Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(12): 28277–28294.
DOI:10.3934/math.20251244
Received: 28 September 2025
Revised: 10 November 2025
Accepted: 14 November 2025
Published: 02 December 2025

Research article

Generalized derivations and their embedding in ω-hom-Lie algebras

Nof T. Alharbi*, **Ishraga A. Mohamed, Halah A. Abd Almeneem and Norah Saleh Barakat**

Mathematics Department, University College in Al-Darb, Jazan University, Jazan 82817, Saudi Arabia

* **Correspondence:** Email: nofalharbi@jazanu.edu.sa, nof20081900@hotmail.com.

Abstract: In this study, we explored the algebraic structure of generalized derivations within finite-dimensional ω-hom-Lie algebras over a field K, emphasizing their symmetry properties in nonassociative settings. We established a novel embedding theorem, proving that every compatible quasiderivation of an ω-hom-Lie algebra can be represented as a compatible derivation in a larger, symmetrically constructed ω-hom-Lie algebra. This result extended classical Lie algebra derivation theory, leveraging the skew-symmetric bilinear form ω and the homomorphism ϕ to preserve structural symmetries. Additionally, we developed a computational algorithm, inspired by Gröbner basis techniques in commutative algebra for solving systems of polynomial equations arising from the derivation conditions, to explicitly calculate compatible generalized derivations and quasiderivations for all 3-dimensional non-Lie complex ω-hom-Lie algebras with $\phi = \text{id}$ (i.e., the corresponding ω-Lie algebras). This approach provided a practical tool for analyzing their structural properties, revealing symmetries in their derivation algebras. Our findings contribute to the broader theory of Hom-Lie algebras, offering new insights into their algebraic and geometric applications, particularly in deformation theory and physics. The results enhance the understanding of symmetry transformations in nonassociative algebras, with potential implications for symmetric structures in mathematical physics.

Keywords: generalized derivations; quasiderivations; ω-Lie algebras; ω-hom-Lie algebras
Mathematics Subject Classification: Primary 16W25, 17B10; Secondary 17B40, 16S99

1. Introduction

The study of Hom-Lie algebras and their generalizations has become a central theme in nonassociative algebra, offering a flexible framework for extending classical Lie theory to structures with twisted identities [1–3]. A Hom-Lie algebra, as defined in [2], is a triple $(L, [\cdot,\cdot], \phi)$, where L is a vector space over a field K of characteristic zero, $[\cdot,\cdot] : L \otimes L \to L$ is a skew-symmetric bilinear map, and $\phi : L \to L$ is an algebra homomorphism satisfying the Hom-Jacobi identity:

$$[[x, y], \phi(z)] + [[y, z], \phi(x)] + [[z, x], \phi(y)] = 0, \tag{1.1}$$

for all $x, y, z \in L$. A Hom-Lie algebra is regular if ϕ is invertible and involutive if $\phi^2 = \mathrm{id}$. This structure, which generalizes classical Lie algebras (recovered when $\phi = \mathrm{id}$), has been extensively explored for its applications in geometry, physics, and deformation theory [4–6]. Extensions include Lie triple derivations [7, 8] and higher derivations in hom-Lie superalgebras [8], which our work complements by focusing on ω-twisted variants. Building on this foundation, ω-hom-Lie algebras introduce an additional skew-symmetric bilinear form $\omega : L \times L \to K$, modifying the Hom-Jacobi identity to the ω-hom-Jacobi identity:

$$[[x, y], \phi(z)] + [[y, z], \phi(x)] + [[z, x], \phi(y)] = \omega(x, y) \cdot z + \omega(y, z) \cdot x + \omega(z, x) \cdot y, \tag{1.2}$$

for all $x, y, z \in L$. This identity, which reduces to the classical Hom-Lie identity when $\omega = 0$, defines an ω-hom-Lie algebra $(L, [\cdot, \cdot], \phi, \omega)$ and provides a rich setting for studying nonassociative structures with geometric and algebraic significance [2,9]. The form ω is often invariant under adjoint action, satisfying $\omega([x, y], z) + \omega(y, [x, z]) = 0$, linking to physical symmetries like Killing forms. Researchers have classified low-dimensional complex ω-hom-Lie algebras, particularly in dimensions up to 5, revealing their structural diversity, with 3-dimensional non-Lie cases being especially insightful [10, 11]. When $\phi = \mathrm{id}$, these reduce to ω-Lie algebras. Derivations and their generalizations are fundamental tools for understanding the structure, representations, and automorphisms of algebraic systems [1,6]. In the context of ω-hom-Lie algebras, a generalized derivation is a linear map $f : L \to L$ for which there exist linear maps $f_1, f_2 : L \to L$ satisfying

$$[f(x), y] = f_2([x, y]) - [x, f_1(y)], \tag{1.3}$$

for all $x, y \in L$. The set of generalized derivations, denoted by $\mathrm{GDer}(L)$, forms a Lie subalgebra of $\mathrm{gl}(L)$ and includes the derivation algebra $\mathrm{Der}(L)$ [12]. The *compatible* condition for a generalized derivation f requires it to preserve the bilinear form ω, i.e., $\omega(f(x), y) + \omega(x, f(y)) = 0$ for all $x, y \in L$. This is the natural infinitesimal analogue of automorphisms ψ satisfying $\omega(\psi(x), \psi(y)) = \omega(x, y)$, ensuring f lies in the Lie algebra of the automorphism group preserving ω. It generalizes adjoint invariance in classical Lie algebras and is crucial for symmetry-preserving extensions in nonassociative settings, as explored in [1] for automorphisms. The set of compatible generalized derivations, denoted by $\mathrm{GDer}_c(L)$, forms a vector space closely tied to the automorphism group of L [1]. This leads to the tower of inclusions $\mathrm{Der}_c(L) \subseteq \mathrm{GDer}_c(L) \subseteq \mathrm{GDer}(L) \subseteq \mathrm{gl}(L)$, which frames our structural analysis. Our embedding theorem extends results on triple derivations [8] to this ω-hom framework. In this paper, we pursue three major objectives in the study of generalized derivations for finite-dimensional ω-hom-Lie algebras: First, we explore the algebraic structure of $\mathrm{GDer}_c(L)$, examining the relationships between the terms in the tower $\mathrm{Der}_c(L) \subseteq \mathrm{GDer}_c(L) \subseteq \mathrm{GDer}(L)$. Second, we extend a classical result from Lie algebras [5] by proving that every compatible quasiderivation of an ω-hom-Lie algebra can be embedded as a compatible derivation in a larger ω-hom-Lie algebra, adapting the framework to account for the ω-hom-Jacobi identity and the homomorphism ϕ. Third, we develop a computational approach, inspired by Gröbner basis techniques in commutative algebra [10, 13, 14], to explicitly calculate compatible generalized derivations and quasiderivations for 3-dimensional non-Lie complex

ω-hom-Lie algebras with $\phi = \mathrm{id}$ (corresponding to the classified ω-Lie algebras [10, 11]), leveraging their classifications.

The paper is structured as follows: In Section 2, we establish key properties of compatible generalized derivations, including lemmas on their structure and decompositions. In Section 3, we present our embedding theorem for compatible quasiderivations and a decomposition result for $\mathrm{Der}_c(L)$. In Section 4, we provide explicit computations for 3-dimensional ω-hom-Lie algebras with $\phi = \mathrm{id}$, illustrating our theoretical findings with concrete examples. Throughout, we assume K is a field of characteristic zero, $\mathbb{C}$ denotes the complex field, and all algebras are finite-dimensional.

2. Generalized derivations

In this section, we establish foundational concepts and results concerning generalized derivations of ω-hom-Lie algebras. While some ideas may echo those in [5] for nonassociative algebras, we adapt and refine them for ω-hom-Lie algebras, providing detailed proofs to ensure clarity and precision in this generalized context. Let K be a field of characteristic zero, and let $(L, [\cdot,\cdot], \phi, \omega)$ be a finite-dimensional ω-hom-Lie algebra over K, where L is a vector space, $[\cdot,\cdot] : L \times L \to L$ is a skew-symmetric bilinear bracket, $\phi : L \to L$ is an algebra homomorphism, and $\omega : L \times L \to K$ is a skew-symmetric bilinear form satisfying the ω-hom-Jacobi identity (1.2) for all $x, y, z \in L$. We denote by $\mathrm{gl}(L)$ the general linear Lie algebra on L, equipped with the commutator $[f, g] = f \circ g - g \circ f$.

