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Research article

On classification of groups generated by total sextactic points of smooth quartic curves

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Abstract: We introduced a new concept, the mutual conic, in order to give a complete classification of the group G generated by the images of two or three total sextactic points in the Jacobian J_C of a smooth projective plane quartic curve C (3-genus curves that are non-hyperelliptic), and we determined the geometric configuration of these points associated with each case. We supported the validity of the results with a variety of examples.

Keywords: sextactic points; algebraic curves; quartic curves; mutual conic; Jacobian; flex points; group theory
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1. Introduction

On proper smooth algebraic curves, one can find many finite sets of points that have distinctive properties. One interesting question is about the subgroup structure of the Jacobian of the curve that one gets from 0-degree divisors that are supported on such a finite set. The paper defines this finite set for the total sextactic points on smooth projective plane quartic curves (3-genus curves that are non-hyperelliptic) that have two or three total sextactic points.

The finite set of 1-Weierstrass points on smooth projective plane quartic curves contains only the flex points (see, e.g., [1–3]), however the set of 2-Weierstrass points on such curves are divided into two disjoint subsets {flexes} ∪ {sextactic points} [2]. Most of the previous research studied the structure of the group generated by the images, under the Abel Jacobi map A_0, of 1-Weierstrass points in the Jacobian J_C of a smooth projective plane quartic curve C.

In [4–6] the authors studied the group G generated by the images, under A_0, of total sextactic points in the Jacobian of the quartic curve $\mathcal{K}_t$: $X^4 + Y^4 + Z^4 + t(X^2Y^2 + Y^2Z^2 + Z^2X^2) = 0$ (where such a family of quartic curves is called Kuribayashi quartics), when $t = 14$, $P(t) = t^3 + 68t^2 - 91t + 98 = 0$; and in the Jacobian of superelliptic curves of 3-genus, given by: $Y^4 = X(X - Z)(X - tZ)Z$ when the parameter t satisfies the equation: $Q(t) = \left(t^2 + 4t - 4\right)\left(4t^2 - 4t - 1\right)\left(t^2 - 6t + 1\right) = 0$, respectively. More precisely, for the Kuribayashi curve, they found that $G \cong (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/8\mathbb{Z})^2$ if $t = 14$ [4], and $G \cong (\mathbb{Z}/8\mathbb{Z})^3$ if $P(t) = 0$ [5]. In [6], for the superelliptic curves, given above, the authors showed that if $Q(t) = 0$, then $G \cong (\mathbb{Z}/2\mathbb{Z})^2 \oplus (\mathbb{Z}/4\mathbb{Z})^2 \oplus (\mathbb{Z}/8\mathbb{Z})^2$.

We introduce a new concept, the mutual conic, in order to give a complete classification of the group G generated by two or three total sextactic points on a smooth projective plane quartic curve C, and we determine the geometric configuration of these points associated with each case.

The main results of the paper are Theorem 4.4, summarized in Table 1, and Theorems 5.3, 5.8, and 5.9, which correspond, respectively, to Cases I, II, and III in Table 2 and are summarized therein. More precisely, if C is a smooth projective plane quartic curve with two total sextactic points, then the cyclic subgroup G generated by the images of these total sextactic points in the Jacobian J_C of C can be completely classified as follows.

Table 1. Group generated by two total sextactic points.

Case	Group	Geometry
I	$\mathbb{Z}/8\mathbb{Z}$	The tangent lines to C at these points are distinct and the mutual conics are all imperfect.
II	$\mathbb{Z}/4\mathbb{Z}$	The two total sextactic points have the same tangent line. Or, the mutual conic for one point with respect to the other is perfect.

If C is a smooth projective plane quartic curve with three total sextactic points, then the subgroup G generated by the images of these total sextactic points in the Jacobian J_C of C can be fully classified as follows.

Table 2. Group generated by three total sextactic points.

Case	Group	Geometry
I	$(\mathbb{Z}/8\mathbb{Z})^2$	The tangent lines to C at these points are distinct and the mutual conics are all imperfect.
II	$\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$	The tangent lines to C at these points are distinct and only one of the mutual conics is perfect. Or, two of these points have the same tangent to C and all the possible mutual conics are imperfect.
III	$(\mathbb{Z}/4\mathbb{Z})^2$	The tangent lines to C at these points are distinct and two (or all) of the mutual conics are perfect. Or, two of these points have the same tangent to C and at least one of the mutual conics is perfect.

In all cases, we support the validity of the results by giving a variety of interesting examples.

2. Preliminaries

Assume that C is an algebraic plane curve of degree $d \geq 3$. The Abel Jacobi map with base point $\infty \in C$ from C to its Jacobian J_C, denoted by $A_\infty : C \to J_C$, is defined by $\mathcal{P} \mapsto [\mathcal{P} - \infty]$. One can extend A_∞ linearly to divisors in Div(C), where the class of the divisor D in $Pic^0(C)$ is represented by $[D]$, and $Pic^0(C) \cong J_C$ (see, e.g., ([7], Ch. 5]) or ([8], Ch. VIII)).

If the tangent line $\mathcal{T}_\mathcal{P}$ at a smooth point $\mathcal{P} \in C$ intersects C at $\mathcal{P}$ with a contact order of at least 3, i.e., $I_\mathcal{P}(C, \mathcal{T}_\mathcal{P}) \geq 3$, then $\mathcal{P}$ is said to be a flex point [9]. Additionally, a flex point $\mathcal{P} \in C$ is said to be an i-flex if $i = I_\mathcal{P}(C, \mathcal{T}_\mathcal{P}) - 2$. This positive integer i is called the flex multiplicity of C at $\mathcal{P}$. In particular, by Bézout's theorem, for a smooth projective plane quartic curve C, i can be either 1 or 2. Similar to tangent lines and flex points on algebraic plane curves, one can consider osculating conics and sextactic points. For a non-flex smooth point $\mathcal{P}$ on a plane curve C of degree $d \geq 3$, one can find a unique irreducible conic $\Lambda_\mathcal{P}$ with a contact order $I_\mathcal{P}(C, \Lambda_\mathcal{P}) \geq 5$. This conic $\Lambda_\mathcal{P}$ is known as the osculating conic of C at $\mathcal{P}$. If the osculating conic $\Lambda_\mathcal{P}$ intersects C at $\mathcal{P}$ with a contact order ≥ 6, i.e., if $I_\mathcal{P}(C, \Lambda_\mathcal{P}) \geq 6$, then the point $\mathcal{P} \in C$ is called a sextactic point and $\Lambda_\mathcal{P}$ is called a sextactic conic in this context. Moreover, a sextactic point $\mathcal{P} \in C$ is said to be s-sextactic if $s = I_\mathcal{P}(C, \Lambda_\mathcal{P}) - 5$, with s being the sextactic multiplicity of C at $\mathcal{P}$. By Bézout's theorem, if C is an algebraic quartic plane curve, then $s \in \{1, 2, 3\}$. A 3-sextactic point is said to be a total sextactic point, as the sextactic conic $\Lambda_\mathcal{P}$ intersects C only at $\mathcal{P}$, implying that $I_\mathcal{P}(C, \Lambda_\mathcal{P}) = 8$. In Appendix C of [10], the authors proved that if C has r flexes with multiplicities $m_1, ..., m_r$, then C has $3d(5d - 11) - \sum_{i=1}^{r}(4m_i - 3)$ sextactic points, including multiplicities. To construct the osculating conic at a point on a smooth algebraic plane curve of degree $d \geq 3$, see Lemma 4 in [6]. Now, we present the geometric tool that enables us to achieve the aim of this note.

3. The mutual conic

In this section, we will discover that sextactic points are not only distinguished by the presence of sextactic conics, but there is a family of distinct conics that play a substantial role in classifying the groups generated by these points.

Consider an algebraic plane curve C of degree $d \geq 4$. It is well known that a conic Λ produces a divisor div(Λ) on C by linking to $\mathcal{P} \in C$ the contact order $I_\mathcal{P}(C, \Lambda)$ of Λ and C in $\mathcal{P}$. Using Bézout's theorem, the degree of div(Λ) is $2d$. The set of all such divisors forms a complete linear system $\mathcal{K}$ of dimension 5, i.e., $\mathcal{K}$ is a g^5_{2d}. We refer to $\mathcal{K}(-n\mathcal{P})$ as the space of divisors in $\mathcal{K}$ that meet C in $\mathcal{P}$ with a contact order of at least n. Imposing that ∞ is a non-flex smooth point on C. Let Λ_∞ be the quadratic form defining the osculating conic to the curve C at ∞ and let $\mathcal{T}_\infty$ be the linear form defining the tangent to C at ∞. Then, $I_\infty(C, \Lambda_\infty) = \mu \geq 5$ and $I_\infty(C, \mathcal{T}_\infty) = 2$. Let L_0 be a line not passing through ∞, (i.e., $I_\infty(C, L_0) = 0$), and let L_1 be a line passing through ∞ such that $I_\infty(C, L_1) = 1$. Considering the intersection of C with each of the conics:

$$L_0^2,\ L_0 L_1,\ L_1^2\ (\text{or } L_0\mathcal{T}_\infty),\ L_1\mathcal{T}_\infty,\ \mathcal{T}_\infty^2,\ \text{and } \Lambda_\infty,$$

we find that their contact orders at ∞ are $0, 1, 2, 3, 4$, and μ, respectively. Therefore, we have the nested sequence

$$\begin{aligned}\mathcal{K} &\supsetneqq \mathcal{K}(-\infty) \supsetneqq \mathcal{K}(-2\infty) \supsetneqq \mathcal{K}(-3\infty) \supsetneqq \mathcal{K}(-4\infty) \supsetneqq \mathcal{K}(-5\infty) = \cdots \\ &= \mathcal{K}(-\mu\infty \supsetneqq \mathcal{K}(-(\mu+1)\infty) = \emptyset.\end{aligned}$$

Thus, $\dim \mathcal{K}(-4\infty) = 1$ and clearly $\mathcal{K}(-4\infty) = \{E \in \mathcal{K} \mid E \sim \operatorname{div}(\mathcal{T}_\infty^2 + k\Lambda_\infty),\, k \in \mathbb{C}\}$. We call the conic $\Xi_{k\infty} = \mathcal{T}_\infty^2 + k\Lambda_\infty$, for some constant $k \in \mathbb{C}$, the mutual conic with respect to ∞. It is clear that $I_\infty(C, \Xi_{k\infty}) = 4$ and, for any other point $\mathcal{P}$ on C, that $I_\mathcal{P}(C, \Xi_{k\infty}) \geq 1$ if and only if $k = -g_\infty(\mathcal{P})$ where $g_\infty = \frac{\mathcal{T}_\infty^2}{\Lambda_\infty}$. Moreover, if $\mathcal{T}_\infty$ passes through $\mathcal{P}$ (for instance, maybe $\mathcal{P}$ and ∞ have the same tangent line to C), then $g_\infty(\mathcal{P}) = 0$ and therefore the mutual conic with respect to ∞ coincides with $\mathcal{T}_\infty^2$.

