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Research article

On a generalized fractional integral equation of a variable-order

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Abstract: The paper considers a class of generalized fractional integral equations with variable-order kernels. By employing estimates based on inequalities for the Gamma function, we establish the existence and uniqueness of solutions to the proposed fractional integral equation. Furthermore, the exponential growth rate of the solution is derived using the classical retarded Gronwall inequality. Moreover, numerical and graphical illustrations of the growth estimates are presented.

Keywords: fractional integral equations; well-posedness; growth estimates; retarded Gronwall-type inequality; fractional variable-order
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1. Introduction

Many real-world phenomena exhibit variable fractional-order behavior with respect to time. The introduction of variable-order fractional differential and integral operators provides greater flexibility in modeling diverse processes and natural phenomena. In particular, such operators are employed to describe mechanical behaviors and diffusion processes, enabling the characterization of time-dependent or concentration-dependent anomalous diffusion. The adaptive nature of variable-order fractional models makes them especially valuable for capturing dynamic, nonlocal, and memory-dependent effects. Consequently, variable-order fractional differential and integral equations have found wide-ranging applications in addressing complex problems across physics, engineering, biology, medicine, economics, finance, environmental sciences, mathematics, and computational sciences.

Applications of variable-order fractional operators are abundant across scientific and engineering disciplines. For instance, in anomalous diffusion, they model processes with non-uniform diffusion rates, such as particle transport in heterogeneous media. In viscoelasticity, they capture materials with time-dependent mechanical properties by allowing the order to vary with changing relaxation behaviors. In wave propagation, they describe phenomena in media with spatially or temporally varying damping properties. Within control systems, variable-order fractional controllers enhance stability and performance, particularly in systems with varying inertia or damping. In signal processing, they facilitate the analysis of signals in systems exhibiting non-stationary behavior. In neuroscience, they represent memory effects in neural networks where the degree of memory evolves over time. In financial modeling, they capture complex market dynamics, including memory effects that vary with economic conditions. For detailed discussions and further applications, see [1–4] and the references therein.

In the literature, many authors have investigated various formulations of the general Cauchy problem

$$^{C}\mathcal{D}^{\alpha(t)}x(t) = g(t, x(t)),\ \ x(0) = x_0,$$

where $^{C}\mathcal{D}^{\alpha(t)}$ denotes the Caputo derivative of variable-order. We highlight a few of the recent developments on the topic. The authors in [1] investigated an initial value problem (IVP) for fractional differential equations with nonlinear variable-order derivatives of Riemann–Liouville (R–L) type given by

$$\mathcal{D}_{0^+}^{\alpha(t,x(t))}x(t) = \psi(t, x(t)),\ \ t \in (0, T],\ 0 < T < \infty,$$

with initial condition $x(0) = 0$, where $\mathcal{D}^{\alpha(t,x(t))}$ denotes the R–L fractional derivative of the variable-order $\alpha(t, x(t))$, ψ is a given function, and α satisfies $0 < \alpha_* \leq \alpha(t, x(t)) \leq \alpha^* < 1$. In 2023, the authors in [5] introduced and investigated general variable-order fractional scale derivatives, considering both the Grünwald–Letnikov (GL) and Hadamard formulations. In [2], the authors considered the retarded fractional linear system with Caputo-type variable-order derivatives and distributed delays in the general form:

$$^{C}\mathcal{D}^{\alpha(t)}x(t)_{a^+} = \int_{-h}^{0}[d_\theta U(t,\theta)]X(t,\theta) + \mathcal{F}_t.$$

In a related development, Almeida [6], in 2025, studied a generalized fractional calculus framework in which the order of the operators is not constant and the integral kernel depends on a given function. See [3, 4, 7–9] and the references therein for other related discussions and results. Motivated by these studies and the wide range of real-world applications of variable-order fractional differential and integral equations, we consider an explicit form of $\alpha(t)$ and analyze a class of generalized fractional equations with variable-order kernels.

The remainder of this paper is structured as follows. Section 2 introduces the necessary preliminary concepts and auxiliary results required for the proofs. Section 3 is devoted to the presentation of the main results. Finally, Section 4 provides a brief summary and concluding remarks.

2. Preliminaries

Let $\alpha(.) : [0, T] \subset \mathbb{R}^+ \to (0, 1)$ be a continuous function and $f : [0, T] \to \mathbb{R}$:

Definition 2.1 (**Variable-order fractional integral).** *The left and right Riemann–Liouville fractional integrals of order* $0 < \alpha(t) < 1$ *are defined by*

$$_0\mathcal{I}_t^{\alpha(t)}f(t) = \frac{1}{\Gamma(\alpha(t))}\int_0^t (t-s)^{\alpha(t)-1}f(s)ds; \quad t > 0,$$

and

$$ {}_{t}\mathcal{I}_{T}^{\alpha(t)} f(t) = \frac{1}{\Gamma(\alpha(t))} \int_t^T (s-t)^{\alpha(t)-1} f(s) ds; \quad t < T. $$

Definition 2.2 (**Variable-order fractional derivative**). *The left and right Riemann–Liouville fractional derivatives of order* $0 < \alpha(t) < 1$ *are defined by*

$$ {}_{0}\mathcal{D}_{t}^{\alpha(t)} f(t) = \frac{1}{\Gamma(1-\alpha(t))} \frac{d}{dt} \int_0^t (t-s)^{-\alpha(t)} f(s) ds; \quad t > 0, $$

and

$$ {}_{t}\mathcal{D}_{T}^{\alpha(t)} f(t) = -\frac{1}{\Gamma(1-\alpha(t))} \frac{d}{dt} \int_t^T (s-t)^{-\alpha(t)} f(s)\, ds; \quad t < T. $$