Lemma 2.1. *Let $a \in K$ be a scalar and $f, g \in \mathrm{gl}(L)$ be two compatible linear maps, i.e., $\omega(f(x), y) + \omega(x, f(y)) = 0$ for all $x, y \in L$. Then $f + g$, $a \cdot f$, and $[f, g]$ are also compatible.*

Proof. Since ω is bilinear, the compatibility of $f + g$ and $a \cdot f$ follows directly:

$$\omega((f+g)(x), y) + \omega(x, (f+g)(y)) = \omega(f(x), y) + \omega(g(x), y) + \omega(x, f(y)) + \omega(x, g(y)) = 0,$$

as f and g are compatible. Similarly,

$$\omega((a \cdot f)(x), y) + \omega(x, (a \cdot f)(y)) = a \cdot \big(\omega(f(x), y) + \omega(x, f(y))\big) = 0.$$

Now consider $[f, g] = f \circ g - g \circ f$. We compute

$$\omega([f, g](x), y) = \omega(f(g(x)) - g(f(x)), y) = \omega(f(g(x)), y) - \omega(g(f(x)), y).$$

Since ω is skew-symmetric,

$$\omega(f(g(x)), y) = -\omega(g(x), f(y)), \qquad \omega(g(f(x)), y) = -\omega(f(x), g(y)).$$

Thus,

$$\omega([f, g](x), y) = -\omega(g(x), f(y)) + \omega(f(x), g(y)).$$

Similarly,

$$\begin{aligned}\omega(x, [f, g](y)) &= \omega(x, f(g(y)) - g(f(y))) = \omega(x, f(g(y))) - \omega(x, g(f(y))) \\ &= -\omega(f(g(y)), x) + \omega(g(f(y)), x) = \omega(g(y), f(x)) - \omega(f(y), g(x)).\end{aligned}$$

Adding these two expressions gives

$$\omega([f, g](x), y) + \omega(x, [f, g](y)) = \big(-\omega(g(x), f(y)) + \omega(f(x), g(y))\big) + \big(\omega(g(y), f(x)) - \omega(f(y), g(x))\big) = 0,$$

since ω is skew-symmetric. Hence, $[f, g]$ is compatible. □

A linear map $f : L \to L$ is a *generalized derivation* of the ω-hom-Lie algebra L if there exist linear maps $f_1, f_2 : L \to L$, such that (1.3), for all $x, y \in L$. The set of all generalized derivations, denoted by $\mathrm{GDer}(L)$, is a Lie subalgebra of $\mathrm{gl}(L)$ [5]. A generalized derivation f is a *quasiderivation* if $f_1 = f$, i.e., there exists a linear map f_2, such that

$$[f(x), y] + [x, f(y)] = f_2([x, y]), \tag{2.1}$$

for all $x, y \in L$. The set of quasiderivations, denoted by $\mathrm{QDer}(L)$, is a nonempty Lie subalgebra of $\mathrm{GDer}(L)$ [5]. A generalized derivation f is *compatible* if it satisfies $\omega(f(x), y) + \omega(x, f(y)) = 0$ for all $x, y \in L$. The set of compatible generalized derivations is denoted by $\mathrm{GDer}_c(L)$, and the set of compatible quasiderivations is denoted by $\mathrm{QDer}_c(L)$. Thus, the derivation algebra $\mathrm{Der}_c(L)$ of compatible derivations (where $f([x, y]) = [f(x), y] + [x, f(y)]$ and $\omega(f(x), y) + \omega(x, f(y)) = 0$) is contained in $\mathrm{QDer}_c(L)$, which is contained in $\mathrm{GDer}_c(L)$, forming the tower:

$$\mathrm{Der}_c(L) \subseteq \mathrm{QDer}_c(L) \subseteq \mathrm{GDer}_c(L) \subseteq \mathrm{GDer}(L).$$

We define the *compatible quasicentroid* of L, denoted by $\mathrm{QCent}_c(L)$, as the set of all compatible linear maps $f : L \to L$, such that

$$[f(x), y] = [x, f(y)],$$

for all $x, y \in L$. Similarly, the *compatible centroid* of L, denoted by $\mathrm{Cent}_c(L)$, consists of all compatible linear maps $f : L \to L$, satisfying:

$$[f(x), y] = [x, f(y)] = f([x, y]),$$

for all $x, y \in L$. These sets are vector spaces, and $\mathrm{Cent}_c(L) \subseteq \mathrm{QCent}_c(L)$, as every map in $\mathrm{Cent}_c(L)$ satisfies the quasicentroid condition. In the following, we explore the structure of $\mathrm{GDer}_c(L)$ and $\mathrm{QDer}_c(L)$, providing detailed characterizations and decompositions, particularly for low-dimensional ω-hom-Lie algebras. Our results extend the framework of [5] by incorporating the homomorphism ϕ and the bilinear form ω, offering new insights into the derivation theory of these algebras, including connections to triple derivations [15, 16].

Proposition 2.1. *Let $(L, [\cdot, \cdot], \phi, \omega)$ be a finite-dimensional ω-hom-Lie algebra over a field K of characteristic zero. The set $\mathrm{GDer}_c(L)$ of compatible generalized derivations is a Lie subalgebra of $\mathrm{GDer}(L)$, and the set $\mathrm{QDer}_c(L)$ of compatible quasiderivations is a Lie subalgebra of $\mathrm{GDer}_c(L)$.*

Proof. We first show that $\mathrm{GDer}_c(L)$ is a Lie subalgebra of $\mathrm{GDer}(L)$. The set $\mathrm{GDer}_c(L)$ is nonempty, as it contains the zero map, which is compatible (since $\omega(0, y) + \omega(x, 0) = 0$) and a generalized derivation (since $[0, y] = 0 = 0 \cdot [x, y] - [x, 0]$). Since $\mathrm{GDer}(L)$ is a Lie subalgebra of $\mathrm{gl}(L)$ [5], for any $f, g \in \mathrm{GDer}_c(L)$ and scalar $a \in K$, the maps $f + g$, $a \cdot f$, and $[f, g] = f \circ g - g \circ f$ are in $\mathrm{GDer}(L)$. By Lemma 2.1, since f and g are compatible (i.e., $\omega(f(x), y) + \omega(x, f(y)) = 0$ and similarly for g), the maps $f + g$, $a \cdot f$, and $[f, g]$ are also compatible. Thus, $f + g$, $a \cdot f$, and $[f, g]$ lie in $\mathrm{GDer}_c(L)$, confirming that $\mathrm{GDer}_c(L)$ is a Lie subalgebra of $\mathrm{GDer}(L)$. For the second statement, we prove that $\mathrm{QDer}_c(L)$ is a Lie subalgebra of $\mathrm{GDer}_c(L)$. The set $\mathrm{QDer}_c(L)$ is nonempty, as it contains all compatible derivations $\mathrm{Der}_c(L) \subseteq \mathrm{QDer}_c(L)$, and $\mathrm{Der}_c(L)$ is nonempty (e.g., the zero map). By [5], $\mathrm{QDer}(L)$ is a Lie subalgebra of $\mathrm{GDer}(L)$, so for any $f, g \in \mathrm{QDer}_c(L)$ and $a \in K$, the maps $f + g$, $a \cdot f$, and $[f, g]$ are in QDer(L). A quasiderivation $f \in \mathrm{QDer}_c(L)$ satisfies (2.1) for some linear map f_2 and is compatible, i.e., $\omega(f(x), y) + \omega(x, f(y)) = 0$. By Lemma 2.1, $f + g$, $a \cdot f$, and $[f, g]$ are compatible. To verify that they are quasiderivations, note that QDer(L) is closed under addition, scalar multiplication, and Lie brackets [5]. Since $f + g$, $a \cdot f$, and $[f, g]$ are in QDer(L) and compatible, they belong to $\mathrm{QDer}_c(L)$. Hence, $\mathrm{QDer}_c(L)$ is a Lie subalgebra of $\mathrm{GDer}_c(L)$. Following [5], we denote a quasiderivation by the pair (f, f_2), where f_2 is the linear map associated with f, satisfying (2.1). The map f_2 will be crucial in Section 3 for addressing the embedding of compatible quasiderivations into compatible derivations of a larger ω-hom-Lie algebra. □

Lemma 2.2. *Let $c \in K$ and $f, g \in \mathrm{QDer}(L)$, where $(L, [\cdot, \cdot], \phi, \omega)$ is a finite-dimensional ω-hom-Lie algebra over a field K of characteristic zero. Then*

(1) $(c \cdot f)_2 = c \cdot f_2$ *and* $(f + g)_2 = f_2 + g_2$.
(2) $[f, g]_2 = [f_2, g_2]$.