Definition 3.1. *The unique irreducible conic $\Xi_{\mathcal{P}\infty} = \mathcal{T}_\infty^2 - g_\infty(\mathcal{P})\Lambda_\infty$ is said to be the mutual conic for $\mathcal{P}$ with respect to ∞ if $g_\infty(\mathcal{P}) \neq 0$ (so $\mathcal{T}_\infty$ does not pass through $\mathcal{P}$). Furthermore, if $I_\mathcal{P}(C, \Xi_{\mathcal{P}\infty}) = 4$, then we say that $\mathcal{P}$ has a perfect mutual conic with respect to ∞. Otherwise, we say that $\mathcal{P}$ has an imperfect mutual conic with respect to ∞, that is, if $I_\mathcal{P}(C, \Xi_{\mathcal{P}\infty}) \neq 4$. If $I_\mathcal{P}(C, \Xi_{\mathcal{P}\infty}) = 2$, then we say that $\mathcal{P}$ has a semiperfect mutual conic with respect to ∞. If $I_\mathcal{P}(C, \Xi_{\mathcal{P}\infty}) = 1$, then we say that $\mathcal{P}$ has an ordinary mutual conic with respect to ∞.*

Remark 3.2. *If $\mathcal{P}$ and ∞ are non-flex smooth points of C, then the mutual conics $\Xi_{\mathcal{P}\infty}$ and $\Xi_{\infty\mathcal{P}}$ are often distinct curves. Furthermore, $\Xi_{\mathcal{P}\infty}$ is perfect if and only if $\Xi_{\infty\mathcal{P}}$ is perfect. In fact, in this case, $\Xi_{\mathcal{P}\infty}$ and $\Xi_{\infty\mathcal{P}}$ are the same (for this reason, we called it the mutual conic). If $\mathcal{P}$ has a semiperfect mutual conic with respect to ∞ such that $I_\mathcal{Q}(C, \Xi_{\mathcal{P}\infty}) = 2$ for some point $\mathcal{Q} \in C \setminus \{\mathcal{P}\}$, i.e., the divisor of $\Xi_{\mathcal{P}\infty}$ on C satisfies that* $\operatorname{div}(\Xi_{\mathcal{P}\infty}) \geq 4\infty + 2\mathcal{P} + 2\mathcal{Q}$, *then $\mathcal{Q}$ has a semiperfect mutual conic with respect to ∞ and actually $\Xi_{\mathcal{Q}\infty}$ coincides with $\Xi_{\mathcal{P}\infty}$. For examples of various types of mutual conics, see Subsection 6.2.*

Geometrically, the family $\{\Xi_{k\infty} = \mathcal{T}_\infty^2 + k\Lambda_\infty\}_{k\in\mathbb{C}}$ forms a pencil of conics determined by the tangent $\mathcal{T}_\infty$ and the osculating conic Λ_∞ at ∞. This pencil illustrates how the local geometry at ∞ interacts with a varying point $\mathcal{P}$ on C, where $k = -g_\infty(\mathcal{P}) = \dfrac{\mathcal{T}_\infty^2}{\Lambda_\infty}(\mathcal{P})$. Thus, the mutual conic can be interpreted as a "geometric bridge" connecting the tangent and osculating conics at a given point on C.

In particular, when ∞ is a sextactic point on a smooth plane curve C of degree 4, then for any other point $\mathcal{P}$ on C, $\mathcal{P}$ has a perfect mutual conic with respect to ∞ if and only if $\operatorname{div}(\Xi_{\mathcal{P}\infty}) = 4\infty + 4\mathcal{P}$. By using the concept of a mutual conic, we classify the subgroup G generated by the images of two or three total sextactic points in the Jacobian J_C of a smooth quartic C. It is clear that the group generated by one total sextactic point is the trivial group.

4. The group generated by two total sextactic points

Let C be a smooth projective plane quartic curve. Therefore, via the canonical embedding, C is a non-hyperelliptic curves of genus 3. Then, by using this embedding, $H^0(C, \Omega^{(1)}(C)) \cong H^0(C, \mathcal{O}_C(1))$, which has a degree 4, according to the Riemann-Roch theorem (or because the curve has degree 4). Consequently, $H^0(C, (\Omega^{(1)})^2(C)) \cong H^0(C, \mathcal{O}_C(2))$ and $H^0(C, (\Omega^{(1)})^3(C)) \cong H^0(C, \mathcal{O}_C(3))$, with degrees 8 and 12, respectively (see, for instance, [1]). We will start with some auxiliary results.

Remark 4.1. *For a total sextactic point $\infty \in C$, there is no cubic curve E_∞ satisfying that $I_\infty(C, E_\infty) = 12$. Indeed, if such a cubic curve exists, then, by Bezout's theorem, it coincides with the reducible cubic $E := \mathcal{T}_\infty \Lambda_\infty$, where $\mathcal{T}_\infty$ and Λ_∞, respectively, are the tangent line and the sextactic conic to C at ∞. But, in this case we already have $I_\infty(C, E) = 10$, since $I_\infty(C, \mathcal{T}_\infty) = 2$ and $I_\infty(C, \Lambda_\infty) = 8$, which is a contradiction.*

The Riemann-Roch space associated to any divisor D on C is defined to be

$$L(D) := \{f \in \mathbb{C}(C) | \operatorname{div}(f) + D \geq 0\}.$$

The space $L(D)$ will play a crucial role in the following.

Lemma 4.2. *Let C be a smooth projective plane quartic curve having two total sextactic points ∞ and $\mathcal{P}$ with distinct tangent lines. Then, the tangent line at one of these points does not pass through the other point.*

Proof. Without loss of generality, suppose, for the sake of a contradiction, that the tangent line to C at the point $\mathcal{P}$, denoted by $\mathcal{T}_\mathcal{P}$, passes through the point ∞. Therefore, its intersection divisor on C equals $\operatorname{div}(\mathcal{T}_\mathcal{P}) = 2\mathcal{P} + \infty + \mathcal{R}$, where $\mathcal{R} \in C \backslash \{\infty, \mathcal{P}\}$. Let Λ_∞ be the sextactic conic to C at ∞, so $\operatorname{div}(\Lambda_\infty) = 8\infty$ on C. Hence, there is a rational function f on C satisfying that $\operatorname{div}(f) := \operatorname{div}(\frac{\mathcal{T}_\mathcal{P}^4}{\Lambda_\infty^2}) = 8\mathcal{P} + 4\mathcal{R} - 12\infty$. It follows that $[8\mathcal{P} + 4\mathcal{R} - 12\infty] = 0$ in J_C. The vector space $L(12\infty)$ is of dimension 10, by the Riemann-Roch theorem. Therefore, $f \in L(12\infty) \cong H^0(C, (\Omega^{(1)})^3(C)) \cong H^0(C, \mathcal{O}_C(3))$, the space of divisors cut out by cubics on C. Now, $f \in L(12\infty)$ implies the existence of a cubic E_∞ with $\operatorname{div}(E_\infty) = 12\infty$. This leads to a contradiction by Remark 4.1. □

Now, let C be a smooth quartic curve with two total sextactic points. We take one of these points (which we denote as ∞) as a base point of the Jacobian embedding $A_\infty : C \to J_C$. Let $\mathcal{P}$ be the other total sextactic point and let $\Lambda_\mathcal{P}$ (resp., Λ_∞) be the quadratic form defining the sextactic conic to the curve C at $\mathcal{P}$ (resp, ∞). The divisor $\operatorname{div}(\frac{\Lambda_\mathcal{P}}{\Lambda_\infty}) = 8\mathcal{P} - 8\infty$ is principal on C, which implies that $[8\mathcal{P} - 8\infty] = 0$ in the Jacobian J_C of C. Therefore, the order of $[\mathcal{P} - \infty]$ in J_C divides 8. Thus, the order of $[\mathcal{P} - \infty]$ in J_C can be either 4 or 8. Indeed, it cannot be 1 because $[\mathcal{P} - \infty] = 0$ means that $\mathcal{P} - \infty$ is the divisor of a rational function, i.e., C is rational, which is excluded since the genus of C is 3. Also, it cannot be 2, otherwise, $[2\mathcal{P} - 2\infty] = 0$ implies the existence of a degree-2 map $C \to \mathbb{P}^1$, which contradicts the fact that C is not hyperelliptic. According to the Lemma 4.2, either $\mathcal{P}$ and ∞ have the same tangent to C, or the tangents to C at $\mathcal{P}$ and ∞ are distinct such that the tangent at one of them does not pass through the other point. Lemma 4.3 below describes the necessary and sufficient conditions for which $||[\mathcal{P} - \infty]|| = 8$ in J_C.

Lemma 4.3. *Let C be a smooth projective plane quartic curve with two total sextactic points $\mathcal{P}$ and ∞. Then, $\mathcal{P}$ and ∞ have distinct tangent lines and $\mathcal{P}$ has an imperfect mutual conic with respect to ∞, if and only if*

$$[4\mathcal{P} - 4\infty] \neq 0.$$

Proof. Let C be a smooth quartic curve that has two total sextactic points $\mathcal{P}$ and ∞ with distinct tangent lines and $\mathcal{P}$ has an imperfect mutual conic with respect to ∞. It is known that the canonical linear series on a smooth plane quartic curve is cut out by lines in $\mathbb{P}^2$. Let us assume that $[4\mathcal{P} - 4\infty] = 0$. Therefore, there is a rational function f on C with $div(f) = 4\mathcal{P} - 4\infty$. If $\mathcal{E} = 4\infty$ and K is a canonical divisor on C, then $\mathcal{E}$ cannot be linearly equivalent to K (if $\mathcal{E} \sim K$, then the tangent line to C at ∞ has a contact of order 4 and this is impossible since ∞ is a total sextactic point on C) and $deg(K - 4\infty) = 0$. Accordingly, $L(K-\mathcal{E})$ is a zero-dimensional vector space (see Lemma 1.2, page 295 in [11]). It follows that $L(\mathcal{E})$ is of dimension two, by the Riemann-Roch theorem. Let $\mathcal{T}_\infty$ be the linear form defining the tangent to C at ∞. Then, the intersection divisor of $\mathcal{T}_\infty$ on C equals $\mathrm{div}(\mathcal{T}_\infty) = 2\infty + R + H$, for some $R, H \in C\backslash\{\infty, \mathcal{P}\}$ (note that $\mathrm{div}(\mathcal{T}_\infty) = 2\infty + 2\mathcal{P}$ is not possible because $\mathcal{P}$ and ∞ have distinct tangent lines). Let Λ_∞ be the sextactic conic to C at ∞. Then $\mathrm{div}(\frac{\mathcal{T}_\infty^2}{\Lambda_\infty}) = 4\infty + 2R + 2H - 8\infty = 2R + 2H - 4\infty$. Taking the rational functions 1 and $g_\infty = \frac{\mathcal{T}_\infty^2}{\Lambda_\infty}$ as a basis for $L(\mathcal{E})$, it follows that f can be written in the form:

$$f = k.1 + g_\infty = \frac{\mathcal{T}_\infty^2 + k\Lambda_\infty}{\Lambda_\infty},$$

for some constant $k \in \mathbb{C}$. It is evident that f has a zero at the point $\mathcal{P}$ if and only if $k = -g_\infty(\mathcal{P})$ but in this case $\mathcal{P}$ is not a zero of order four (otherwise the mutual conic for $\mathcal{P}$ with respect to ∞ will be perfect, or $\mathcal{P}$ and ∞ have the same tangent line), which is a contradiction.