Definition 2.3 (**Variable-order Caputo fractional derivative**). *The left and right Caputo derivatives of order* $0 < \alpha(t) < 1$ *are defined by*

$$ {}_{0}^{C}\mathcal{D}_{t}^{\alpha(t)} f(t) = \frac{1}{\Gamma(1-\alpha(t))} \int_0^t (t-s)^{-\alpha(t)} f'(s) ds; \quad t > 0, $$

and

$$ {}_{t}^{C}\mathcal{D}_{T}^{\alpha(t)} f(t) = \frac{-1}{\Gamma(1-\alpha(t))} \int_t^T (s-t)^{-\alpha(t)} f'(s)\, ds; \quad t < T. $$

Remark 2.4. The fractional integral and differential operators defined above present significant technical challenges. For instance, the operator ${}_{0}\mathcal{D}_{t}^{\alpha(t)}$ does not, in general, serve as the left inverse of ${}_{0}\mathcal{I}_{t}^{\alpha(t)}$; see [7]. In this paper, our focus is directed toward analyzing the properties of the Riemann–Liouville fractional integral of variable-order and studying a general class of variable-order fractional integral equations involving a suitably chosen variable-order function.

2.1. Inequalities for Gamma function

Here, we present some properties and estimates of Gamma functions.

Theorem 2.5. [10] *Let* $\gamma = 0.577...$ *be the Euler–Mascheroni constant. Then, for* $0 \le t \le 1$,

$$ e^{(1-\gamma)(t-1)} \le \Gamma(t+1) \le 1, \tag{2.1} $$

where in the lower bound, equality occurs if and only if $t = 1$.

Theorem 2.6. [11] *The function* $f(t) = \frac{\log \Gamma(t+1)}{t \log t}$ *is strictly increasing from* $[2, \infty)$ *onto* $(1-\gamma, 1)$, *where* γ *is the Euler–Mascheroni constant.*

In particular, for $t \in (1, \infty)$,

$$ t^{(1-\gamma)t-1} < \Gamma(t) < t^{t-1}. \tag{2.2} $$

Other versions of the double inequalities are as follows:

Theorem 2.7. [12] *For* $0 \le t \le 1$,

$$ \frac{t^2+1}{t+1} \le \Gamma(t+1) \le \frac{t^2+2}{t+2}. \tag{2.3} $$

The above inequality (2.3) was improved as follows:

Theorem 2.8. [13] *Let* $0 \le t \le 1$ *and* γ *is the Euler–Mascheroni constant, then*

$$ \left(\frac{t^2+1}{t+1}\right)^{2(1-\gamma)} \le \Gamma(t+1) \le \left(\frac{t^2+2}{t+2}\right)^{\gamma}. $$

3. Main results

In this paper, we investigate a general class of fractional integral equations of variable order $\alpha(.)$, where λ and μ are given real parameters, of the form

$$f(t) = \lambda \int_0^t \mathcal{K}_{\alpha(t)}(t,s)\sigma(s,f(s))ds + \mu \int_t^T \mathcal{K}_{\alpha(t)}(s,t)\sigma(s,f(s))ds. \tag{3.1}$$

In this work, we focus on a particular case of (3.1), which serves as the basis for our analysis and results, and let

$$\mathcal{K}_{\alpha(t)}(t,s) = \frac{(t-s)^{\alpha(t)-1}}{\Gamma(\alpha(t))}, \ \alpha(t) > 0, \forall\, s,t \in [1,T], \quad T < \infty$$

with variable function $\alpha(t) = \alpha t,\ \alpha > 0,\ \forall\, t \in [1,T],\ T < \infty$, and define

$$f(t) = \frac{\lambda}{\Gamma(\alpha t)} \int_1^t (t-s)^{\alpha t-1}\sigma(s,f(s))ds + \frac{\mu}{\Gamma(\alpha t)} \int_t^T (s-t)^{\alpha t-1}\sigma(s,f(s))ds. \tag{3.2}$$

Remark 3.1. The main challenge encountered in this research lies in the absence of a unified inequality or closed-form estimate capable of bounding the Gamma function over the entire interval $[0,\infty)$. Consequently, we restrict our analysis to the interval of interest $[1,T]$, with $T < \infty$. Although the segment $[0,1]$ can be estimated using inequality 2.1 of Theorem 2.5, it is treated as a negligible subinterval for the purposes of this study.

Therefore, the proofs of our results will rely on the estimate for the Gamma function provided by inequality (2.2) of Theorem 2.6.