Proof. Let $x, y \in L$ be arbitrary. (1) Scalar and sum: Since (2.1), we have

$$[(cf)(x), y] + [x, (cf)(y)] = [cf(x), y] + [x, cf(y)] = c([f(x), y] + [x, f(y)]) = c\, f_2([x, y]),$$

so $(cf)_2 = cf_2$. Similarly,

$$\begin{aligned}&[(f + g)(x), y] + [x, (f + g)(y)]\\ =&[f(x) + g(x), y] + [x, f(y) + g(y)] = \big([f(x), y] + [x, f(y)]\big) + \big([g(x), y] + [x, g(y)]\big)\\ =&f_2([x, y]) + g_2([x, y]) = (f_2 + g_2)([x, y]),\end{aligned}$$

hence $(f + g)_2 = f_2 + g_2$. (2) Bracket: $[f, g]_2 = [f_2, g_2]$ Recall $[f, g] = f \circ g - g \circ f$. Compute

$$[f, g]_2([x, y]) = [[f, g](x), y] + [x, [f, g](y)] = [f(g(x)), y] - [g(f(x)), y] + [x, f(g(y))] - [x, g(f(y))].$$

Add and subtract the convenient terms $[g(x), f(y)]$ and $[f(x), g(y)]$ to form pairs to which the quasiderivation identities apply:

$$\begin{aligned}[f, g]_2([x, y]) = &\big([f(g(x)), y] + [g(x), f(y)]\big) + \big([x, f(g(y))] + [f(x), g(y)]\big)\\ &- \big([g(f(x)), y] + [f(x), g(y)]\big) - \big([x, g(f(y))] + [g(x), f(y)]\big).\end{aligned}$$

Now use the quasiderivation identities

$$[f(u), v] + [u, f(v)] = f_2([u, v]), \qquad [g(u), v] + [u, g(v)] = g_2([u, v])$$

with appropriate choices of u, v. Applying them to each grouped pair yields

$$\begin{aligned}[f, g]_2([x, y]) &= f_2([g(x), y]) + f_2([x, g(y)]) - g_2([f(x), y]) - g_2([x, f(y)])\\ &= f_2([g(x), y] + [x, g(y)]) - g_2([f(x), y] + [x, f(y)])\\ &= f_2(g_2([x, y])) - g_2(f_2([x, y]))\\ &= [f_2, g_2]([x, y]).\end{aligned}$$

Since this holds for all $[x, y] \in [L, L]$ (and both sides define linear maps on $[L, L]$), we conclude $[f, g]_2 = [f_2, g_2]$. □

Remark 2.1. *The following statements hold, assuming results from [1] and Lemma 2.1:*

(1) $[\mathrm{Der}_c(L), \mathrm{Cent}_c(L)] \subseteq \mathrm{Cent}_c(L)$.
(2) $[\mathrm{QDer}_c(L), \mathrm{QCent}_c(L)] \subseteq \mathrm{QCent}_c(L)$.
(3) $\mathrm{Cent}_c(L) \subseteq \mathrm{QCent}_c(L)$.
(4) $[\mathrm{QCent}_c(L), \mathrm{QCent}_c(L)] \subseteq \mathrm{QDer}_c(L)$.

Proof. We verify each statement, relying on [1] for analogous results on non-compatible sets and Lemma 2.1 for compatibility.

(1) $[\mathrm{Der}_c(L), \mathrm{Cent}_c(L)] \subseteq \mathrm{Cent}_c(L)$: Let $d \in \mathrm{Der}_c(L)$, so $d([x, y]) = [d(x), y]+[x, d(y)]$ and $\omega(d(x), y)+\omega(x, d(y)) = 0$. Let $f \in \mathrm{Cent}_c(L)$, so $[f(x), y] = [x, f(y)] = f([x, y])$ and $\omega(f(x), y)+\omega(x, f(y))=0$. The Lie bracket is $[d, f] = d \circ f - f \circ d$. We need $[d, f] \in \mathrm{Cent}_c(L)$, which follows from computations similar to those in [1] and the compatibility conditions ensured by Lemma 2.1.
(2) $[\mathrm{QDer}_c(L), \mathrm{QCent}_c(L)] \subseteq \mathrm{QCent}_c(L)$: Let $f \in \mathrm{QDer}_c(L)$, so $[f(x), y] + [x, f(y)] = f_2([x, y])$ for some f_2, and $\omega(f(x), y) + \omega(x, f(y)) = 0$. Let $g \in \mathrm{QCent}_c(L)$, so $[g(x), y] = [x, g(y)]$ and $\omega(g(x), y) + \omega(x, g(y)) = 0$. Compute

$$[[f, g](x), y] = [f(g(x)), y] - [g(f(x)), y],$$

$$[x, [f, g](y)] = [x, f(g(y))] - [x, g(f(y))].$$

Using the properties of f and g, we find

$$[[f, g](x), y] + [x, [f, g](y)] = f_2([x, g(y)]) - f_2([x, g(y)]) = 0.$$

Thus, $[f, g]$ satisfies the quasicentroid condition. Compatibility of $[f, g]$ follows from Lemma 2.1, so $[f, g] \in \mathrm{QCent}_c(L)$.
(3) $\mathrm{Cent}_c(L) \subseteq \mathrm{QCent}_c(L)$: If $f \in \mathrm{Cent}_c(L)$, then $[f(x), y] = [x, f(y)] = f([x, y])$ and $\omega(f(x), y) + \omega(x, f(y)) = 0$. Hence, f satisfies the condition for $\mathrm{QCent}_c(L)$, and compatibility is preserved. This inclusion is immediate.
(4) $[\mathrm{QCent}_c(L), \mathrm{QCent}_c(L)] \subseteq \mathrm{QDer}_c(L)$: Let $f, g \in \mathrm{QCent}_c(L)$, so $[f(x), y] = [x, f(y)]$ and similarly for g, with both maps compatible. Then

$$[[f, g](x), y] + [x, [f, g](y)] = [f(g(x)), y] - [g(f(x)), y] + [x, f(g(y))] - [x, g(f(y))].$$

Using identities from [1] and the symmetry of quasicentroids, we get

$$[[f, g](x), y] + [x, [f, g](y)] = h([x, y]),$$

for some linear map h, showing $[f, g] \in \mathrm{QDer}(L)$. Compatibility follows from Lemma 2.1, hence $[f, g] \in \mathrm{QDer}_c(L)$.

□

Proposition 2.2. *If* $\mathrm{QDer}_c(L) = \mathrm{QDer}(L)$ *or* $\mathrm{QCent}_c(L) = \mathrm{QCent}(L)$, *then* $\mathrm{GDer}_c(L) = \mathrm{QDer}_c(L) + \mathrm{QCent}_c(L)$.

Proof. Assume that either $\mathrm{QDer}_c(L) = \mathrm{QDer}(L)$ or $\mathrm{QCent}_c(L) = \mathrm{QCent}(L)$. We show

$$\mathrm{GDer}_c(L) = \mathrm{QDer}_c(L) + \mathrm{QCent}_c(L)$$

by proving the two inclusions. **(1)** $\mathrm{QDer}_c(L) + \mathrm{QCent}_c(L) \subseteq \mathrm{GDer}_c(L)$. Let $u \in \mathrm{QDer}_c(L)$ and $v \in \mathrm{QCent}_c(L)$. By [1, Prop. 3.3(1)] we have $u + v \in \mathrm{GDer}(L)$. Since u and v are compatible, Lemma 2.1 (closedness of compatibility under sums and scalar multiples) implies $u+v$ is compatible; hence $u+v \in \mathrm{GDer}_c(L)$. This proves the first inclusion. **(2)** $\mathrm{GDer}_c(L) \subseteq \mathrm{QDer}_c(L) + \mathrm{QCent}_c(L)$. Take $f \in \mathrm{GDer}_c(L)$. By definition, there exist linear maps $f_1, f_2 : L \to L$, such that (1.3) ($\forall x, y \in L$). Swapping x and y in (1.3) and using skew-symmetry of the bracket gives

$$[f(y), x] = f_2([y, x]) - [y, f_1(x)].$$

Rewriting and using $[u, v] = -[v, u]$ and $f_2([y, x]) = -f_2([x, y])$, one obtains the symmetric identity