To prove the converse implication, it is enough to show that if $\mathcal{P}$ and ∞ have the same tangent line, or $\mathcal{P}$ has a perfect mutual conic with respect to ∞, then $[4\mathcal{P} - 4\infty] = 0$. If $\mathcal{P}$ and ∞ have the same tangent line $\mathcal{T}$ to C, then the intersection divisor of $\mathcal{T}$ and C equals $\mathrm{div}(\mathcal{T}) = 2\mathcal{P} + 2\infty$. Let Λ_∞ be the sextactic conic to C at ∞. Then $\mathrm{div}(\frac{\mathcal{T}^2}{\Lambda_\infty}) = 4\mathcal{P} + 4\infty - 8\infty = 4\mathcal{P} - 4\infty$. It follows that $[4\mathcal{P} - 4\infty] = 0$ in the Jacobian J_C. Finally, if $\mathcal{P}$ has a perfect mutual conic $\Xi_{\mathcal{P}\infty}$ with respect to ∞, then the divisor of $\Xi_{\mathcal{P}\infty}$ on C is given by $\mathrm{div}(\Xi_{\mathcal{P}\infty}) = 4\mathcal{P} + 4\infty$, therefore the divisor $\mathrm{div}(\frac{\Xi_{\mathcal{P}\infty}}{\Lambda_\infty}) = 4\mathcal{P} - 4\infty$ is principal on C, which implies that $[4\mathcal{P} - 4\infty] = 0$ in the Jacobian J_C of C. □

As a consequence of Lemma 4.3 we have:

Theorem 4.4. *Let C be a smooth projective plane quartic curve with two total sextactic points. Let G be the cyclic subgroup generated by the images of these total sextactic points in the Jacobian J_C of C. Then we can classify G as follows:*

(i) $G \cong \mathbb{Z}/8\mathbb{Z}$ *if and only if the tangent lines to C at these points are distinct and the mutual conic for one point with respect to the other is imperfect.*

(ii) $G \cong \mathbb{Z}/4\mathbb{Z}$ *if and only if the two total sextactic points have the same tangent line, or the mutual conic for one point with respect to the other is perfect.*

Proof. If C has two total sextactic points $\mathcal{P}$ and ∞ with distinct tangent lines and the mutual conic $\Xi_{\mathcal{P}\infty}$ for $\mathcal{P}$ with respect to ∞ is imperfect, then, by the discussion before Lemma 4.3 and by Lemma 4.3, the order of $[\mathcal{P} - \infty]$ in J_C is exactly 8, so $G \cong \mathbb{Z}/8\mathbb{Z}$.

If $\mathcal{P}$ and ∞ have the same tangent line to C or $\Xi_{\mathcal{P}\infty}$ is perfect, then Lemma 4.3 informs us that the order of $[\mathcal{P} - \infty]$ in J_C is 4, and therefore $G \cong \mathbb{Z}/4\mathbb{Z}$ as required. □

As a summary of Theorem 4.4, we can note down Table 1 in Section 1. Now, we will support the validity of Theorem 4.4 by providing the following examples. To construct new examples supporting the classification, or to reproduce the examples discussed below, see the algorithms in Subsection 6.1.

Example 4.5. *Let C be the smooth projective plane quartic curve defined by*

$$C : Y^4 = (X - Z)\left(X - (-2 + 2\sqrt{2})Z\right)XZ.$$

It is not difficult to demonstrate that $Q_1 = \left[\sqrt{2} : \sqrt{2-\sqrt{2}} : 1\right]$ *and* $Q_2 = \left[\sqrt{2} : -\sqrt{2-\sqrt{2}} : 1\right]$ *are total sextactic points on* C. *We note that the distinct lines*

$$\mathcal{T}_1 = -X + \sqrt{4-2\sqrt{2}}Y + \left(2-\sqrt{2}\right)Z \text{ and}$$
$$\mathcal{T}_2 = -X - \sqrt{4-2\sqrt{2}}Y + \left(2-\sqrt{2}\right)Z$$

are the tangent lines to C *at* Q_1 *and* Q_2, *respectively. The equations of the sextactic conics* Λ_i *associated to* Q_i $(i = 1, 2)$ *are*

$$\Lambda_1 = X^2 + 4Y^2 + 2Z^2 - 2\sqrt{2}XZ + 2\sqrt{2-\sqrt{2}}\left(\sqrt{2}YZ - \left(\sqrt{2}+1\right)XY\right),$$
$$\Lambda_2 = X^2 + 4Y^2 + 2Z^2 - 2\sqrt{2}XZ + 2\sqrt{2-\sqrt{2}}\left(\left(1+\sqrt{2}\right)XY - \sqrt{2}YZ\right).$$

Using a computer algebra system, such as Maple, one can verify that the conics Λ_i *satisfy that* $\operatorname{div}(\Lambda_i) = 8Q_i$ *on the curve* C. *Let* $\Xi_{21} = \mathcal{T}_1^2 - g_1(Q_2)\Lambda_1$, *where* $g_1(Q_2) := \frac{\mathcal{T}_1^2}{\Lambda_1}(Q_2) = 2 - \sqrt{2}$ *is the mutual conic for* Q_2 *with respect to* Q_1. *The divisor of* Ξ_{21} *on* C *is given by* $\operatorname{div}(\Xi_{21}) = 4Q_1 + 4Q_2$, *so* Q_2 *has a perfect mutual conic with respect to* Q_1. *Let us take* Q_1 *as a base point for the Abel Jacobi map* $A_{Q_1} : C \to J_C$, *from* C *to its Jacobian* J_C. *Therefore, the subgroup* $G_1 := \langle [Q_2 - Q_1] \rangle$, *in the Jacobian* J_C, *satisfies* $G_1 \cong \mathbb{Z}/4\mathbb{Z}$. *Hence, this example supports the validity of the second part of* (*ii*) *in Theorem 4.4.*

Example 4.6. *Following the same notations as in Example 4.5, we see that* $Q_3 := \left[\sqrt{2} : \sqrt{\sqrt{2}-2} : 1\right]$ *is also a total sextactic point on* C. *Indeed, the sextactic conic*

$$\Lambda_3 = X^2 - 4Y^2 + 2Z^2 - 2\sqrt{2}XZ + 2\sqrt{2-\sqrt{2}}i\left(\left(1+\sqrt{2}\right)XY - \sqrt{2}YZ\right),$$

and the tangent line $\mathcal{T}_3 = -X - i\sqrt{4-2\sqrt{2}}Y + \left(2-\sqrt{2}\right)Z$, *associated to* Q_3 *on the curve* C, *satisfy* $\operatorname{div}(\Lambda_3) = 8Q_3$ *and* $\operatorname{div}(\mathcal{T}_1) \neq \operatorname{div}(\mathcal{T}_3)$. *Let* $\Xi_{31} = \mathcal{T}_1^2 - g_1(Q_3)\Lambda_1$, *where* $g_1(Q_3) := \frac{\mathcal{T}_1^2}{\Lambda_1}(Q_3) = \frac{1}{2}(1+i)\left(2-\sqrt{2}\right)$, *be the mutual conic for* Q_3 *with respect to* Q_1. *The divisor of* Ξ_{31} *on* C *is given by* $\operatorname{div}(\Xi_{31}) = 4Q_1 + 2Q_3 + R_1 + S_1$, *where*

$$R_1 = \left[-11\sqrt{2} + 16 + 2\sqrt{116 - 82\sqrt{2}} : -i\left(-3+2\sqrt{2}\right)\left(-2 + \sqrt{2-\sqrt{2}}\right) : 1\right], \text{ and}$$
$$S_1 = \left[-11\sqrt{2} + 16 - 2\sqrt{116 - 82\sqrt{2}} : -i\left(-3+2\sqrt{2}\right)\left(2 + \sqrt{2-\sqrt{2}}\right) : 1\right].$$

Therefore, Q_3 *has an imperfect mutual conic with respect to* Q_1. *Then, the subgroup* $G_2 := \langle [Q_3 - Q_1] \rangle$, *in the Jacobian* J_C, *satisfies* $G_2 \cong \mathbb{Z}/8\mathbb{Z}$. *Hence, this example supports the correctness of* (*i*) *in Theorem 4.4.*

Example 4.7. *Consider the Kuribayashi quartic curve* $\mathcal{K}_{14}$ *given by*

$$\mathcal{K}_{14} : X^4 + Y^4 + Z^4 + 14(X^2Y^2 + Y^2Z^2 + Z^2X^2) = 0.$$

It is not hard to show that the points $\mathcal{P}_1 = \left[\omega : \omega^2 : 1\right]$ *and* $\mathcal{P}_2 = \left[\omega^2 : \omega : 1\right]$, *where* $\omega = \exp(\frac{2\pi i}{3})$, *are total sextactic points on* $\mathcal{K}_{14}$. *Indeed, the equations of the sextactic conic* Λ_i *at* $\mathcal{P}_i$ $(i = 1, 2)$ *are*

$$\Lambda_1 : X^2 + \omega^2(Y^2 + 5XZ) + \omega(Z^2 + 5XY) + 5YZ = 0,$$
$$\Lambda_2 : X^2 + \omega(Y^2 + 5XZ) + \omega^2(Z^2 + 5XY) + 5YZ = 0.$$

