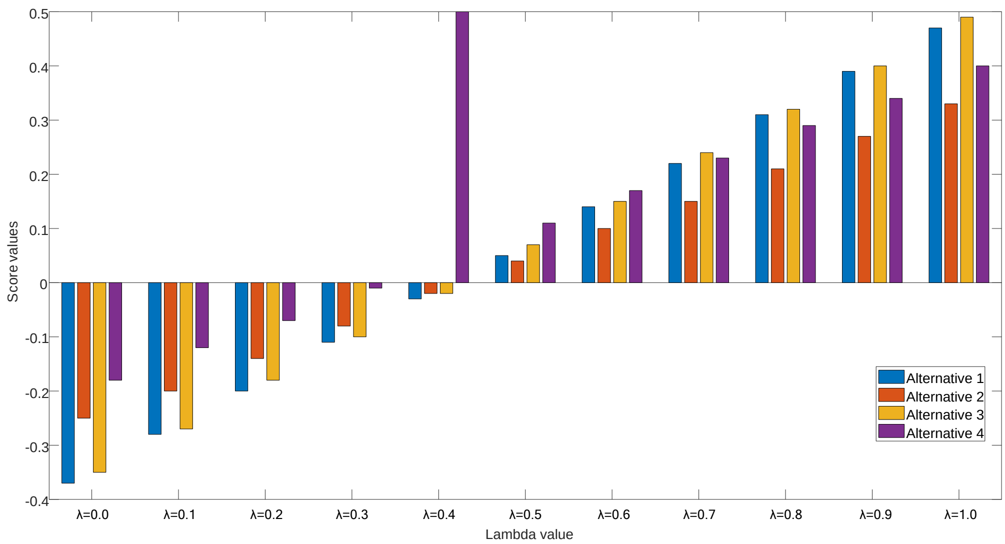

Figure 17. Scores of alternatives based on the $\mathcal{L}^3 - IFWGA_{\sigma}$ w.r.t. λ.

9. Comparative analysis

In this section, the analytical comparison of $\mathcal{L}^p$-$IFSs$ is discussed in Table 14.

Table 14. The analytical comparison of $\mathcal{L}^p$-$IFSs$ with alternative fuzzy methodologies.

Collections	Remarks	₦$_{\emptyset}$
FS [72]	Unable to handle non-membership $s(\omega)$	No
IFS [9]	just deal with the single value	No
Interval IFS [10]	cannot deal with the condition value in the $\mathcal{L}^p$-shapes for $2 > p > 1$, $p > 2$	No
$CIFS$ [11]	cannot deal with the condition value in the $\mathcal{L}^p$-shapes for $2 > p > 1$, $p > 2$	No
$\mathcal{D}$-IFS [40]	cannot deal with the condition value in the $\mathcal{L}^p$-shapes for $2 > p > 1$, $p > 2$	No
$\mathcal{L}^p - IFS$	deal with the condition value in the $\mathcal{L}^p$-shapes for $2 > p > 1$, $p > 2$	Yes

These fuzzy sets influence the optimal selection and impose limitations on decision-makers. We introduce the innovative concept of $\mathcal{L}^p$-$IFSs$, enabling decision-makers to achieve improved outcomes through this advanced approach.

10. Conclusions

Our main aim of this paper is to present the concept of $\mathcal{L}^p$ −iintuitionistic fuzzy Set ($\mathcal{L}^p - IFS$), where the membership and non-membership degrees are depicted by different $\mathcal{L}^p$-shapes with a norm $\aleph$ and a center composed of two components. These components must satisfy the condition that the sum of their squares is less than or equal to one. In this fuzzy set representation, the membership and non-membership degrees are visualized through the circular structure. Circular Intuitionistic Fuzzy Sets ($C - IFS$s) and interval-intuitionistic Fuzzy Sets ($IVIFS$s) are extended by a $\mathcal{L}^p - IFS$. Compared to IFS, $C - IFS$s, and $IVIFS$s, $\mathcal{L}^p - IFS$ offer decision-makers or specialists a more flexible and comprehensive framework for analyzing items. This flexibility makes it possible to modify the degrees of membership and non-membership, which makes it easier to communicate doubt and promotes more thoughtful decision-making. To establish a novel scoring function and an accuracy function that incorporates the decision-makers' attitude (λ), the set's optimistic and pessimistic points are also defined. When the decision-maker's viewpoint (λ) approaches 1, the defuzzification of $\mathcal{L}^p$-IFS occurs near its optimistic point, while it occurs near its pessimistic point as (λ) approaches 0. A technique for converting intuitionistic fuzzy values (IFV) into $\mathcal{L}^p - IFV$ is presented in this study. Algebraic operations for $\mathcal{L}^p - IFSs$ using continuous Archimedean $\mathcal{t}$-norms and $\mathcal{t}$-conorms are also introduced, as well as basic set-theoretic operations for $\mathcal{L}^p - IFSs$. A number of weighted aggregation procedures for $\mathcal{L}^p - IFSs$ are presented using these algebraic techniques. Finally, based on the concepts discussed, we propose a $MADM$ approach within a $\mathcal{L}^p$ −iintuitionistic fuzzy framework, applying it to a real-world $MADM$ problem from the literature concerning the selection of the optimal agricultural field robots $MADM$ framework. In the future, researchers may explore alternative aggregation operators and similarity measures. Additionally, tools like fuzzy integrals or other aggregation operators could be utilized when transforming $IFVs$ into $\mathcal{L}^p - IFV$s. The proposed approach could also be applied to $MADM$ problems such as classification, machine learning, pattern recognition, data mining, clustering, and medical diagnostics.

Author contributions

Conceptualization, M.B.K.; methodology, J.T., N.Z., and L.C.; software, A.A.A.A., N.Z., and L.C.; validation, A.M.D., J.T., N.Z., and L.C.; formal analysis, A.M.D., N.Z., and L.C.; investigation, A.A.A.A., M.B.K., and A.M.D.; resources, N.Z., and L.C.; data curation, J.T., N.Z. and L.C.; writing—original draft preparation, M.B.K., N.Z., and L.C.; writing—review and editing, A.A.A.A. and M.B.K.; visualization, N.Z., and L.C.; supervision, M.B.K. and A.M.D.; project administration, A.A.A.A., and M.B.K.; funding acquisition, A.M.D., and L.C. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors claim to have no conflicts of interest.

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