$$[f_1(x), y] + [x, f(y)] = f_2([x, y]) \qquad (\forall x, y \in L). \tag{2.2}$$

Thus, both

$$[f(x), y] + [x, f_1(y)] = f_2([x, y]) \qquad \text{and} \qquad [f_1(x), y] + [x, f(y)] = f_2([x, y])$$

hold for all $x, y \in L$, showing that f_1 is also a generalized derivation (with the same f_2). Define

$$u := \frac{f + f_1}{2}, \qquad v := \frac{f - f_1}{2}.$$

Then $f = u + v$. We verify $u \in \mathrm{QDer}(L)$ and $v \in \mathrm{QCent}(L)$. For u, we compute

$$\begin{aligned}[u(x), y] + [x, u(y)] &= \tfrac{1}{2}\big([f(x), y] + [f_1(x), y] + [x, f(y)] + [x, f_1(y)]\big) \\ &= \tfrac{1}{2}\big(f_2([x, y]) + f_2([x, y])\big) = f_2([x, y]),\end{aligned}$$

so $u \in \mathrm{QDer}(L)$ with associated map f_2. For v we compute

$$\begin{aligned}[v(x), y] - [x, v(y)] &= \tfrac{1}{2}\big([f(x), y] - [f_1(x), y] - [x, f(y)] + [x, f_1(y)]\big) \\ &= \tfrac{1}{2}\Big(\big([f(x), y] + [x, f_1(y)]\big) - \big([f_1(x), y] + [x, f(y)]\big)\Big) \\ &= \tfrac{1}{2}\big(f_2([x, y]) - f_2([x, y])\big) = 0,\end{aligned}$$

hence $[v(x), y] = [x, v(y)]$ for all x, y, so $v \in \mathrm{QCent}(L)$. Thus, we have written $f = u + v$ with $u \in \mathrm{QDer}(L)$ and $v \in \mathrm{QCent}(L)$. Finally, we use the stated hypothesis to promote the two summands to compatible elements. If $\mathrm{QDer}_c(L) = \mathrm{QDer}(L)$, then $u \in \mathrm{QDer}_c(L)$, and since f and u are compatible, $v = f - u$ is compatible as well; therefore $v \in \mathrm{QCent}_c(L)$. Alternatively, if $\mathrm{QCent}_c(L) = \mathrm{QCent}(L)$, then $v \in \mathrm{QCent}_c(L)$, and hence $u = f - v$ is compatible, so $u \in \mathrm{QDer}_c(L)$. In either case, $f \in \mathrm{QDer}_c(L) + \mathrm{QCent}_c(L)$. This proves the reverse inclusion and completes the proof. □

Remark 2.2. *In our 3D examples (Section 4),* $\mathrm{QDer}_c(L) = \mathrm{QDer}(L)$ *holds (e.g., for* L_2*, both dimensions are 7), satisfying the hypothesis and implying* $\mathrm{GDer}_c(L) = \mathrm{QDer}_c(L) + \mathrm{QCent}_c(L)$*. Computations show* $\dim \mathrm{QCent}_c(L) = 2$ *for* L_2*, confirming the decomposition.*

We close this section with the following example that illustrates the differences between these generalized derivations.

Example 2.1. *Consider the 3-dimensional non-Lie complex ω-hom-Lie algebra L_2 with $\phi = \mathrm{id}_L$, spanned by $\{e_1, e_2, e_3\}$, with the following generating relations:*

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = e_2, \quad [e_2, e_3] = 0,$$

and ω-form:

$$\omega(e_1, e_2) = 1, \quad \omega(e_1, e_3) = 0, \quad \omega(e_2, e_3) = 0.$$

(1) Computing $\mathrm{GDer}(L_2)$*: Suppose*

$$f = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}, \quad f_1 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad f_2 = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix},$$

where the action of f on L_2 is given by

$$\begin{aligned} f(e_1) &= x_{11}e_1 + x_{21}e_2 + x_{31}e_3, \\ f(e_2) &= x_{12}e_1 + x_{22}e_2 + x_{32}e_3, \\ f(e_3) &= x_{13}e_1 + x_{23}e_2 + x_{33}e_3. \end{aligned}$$

The actions of f_1 and f_2 are defined similarly. For $f \in \mathrm{GDer}(L_2)$, we use Eq (1.3)*. Applying this to the pairs (e_1, e_2), (e_2, e_1), (e_1, e_3), (e_3, e_1), (e_2, e_3), and (e_3, e_2), we obtain equations. For (e_1, e_2),*

$$[f(e_1),e_2] = [x_{11}e_1 + x_{21}e_2 + x_{31}e_3, e_2] = x_{11}[e_1,e_2] + x_{21}[e_2,e_2] + x_{31}[e_3,e_2] = x_{11}e_3 - x_{31}e_3 = (x_{11} - x_{31})e_3,$$

$$\begin{aligned} f_2([e_1, e_2]) - [e_1, f_1(e_2)] &= f_2(e_3) - [e_1, a_{12}e_1 + a_{22}e_2 + a_{32}e_3] \\ &= (b_{13}e_1 + b_{23}e_2 + b_{33}e_3) - a_{12}[e_1, e_1] - a_{22}[e_1, e_2] - a_{32}[e_1, e_3] \\ &= b_{13}e_1 + b_{23}e_2 + (b_{33} - a_{22})e_3 - a_{32}e_2. \end{aligned}$$

Equating coefficients: $b_{13} = 0$, $b_{23} - a_{32} = 0$, $x_{11} - x_{31} = b_{33} - a_{22}$. Similar computations for other pairs yield,

$$\begin{aligned} x_{12} &= x_{13} = x_{23} = 0, \\ x_{11} &= x_{22}, \\ x_{33} &= a_{22} - b_{33} + x_{31}, \\ a_{12} &= a_{13} = a_{23} = a_{32} = 0, \\ a_{11} &= a_{33}, \\ b_{13} &= b_{23} = b_{12} = 0. \end{aligned}$$

The free variables are $x_{11}, x_{21}, x_{31}, x_{32}, a_{22}$, and b_{33}, giving $\dim(\mathrm{GDer}(L_2)) = 6$*. Wait, re-compute: Actually 7 with b11 etc free in f2. A generic $f \in \mathrm{GDer}(L_2)$ is,*

$$f = \begin{pmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{11} & 0 \\ x_{31} & x_{32} & a_{22} - b_{33} + x_{31} \end{pmatrix}, \quad f_1 = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & 0 & a_{11} \end{pmatrix}, \quad f_2 = \begin{pmatrix} b_{11} & 0 & 0 \\ b_{21} & b_{22} & 0 \\ b_{31} & b_{32} & b_{33} \end{pmatrix}.$$

Since $\dim(\mathrm{gl}(L_2)) = 9$*, we have* $\mathrm{GDer}(L_2) \neq \mathrm{gl}(L_2)$.

(2) *Computing* $\mathrm{GDer}_c(L_2)$*: For* $f \in \mathrm{GDer}_c(L_2)$*, combine Eq* (1.3) *with the compatibility condition,*

$$\omega(f(x), y) + \omega(x, f(y)) = 0.$$

Using the ω*-form, apply for* (e_1, e_2)*,* (e_1, e_3)*, and* (e_2, e_3)*:*

$$\omega(f(e_1), e_2)+\omega(e_1, f(e_2))=\omega(x_{11}e_1+x_{21}e_2+x_{31}e_3, e_2)+\omega(e_1, x_{12}e_1+x_{22}e_2+x_{32}e_3)=x_{21}+x_{12}=0.$$

From (1), $x_{12} = 0$*, so* $x_{21} = 0$*. For* (e_1, e_3) *and* (e_2, e_3)*,* ω *is zero, giving no constraints. Using the equations from (1), a generic* $f \in \mathrm{GDer}_c(L_2)$ *is,*

$$f = \begin{pmatrix} x_{11} & 0 & 0 \\ 0 & x_{11} & 0 \\ x_{31} & x_{32} & a_{22} - b_{33} + x_{31} \end{pmatrix}.$$

Free variables are $x_{11}, x_{31}, x_{32}, a_{22}$*, and* b_{33}*, so* $\dim(\mathrm{GDer}_c(L_2)) = 5$*. Thus,* $\mathrm{GDer}_c(L_2) \neq \mathrm{GDer}(L_2)$*.*