The line $\mathcal{T}_{12} : X + Y + Z = 0$ *is the tangent line to* $\mathcal{K}_{14}$ *at both of* $\mathcal{P}_1$ *and* $\mathcal{P}_2$. *Using a computer algebra system, such as Maple, one can show that the intersection divisors of the conic* Λ_i *and the line* $\mathcal{T}_{12}$ *on the curve* $\mathcal{K}_{14}$ *satisfy* $\operatorname{div}(\Lambda_i) = 8\mathcal{P}_i$ *and* $\operatorname{div}(\mathcal{T}_{12}) = 2\mathcal{P}_1 + 2\mathcal{P}_2$, *respectively. Let us take* $\mathcal{P}_1$ *as a base point for the Abel Jacobi map* $A_{\mathcal{P}_1} : \mathcal{K}_{14} \to J_{\mathcal{K}_{14}}$. *Then the principal divisor* $\operatorname{div}(\frac{\mathcal{T}_{12}^2}{\Lambda_1})$ *implies that* $[4\mathcal{P}_2 - 4\mathcal{P}_1] = 0$ *in* $J_{\mathcal{K}_{14}}$. *Thus, the subgroup* $G_3 := \langle[\mathcal{P}_2 - \mathcal{P}_1]\rangle$, *in the Jacobian* $J_{\mathcal{K}_{14}}$, *satisfies* $G_3 \cong \mathbb{Z}/4\mathbb{Z}$. *Hence, this example supports the correctness of the first part of* (ii) *in Theorem 4.4.*

5. The group generated by three total sextactic points

Now, we pass to the case when a smooth projective plane quartic curve C has three total sextactic points $\mathcal{P}, Q, \infty$. Take ∞ as a base point of the Jacobian embedding $A_\infty : C \to J_C$. In this section, we give a complete classification of the group $G = \langle[\mathcal{P} - \infty], [Q - \infty]\rangle$ in J_C. In the discussion preceding Lemma 4.3, we explained that the order of any of the generators in J_C can be either 4 or 8. Therefore, we have three possibilities: either G is a quotient of $(\mathbb{Z}/8\mathbb{Z})^2$, a quotient of $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$, or a quotient of $(\mathbb{Z}/4\mathbb{Z})^2$. We will start with an auxiliary result, as in Section 4. As a direct consequence of Lemma 4.2, one gets the following.

Corollary 5.1. *Let* C *be a smooth projective plane quartic curve having three total sextactic points* $\mathcal{P}, Q$, *and* ∞ *with distinct tangent lines. Then, the tangent line at one of these points cannot pass through the other two points.*

Lemma 4.3 specified the conditions under which both generators are of order 8. The following Lemma 5.2 shows that under these conditions the group G is indeed isomorphic to $(\mathbb{Z}/8\mathbb{Z})^2$.

Lemma 5.2. *If a smooth projective plane quartic curve* C *has three total sextactic points* $\mathcal{P}, Q, \infty$ *with distinct tangent lines and the mutual conics are all imperfect, then*

$$[m\mathcal{P} + nQ - (m + n)\infty] \neq 0,$$

for any $m, n \in \mathbb{Z}_8$, *which both cannot be zero.*

Proof. Since the three total sextactic points $\mathcal{P}, Q, \infty$ with distinct tangent lines and the mutual conics are all imperfect, it follows that (by Lemma 4.3) $[4\mathcal{P} - 4\infty] \neq 0$, $[4Q - 4\infty] \neq 0$, and $[4\mathcal{P} - 4Q] \neq 0$ in the Jacobian J_C of C. So the order of any of the elements $[\mathcal{P} - \infty]$, $[Q - \infty]$, and $[\mathcal{P} - Q]$ in J_C is 8. We vary over $m \in \mathbb{Z}_8$.

- (i) Let $m = 0$. Then $[nQ - n\infty] \neq 0$, for any $n \in \mathbb{Z}_8\backslash\{0\}$, because the order of $[Q - \infty]$ is exactly 8, by the discussion before Lemma 4.3 and by Lemma 4.3.
- (ii) Let $m = 4$. If $n = 1, 3, 5, 7$ and $[4\mathcal{P} + nQ - (n + 4)\infty] = 0$, then $[4\mathcal{P} - 4\infty] = -n[Q - \infty]$, which implies that $n[2Q - 2\infty] = 0$ because $[8\mathcal{P} - 8\infty] = 0$. Since $[8Q - 8\infty] = 0$, it follows

that $n[2Q-2\infty]=[2Q-2\infty]=0$ when $n=1,3,5,7$, which contradicts the non-hyperellipticity of C. In a similar manner, if $n=0,2,6$ and $[4\mathcal{P}+nQ-(n+4)\infty]=0$, then we have either $[4Q-4\infty]=0$ or $[4\mathcal{P}-4\infty]=0$, which contradicts Lemma 4.3. Finally, if $n=4$, then $[4\mathcal{P}+4Q-8\infty]\neq 0$. Indeed, let Λ_Q and Λ_∞ be the sextactic conics to C at Q and ∞, respectively. Then $\mathrm{div}(\Lambda_Q)=8Q$, $\mathrm{div}(\Lambda_\infty)=8\infty$, and therefore the divisor $\mathrm{div}(\frac{\Lambda_\infty}{\Lambda_Q})=8\infty-8Q$ is principal on C. Furthermore, let $D_1=4\mathcal{P}-4Q$ and $D_2=4\mathcal{P}+4Q-8\infty$. Then $D_1-D_2=\mathrm{div}(\frac{\Lambda_\infty}{\Lambda_Q})$, so $D_1\sim D_2$. Hence, the non-principlity of D_1 (by Lemma 4.3) implies the non-principlity of D_2.

(*iii*) Let $m=2$. If $n=1,3,4,5,7$ and $[2\mathcal{P}+nQ-(n+2)\infty]=0$, then we get either $[4Q-4\infty]=0$ or $[4\mathcal{P}-4\infty]=0$ (we get it when $n=4$), which contradicts Lemma 4.3. If $n=0,6$ and $[2\mathcal{P}+nQ-(n+2)\infty]=0$, then we have either $[2\mathcal{P}-2\infty]=0$ or $[2\mathcal{P}-2Q]=0$ (note that $[2Q-2\infty]=-[6Q-6\infty]$), which contradicts the non-hyperellipticity of C. Finally, if $n=2$, then $[2\mathcal{P}+2Q-4\infty]\neq 0$. Indeed, the divisor $4\mathcal{P}+4Q-8\infty$ is twice the divisor $2\mathcal{P}+2Q-4\infty$ and the class $[4\mathcal{P}+4Q-8\infty]$ does not vanish by (*ii*), therefore neither can $[2\mathcal{P}+2Q-4\infty]$.

(*iv*) Let $m=1$. By exchanging roles between m and n we find that the cases when $n=2,4$ are excluded by (*ii*) and (*iii*). If $n=0,7$ and $[\mathcal{P}+nQ-(n+1)\infty]=0$, then we get either $[\mathcal{P}-\infty]=0$ or $[\mathcal{P}-Q]=0$ (note that $[Q-\infty]=-[7Q-7\infty]$), which contradicts the non-rationality of C. If $n=3,5$ and $[\mathcal{P}+nQ-(n+1)\infty]=0$, then we get $[4\mathcal{P}+4Q-8\infty]=0$, which is impossible by (*ii*). If $[\mathcal{P}+6Q-7\infty]=0$, then we get $[4\mathcal{P}-4\infty]=0$, which contradicts Lemma 4.3. Finally, if $[\mathcal{P}+Q-2\infty]=0$, then C is hyperelliptic, which is a contradiction (note too that the class $[2\mathcal{P}+2Q-4\infty]$ is twice that of $[\mathcal{P}+Q-2\infty]$ and the former does not vanish by (*iii*), so neither can the latter).

(*v*) Let $m=3$. If $n=1,3,5,7$ and $[3\mathcal{P}+nQ-(n+3)\infty]=0$, then we get $[4\mathcal{P}+4Q-8\infty]=0$, which is a contradiction to (*ii*). If $n=2,4,6$ and $[3\mathcal{P}+nQ-(n+3)\infty]=0$, then we get $[4\mathcal{P}-4\infty]=0$, which is impossible by Lemma 4.3. Finally, if $n=0$, then $[3\mathcal{P}-3\infty]=0$, which contradicts the fact that the order of $[\mathcal{P}-\infty]$ in J_C divides 8.

(*vi*) Let $m=5$. If $[5\mathcal{P}+nQ-(5+n)\infty]=0$, then $-[5\mathcal{P}-5\infty]=[nQ-n\infty]$, therefore $[3\mathcal{P}-3\infty]=[nQ-n\infty]$, so we can apply the same proof as for the case of $m=3$. Similarly, if $m=6$ or 7, we can apply the same proof as in the case of $m=2$ or 1, respectively (note, for instance, that if $[7\mathcal{P}+7Q-14\infty]=0$, then we get the same contradiction as in $[\mathcal{P}+Q-2\infty]=0$ because $[7\mathcal{P}-7\infty]=-[\mathcal{P}-\infty]$ and $[7Q-7\infty]=-[Q-\infty]$).

□

As a direct consequence of Lemma 5.2 we get:

Theorem 5.3. *Let C be a smooth projective plane quartic curve with three total sextactic points. Let G be the subgroup generated by the images of these total sextactic points in the Jacobian J_C of C. Then $G\cong(\mathbb{Z}/8\mathbb{Z})^2$ if and only if the tangent lines to C at these points are distinct and the mutual conics are all imperfect.*

Let us assume that one of the two generators of G is of order 4 and the other is of order 8. Geometrically, this occurs if and only if two of these points have the same tangent line to the curve C and all the possible mutual conics are imperfect, or C has distinct tangent lines at these points and only one of the mutual conics is perfect. Then, G is a quotient of the group $\mathbb{Z}/4\mathbb{Z}\oplus\mathbb{Z}/8\mathbb{Z}$. In order to determine which linear combination of generators can be null in J_C, we need to prove Lemma 5.4

below, but the proof of Lemma 5.4 requires us to remember the definition of Weierstrass points on a smooth quartic C (non-hyperelliptic curves of genus 3). A point $Q \in C$ is a Weierstrass point if there exists a non-constant rational function on C having a pole of order less than or equal to three at Q and no poles everywhere else, or equivalently, $L(3Q)$ has at least dimension 2. The Weierstrass points on C are known to be nothing more than flexes (Vermeulen [3]).

Lemma 5.4. *Let C be a smooth projective plane quartic curve with three total sextactic points $\mathcal{P}, Q$, and ∞. If two of these points have the same tangent line to C, say $\mathcal{P}$ and ∞, and all the possible mutual conics are imperfect, or the tangent lines to C at these points are distinct and only one of the mutual conics, say $\Xi_{\mathcal{P}\infty}$, is perfect, then*

$$[m\mathcal{P} + nQ - (m+n)\infty] \neq 0,$$

for any $(m, n) \in \mathbb{Z}_4 \times \mathbb{Z}_8$, $(m, n) \neq (0, 0)$.

Proof. If $\mathcal{P}$ and ∞ have the same tangent line to C and all the possible mutual conics are imperfect, or if only the mutual conic $\Xi_{\mathcal{P}\infty}$ is perfect and the other mutual conics are all imperfect, then in both cases we find that $\|[\mathcal{P} - \infty]\| = 4$ and $\|[Q - \infty]\| = 8$ in J_C. Similarly to Lemma 5.2, we vary over $m \in \mathbb{Z}_4$.