Assume that the non-linear function σ is Lipschitz continuous with respect to its second argument:

Assumption 3.2 (Lipschitz continuity). Let $0 < \mathrm{Lip}_\sigma < \infty$, and $f,g : [1,T] \to \mathbb{R}$; we assume

$$|\sigma(s,f) - \sigma(s,g)| \leq \mathrm{Lip}_\sigma |f-g|, \ \ \forall\ s \in [1,T]. \tag{3.3}$$

So,

$$|\sigma(s,f)| \leq |\sigma(s,0)| + \mathrm{Lip}_\sigma |f| \leq c_1 + \mathrm{Lip}_\sigma |f|, \ \ \forall\ s \in [1,T]. \tag{3.4}$$

Let the supremum norm of the solution be given by

$$\|f\| = \sup_{t\in[1,T]} |f(t)|. \tag{3.5}$$

3.1. Existence and uniqueness result

We define the operator $\mathcal{K}$ as follows:

$$\mathcal{K}f(t) = \frac{\lambda}{\Gamma(\alpha t)} \int_1^t (t-s)^{\alpha t-1}\sigma(s,f(s))ds + \frac{\mu}{\Gamma(\alpha t)} \int_t^T (s-t)^{\alpha t-1}\sigma(s,f(s))ds, \tag{3.6}$$

where λ and μ are real numbers, and use the fixed point theorem to show that the fixed point of (3.6) solves (3.2).

Lemma 3.3. *Let $f(t)$ be the solution of (3.2). Suppose that Assumption 3.2 is satisfied; then one obtains*

$$\|\mathcal{K}f\| \leq c_2 + c_3\mathrm{Lip}_\sigma\|f\|, \tag{3.7}$$

with positive constants $c_2 := \frac{c_1}{\alpha}[\lambda T^{\alpha\gamma T} + \mu T^{\alpha T}]$ and $c_3 := \frac{1}{\alpha}[\lambda T^{\alpha\gamma T} + \mu T^{\alpha T}]$.

Proof. By absolute value of Eq (3.6) to obtain

$$|\mathcal{K}f(t)| \leq \frac{\lambda}{\Gamma(\alpha t)}\int_1^t (t-s)^{\alpha t-1}|\sigma(s,f(s))|ds + \frac{\mu}{\Gamma(\alpha t)}\int_t^T (s-t)^{\alpha t-1}|\sigma(s,f(s))|ds.$$

Then by Eq (3.4) of Assumption 3.2, one obtains

$$\begin{aligned}|\mathcal{K}f(t)| &\leq \frac{\lambda}{\Gamma(\alpha t)}\int_1^t (t-s)^{\alpha t-1}(c_1 + \mathrm{Lip}_\sigma|f(s)|)ds + \frac{\mu}{\Gamma(\alpha t)}\int_t^T (s-t)^{\alpha t-1}(c_1 + \mathrm{Lip}_\sigma|f(s)|)ds \\ &\leq (c_1 + \mathrm{Lip}_\sigma \sup_{s\in[1,t]}|f(s)|)\frac{\lambda}{\Gamma(\alpha t)}\int_1^t (t-s)^{\alpha t-1}ds + (c_1 + \mathrm{Lip}_\sigma \sup_{s\in[t,T]}|f(s)|)\frac{\mu}{\Gamma(\alpha t)}\int_t^T (s-t)^{\alpha t-1}ds.\end{aligned}$$

Evaluating the integrals, we have

$$\begin{aligned}|\mathcal{K}f(t)| &\leq (c_1 + \mathrm{Lip}_\sigma \sup_{s\in[1,t]}|f(s)|)\frac{\lambda}{\Gamma(\alpha t)}\frac{(t-1)^{\alpha t}}{\alpha t} + (c_1 + \mathrm{Lip}_\sigma \sup_{s\in[t,T]}|f(s)|)\frac{\mu}{\Gamma(\alpha t)}\frac{(T-t)^{\alpha t}}{\alpha t} \\ &\leq (c_1 + \mathrm{Lip}_\sigma \sup_{s\in[1,t]}|f(s)|)\frac{\lambda}{\Gamma(\alpha t)}\frac{t^{\alpha t}}{\alpha t} + (c_1 + \mathrm{Lip}_\sigma \sup_{s\in[t,T]}|f(s)|)\frac{\mu}{\Gamma(\alpha t)}\frac{T^{\alpha t}}{\alpha t}.\end{aligned}$$

Choose $\alpha > 0$ such that $\alpha t \in (1,\infty)$ for all $t \in (1,\infty)$. Then by (2.2) of Theorem 2.6, one arrives at $(\alpha t)^{(1-\gamma)\alpha t-1} < \Gamma(\alpha t)$, and thus,

$$\begin{aligned}|\mathcal{K}f(t)| &\leq \frac{1}{\Gamma(\alpha t)}\Big[\lambda(c_1 + \mathrm{Lip}_\sigma \sup_{s\in[1,t]}|f(s)|)\frac{t^{\alpha t}}{\alpha t} + \mu(c_1 + \mathrm{Lip}_\sigma \sup_{s\in[t,T]}|f(s)|)\frac{T^{\alpha t}}{\alpha t}\Big] \\ &\leq (\alpha t)^{1-(1-\gamma)\alpha t}\Big[\lambda(c_1 + \mathrm{Lip}_\sigma \sup_{s\in[1,t]}|f(s)|)\frac{t^{\alpha t}}{\alpha t} + \mu(c_1 + \mathrm{Lip}_\sigma \sup_{s\in[t,T]}|f(s)|)\frac{T^{\alpha t}}{\alpha t}\Big] \\ &= (\alpha t)^{-(1-\gamma)\alpha t}\Big[\lambda(c_1 + \mathrm{Lip}_\sigma \sup_{s\in[1,t]}|f(s)|)t^{\alpha t} + \mu(c_1 + \mathrm{Lip}_\sigma \sup_{s\in[t,T]}|f(s)|)T^{\alpha t}\Big] \\ &= \alpha^{-(1-\gamma)\alpha t}\Big[\lambda(c_1 + \mathrm{Lip}_\sigma \sup_{s\in[1,t]}|f(s)|)t^{\alpha\gamma t} + \mu(c_1 + \mathrm{Lip}_\sigma \sup_{s\in[t,T]}|f(s)|)t^{-(1-\gamma)\alpha t}T^{\alpha t}\Big].\end{aligned}$$