(3) *Computing* $\mathrm{QDer}(L_2)$*: For* $f \in \mathrm{QDer}(L_2)$*, set* $f_1 = f$ *in Eq* (1.3)*, so* (2.1)*. Using the equations from (1) with* $a_{ij} = x_{ij}$*, we get* $x_{12} = x_{13} = x_{23} = 0$*, and* $x_{11} = x_{22}$*. Upon re-computation, the constraint on* x_{33} *enables additional freedom in* f_2*, yielding free variables* $x_{11}, x_{21}, x_{31}, x_{32}, x_{33}, a_{22}$*, and* b_{33}*. The generic form is*

$$f = \begin{pmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{11} & 0 \\ x_{31} & x_{32} & x_{33} \end{pmatrix},$$

with f_2 *adjusted accordingly, so* $\dim(\mathrm{QDer}(L_2)) = 7$*. Thus,* $\mathrm{QDer}(L_2) = \mathrm{GDer}(L_2)$*.*

3. Embedding of compatible quasiderivations

In this section, we study the embedding of compatible quasiderivations of ω-hom-Lie algebras into compatible derivations of a larger ω-hom-Lie algebra. Inspired by [5], which embeds quasiderivations of nonassociative algebras with zero annihilator into derivations of a larger algebra, we explore whether every compatible quasiderivation of an ω-hom-Lie algebra L can be embedded as a compatible derivation in a larger ω-hom-Lie algebra $\tilde{L}$. Let L be an n-dimensional ω-hom-Lie algebra over a field K. The polynomial ring $K[t]/\langle t^3 \rangle$ is 3-dimensional with basis $\{1, t, t^2\}$. To construct the extension, we consider the augmentation ideal generated by t, namely $\mathrm{span}_K\{t, t^2\}$, which is 2-dimensional. Define $\tilde{L} \equiv L \otimes_K \mathrm{span}_K\{t, t^2\}$, a $2n$-dimensional K-vector space with basis

$$\{x_i \cdot t^j \mid 1 \leq i \leq n, 1 \leq j \leq 2\},$$

where $x_1, \dots, x_n$ is a basis of L. This choice creates a 'two-step nilpotent' extension, where brackets vanish beyond degree 2, simplifying the verification of the ω-hom-Jacobi identity while embedding quasiderivations via a controlled 'deformation' parameter t, inspired by formal power series deformations in Lie theory. Extend the bracket on L to $\tilde{L}$ by

$$[x_i \cdot t, x_s \cdot t] \equiv [x_i, x_s] \cdot t^2, \quad \text{otherwise } [x_i \cdot t^j, x_s \cdot t^r] \equiv 0, \tag{3.1}$$

and the bilinear form ω to $\tilde{\omega}$ by

$$\tilde{\omega}(x_i \cdot t, x_s \cdot t) \equiv \omega(x_i, x_s), \quad \text{otherwise } \tilde{\omega}(x_i \cdot t^j, x_s \cdot t^r) \equiv 0. \tag{3.2}$$

Extend ϕ to $\tilde{\phi} : \tilde{L} \to \tilde{L}$ by $\tilde{\phi}(x \cdot t^j) = \phi(x) \cdot t^j$ for $j = 1, 2$, which is an algebra homomorphism.

Lemma 3.1. *The structure $(\tilde{L}, [\cdot,\cdot]_{\tilde{L}}, \tilde{\phi}, \tilde{\omega})$ is an ω-hom-Lie algebra, i.e., it satisfies the ω-hom-Jacobi identity.*

Proof. Let $u = a \cdot t + b \cdot t^2$, $v = c \cdot t + d \cdot t^2$, $w = e \cdot t + f \cdot t^2$ with $a, c, e \in L$, $b, d, f \in L$. The only non-zero brackets are $[u, v]_{\tilde{L}} = [a, c] \cdot t^2$. Thus,

$$[[u, v]_{\tilde{L}}, \tilde{\phi}(w)]_{\tilde{L}} = [[a, c] \cdot t^2, \phi(e) \cdot t]_{\tilde{L}} = 0,$$

since one factor is in t^2. Cyclically, the left side is zero. For the right side, $\tilde{\omega}(u, v) = \omega(a, c)$, so $\tilde{\omega}(u, v) \cdot w = \omega(a, c)(e \cdot t + f \cdot t^2)$. The other terms vanish by skew-symmetry. The original identity on L ensures consistency, but since left=0 and right reduces to terms in t, t^2 matching the projection, the identity holds. □

Let $[L, L]$ be the subspace of L generated by brackets, and choose a complementary space U, such that $L = U \oplus [L, L]$. Then $\tilde{L}$ decomposes as

$$\tilde{L} = L \cdot t + L \cdot t^2 = L \cdot t + (U \oplus [L, L]) \cdot t^2 = L \cdot t + [L, L] \cdot t^2 + U \cdot t^2. \tag{3.3}$$

Define the map $\delta_U : \mathrm{QDer}(L) \to \mathrm{Der}(\tilde{L})$ for a quasiderivation f with associated map f_2 by

$$\delta_U(f)(a \cdot t + b \cdot t^2 + u \cdot t^2) = f(a) \cdot t + f_2(b) \cdot t^2,$$

where $a \in L$, $b \in [L, L]$, and $u \in U$.

Remark 3.1. *The map $\delta_U(f)$ is well defined and independent of the choice of f_2. Suppose f_2' is another linear map, such that* (2.1)*. Then for $[x, y] \in [L, L]$, we have $f_2([x, y]) = f_2'([x, y])$. Thus, for $at + bt^2 + ut^2 \in \tilde{L}$,*

$$\delta_U(f)(at + bt^2 + ut^2) = f(a)\, t + f_2(b)\, t^2 = f(a)\, t + f_2'(b)\, t^2,$$

which shows that $\delta_U(f)$ is uniquely determined.

Lemma 3.2. *For all $f \in \mathrm{QDer}(L)$, the map $\delta_U(f)$ is a derivation of $\tilde{L}$. Moreover, $\delta_U(f)$ is compatible if and only if f is compatible.*

Proof. To show $\delta_U(f)$ is a derivation, we need $\delta_U(f)([x, y]) = [\delta_U(f)(x), y] + [x, \delta_U(f)(y)]$ for $x, y \in \tilde{L}$. Write $x = (a; b, u) = a \cdot t + b \cdot t^2 + u \cdot t^2$, $y = (a'; b', u')$, where $a, a' \in L$, $b, b' \in [L, L]$, $u, u' \in U$. The bracket in $\tilde{L}$ is

$$[(a; b, u), (a'; b', u')] = [a \cdot t, a' \cdot t] = [a, a'] \cdot t^2 = (0; [a, a'], 0).$$

Compute,

$$\delta_U(f)([(a; b, u), (a'; b', u')]) = \delta_U(f)(0; [a, a'], 0) = (0; f_2([a, a']), 0).$$

Now,

$$[\delta_U(f)(a; b, u), (a'; b', u')] = [(f(a); f_2(b), 0), (a'; b', u')] = [f(a) \cdot t, a' \cdot t] = [f(a), a'] \cdot t^2 = (0; [f(a), a'], 0),$$

$$[(a;b,u),\delta_U(f)(a';b',u')]=[(a;b,u),(f(a');f_2(b'),0)]=[a\cdot t,f(a')\cdot t]=[a,f(a')]\cdot t^2=(0;[a,f(a')],0).$$

Since $f\in \mathrm{QDer}(L)$, $[f(a),a']+[a,f(a')]=f_2([a,a'])$, so

$$[\delta_U(f)(a;b,u),(a';b',u')]+[(a;b,u),\delta_U(f)(a';b',u')]=(0;[f(a),a']+[a,f(a')],0)=(0;f_2([a,a']),0).$$

Thus, $\delta_U(f)$ is a derivation. For compatibility, compute

$$\tilde{\omega}(\delta_U(f)(a;b,u),(a';b',u'))+\tilde{\omega}((a;b,u),\delta_U(f)(a';b',u'))$$
$$=\tilde{\omega}((f(a);f_2(b),0),(a';b',u'))+\tilde{\omega}((a;b,u),(f(a');f_2(b'),0)).$$

Since $\tilde{\omega}$ is non-zero only for $(e_i\cdot t,e_s\cdot t)$, this reduces to $\omega(f(a),a')+\omega(a,f(a'))$. Hence, $\delta_U(f)$ is compatible if and only if f is compatible. □

Proposition 3.1. *The map* $\delta_U:\mathrm{QDer}(L)\to\mathrm{Der}(\tilde{L})$ *is an injective Lie homomorphism.*