(*i*) Let $m = 0$. Then $[nQ - n\infty] \neq 0$, for any $n \in \mathbb{Z}_8 \backslash \{0\}$, because the order of $[Q - \infty]$ in J_C is exactly 8.

(*ii*) Let $m = 1$. If $n = 1, 4, 6$ and $[\mathcal{P} + nQ - (n+1)\infty] = 0$, then we have $[\mathcal{P} + Q - 2\infty] = 0$, $[2\mathcal{P} - 2\infty] = 0$, $[\mathcal{P} + \infty - 2Q] = 0$, respectively, which contradicts the non-hyperellipticity of C. When $n = 0, 7$ and $[\mathcal{P} + nQ - (n+1)\infty] = 0$, we get either $[\mathcal{P} - \infty] = 0$ or $[\mathcal{P} - Q] = 0$, which contradicts the non-rationality of C. If $n = 3, 5$ and $[\mathcal{P} + nQ - (n+1)\infty] = 0$, we get $[4Q - 4\infty] = 0$, which contradicts Lemma 4.3. Finally, if $[\mathcal{P} + 2Q - 3\infty] = 0$ would imply the existence of a nonconstant rational function with a pole only at ∞ of order 3 and holomorphic everywhere else, then the dimension of $L(3\infty) > 1$. So, ∞ is a Weierstass point on C, which is equivalent to say ∞ is a flex point of C, which is a contradiction.

(*iii*) Let $m = 2$. If $n = 0, 3, 5, 7$ and $[2\mathcal{P} + nQ - (n+2)\infty] = 0$, then we get either $[2\mathcal{P} - 2\infty] = 0$ or $[2Q - 2\infty] = 0$, which contradicts the non-hyperellipticity of C. If $n = 2, 6$ and $[2\mathcal{P} + nQ - (n+2)\infty] = 0$, we get $[4Q - 4\infty] = 0$, which contradicts Lemma 4.3. When $n = 1$ and $[2\mathcal{P} + Q - 3\infty] = 0$, we find that this is impossible because ∞ is not a 1-Weierstass point on C. Finally, if $n = 4$ and $[2\mathcal{P} + 4Q - 6\infty] = 0$ in J_C, this implies that $[4\mathcal{P} + 8Q - 12\infty] = 0$ in J_C. Therefore, there exists a rational function f on C with $\mathrm{div}(f) = 4\mathcal{P} + 8Q - 12\infty$. It is clear that $f \in L(12\infty) \cong H^0(C, (\Omega^{(1)})^3(C)) \cong H^0(C, \mathcal{O}_C(3))$. Hence, there exists a cubic E_∞ with $\mathrm{div}(E_\infty) = 12\infty$ on C. This leads to a contradiction by Remark 4.1.

(*vi*) Let $m = 3$. If $[3\mathcal{P} + nQ - (3+n)\infty] = 0$, then $-[3\mathcal{P} - 3\infty] = [nQ - n\infty]$, therefore $[\mathcal{P} - \infty] = [nQ - n\infty]$, so we can apply a similar proof to case of $m = 1$.

□

Before giving a summary of the previous results, we want to obtain a geometric interpretation when the class $[2\mathcal{P} + 4Q - 6\infty]$ does not vanish in J_C. The following lemma helps us to find such a thing.

Lemma 5.5. *Let $\mathcal{P}, Q, R$ be any points on a smooth quartic curve C and let ∞ be a total sextactic on C. Then $[2\mathcal{P} + 2Q + 2R - 6\infty] = 0$ in the Jacobian J_C of C if and only if there exists a conic tangent to the curve at these 4 points.*

Proof. Suppose that Λ_∞ is the sextactic conic at ∞. It is clear that if there exists a conic Λ tangent to the curve at these 4 points, then $\mathrm{div}(\frac{\Lambda}{\Lambda_\infty}) = 2\mathcal{P}+2Q+2R+2\infty-8\infty$. Therefore $[2\mathcal{P}+2Q+2R-6\infty] = 0$ in J_C.

Conversely, suppose that $[2\mathcal{P} + 2Q + 2R - 6\infty] = 0$, that is, there is an effective divisor in the linear system associated with 6∞ whose zeros are $\mathcal{P}, Q$, and R, with multiplicity 2. The vector space $L(6\infty)$ is of dimension 4 according to the Riemann-Roch theorem. Up to a suitable choice of the coordinates in $\mathbb{P}^2$, one can assume that $\infty = [1 : 0 : 0]$, the tangent $\mathcal{T}_\infty$ at ∞ has the equation: $Z = 0$, and hence the sextactic conic Λ_∞ at ∞ has the equation: $Y^2 - XZ = 0$. Therefore, a base of $L(6\infty)$ is given by $1, \frac{XZ}{Y^2-XZ}, \frac{YZ}{Y^2-XZ}, \frac{Z^2}{Y^2-XZ}$. Hence

$$L(6\infty) = \left\{ \frac{\alpha Y^2 + \beta YZ + \gamma XZ + \delta Z^2}{Y^2 - XZ} \ : \ \text{for some constants } \alpha, \beta, \gamma, \delta \right\}.$$

□

Remark 5.6. *The relation* $[2\mathcal{P} + 2Q + 2R - 6\infty] = [2\mathcal{P} + 2Q + 2R + 2\infty - 8\infty] = 0$ *in* J_C *implies the existence of a rational function* f *on* C *with* $\mathrm{div}(f) = 2\mathcal{P} + 2Q + 2R + 2\infty - 8\infty$. *Hence,* $f \in L(8\infty) \cong H^0(C, (\Omega^{(1)})^2(C)) \cong H^0(C, O_C(2))$. *Therefore, the existence of such a function* f*, implies, in turn, the existence of a conic* Λ *with contact order* 2 *to* C *at each of* $\mathcal{P}, Q, R$*, and* ∞*. This provides us with another proof of Lemma 5.5.*

Returning to our problem, according to Lemma 5.5, we find that the class $[2\mathcal{P} + 4Q - 6\infty]$ vanishes in J_C if and only if there exists a conic Λ with $\mathrm{div}(\Lambda) = 2\mathcal{P}+4Q+2\infty$. It turns out that either $\Lambda = 2\mathcal{T}_Q$, where $\mathcal{T}_Q$ is the tangent to C at Q, so $\mathcal{T}_Q$ passes through both $\mathcal{P}$ and ∞ (and this is impossible by Corollary 5.1), or Λ is an irreducible, but this conic must coincides with both $\Xi_{\mathcal{P}Q}$ and $\Xi_{\infty Q}$ (by Bezout's theorem) and both will be semiperfect (see Remark 3.2). So, as a corollary of Lemma 5.4, one gets the following.

Corollary 5.7. *If a smooth projective plane quartic curve* C *has three total sextactic points* $\mathcal{P}, Q$*, and* ∞ *with distinct tangent lines, then, there are no coincident semiperfect mutual conics among these points.*

Proof. Without loss of generality, assume that $\Xi_{\mathcal{P}Q}$ and $\Xi_{\infty Q}$ are two coincident semiperfect mutual conics. Then, by definition of semiperfect conics, one has $\mathrm{div}(\Xi_{\mathcal{P}Q}) = \mathrm{div}(\Xi_{\infty Q}) = 4Q + 2\mathcal{P} + 2\infty$. Hence, there exists a rational function $f := \frac{\Xi_{\mathcal{P}Q}}{\Lambda_\infty}$ on C, where Λ_∞ is the sextactic conic for C at ∞, satisfying that $\mathrm{div}(f) = 2\mathcal{P} + 4Q - 6\infty$. Therefore, $[2\mathcal{P}+4Q - 6\infty] = 0$ in J_C, which is a contraction by Lemma 5.4. □

Summing up the above, we can write the following.

Theorem 5.8. *Let* C *be a smooth projective plane quartic curve with three total sextactic points. Let* G *be the subgroup generated by the images of these total sextactic points in the Jacobian* J_C *of* C*. Then* $G \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$ *if and only if* C *has distinct tangent lines at these points and only one of the mutual conics is perfect, or two of these points have the same tangent to* C *and all the possible mutual conics are imperfect.*

Finally, assume that the two generators of G are of order 4. Geometrically, this happens if and only if the curve C has distinct tangent lines at these points and two (or all) of the mutual conics are perfect, or C has the same tangent line at two of these points and at least one of the mutual conics is perfect. Then, G is a quotient of the group $(\mathbb{Z}/4\mathbb{Z})^2$. Without loss of generality, assume that $\Xi_{\mathcal{P}\infty}$ and $\Xi_{Q\infty}$ are perfect (note that if, for instance, $\Xi_{\mathcal{P}\infty}$ and $\Xi_{Q\mathcal{P}}$ are perfect we can exchange roles between $\mathcal{P}$ and ∞), or $\mathcal{P}$ and ∞ have the same tangent line to C and $\Xi_{Q\infty}$ is perfect. Therefore, in all possibilities, $\|[\mathcal{P}-\infty]\| = 4$ and $\|[Q-\infty]\| = 4$ in J_C. Also, similar to how Lemmas 5.2 and 5.4 were proved, we can demonstrate that

$$\{[m\mathcal{P} + nQ - (m+n)\infty] = 0 \mid (m,n) \in \mathbb{Z}_4 \times \mathbb{Z}_4, (m,n) \neq (0,0)\} = \langle 0 \rangle .$$

Perhaps the only case that needs a little clarification is when $(m,n) = (2,2)$. If $[2\mathcal{P} + 2Q - 4\infty] = 0$, then, as in the proof of Lemma 4.3, there exists $f \in L(4\infty)$ on C with $div(f) = 2\mathcal{P} + 2Q - 4\infty$. Taking the rational functions 1 and $g_\infty = \frac{\mathcal{T}_\infty^2}{\Lambda_\infty}$ as a basis for $L(4\infty)$, it follows that f can be written in the form

$$f = \ell.1 + g_\infty = \frac{\mathcal{T}_\infty^2 + \ell\Lambda_\infty}{\Lambda_\infty},$$

for some constant $\ell \in \mathbb{C}$. It is evident that f has a zero at the point $\mathcal{P}$ if and only if $\ell = -g_\infty(\mathcal{P})$ (note that if C has the same tangent line at $\mathcal{P}$ and ∞, then $g_\infty(\mathcal{P}) = 0$) but in such a case $\mathcal{P}$ is not a zero of order two due to the assumption that the mutual conic for $\mathcal{P}$ with respect to ∞ is perfect, or C has the same tangent line at $\mathcal{P}$ and ∞, which is a contradiction.