For $t \geq 1$ and $\alpha > 0$, then $-\alpha(1-\gamma)t \leq -1$ and $\alpha^{-\alpha(1-\gamma)t} \leq \frac{1}{\alpha}$. In the same vein, $t^{-\alpha(1-\gamma)t} \leq \frac{1}{t} \leq 1$, since $t \geq 1$, and therefore,

$$|\mathcal{K}f(t)| \leq \frac{1}{\alpha}\Big[\lambda(c_1 + \mathrm{Lip}_\sigma \sup_{s\in[1,t]}|f(s)|)t^{\alpha\gamma t} + \mu(c_1 + \mathrm{Lip}_\sigma \sup_{s\in[t,T]}|f(s)|)T^{\alpha t}\Big].$$

Now, take the supremum of both sides for $t \in [1,T]$ and Eq (3.5) to arrive at

$$\|\mathcal{K}f\| \leq \frac{1}{\alpha}(c_1 + \mathrm{Lip}_\sigma\|f\|)[\lambda T^{\alpha\gamma T} + \mu T^{\alpha T}] = c_2 + c_3\mathrm{Lip}_\sigma\|f\|.$$

□

Lemma 3.4. *Suppose that f and g are the solutions of* (3.2)*. Given that Assumption 3.2 holds, then one obtains*

$$\|\mathcal{K}f - \mathcal{K}g\| \leq c_3 \mathrm{Lip}_\sigma \|f - g\|. \tag{3.8}$$

Proof. Take the absolute value of both sides of Eq (3.6) and by Eq (3.3) of Assumption 3.2 to get

$$\begin{aligned}
|\mathcal{K}f(t) - \mathcal{K}g(t)| &\leq \frac{\lambda}{\Gamma(\alpha t)} \int_1^t (t-s)^{\alpha t-1} |\sigma(s, f(s)) - \sigma(s, g(s))| ds \\
&\quad + \frac{\mu}{\Gamma(\alpha t)} \int_t^T (s-t)^{\alpha t-1} |\sigma(s, f(s)) - \sigma(s, g(s))| ds \\
&\leq \frac{\lambda}{\Gamma(\alpha t)} \int_1^t (t-s)^{\alpha t-1} \mathrm{Lip}_\sigma |f(s) - g(s)| ds + \frac{\mu}{\Gamma(\alpha t)} \int_t^T (s-t)^{\alpha t-1} \mathrm{Lip}_\sigma |f(s) - g(s)| ds \\
&\leq \frac{\lambda}{\Gamma(\alpha t)} \mathrm{Lip}_\sigma \sup_{s\in[0,t]} |f(s) - g(s)| \int_0^t (t-s)^{\alpha t-1} ds \\
&\quad + \frac{\mu}{\Gamma(\alpha t)} \mathrm{Lip}_\sigma \sup_{s\in[t,T]} |f(s) - g(s)| \int_t^T (s-t)^{\alpha t-1} ds.
\end{aligned}$$

Therefore, by Eq (2.1), one arrives at

$$\begin{aligned}
|\mathcal{K}f(t) - \mathcal{K}g(t)| &\leq \frac{\mathrm{Lip}_\sigma}{\Gamma(\alpha t)} \Big[\lambda \sup_{s\in[1,t]} |f(s) - g(s)| \frac{(t-1)^{\alpha t}}{\alpha t} + \mu \sup_{s\in[t,T]} |f(s) - g(s)| \frac{(T-t)^{\alpha t}}{\alpha t}\Big] \\
&\leq \mathrm{Lip}_\sigma \frac{1}{\alpha} \Big[\lambda \sup_{s\in[1,t]} |f(s) - g(s)| t^{\alpha\gamma t} + \mu \sup_{s\in[t,T]} |f(s) - g(s)| T^{\alpha t}\Big].
\end{aligned}$$

Take the supremum of both sides over $t \in [1, T]$ and use Eq (3.5) to write

$$\|\mathcal{K}f - \mathcal{K}g\| \leq \frac{1}{\alpha}(\lambda T^{\alpha\gamma T} + \mu T^{\alpha T}) \mathrm{Lip}_\sigma \|f - g\|. \tag{3.9}$$

□

One now establishes the existence and uniqueness of the solution to the fractional integral equation by employing the above Lemma(s).

Theorem 3.5. *If Assumption 3.2 holds, then there is a constant $0 < \mathrm{Lip}_\sigma < \frac{1}{c_3}$ such that* (3.2) *has a unique solution f.*

Proof. By the fixed point theorem we have $\mathcal{K}f = f$, then from (3.7) of Lemma 3.3

$$\|f\| \leq c_2 + c_3 \mathrm{Lip}_\sigma \|f\|,$$

and $\|f\| \, [1 - c_3 \mathrm{Lip}_\sigma] \leq c_2$. Thus, $\|f\| < \infty$ if and only if $c_3 \mathrm{Lip}_\sigma < 1$. Also, suppose by contradiction that (3.2) has the following solutions: $f \neq g$, then by (3.8) of Lemma 3.4, one gets

$$\|f - g\| \leq c_3 \mathrm{Lip}_\sigma \|f - g\|.$$

This gives $\|f - g\|[1 - c_3 \mathrm{Lip}_\sigma] \leq 0$, and since $1 - c_3 \mathrm{Lip}_\sigma > 0$, it must follow that $\|f - g\| \leq 0$. Hence, $\|f - g\| = 0$ by the property of a norm, and consequently, $f - g = 0$, contradicting the assumption that $f \neq g$. □