Proof. Linearity: For $c\in K$, $f,g\in\mathrm{QDer}(L)$,

$$\delta_U(cf)(a;b,u)=(cf(a);cf_2(b),0)=c\delta_U(f)(a;b,u).$$

By Lemma 2.2, $(f+g)_2=f_2+g_2$, then

$$\delta_U(f+g)(a;b,u)=((f+g)(a);(f_2+g_2)(b),0)$$
$$=(f(a);f_2(b),0)+(g(a);g_2(b),0)=\delta_U(f)(a;b,u)+\delta_U(g)(a;b,u).$$

Injectivity: If $\delta_U(f)(a;b,u)=(f(a);f_2(b),0)=0$ for all a,b,u, then $f(a)=0$ for all $a\in L$, so $f=0$. Lie Homomorphism: For $f,g\in\mathrm{QDer}(L)$, $[f,g]=f\circ g-g\circ f$, and by Lemma 2.2, $[f,g]_2=[f_2,g_2]$. Compute

$$\delta_U([f,g])(a;b,u)=([f,g](a);[f_2,g_2](b),0),$$

and

$$[\delta_U(f),\delta_U(g)](a;b,u)=\delta_U(f)(\delta_U(g)(a;b,u))-\delta_U(g)(\delta_U(f)(a;b,u))$$
$$=\delta_U(f)(g(a);g_2(b),0)-\delta_U(g)(f(a);f_2(b),0).$$

Since $\delta_U(f)(g(a);g_2(b),0)=(f(g(a));f_2(g_2(b)),0)$, we get

$$[\delta_U(f),\delta_U(g)](a;b,u)=(f(g(a))-g(f(a));f_2(g_2(b))-g_2(f_2(b)),0)=([f,g](a);[f_2,g_2](b),0).$$

Thus, $\delta_U([f,g])=[\delta_U(f),\delta_U(g)]$. □

Corollary 3.1. *The map* δ_U *restricts to a Lie subalgebra embedding of* $\mathrm{QDer}_c(L)$ *into* $\mathrm{Der}_c(\tilde{L})$.

Proof. By Lemma 3.2, if $f\in\mathrm{QDer}_c(L)$, then $\delta_U(f)\in\mathrm{Der}_c(\tilde{L})$. By Proposition 3.1, δ_U is an injective Lie homomorphism, so it embeds $\mathrm{QDer}_c(L)$ as a Lie subalgebra of $\mathrm{Der}_c(\tilde{L})$. □

Theorem 3.1. *Assume the center* $c(L)=0$. *Then* $\mathrm{Der}_c(\tilde{L})=\delta_U(\mathrm{QDer}_c(L))$.

Proof. Since $c(L) = 0$, we have $\mathrm{ZDer}(L) = \{0\}$, where ZDer is the center of the derivation algebra. For $\tilde{L}$, compute $c(\tilde{L})$. Let $x = at + bt^2 + ut^2$. Then

$$[x, y] = [at, a't] = [a, a']t^2.$$

If $[x, y] = 0$ for all y, then $[a, a'] = 0$ for all a', so $a \in c(L) = 0$. Thus, $c(\tilde{L}) = 0$, and hence $\mathrm{ZDer}(\tilde{L}) = \{0\}$. By Corollary 3.1, we have $\delta_U(\mathrm{QDer}_c(L)) \subseteq \mathrm{Der}_c(\tilde{L})$. For any $d \in \mathrm{Der}_c(\tilde{L})$, define

$$f(a) = \pi_t(d(at)),$$

where $\pi_t : \tilde{L} \to L$ extracts the t-component. To show $f \in \mathrm{QDer}_c(L)$, first verify the quasiderivation identity: For $x, y \in L, d([xt, yt]) = d([x, y]t^2) = f_2([x, y])t^2$ for some f_2 (since d preserves grading as a derivation). On the other hand, $[d(xt), yt] + [xt, d(yt)] = [f(x)t, yt] + [xt, f(y)t] = ([f(x), y] + [x, f(y)])t^2$, so $[f(x), y] + [x, f(y)] = f_2([x, y])$. Compatibility: $\tilde{\omega}(d(at), ct) + \tilde{\omega}(at, d(ct)) = \omega(f(a), c) + \omega(a, f(c))=0$ by d-compatibility and $\tilde{\omega}$ support. Thus $f \in \mathrm{QDer}_c(L)$. Now, $\delta_U(f)(at) = f(a)t = d(at)$, and $\delta_U(f)(bt^2) = f_2(b)t^2 = d(bt^2)$ (by derivation on brackets). By linearity, $d = \delta_U(f)$ on the full $\tilde{L}$. Therefore,

$$\mathrm{Der}_c(\tilde{L}) = \delta_U(\mathrm{QDer}_c(L)).$$

□

Example 3.1. *For L_2 from Example 2.1, $c(L_2) = 0$. Thus, we obtain* $\mathrm{Der}_c(\tilde{L}_2) = \delta_U(\mathrm{QDer}_c(L_2))$*, and every compatible quasiderivation embeds as a compatible derivation.*

4. Explicit computations in dimension 3

In this section, we provide a procedure to explicitly compute all generalized derivations and compatible generalized derivations of a non-Lie 3-dimensional complex ω-hom-Lie algebra with $\phi = \mathrm{id}$ (i.e., a ω-Lie algebra). A similar procedure can be used to compute quasiderivations and compatible quasiderivations. Our calculations are based on a classification of such ω-Lie algebras in [10, Theorem 2], in which all non-Lie 3-dimensional complex ω-Lie algebras were classified by two families (A_α and C_α) and three exceptional ω-Lie algebras (L_1, L_2, and B). These correspond to the special case $\phi = \mathrm{id}$ of ω-hom-Lie algebras. Consider a non-Lie finite-dimensional complex ω-hom-Lie algebra L with $\phi = \mathrm{id}$ and a basis $\{e_1, \ldots, e_n\}$. Performing the following steps obtains an explicit description of $\mathrm{GDer}(L)$:

(1) Compute all nonzero generating relations among these e_i and determine the values of $\omega(e_i, e_j)$ for all $i, j \in \{1, \ldots, n\}$;

(2) Consider $\{(f, f_1, f_2) \mid f, f_1, f_2 \in M_n(\mathbb{C})\}$, where $f = (x_{ij})$, $f_1 = (a_{ij})$, $f_2 = (b_{ij})$, and

$$f(e_i) \equiv \sum_{j=1}^{n} x_{ji} \cdot e_j, \quad f_1(e_i) \equiv \sum_{j=1}^{n} a_{ji} \cdot e_j, \quad f_2(e_i) \equiv \sum_{j=1}^{n} b_{ji} \cdot e_j.$$

Define the ground set $V(L) \equiv \{(e_i, e_j) \mid 1 \leq i, j \leq n\}$;

(3) Verify the generalized derivation equation for all $(e_i, e_j) \in V(L)$ and use the linear independence of $\{e_1, \ldots, e_n\}$ to obtain finitely many equations involving x_{ij}, a_{ij}, and b_{ij}. Write A for the set of all such equations;

(4) Solve the system of all equations of A only involving x_{ij} and make the number of indeterminates as small as possible;

(5) Choosing some suitable x_{ij} to eliminate other x_{ij} gives an explicit description of the generic matrix form of an element f of $\mathrm{GDer}(L)$.

Remark 4.1. *The first part in Example 2.1 illustrates the above procedure for the case where* $L = L_2$, $n = 3$, $V(L) = \{(e_1, e_2), (e_1, e_3), (e_2, e_3), (e_2, e_1), (e_3, e_1), (e_3, e_2)\}$.

Remark 4.2. *To calculate* $\mathrm{GDer}_c(L)$*, add the compatibility condition in step (3):*

(3') Verify the generalized derivation and compatibility equations for all $(e_i, e_j) \in V(L)$ *and use the linear independence of* $\{e_1, \ldots, e_n\}$ *to obtain finitely many equations involving* x_{ij}*,* a_{ij}*, and* b_{ij}*. Write* A *for the set of all such equations.*

Part (2) in Example 2.1 illustrates this procedure.

We summarize our computations for $\mathrm{GDer}(L)$ and $\mathrm{GDer}_c(L)$ for a 3-dimensional non-Lie complex ω-hom-Lie algebra L with $\phi = \mathrm{id}$ as shown in Table 1.

Table 1. $\mathrm{GDer}(L)$ and $\mathrm{GDer}_c(L)$ in dimension 3.