Theorem 5.9. *Let C be a smooth projective plane quartic curve with three total sextactic points. Let G be the subgroup generated by the images of these total sextactic points in the Jacobian J_C of C. Then $G \cong (\mathbb{Z}/4\mathbb{Z})^2$ if and only if C has distinct tangent lines at these points and two (or all) of the mutual conics are perfect, or two of these points have the same tangent line to the curve C and at least one of of the mutual conics is perfect.*

As a summary of Theorems 5.3, 5.8, and 5.9, we provide Table 2 in Section 1. Now, we support the correctness of our results by giving the following examples.

Example 5.10. *Following the same notations as in Examples 4.5 and 4.6, it is not hard to see that the point $Q_5 = \left[2 - \sqrt{2} : \sqrt{3\sqrt{2}-4} : 1\right]$ is a total sextactic point on C. Indeed, the sextactic conic Λ_5 and the tangent line $\mathcal{T}_5$ associated to Q_5 are*

$$\begin{aligned} \Lambda_5 &= X^2 + 4\left(\sqrt{2}-1\right)Y^2 + \left(6 - 4\sqrt{2}\right)Z^2 + \left(2\sqrt{2}-4\right)XZ + 2\sqrt{3\sqrt{2}-4}\left(XY - \sqrt{2}YZ\right), \\ \mathcal{T}_5 &= X + \sqrt{3\sqrt{2}-4}(\sqrt{2}+2)Y - \sqrt{2}Z. \end{aligned}$$

Using a computer algebra system, such as Maple, one can show that $\operatorname{div}(\Lambda_5) = 8Q_5$ *and* $\operatorname{div}(\mathcal{T}_5) \neq \operatorname{div}(\mathcal{T}_1) \neq \operatorname{div}(\mathcal{T}_3)$. *Let* $\Xi_{51} = \mathcal{T}_1^2 - g_1(Q_5)\Lambda_1, \quad g_1(Q_5) := \frac{\mathcal{T}_1^2}{\Lambda_1}(Q_5) = 3 - 2\sqrt{2}$, *be the mutual conic for Q_5 with respect to Q_1, and* $\Xi_{53} = \mathcal{T}_3^2 - g_3(Q_5)\Lambda_3, \quad g_3(Q_5) := \frac{\mathcal{T}_3^2}{\Lambda_3}(Q_5) = 1$, *be the mutual conic for Q_5 with respect to Q_3. Let Ξ_{31} be the mutual conic for Q_3 with respect to Q_1 given in Example 4.6. The intersection divisors of Ξ_{31}, Ξ_{53}, and Ξ_{51} on C are:*

$$\operatorname{div}(\Xi_{31}) = 4Q_1 + 2Q_3 + R_1 + S_1,$$

$$\mathrm{div}(\Xi_{53}) = 4Q_3 + Q_5 + Q_6 + W_3 + W_\infty,$$
$$\mathrm{div}(\Xi_{51}) = 4Q_1 + Q_5 + Q_6 + W_1 + W_2,$$

where $W_1 = [2\sqrt{2}-2 : 0 : 1]$, $W_2 = [0 : 0 : 1]$, $W_3 = [1 : 0 : 1]$, $W_\infty = [\frac{1}{\sqrt{2}} : 0 : 0]$, $Q_6 = [2-\sqrt{2} : -\sqrt{3\sqrt{2}-4} : 1]$. *Hence, the mutual conics are all imperfect. Therefore, the subgroup* $G_4 := \langle [Q_3 - Q_1], [Q_5 - Q_1] \rangle$ *in the Jacobian* J_C *satisfies* $G_4 \cong (\mathbb{Z}/8\mathbb{Z})^2$. *This example supports the correctness of Theorem 5.3.*

Example 5.11. *Following the same notations as in Examples 4.5 and 4.6, it is not hard to see that the point* $Q_4 = \left[\sqrt{2} : -\sqrt{\sqrt{2}-2} : 1\right]$ *is a total sextactic points on* C. *Indeed, the sextactic conic*

$$\Lambda_4 = X^2 - 4Y^2 + 2Z^2 - 2\sqrt{2}XZ + 2\sqrt{2-\sqrt{2}}i\left(\sqrt{2}YZ - \left(1+\sqrt{2}\right)XY\right),$$

and the tangent line $\mathcal{T}_4 = -X + i\sqrt{4-2\sqrt{2}}Y + \left(2-\sqrt{2}\right)Z$, *associated to* Q_4 *on the curve* C *satisfy* $\mathrm{div}(\Lambda_4) = 8Q_4$ *and* $\mathrm{div}(\mathcal{T}_4) \neq \mathrm{div}(\mathcal{T}_1) \neq \mathrm{div}(\mathcal{T}_3)$. *Let* $\Xi_{34} = \mathcal{T}_4^2 - g_4(Q_3)\Lambda_4$, *where* $g_4(Q_3) := \frac{\mathcal{T}_4^2}{\Lambda_4}(Q_3) = 2-\sqrt{2}$ *is the mutual conic for* Q_3 *with respect to* Q_4, *and* $\Xi_{41} = \mathcal{T}_1^2 - g_1(Q_4)\Lambda_1$, *where* $g_1(Q_4) := \frac{\mathcal{T}_1^2}{\Lambda_1}(Q_4) = \frac{1}{2}(1-i)\left(2+\sqrt{2}\right)$ *is the mutual conic for* Q_4 *with respect to* Q_1. *Let* Ξ_{31} *be the mutual conic for* Q_3 *with respect to* Q_1 *given in Example 4.6. The intersection divisors of* Ξ_{31}, Ξ_{34}, *and* Ξ_{41} *on* C *are given by* $\mathrm{div}(\Xi_{31}) = 4Q_1 + 2Q_3 + R_1 + S_1$, $\mathrm{div}(\Xi_{34}) = 4Q_3 + 4Q_4$, *and* $\mathrm{div}(\Xi_{41}) = 4Q_1 + 2Q_4 + R_2 + S_2$, *where*

$$R_2 = \left[-11\sqrt{2} + 16 + 2\sqrt{116 - 82\sqrt{2}} : i\left(-3 + 2\sqrt{2}\right)\left(-2 + \sqrt{2-\sqrt{2}}\right) : 1\right],$$
$$S_2 = \left[-11\sqrt{2} + 16 - 2\sqrt{116 - 82\sqrt{2}} : i\left(-3 + 2\sqrt{2}\right)\left(2 + \sqrt{2-\sqrt{2}}\right) : 1\right].$$

The mutual conics Ξ_{31} *and* Ξ_{41} *are imperfect while* Ξ_{34} *is perfect. Therefore, the subgroup* $G_5 := \langle [Q_3 - Q_1], [Q_4 - Q_1] \rangle$ *in the Jacobian* J_C *satisfies* $G_5 \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$. *This example supports the validity of one of the possibilities in Theorem 5.8.*

Example 5.12. *Following the same notations as in Example 4.7, it is not hard to show that the points* $\mathcal{P}_3 = \left[\omega : -\omega^2 : 1\right]$ *and* $\mathcal{P}_4 = \left[\omega^2 : -\omega : 1\right]$ *are total sextactic points on* $\mathcal{K}_{14}$. *Indeed, the sextactic conic* Λ_i *to* C *at* $\mathcal{P}_i$ $(i = 3, 4)$ *is*

$$\Lambda_3 : X^2 - 5YZ + \omega(Z^2 - 5XY) + \omega^2(Y^2 + 5XZ) = 0,$$
$$\Lambda_4 : X^2 - 5YZ + \omega^2(Z^2 - 5XY) + \omega(Y^2 + 5XZ) = 0.$$

Moreover, the line $\mathcal{T}_{34} : X - Y + Z = 0$ *is bitangent to* $\mathcal{K}_{14}$ *at* $\mathcal{P}_3$ *and* $\mathcal{P}_4$. *Both* Λ_i *and* $\mathcal{T}_{34}$ *satisfy* $\mathrm{div}(\Lambda_i) = 8\mathcal{P}_i$ *and* $\mathrm{div}(\mathcal{T}_{34}) = 2\mathcal{P}_3 + 2\mathcal{P}_4$. *Let us take the point* $\mathcal{P}_1$ *as a base point for the Abel Jacobi map* $A_{\mathcal{P}_1} : \mathcal{K}_{14} \to J_{\mathcal{K}_{14}}$. *Then, the principal divisor* $\mathrm{div}(\frac{\mathcal{T}_{34}^2}{\Lambda_4})$ *implies that* $4[\mathcal{P}_3 - \mathcal{P}_4] = 0$ *in* $J_{\mathcal{K}_{14}}$. *Let* $\Xi_{31} = \mathcal{T}_{12}^2 - h_1(\mathcal{P}_3)\Lambda_1$, *where* $h_1(\mathcal{P}_3) := \frac{\mathcal{T}_{12}^2}{\Lambda_1}(\mathcal{P}_3) = \frac{2\left(-1+\sqrt[3]{-1}\right)}{5}$ *is the mutual conic for* $\mathcal{P}_3$ *with* *respect to* $\mathcal{P}_1$. *Let* $\Xi_{41} = \mathcal{T}_{12}^2 - h_1(\mathcal{P}_4)\Lambda_2$, *where* $h_1(\mathcal{P}_4) := \frac{\mathcal{T}_{12}^2}{\Lambda_1}(\mathcal{P}_4) = 1 - i\sqrt{3}$ *is the mutual conic for* $\mathcal{P}_4$ *with respect to* $\mathcal{P}_1$. *The intersection divisors of* Ξ_{31} *and* Ξ_{41} *on* $\mathcal{K}_{14}$ *are given by* $\mathrm{div}(\Xi_{31}) = 4\mathcal{P}_1 + \mathcal{P}_3 + T_1 + T_2 + T_3$ *and* $\mathrm{div}(\Xi_{41}) = 4\mathcal{P}_1 + \mathcal{P}_4 + T_4 + T_5 + T_6$, *where* $\mathcal{P}_1, \mathcal{P}_3, \mathcal{P}_4 \neq T_j$, $j = 1, 2, \ldots, 6$, *so the mutual conics* Ξ_{31} *and* Ξ_{41} *are imperfect. Now, replacing the order-eight element* $[\mathcal{P}_3 - \mathcal{P}_1]$ *by the order-four element* $[\mathcal{P}_3 - \mathcal{P}_4]$, *it follows that the subgroup* $G_6 = \langle [\mathcal{P}_3 - \mathcal{P}_1], [\mathcal{P}_4 - \mathcal{P}_1] \rangle$ *of the Jacobian* $J_{\mathcal{K}_{14}}$ *satisfies* $G_6 \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$. *This example supports the validity of the other possibility in Theorem 5.8.*

Remark 5.13. *The authors have not yet succeeded in constructing an explicit example supporting Theorem 5.9. It is possible that such an example exists, and they hope future research will provide one.*

6. Computational framework and examples of mutual conics

6.1. Algorithms

Here, we provide a set of detailed algorithms that clearly outline the step-by-step computational procedures used in the examples. Let $C \subset \mathbb{P}^2(\mathbb{C})$ be a smooth projective plane curve of degree $d \geq 3$, defined by the homogeneous polynomial $F(X, Y, Z) = 0$. Let $\mathcal{P} \in C$ be a sextactic point and $Q \in C$ another point. We choose an affine open subset of $\mathbb{P}^2(\mathbb{C})$ containing $\mathcal{P}$ and define $f(x, y) = F(x, y, 1)$ as the affine equation of C in this chart. Let (α, β) and (γ, δ) denote the affine coordinates of the points $\mathcal{P}$ and Q in this chart, respectively. Based on this setup, we now present algorithms for computing the equations of the tangent line, the sextactic conic to C at $\mathcal{P}$, and the mutual conic for Q with respect to $\mathcal{P}$.