3.2. Growth estimate

To establish this result, one first introduces the following variant of a Gronwall-type inequality:

Proposition 3.6. [14] *Let $x, g, h \in C([t_0, T), \mathbb{R}_+)$, and $w \in C(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $w(x) > 0$ for $x > 0$, and $b \in C^1([t_0, T), [t_0, T))$ be nondecreasing with $b(t) \le t$ on $[t_0, T)$. If*

$$x(t) \le k + \int_{t_0}^{t} g(s)w(x(s))ds + \int_{b(t_0)}^{b(t)} h(s)w(x(s))ds, \ \ t_0 \le t < T,$$

where k is a nonnegative constant, then for $t_0 \le t < t_1$,

$$x(t) \le G^{-1}\Big(G(k) + \int_{t_0}^{t} g(s)ds + \int_{b(t_0)}^{b(t)} h(s)ds\Big),$$

with $G(r) = \int_1^r \frac{ds}{w(s)}$, $r > 0$ and $t_1 \in (t_0, T)$ chosen so that the right-hand side is well-defined.

Remark 3.7. The above Proposition 3.6 is not sufficient to obtain the desired result, as the functions $g(s)$ and $h(s)$ must depend on two variables.

Consequently, in 2005, Agarwal et al., in [15], generalized the above retarded Gronwall-type inequality to:

$$x(t) \le a(t) + \sum_{i=1}^{n} \int_{b_i(t_0)}^{b_i(t)} g_i(t, s)w_i(x(s))ds, \ \ t_0 \le t < t_1. \tag{3.10}$$

Theorem 3.8 (Theorem 2.1 of [15])**.** *Suppose that the hypotheses of (Theorem 2.1 of [15]) hold and $x(t)$ is a continuous and nonnegative function on $[t_0, t_1)$ satisfying* (3.10)*. Then*

$$x(t) \le W_n^{-1}\Big[W_n(r_n(t)) + \int_{b_n(t_0)}^{b_n(t)} \max_{t_0 \le \tau \le t} g_n(\tau, s)ds\Big], \ \ t_0 \le t \le T_1,$$

where $r_n(t)$ is determined recursively by

$$r_1(t) := a(t_0) + \int_{t_0}^{t} |a'(s)|ds,$$

$$r_{i+1} := W_i^{-1}\Big[W_i(r_i(t)) + \int_{b_i(t_0)}^{b_i(t)} \max_{t_0 \le \tau \le t} g_i(\tau, s)ds\Big], \ \ i = 1, ..., n-1,$$

and $W_i(x, x_i) := \int_{x_i}^{x} \frac{dz}{w_i(z)}$.

Remark 3.9. Now, consider the case where $n = 2$ in (3.10): if

$$x(t) \le a(t) + \int_{b_1(t_0)}^{b_1(t)} g_1(t, s)w_1(x(s))ds + \int_{b_2(t_0)}^{b_2(t)} g_2(t, s)w_2(x(s))ds,$$

then

$$x(t) \le W_2^{-1}\Big[W_2(r_2(t)) + \int_{b_2(t_0)}^{b_2(t)} \max_{t_0 \le \tau \le t} g_2(\tau, s)ds\Big],$$

with $r_2(t) = W_1^{-1}\Big[W_1(r_1(t)) + \int_{b_1(t_0)}^{b_1(t)} \max_{t_0 \le \tau \le t} g_1(\tau, s)ds\Big]$.

In what follows, we take $w_1(x(s)) = w_2(x(s)) = x(s)$, $b_1(t_0) = 1$, $b_1(t) = b_2(t_0) = t$, and $b_2(t) = T$ to get the inequality

$$x(t) \le a(t) + \int_1^t g_1(t, s)x(s)ds + \int_t^T g_2(t, s)x(s)ds.$$

Next, one establishes the growth bound for a solution to (3.2).

Theorem 3.10. *Suppose that Assumption 3.2 holds; then there exists $\alpha > 0$ such that the solution to Eq (3.2) satisfies the growth bound*

$$|f(t)| \le \frac{c_1\mu}{\alpha} T^{\alpha t} \exp\left[\frac{1}{\alpha}\Big(\lambda \mathrm{Lip}_\sigma t^{\alpha\gamma t} + \mu \mathrm{Lip}_\sigma T^{\alpha t}\Big)\right],$$

for all $t \in [1, T]$, $T < \infty$, and some positive constant c_1.