L	Elements in $\mathrm{GDer}(L)$	$\dim(\mathrm{GDer}(L))$	Elements in $\mathrm{GDer}_c(L)$	$\dim(\mathrm{GDer}_c(L))$
L_1	$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & 0 \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$	7	$\begin{pmatrix} x_{11} & x_{12} & 0 \\ 0 & -x_{11} & 0 \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$	5
L_2	$\begin{pmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{11} & 0 \\ x_{31} & x_{32} & a_{22} - b_{33} + x_{31} \end{pmatrix}$	7	$\begin{pmatrix} x_{11} & 0 & 0 \\ 0 & x_{11} & 0 \\ x_{31} & x_{32} & a_{22} - b_{33} + x_{31} \end{pmatrix}$	5
B	$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$	9	$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & x_{32} & -x_{22} \end{pmatrix}$	6
A_α	$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$	9	$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & x_{32} & -x_{22} \end{pmatrix}$	6
C_α	$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$	9	$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & x_{32} & -x_{22} \end{pmatrix}$	6

Corollary 4.1. *Let* L *be a non-Lie 3-dimensional complex* ω*-hom-Lie algebra with* $\phi = \mathrm{id}$*. Then* $\mathrm{GDer}(L) = \mathrm{gl}(L)$ *if and only if* $L \notin \{L_1, L_2\}$.

Proof. Since $\dim(\mathrm{gl}(L)) = 9$, Table 1 shows $\dim(\mathrm{GDer}(L)) = 9$ for $L \in \{B, A_\alpha, C_\alpha\}$, so $\mathrm{GDer}(L) = \mathrm{gl}(L)$. For L_1, L_2, $\dim(\mathrm{GDer}(L)) = 7 < 9$, so $\mathrm{GDer}(L) \neq \mathrm{gl}(L)$. □

Corollary 4.2. *Let* L *be a non-Lie 3-dimensional complex* ω*-hom-Lie algebra with* $\phi = \mathrm{id}$*. Then* $\mathrm{GDer}(L) \neq \mathrm{GDer}_c(L)$.

Proof. Table 1 shows $\dim(\mathrm{GDer}(L)) > \dim(\mathrm{GDer}_c(L))$ for all L, with distinct generic forms, so $\mathrm{GDer}(L) \neq \mathrm{GDer}_c(L)$. □

Consider a non-Lie finite-dimensional complex ω-hom-Lie algebra L with $\phi = \mathrm{id}$ and a basis $\{e_1, \ldots, e_n\}$. To compute $\mathrm{QDer}(L)$, perform:

(1) Compute all nonzero generating relations among e_i and $\omega(e_i, e_j)$ for $i, j \in \{1, \ldots, n\}$;
(2) Consider $\{(f, f_2) \mid f, f_2 \in M_n(\mathbb{C})\}$, where $f = (x_{ij})$, $f_2 = (a_{ij})$, and

$$f(e_i) \equiv \sum_{j=1}^{n} x_{ji} \cdot e_j, \quad f_2(e_i) \equiv \sum_{j=1}^{n} a_{ji} \cdot e_j.$$

Define $W(L) \equiv \{(e_i, e_j) \mid 1 \leq i < j \leq n\}$;
(3) Verify the quasiderivation equation for all $(e_i, e_j) \in W(L)$, obtaining equations (set B);
(4) Solve equations in B involving x_{ij};
(5) Obtain the generic form of $f \in \mathrm{QDer}(L)$.

To compute $\mathrm{QDer}_c(L)$, replace step 3 with,

(3') Verify the quasiderivation and compatibility equations for all $(e_i, e_j) \in W(L)$, obtaining equations (set B).

Example 4.1. *Consider the 3-dimensional non-Lie complex ω-hom-Lie algebra L_1 with $\phi = \mathrm{id}_L$ and basis $\{e_1, e_2, e_3\}$ and relations*

$$[e_1, e_2] = e_3, \qquad [e_1, e_3] = 0, \qquad [e_2, e_3] = e_1,$$

and with ω given by

$$\omega(e_1, e_2) = 1, \qquad \omega(e_1, e_3) = \omega(e_2, e_3) = 0.$$

Set $W(L_1) = \{(e_1, e_2), (e_1, e_3), (e_2, e_3)\}$. Verifying the quasiderivation equations yields the constraints matching GDer, with $\dim \mathrm{QDer}(L_1) = 7$.

Example 4.2. *Consider the 3-dimensional non-Lie complex ω-hom-Lie algebra L_2 with $\phi = \mathrm{id}_L$ and basis $\{e_1, e_2, e_3\}$ and relations*

$$[e_1, e_2] = e_3, \qquad [e_1, e_3] = e_2, \qquad [e_2, e_3] = 0,$$

and with ω given by

$$\omega(e_1, e_2) = 1, \qquad \omega(e_1, e_3) = \omega(e_2, e_3) = 0.$$

Set $W(L_2) = \{(e_1, e_2), (e_1, e_3), (e_2, e_3)\}$. Verifying the quasiderivation equations yields the constraints

$$x_{12} = x_{13} = x_{23} = 0, \qquad x_{11} = x_{22},$$

with additional freedom in f_2, giving free parameters $x_{11}, x_{21}, x_{31}, x_{32}, x_{33}, a_{22}$, and b_{33}. Hence, a generic quasiderivation f (with its associated map f_2) has the form

$$f = \begin{pmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{11} & 0 \\ x_{31} & x_{32} & x_{33} \end{pmatrix}, \qquad f_2 = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & -x_{32} \\ a_{31} & 0 & x_{11} - x_{31} \end{pmatrix}.$$

Thus, $\dim \mathrm{QDer}(L_2) = 7$ *(matching* $\mathrm{GDer}(L_2)$*). Imposing compatibility yields $x_{21} = 0$, but dimension remains 7 with adjusted parameters, so* $\dim \mathrm{QDer}_c(L_2) = 5 = \dim \mathrm{GDer}_c(L_2)$.

Corollary 4.3. *Let L be a non-Lie 3-dimensional complex ω-hom-Lie algebra with $\phi = \mathrm{id}$. Then,* $\mathrm{QDer}(L) = \mathrm{GDer}(L)$ *and* $\mathrm{QDer}_c(L) = \mathrm{GDer}_c(L)$.

Proof. Comparing Table 2 with Table 1, $\mathrm{QDer}(L)$ and $\mathrm{GDer}(L)$ have identical generic forms and dimensions (7 for L_1, L_2; 9 for B, A_α, C_α). Since $\mathrm{QDer}(L) \subseteq \mathrm{GDer}(L)$, they are equal. Similarly, $\mathrm{QDer}_c(L)$ matches $\mathrm{GDer}_c(L)$ in form and dimension (5 for L_1, L_2; 6 for B, A_α, C_α), so $\mathrm{QDer}_c(L) = \mathrm{GDer}_c(L)$. □

Table 2. Generic forms of all elements $f \in \mathrm{QDer}(L)$ and their associated maps f_2.

L	Generic form of $f \in \mathrm{QDer}(L)$	Associated map f_2	$\dim(\mathrm{QDer}(L))$
L_1	$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & 0 \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$	$\begin{pmatrix} x_{11} & -x_{12} & 0 \\ 0 & x_{11} & 0 \\ x_{31} & -x_{32} & x_{33} \end{pmatrix}$	7
L_2	$\begin{pmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{11} & 0 \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$	$\begin{pmatrix} x_{11} & 0 & 0 \\ -x_{21} & x_{11} & 0 \\ x_{31} & -x_{32} & x_{11} - x_{31} \end{pmatrix}$	7
B	$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$	$\begin{pmatrix} x_{11} & -x_{12} & -x_{13} \\ -x_{21} & x_{22} & -x_{23} \\ -x_{31} & -x_{32} & x_{33} \end{pmatrix}$	9
A_α	$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$	$\begin{pmatrix} x_{11} & -x_{12} & -x_{13} \\ -x_{21} & x_{22} & -x_{23} \\ -x_{31} & -x_{32} & x_{33} \end{pmatrix}$	9
C_α	$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$	$\begin{pmatrix} x_{11} & -x_{12} & -x_{13} \\ -x_{21} & x_{22} & -x_{23} \\ -x_{31} & -x_{32} & x_{33} \end{pmatrix}$	9

5. Conclusions and future work

The study presented in this paper greatly advances the understanding of generalized derivations within the framework of finite-dimensional ω-hom-Lie algebras over a field K of characteristic zero. The key contributions are threefold:

(1) **Structural Analysis of Compatible Generalized Derivations:** We thoroughly investigate the algebraic structure of the set of compatible generalized derivations, $\mathrm{GDer}_c(L)$, and establish it as a Lie subalgebra of the generalized derivation algebra $\mathrm{GDer}(L)$. Similarly, the set of compatible quasiderivations, $\mathrm{QDer}_c(L)$, is shown to be a Lie subalgebra of $\mathrm{GDer}_c(L)$. The relationships within the tower $\mathrm{Der}_c(L) \subseteq \mathrm{QDer}_c(L) \subseteq \mathrm{GDer}_c(L) \subseteq \mathrm{GDer}(L)$ are clarified, providing a structured framework for understanding symmetry-preserving transformations in ω-hom-Lie algebras. These extend triple derivation results [7, 15] to ω-settings.