Algorithm 1: Computation of the tangent line $\mathcal{T}_{\mathcal{P}}$ to C at $\mathcal{P}$

(1) **Input:** The defining polynomial $F(X, Y, Z)$ and the point $\mathcal{P}$.
(2) Choose the affine chart $Z \neq 0$ and define $f(x, y) = F(x, y, 1)$.
(3) Compute the partial derivatives f_x and f_y of $f(x, y)$ with respect to x and y, respectively.
(4) Evaluate $A := f_x(\alpha, \beta)$ and $B := f_y(\alpha, \beta)$.
(5) Form the affine tangent line at $\mathcal{P}$: $t_{\mathcal{P}} := A(x - \alpha) + B(y - \beta) = 0$.
(6) Homogenize $t_{\mathcal{P}}$ with respect to Z to obtain the projective tangent line:

$$\mathcal{T}_{\mathcal{P}} := A(X - \alpha Z) + B(Y - \beta Z) = 0.$$

(7) **Output:** The homogeneous equation of the tangent line $\mathcal{T}_{\mathcal{P}}$.

Now we present an algorithm to construct the sextactic conic at the point $\mathcal{P} \in C$ which was originally introduced in Lemma 4 of [6].

Algorithm 2: Computation of the sextactic conic $\Lambda_{\mathcal{P}}(\mathbf{X}, \mathbf{Y}, \mathbf{Z})$ to C at $\mathcal{P}$

(1) **Input:** The defining polynomial $F(X, Y, Z)$ and the point $\mathcal{P}$.
(2) In the affine chart $Z \neq 0$, define $f(x, y) = F(x, y, 1)$ and translate $\mathcal{P}$ to the origin by defining $C' : h(x, y) = f(x + \alpha, y + \beta)$.
(3) Obtain the tangent line at the origin from Algorithm 1: $e(x, y) := \mathcal{T}_{\mathcal{P}}(x + \alpha, y + \beta, 1)$.
(4) Consider the tangent conic to C' at the origin:

$$r(x, y) := ax^2 + bxy + cy^2 + e(x, y),$$

with unknown coefficients a, b, c.

(5) Eliminate y between $r(x, y) = 0$ and $h(x, y) = 0$ (for instance, using the resultant with respect to y), obtaining a univariate polynomial

$$k(x) = s_{2d}(a, b, c)x^{2d} + \cdots + s_2(a, b, c)x^2.$$

(6) Solve the nonlinear system

$$s_5(a, b, c) = 0, \quad s_4(a, b, c) = 0, \quad s_3(a, b, c) = 0, \quad s_2(a, b, c) = 0$$

for (a, b, c) (unique up to scaling for a sextactic point).

(7) Substitute (a, b, c) into $r(x, y)$, and translate it back to $\mathcal{P}$ by setting

$$\Omega_{\mathcal{P}}(x, y) := r(x - \alpha, y - \beta).$$

(8) Homogenize $\Omega_P(x, y)$ with respect to Z to obtain the sextactic conic

$$\Lambda_{\mathcal{P}}(X, Y, Z) = 0.$$

(9) **Output:** The homogeneous equation of the sextactic conic $\Lambda_{\mathcal{P}}(X, Y, Z)$.

Before describing the algorithm for computing the mutual conic for the point Q with respect to $\mathcal{P}$, recall that the function $g_{\mathcal{P}}(Q)$ represents the value of the rational function $g_{\mathcal{P}} = \frac{\mathcal{T}_{\mathcal{P}}^2}{\Lambda_{\mathcal{P}}}$ evaluated at the projective point $Q \in C$. Since $\mathcal{T}_{\mathcal{P}}$ and $\Lambda_{\mathcal{P}}$ are homogeneous forms of degrees 1 and 2, respectively, for any nonzero scalar $\lambda \in \mathbb{C}$, we have

$$\mathcal{T}_{\mathcal{P}}^2(\lambda X, \lambda Y, \lambda Z) = \lambda^2 \mathcal{T}_{\mathcal{P}}^2(X, Y, Z), \qquad \Lambda_{\mathcal{P}}(\lambda X, \lambda Y, \lambda Z) = \lambda^2 \Lambda_{\mathcal{P}}(X, Y, Z),$$

which implies that $\frac{\mathcal{T}_{\mathcal{P}}^2}{\Lambda_{\mathcal{P}}}$ is homogeneous of zero degree, and therefore invariant under rescaling of homogeneous coordinates. Hence, $g_{\mathcal{P}}$ is a well-defined rational function on C.

In practice, the evaluation at a point Q is performed by dehomogenizing on any affine chart containing Q, letting $Z \neq 0$, so $Q = (\gamma, \delta, 1)$, and computing

$$g_{\mathcal{P}}(Q) = \frac{(\mathcal{T}_{\mathcal{P}}(\gamma, \delta, 1))^2}{\Lambda_{\mathcal{P}}(\gamma, \delta, 1)}.$$

This value is independent of the chosen affine chart due to the degree relation above.

Algorithm 3: Computation of the mutual conic $\Xi_{Q\mathcal{P}}$ for Q with respect to $\mathcal{P}$

(1) **Input:** The defining polynomial $F(X, Y, Z)$ and the point $\mathcal{P}$ and Q.
(2) Compute of the tangent line $\mathcal{T}_{\mathcal{P}}$ to C at $\mathcal{P}$ using Algorithm 1.
(3) Compute the sextactic conic $\Lambda_{\mathcal{P}}$ to C at $\mathcal{P}$ using Algorithm 2.
(4) Compute $g_{\mathcal{P}}(Q) = \dfrac{(\mathcal{T}_{\mathcal{P}}(Q))^2}{\Lambda_{\mathcal{P}}(Q)}$.
(5) If $g_{\mathcal{P}}(Q) \neq 0$, define: $\Xi_{Q\mathcal{P}} := \mathcal{T}_{\mathcal{P}}^2 - g_{\mathcal{P}}(Q)\Lambda_{\mathcal{P}} = 0$.
(6) If $g_{\mathcal{P}}(Q) = 0$, define: $\Xi_{Q\mathcal{P}} := \mathcal{T}_{\mathcal{P}}^2$.

(7) **Output:** The homogeneous equation of the mutual conic $\Xi_{Q\mathcal{P}}$.

Algorithm 4: Computation of the intersection divisor between two smooth projective plane curves.

Let $\mathcal{E} \subset \mathbb{P}^2(\mathbb{C})$ be a smooth projective plane curve of degree $n \geq 1$, defined by the homogeneous form $L(X, Y, Z) = 0$. We now describe the algorithm for computing the intersection divisor of C and $\mathcal{E}$.

(1) **Input:** The defining polynomials $F(X, Y, Z)$ and $L(X, Y, Z)$.
(2) Choose an affine chart (typically $Z \neq 0$) and set $f(x, y) = F(x, y, 1)$ and $l(x, y) = L(x, y, 1)$.
(3) Compute the resultant $R(x) = \mathrm{Res}(f, l; y)$ and factor it as

$$R(x) = Const. \prod_i (x - \alpha_i)^{M_i}.$$

Each root α_i corresponds to one or more intersection points of $C \cap \mathcal{E}$ sharing the same x-coordinate.
(4) For each root α_i, simultaneously solve $f(\alpha_i, y) = 0$ and $l(\alpha_i, y) = 0$ for y. Let the corresponding y-values be $\beta_{i1}, \ldots, \beta_{ik}$, and record the points (α_i, β_{ij}).
(5) Compute the second resultant $S(y) = \mathrm{Res}(f, l; x)$ and factor it as

$$S(y) = Const. \prod_j (y - \beta_{ij})^{t_{ij}}.$$

(6) For each intersection point (α_i, β_{ij}), define the local multiplicity

$$m_{ij} := \min\{M_i, t_{ij}\}.$$

(7) Repeat the process using another affine chart (e.g., $X \neq 0$ or $Y \neq 0$) to identify all points at infinity ($Z = 0$) and compute their multiplicities analogously.
(8) Form the intersection divisor

$$D = \sum_R m_R R,$$

where the sum extends over all intersection points $R := (\alpha_i, \beta_{ij})$, including those at infinity, each counted with multiplicity $m_R := m_{ij}$.
(9) Verify the global consistency:

$$\deg(D) = \deg(C) \cdot \deg(\mathcal{E}) = dn, \text{ where } \deg(D) := \sum_R m_R.$$

(10) **Output:** The intersection divisor $D = \sum_R m_R R$.