Proof. Following the proof of Lemma 3.3,

$$\begin{aligned}
|f(t)| &\le c_1 \frac{\lambda}{\Gamma(\alpha t)} \int_1^t (t-s)^{\alpha t-1} ds + \mathrm{Lip}_\sigma \frac{\lambda}{\Gamma(\alpha t)} \int_1^t (t-s)^{\alpha t-1} |f(s)| ds \\
&\quad + \ c_1 \frac{\mu}{\Gamma(\alpha t)} \int_t^T (s-t)^{\alpha t-1} ds + \mathrm{Lip}_\sigma \frac{\mu}{\Gamma(\alpha t)} \int_t^T (s-t)^{\alpha t-1} |f(s)| ds \\
&= c_1 \frac{\lambda}{\Gamma(\alpha t)} \frac{(t-1)^{\alpha t}}{\alpha t} + \mathrm{Lip}_\sigma \frac{\lambda}{\Gamma(\alpha t)} \int_1^t (t-s)^{\alpha t-1} |f(s)| ds \\
&\quad + \ c_1 \frac{\mu}{\Gamma(\alpha t)} \frac{(T-t)^{\alpha t}}{\alpha t} + \mathrm{Lip}_\sigma \frac{\mu}{\Gamma(\alpha t)} \int_t^T (s-t)^{\alpha t-1} |f(s)| ds \\
&\le \frac{c_1}{\alpha}\Big[\lambda t^{\alpha\gamma t} + \mu T^{\alpha t}\Big] + \mathrm{Lip}_\sigma \frac{\lambda}{\Gamma(\alpha t)} \int_1^t (t-s)^{\alpha t-1} |f(s)| ds + \mathrm{Lip}_\sigma \frac{\mu}{\Gamma(\alpha t)} \int_t^T (s-t)^{\alpha t-1} |f(s)| ds.
\end{aligned}$$

Define $\Phi(t) = |f(t)|$ for all $t \in [1, T]$ so that

$$\begin{aligned}
\Phi(t) &\le \frac{c_1}{\alpha}\Big[\lambda t^{\alpha\gamma t} + \mu T^{\alpha t}\Big] + \frac{\lambda \mathrm{Lip}_\sigma}{\Gamma(\alpha t)} \int_1^t (t-s)^{\alpha t-1} \Phi(s) ds + \frac{\mu \mathrm{Lip}_\sigma}{\Gamma(\alpha t)} \int_t^T (s-t)^{\alpha t-1} \Phi(s) ds \\
&\le \frac{c_1}{\alpha}\Big[\lambda + \mu T^{\alpha t}\Big] + \frac{\lambda \mathrm{Lip}_\sigma}{\Gamma(\alpha t)} \int_1^t (t-s)^{\alpha t-1} \Phi(s) ds + \frac{\mu \mathrm{Lip}_\sigma}{\Gamma(\alpha t)} \int_t^T (s-t)^{\alpha t-1} \Phi(s) ds.
\end{aligned}$$

To apply Theorem 3.8 to the our inequality, let $x(t) = \Phi(t)$, and $a(t) = \frac{c_1}{\alpha}\Big[\lambda + \mu T^{\alpha t}\Big]$ so that $a'(t) = c_1 \mu T^{\alpha t} \ln(T)$, and therefore,

$$r_1(t) = c_1\mu \ln(T) \int_1^t T^{\alpha s} ds = c_1 \mu \ln(T) \frac{T^{\alpha t} - T^{\alpha}}{\alpha \ln(T)} = \frac{c_1 \mu}{\alpha}(T^{\alpha t} - T^{\alpha}).$$

For W_2, one obtains that

$$W_2(x, x_2) = \int_{x_2}^x \frac{dz}{z} = \ln(x) - \ln(x_2).$$

We take $x_2 = 1$ for convenience and $W_2(x) = \ln(x)$ with the inverse $W_2^{-1}(x) = e^x$. Similarly, $W_1(x) = \ln x$ with its inverse $W_1^{-1}(x) = e^x$.

Next, define non-negative functions $g_1,\ g_2 : [1,T] \times [1,T] \to \mathbb{R}_+$ for a fixed t as follows:

$$g_1(\tau, s) := \begin{cases} \frac{\lambda \mathrm{Lip}_\sigma}{\Gamma(\alpha t)}(\tau - s)^{\alpha t - 1},\ 1 \le s < \tau \\ \\ \frac{\lambda \mathrm{Lip}_\sigma}{\Gamma(\alpha t)}(s - \tau)^{\alpha t - 1},\ 1 \le \tau < s, \end{cases}$$

and

$$g_2(\tau, s) := \begin{cases} \frac{\mu \mathrm{Lip}_\sigma}{\Gamma(\alpha t)}(\tau - s)^{\alpha t - 1},\ s \le \tau < T \\ \\ \frac{\mu \mathrm{Lip}_\sigma}{\Gamma(\alpha t)}(s - \tau)^{\alpha t - 1},\ t \le \tau < s. \end{cases}$$

Note that the Gamma function is strictly positive and increasing. That is, $\Gamma(t)$ is decreasing on $(0, t_0)$ and increasing on $(t_0, \infty,)$ where $1 < t_0 < 2$. Now, consider two cases.

- Case 1: Consider when $1 \le s < \tau$. Not that $\Gamma(\alpha\tau)$ is increasing, so g_1 is continuous and increasing (continuously increasing); hence,

$$\max_{1 \le \tau \le t} g_1(t, s) = \frac{\lambda \mathrm{Lip}_\sigma}{\Gamma(\alpha t)}(t - s)^{\alpha t - 1},$$

  and we have

$$\begin{aligned} r_2(t) &= \exp\left[\ln\left(\frac{c_1\mu}{\alpha}(T^{\alpha t} - T^{\alpha})\right) + \frac{\lambda \mathrm{Lip}_\sigma}{\Gamma(\alpha t)}\int_1^t (t - s)^{\alpha t - 1} ds\right] \\ &= \exp\left[\ln\left(\frac{c_1\mu}{\alpha}(T^{\alpha t} - T^{\alpha})\right) + \frac{\lambda \mathrm{Lip}_\sigma}{\Gamma(\alpha t)}\frac{(t-1)^{\alpha t}}{\alpha t}\right]. \end{aligned}$$