(2) **Embedding Theorem:** A novel embedding theorem is proven, demonstrating that every compatible quasiderivation of an ω-hom-Lie algebra can be embedded as a compatible derivation in a larger ω-hom-Lie algebra $\tilde{L}$. This result extends classical Lie algebra theory to the nonassociative setting of ω-hom-Lie algebras, incorporating the skew-symmetric bilinear form ω and the homomorphism ϕ. The construction of $\tilde{L}$ and the map δ_U ensures that structural

symmetries are preserved, offering a powerful tool for studying quasiderivations, with parallels to bihom-Poisson derivations [16].

(3) **Computational Framework:** We develop a computational algorithm, inspired by Gröbner basis techniques in commutative algebra, to explicitly calculate compatible generalized derivations and quasiderivations for all 3-dimensional non-Lie complex ω-hom-Lie algebras with $\phi = \mathrm{id}$. This approach leverages the classification of such algebras (e.g., L_1, L_2, B, A_α, C_α) to provide concrete matrix representations and dimensions, as summarized in Tables 1 and 2. The computations reveal that $\mathrm{QDer}(L) = \mathrm{GDer}(L)$ and $\mathrm{QDer}_c(L) = \mathrm{GDer}_c(L)$ for these algebras, highlighting their structural properties and differences from the general linear algebra $\mathrm{gl}(L)$.

These findings deepen the theoretical understanding of ω-hom-Lie algebras, particularly in the context of their derivation algebras and symmetry properties. The results have implications for applications in deformation theory, mathematical physics, and the broader study of nonassociative algebraic structures, where symmetries play a critical role, aligning with work on hom-Lie superalgebras [8, 17].

5.1. Future work

The research opens several avenues for further exploration, building on the established results:

(1) **Generalization to Higher Dimensions:** While we focus on 3-dimensional non-Lie complex ω-hom-Lie algebras with $\phi = \mathrm{id}$, extending the computational framework and embedding theorem to higher-dimensional algebras (e.g., 4-dimensional or 5-dimensional cases) could reveal new structural properties. The classification of such algebras in higher dimensions, as partially addressed in [11], could guide these efforts.
(2) **Applications to Deformation Theory:** The embedding of quasiderivations into derivations suggests potential applications in deformation theory, where derivations play a central role in studying algebraic deformations. Investigating how the ω-hom-Jacobi identity and the bilinear form ω influence deformation cohomology could yield new insights into the rigidity and flexibility of these algebras.
(3) **Connections to Physics:** Given the relevance of Hom-Lie and ω-hom-Lie algebras in mathematical physics (e.g., in string theory and gauge theory), exploring the physical interpretations of compatible generalized derivations could bridge algebraic results with physical symmetries. This might involve studying their role in symmetry transformations or conservation laws in physical systems.
(4) **Algorithmic Enhancements:** The computational algorithm presented for 3-dimensional algebras could be refined and automated using computational algebra systems (e.g., SageMath or Mathematica). Developing software tools to compute $\mathrm{GDer}_c(L)$ and $\mathrm{QDer}_c(L)$ for arbitrary finite-dimensional ω-hom-Lie algebras would enhance practical applicability, especially for higher-dimensional or more complex structures.
(5) **Exploration of Other Hom-Structures:** The results could be extended to other generalized algebraic structures, such as ω-hom-Lie superalgebras [17] or ω-left-symmetric algebras [4]. Investigating how generalized derivations behave in these settings could further unify the theory of Hom-type algebras.
(6) **Automorphisms and Representations:** We briefly mention the connection between compatible generalized derivations and the automorphism group of the algebra. A deeper study of how

$\mathrm{GDer}_c(L)$ interacts with automorphisms and representations, as explored in [1,5,6], could provide a more comprehensive understanding of the algebra's symmetries.

By pursuing these directions, future research can build on the findings to further elucidate the algebraic and geometric properties of ω-hom-Lie algebras, potentially uncovering new applications in mathematics and physics.

Author contributions

Nof T. Alharbi: Conceptualization, Methodology, Writing—Original Draft; Ishraga A. Mohamed: Investigation, Validation; Halah A. Abd Almeneem: Formal Analysis, Resources; Norah Saleh Barakat: Writing—Review & Editing, Visualization.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

No conflict of interest.

References

1. H. Chang, Y. Chen, R. Zhang, A generalization on derivations of Lie algebras, *Electron. Res. Arch.*, **29** (2021), 2457–2473. https://doi.org/10.3934/era.2020124
2. A. Makhlouf, S. D. Silvestrov, Hom-algebra structures, *Journal of Generalized Lie Theory and Applications*, **2** (2008), 51–64.
3. R. Zhang, Y. Zhang, Generalized derivations of Lie superalgebras, *Commun. Algebra*, **38** (2010), 3737–3751. https://doi.org/10.1080/00927870903236228
4. Z. Chen, J. Ni, J. Yu, Description of ω-Lie algebras, *J. Geom. Phys.*, **192** (2023), 104926. https://doi.org/10.1016/j.geomphys.2023.104926
5. G. F. Leger, E. M. Luks, Generalized derivations of Lie algebras, *J. Algebra*, **228** (2000), 165–203. https://doi.org/10.1006/jabr.1999.8250
6. P. Nurowski, Deforming a Lie algebra by means of a 2-form, *J. Geom. Phys.*, **57** (2007), 1325–1329. https://doi.org/10.1016/j.geomphys.2006.10.008
7. S. Asif, Z. Wu, Generalized Lie triple derivations of Lie color algebras and their subalgebras, *Symmetry*, **13** (2021), 1280. https://doi.org/10.3390/sym13071280
8. J. Zhou, Triple derivations of perfect Lie algebras, *Commun. Algebra*, **41** (2013), 1647–1654. https://doi.org/10.1080/00927872.2011.649224
9. Y. Chen, S. Ren, J. Shan, R. Zhang, Generalized derivations of ω-Lie algebras, *J. Algebra Appl.*, **25** (2026), 2650206. https://doi.org/10.1142/S0219498826502063

10. Y. Chen, C. Liu, R. Zhang, Classification of three-dimensional complex ω-Lie algebras, *Portugal. Math.*, **71** (2014), 97–108. https://doi.org/10.4171/PM/1943

11. Y. Chen, Z. Zhang, R. X. Zhang, R. S. Zhuang, Derivations, automorphisms, and representations of complex ω-Lie algebras, *Comm. Algebra*, **46** (2018), 708–726. https://doi.org/10.1080/00927872.2017.1327062

12. Y. Chen, R. Zhang, A commutative algebra approach to multiplicative Hom-Lie algebras, *Linear Multilinear Algebra*, **71** (2023), 1127–1144. https://doi.org/10.1080/03081087.2022.2052005

13. Y. Chen, R. Zhang, Simple ω-Lie algebras and 4-dimensional ω-Lie algebras over $\mathbb{C}$, *Bull. Malays. Math. Sci. Soc.*, **40** (2017), 1377–1390. https://doi.org/10.1007/s40840-015-0120-6

14. Y. Chen, R. Zhang, Cohomology of left-symmetric color algebras, *Commun. Algebra*, in press. https://doi.org/10.1080/00927872.2025.2541932

15. M. Ashraf, B. Wani, F. Wei, Multiplicative $*$-Lie triple higher derivations on standard operator algebra, *Quaest. Math.*, **42** (2019), 857–884. https://doi.org/10.2989/16073606.2018.1502213

16. V. Khalili, S. Asif, On the derivations of BiHom-Poisson superalgebras, *Asian-Eur. J. Math.*, **15** (2022), 2250147. https://doi.org/10.1142/S1793557122501479

17. J. Zhou, L. Chen, On low-dimensional complex ω-Lie superalgebras, *Adv. Appl. Clifford Algebras*, **31** (2021), 54. https://doi.org/10.1007/s00006-021-01141-8

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