  Consider also, $t \le \tau < s$. Given that g_2 is continuously increasing, and

$$\max_{t \le \tau \le s} g_2(\tau, s) = \frac{\mu \mathrm{Lip}_\sigma}{\Gamma(\alpha t)}(s - t)^{\alpha t - 1}.$$

  Thus,

$$\begin{aligned} \Phi(t) &\le \exp\left[\ln(r_2(t)) + \frac{\mu \mathrm{Lip}_\sigma}{\Gamma(\alpha t)}\int_t^T (s - t)^{\alpha t - 1} ds\right] \\ &= \exp\left[\ln\left(\frac{c_1\mu}{\alpha}(T^{\alpha t} - T^{\alpha})\right) + \frac{\lambda \mathrm{Lip}_\sigma}{\Gamma(\alpha t)}\frac{(t-1)^{\alpha t}}{\alpha t} + \frac{\mu \mathrm{Lip}_\sigma}{\Gamma(\alpha t)}\frac{(T-t)^{\alpha t}}{\alpha t}\right] \\ &= \left(\frac{c_1\mu}{\alpha}(T^{\alpha t} - T^{\alpha})\right)\exp\left[\frac{\lambda \mathrm{Lip}_\sigma}{\Gamma(\alpha t)}\frac{(t-1)^{\alpha t}}{\alpha t} + \frac{\mu \mathrm{Lip}_\sigma}{\Gamma(\alpha t)}\frac{(T-t)^{\alpha t}}{\alpha t}\right]. \end{aligned}$$

- Case 2: For $1 \le \tau < s$, one follows similar steps as in case 1, which gives that g_1 is continuously increasing and

$$\max_{1 \le s \le t} g_1(\tau, s) = \frac{\lambda \mathrm{Lip}_\sigma}{\Gamma(\alpha t)}(t - \tau)^{\alpha t - 1}.$$

Hence,

$$\begin{aligned} r_2(t) &= \exp\Big[\ln\Big(\frac{c_1\mu}{\alpha}(T^{\alpha t}-T^{\alpha})\Big)+\frac{\lambda\mathrm{Lip}_\sigma}{\Gamma(\alpha t)}\int_1^t (t-\tau)^{\alpha t-1}d\tau\Big] \\ &= \exp\Big[\ln\Big(\frac{c_1\mu}{\alpha}(T^{\alpha t}-T^{\alpha})\Big)+\frac{\lambda\mathrm{Lip}_\sigma}{\Gamma(\alpha t)}\frac{(t-1)^{\alpha t}}{\alpha t}\Big]. \end{aligned}$$

On the other hand, g_2 is continuously increasing, and

$$\max_{s\le\tau\le T} g_2(\tau,s)=\frac{\mu\mathrm{Lip}_\sigma}{\Gamma(\alpha t)}(T-s)^{\alpha t-1}.$$

Therefore,

$$\begin{aligned} \Phi(t) &\le \exp\Big[\ln(r_2(t))+\frac{\mu\mathrm{Lip}_\sigma}{\Gamma(\alpha t)}\int_t^T (T-s)^{\alpha t-1}ds\Big] \\ &= \exp\Big[\ln\Big(\frac{c_1\mu}{\alpha}(T^{\alpha t}-T^{\alpha})\Big)+\frac{\lambda\mathrm{Lip}_\sigma}{\Gamma(\alpha t)}\frac{(t-1)^{\alpha t}}{\alpha t}+\frac{\mu\mathrm{Lip}_\sigma}{\Gamma(\alpha t)}\frac{(T-t)^{\alpha t}}{\alpha t}\Big] \\ &= \Big(\frac{c_1\mu}{\alpha}(T^{\alpha t}-T^{\alpha})\Big)\exp\Big[\frac{\lambda\mathrm{Lip}_\sigma}{\Gamma(\alpha t)}\frac{(t-1)^{\alpha t}}{\alpha t}+\frac{\mu\mathrm{Lip}_\sigma}{\Gamma(\alpha t)}\frac{(T-t)^{\alpha t}}{\alpha t}\Big] \\ &\le \frac{c_1\mu}{\alpha}T^{\alpha t}\exp\Big[\frac{\lambda\mathrm{Lip}_\sigma}{\Gamma(\alpha t)}\frac{t^{\alpha t}}{\alpha t}+\frac{\mu\mathrm{Lip}_\sigma}{\Gamma(\alpha t)}\frac{T^{\alpha t}}{\alpha t}\Big]\le\frac{c_1\mu}{\alpha}T^{\alpha t}\exp\Big[\frac{1}{\alpha}\Big(\lambda\mathrm{Lip}_\sigma t^{\alpha\gamma t}+\mu\mathrm{Lip}_\sigma T^{\alpha t}\Big)\Big], \end{aligned}$$

and this completes the proof. □

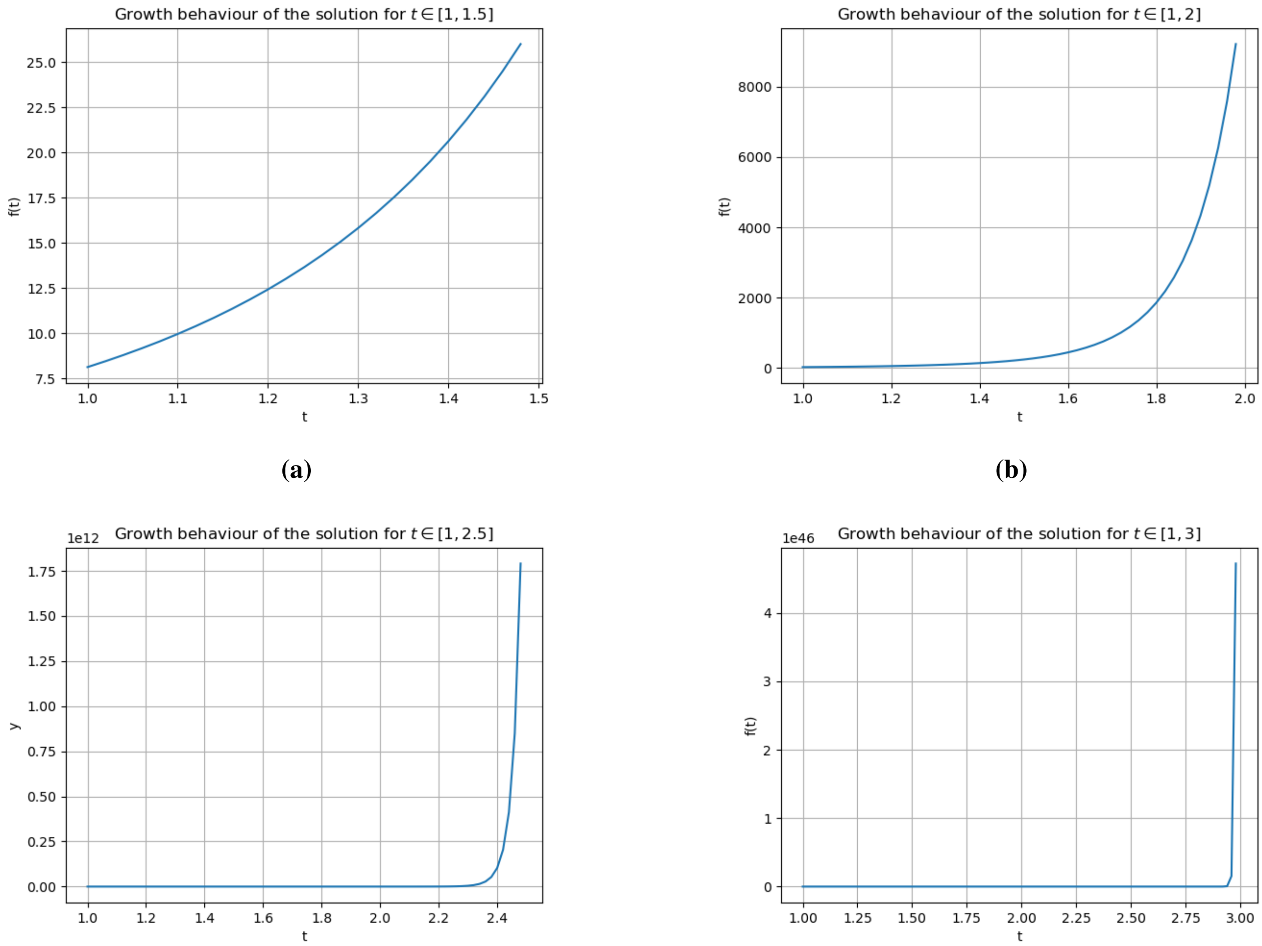
Example 3.11. Let $\alpha=\frac{3}{2}$ and the Lipschitz function $\sigma:\mathbb{R}\to\mathbb{R}$ be given by $\sigma(t,f(t))=\sin(f(t))$, $\forall\, t\in[1,T]$ with $\mathrm{Lip}_\sigma=1$. The fractional integral equation becomes

$$f(t)=\frac{\lambda}{\Gamma(\frac{3}{2}t)}\int_1^t (t-s)^{\frac{3}{2}t-1}\sin(f(s))ds+\frac{\mu}{\Gamma(\frac{3}{2}t)}\int_t^T (s-t)^{\frac{3}{2}t-1}\sin(f(s))ds.$$

For convenience, let $c_1=\mu=\lambda=1$ and $\gamma=\frac{6}{10}$ to obtain the estimate on the growth bound as

$$|f(t)|\le\frac{2}{3}T^{\frac{3}{2}t}\exp\Big[\frac{2}{3}\Big(t^{\frac{9}{10}t}+T^{\frac{3}{2}t}\Big)\Big],\ \ \forall\, t\in[1,T].$$

Next, we present graphical representations of the growth estimates of the solution for various values of t. The plots in Figure 1 below indicate that the solution exhibits standard exponential growth for small values of t, and as t increases or the interval $[1,T]$ becomes larger, the growth behavior becomes progressively steeper.



(a) **(b)** **(c)** **(d)**

Figure 1. Exponential growth behavior of the solution for values of $t \in [1, T]$.

4. Conclusions

Variable-order fractional differential and integral equations play a significant role in modeling complex biological and financial phenomena. In this paper, we established existence and uniqueness results for a class of fractional integral equations with variable-order kernels and, moreover, derived the exponential growth rate of their solutions. Lastly, we presented some numerical examples and graphical representations of the growth estimates. Future research may focus on analyzing the behavior of solutions under specific choices of kernel functions and exploring broader applications in real-world systems.

Use of Generative-AI tools declaration

The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no potential conflicts of interest.

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