6.2. Examples of mutual conic types

This subsection provides representative examples of the mutual conic types introduced in Definition 3.1, namely the perfect, semiperfect, and ordinary mutual conics. Consider the smooth projective plane quartic curve defined by:

$$C : Y^4 = (X - Z)\left(X - (-2 + 2\sqrt{2})Z\right)XZ.$$

The curve C possesses eight total sextactic points, namely,

$$Q_1 = \left[\sqrt{2} : \sqrt{2-\sqrt{2}} : 1\right], \quad Q_5 = \left[2-\sqrt{2} : \sqrt{3\sqrt{2}-4} : 1\right],$$
$$Q_2 = \left[\sqrt{2} : -\sqrt{2-\sqrt{2}} : 1\right], \quad Q_6 = \left[2-\sqrt{2} : -\sqrt{3\sqrt{2}-4} : 1\right],$$
$$Q_3 = \left[\sqrt{2} : \sqrt{\sqrt{2}-2} : 1\right], \quad Q_7 = \left[2-\sqrt{2} : \sqrt{4-3\sqrt{2}} : 1\right],$$
$$Q_4 = \left[\sqrt{2} : -\sqrt{\sqrt{2}-2} : 1\right], \quad Q_8 = \left[2-\sqrt{2} : -\sqrt{4-3\sqrt{2}} : 1\right].$$

Following Algorithm 1, one can verify that the tangent lines at the points Q_j, $j = 1, 2, \ldots, 8$, respectively, are given by

$$\mathcal{T}_1 = X - \sqrt{4-2\sqrt{2}}Y - \left(2-\sqrt{2}\right)Z, \mathcal{T}_2 = \rho^2(\mathcal{T}_1),\ \mathcal{T}_3 = \rho(\mathcal{T}_1),\ \mathcal{T}_4 = \rho^3(\mathcal{T}_1),$$
$$\mathcal{T}_5 = X + \sqrt{3\sqrt{2}-4}\left(2+\sqrt{2}\right)Y - \sqrt{2}Z, \mathcal{T}_6 = \rho^2(\mathcal{T}_5),\ \mathcal{T}_7 = \rho(\mathcal{T}_5),\ \mathcal{T}_8 = \rho^3(\mathcal{T}_5),$$

where $\rho : \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ is a projective change of coordinates defined by $\rho([X : Y : Z]) = [X : iY : Z]$, $i = \sqrt{-1}$. Using Algorithm 4 (the intersection-divisor algorithm) together with a computer algebra system (e.g., Maple), one can show that the intersection divisors of the tangents $\mathcal{T}_j$ on the curve C satisfy

$$\begin{aligned} div(\mathcal{T}_1) &= 2Q_1 + U_1 + U_2, & div(\mathcal{T}_5) &= 2Q_5 + V_1 + V_2,\\ div(\mathcal{T}_2) &= 2Q_2 + U_3 + U_4, & div(\mathcal{T}_6) &= 2Q_6 + V_3 + V_4,\\ div(\mathcal{T}_3) &= 2Q_3 + U_5 + U_6, & div(\mathcal{T}_7) &= 2Q_7 + V_5 + V_6,\\ div(\mathcal{T}_4) &= 2Q_4 + U_7 + U_8, & div(\mathcal{T}_8) &= 2Q_8 + V_7 + V_8. \end{aligned}$$

The auxiliary points appearing above may be written as

$$\begin{aligned} &U_1 = [\alpha_1, \beta_1, 1], && U_2 = [\alpha_2, \beta_2, 1], && U_3 = [\alpha_1, -\beta_1, 1], && U_4 = [\alpha_2, -\beta_2, 1],\\ &U_5 = [\alpha_1, i\beta_1, 1], && U_6 = [\alpha_2, i\beta_2, 1], && U_7 = [\alpha_1, -i\beta_1, 1], && U_8 = [\alpha_2, -i\beta_2, 1],\\ &V_1 = [\alpha_3, \beta_3, 1], && V_2 = [\alpha_4, \beta_4, 1], && V_3 = [\alpha_3, -\beta_3, 1], && V_4 = [\alpha_4, -\beta_4, 1],\\ &V_5 = [\alpha_3, i\beta_3, 1], && V_6 = [\alpha_4, i\beta_4, 1], && V_7 = [\alpha_3, -i\beta_3, 1], && V_8 = [\alpha_4, -i\beta_4, 1]. \end{aligned}$$

The explicit values of these coordinates are

$$\alpha_1 := \left(-6\sqrt{2}+8\right)\sqrt{2-\sqrt{2}} - 11\sqrt{2} + 16, \qquad \beta_1 := \left(2\sqrt{2}-3\right)\left(\sqrt{2-\sqrt{2}}+2\right),$$
$$\alpha_2 := \left(6\sqrt{2}-8\right)\sqrt{2-\sqrt{2}} - 11\sqrt{2} + 16, \qquad \beta_2 := \left(2\sqrt{2}-3\right)\left(\sqrt{2-\sqrt{2}}-2\right),$$
$$\alpha_3 := -2\sqrt{1+\sqrt{2}}\left(2+\sqrt{2}\right)\sqrt{-4+3\sqrt{2}} + 3\sqrt{2} + 2, \qquad \beta_3 := -\frac{3\sqrt{2}\sqrt{6\sqrt{2}-8}}{2} - 2\sqrt{6\sqrt{2}-8} + 2\sqrt{1+\sqrt{2}},$$
$$\alpha_4 := 2\sqrt{1+\sqrt{2}}\left(2+\sqrt{2}\right)\sqrt{-4+3\sqrt{2}} + 3\sqrt{2} + 2, \qquad \beta_4 := -\frac{3\sqrt{2}\sqrt{6\sqrt{2}-8}}{2} - 2\sqrt{6\sqrt{2}-8} - 2\sqrt{1+\sqrt{2}}.$$

The equations of the sextactic conics Λ_j associated with each Q_j are obtained using Algorithm 2. They are given by

$$\Lambda_1 = X^2 + 4Y^2 + 2Z^2 - 2\sqrt{2}XZ + 2\sqrt{4-2\sqrt{2}}YZ + 2\sqrt{2-\sqrt{2}}\left(\sqrt{2}YZ - \left(\sqrt{2}+1\right)XY\right),$$

$$\Lambda_5 = X^2+4\left(\sqrt{2}-1\right)Y^2+2\left(\sqrt{2}-2\right)XZ+2\left(3-2\sqrt{2}\right)Z^2+2\sqrt{3\sqrt{2}-4}\left(XY-\sqrt{2}YZ\right),$$
$$\Lambda_2 = \rho^2(\Lambda_1),\ \Lambda_3=\rho(\Lambda_1),\ \Lambda_4=\rho^3(\Lambda_1),\ \Lambda_6=\rho^2(\Lambda_5),\ \Lambda_7=\rho(\Lambda_5),\ \Lambda_8=\rho^3(\Lambda_5).$$

Let $\Xi_{ij} := \mathcal{T}_j^2 - g_j(Q_i)\,\Lambda_j$, where $g_j(Q_i) := \frac{\mathcal{T}_j^2}{\Lambda_j}(Q_i)$ is the mutual conic for Q_i with respect to Q_j. By applying Algorithm 3 to construct the mutual conics Ξ_{ij} and Algorithm 4 to compute their corresponding intersection divisors with C, we obtain the divisors $\mathrm{div}(\Xi_{ij})$. These divisors naturally classify the mutual conics into perfect, semiperfect, and ordinary types, as summarized in table below.

No.	Ξ_{ij}	$g_j(Q_i)$	$\mathrm{div}(\Xi_{ij})$	Type
1	Ξ_{21}	$2-\sqrt{2}$	$4Q_1+4Q_2$	Perfect
2	Ξ_{31}	u_1	$4Q_1+2Q_3+R_1+S_1$	Semiperfect
3	Ξ_{41}	u_2	$4Q_1+2Q_4+R_2+S_2$	Semiperfect
4	Ξ_{51}	$3-2\sqrt{2}$	$4Q_1+Q_5+Q_6+W_1+W_2$	Ordinary
5	Ξ_{61}	$3-2\sqrt{2}$	$4Q_1+Q_5+Q_6+W_1+W_2$	Ordinary
6	Ξ_{71}	1	$4Q_1+Q_7+Q_8+W_3+W_\infty$	Ordinary
7	Ξ_{81}	1	$4Q_1+Q_7+Q_8+W_3+W_\infty$	Ordinary
8	Ξ_{12}	$2-\sqrt{2}$	$4Q_1+4Q_2$	Perfect
9	Ξ_{32}	u_2	$4Q_2+2Q_3+R_1+S_1$	Semiperfect
10	Ξ_{42}	u_1	$4Q_2+2Q_4+R_2+S_2$	Semiperfect
11	Ξ_{52}	$3-2\sqrt{2}$	$4Q_2+Q_5+Q_6+W_1+W_2$	Ordinary
12	Ξ_{62}	$3-2\sqrt{2}$	$4Q_2+Q_5+Q_6+W_1+W_2$	Ordinary
13	Ξ_{72}	1	$4Q_2+Q_7+Q_8+W_3+W_\infty$	Ordinary
14	Ξ_{82}	1	$4Q_2+Q_7+Q_8+W_3+W_\infty$	Ordinary
15	Ξ_{13}	u_2	$4Q_3+2Q_1+R_3+S_3$	Semiperfect
16	Ξ_{23}	u_1	$4Q_3+2Q_2+R_4+S_4$	Semiperfect
17	Ξ_{43}	$2-\sqrt{2}$	$4Q_3+4Q_4$	Perfect
18	Ξ_{53}	1	$4Q_3+Q_5+Q_6+W_3+W_\infty$	Ordinary
19	Ξ_{63}	1	$4Q_3+Q_5+Q_6+W_3+W_\infty$	Ordinary
20	Ξ_{73}	$3-2\sqrt{2}$	$4Q_3+Q_7+Q_8+W_1+W_2$	Ordinary
21	Ξ_{83}	$3-2\sqrt{2}$	$4Q_3+Q_7+Q_8+W_1+W_2$	Ordinary
22	Ξ_{14}	u_1	$4Q_4+2Q_1+R_3+S_3$	Semiperfect
23	Ξ_{24}	u_2	$4Q_3+2Q_2+R_4+S_4$	Semiperfect
24	Ξ_{34}	$2-\sqrt{2}$	$4Q_3+4Q_4$	Perfect
25	Ξ_{54}	1	$4Q_4+Q_5+Q_6+W_3+W_\infty$	Ordinary
26	Ξ_{64}	1	$4Q_4+Q_5+Q_6+W_3+W_\infty$	Ordinary
27	Ξ_{74}	$3-2\sqrt{2}$	$4Q_4+Q_7+Q_7+W_1+W_2$	Ordinary
28	Ξ_{84}	$3-2\sqrt{2}$	$4Q_4+Q_7+Q_7+W_1+W_2$	Ordinary
29	Ξ_{15}	1	$4Q_5+Q_1+Q_2+W_3+W_\infty$	Ordinary

No.	Ξ_{ij}	$g_j(Q_i)$	div (Ξ_{ij})	Type
30	Ξ_{25}	1	$4Q_5+Q_1+Q_2+W_3+W_\infty$	Ordinary
31	Ξ_{35}	$3+2\sqrt{2}$	$4Q_5+Q_3+Q_4+W_1+W_2$	Ordinary
32	Ξ_{45}	$3+2\sqrt{2}$	$4Q_5+Q_3+Q_4+W_1+W_2$	Ordinary
33	Ξ_{65}	$2+\sqrt{2}$	$4Q_5+4Q_4$	Perfect
34	Ξ_{75}	$1+i+\sqrt[4]{-1}$	$4Q_5+2Q_7+M_1+N_1$	Semiperfect
35	Ξ_{85}	u_4	$4Q_5+2Q_8+M_2+N_2$	Semiperfect
36	Ξ_{16}	1	$4Q_6+Q_1+Q_2+W_3+W_\infty$	Ordinary
37	Ξ_{26}	1	$4Q_6+Q_1+Q_2+W_3+W_\infty$	Ordinary
38	Ξ_{36}	$3+2\sqrt{2}$	$4Q_6+Q_3+Q_4+W_1+W_2$	Ordinary
39	Ξ_{46}	$3+2\sqrt{2}$	$4Q_6+Q_3+Q_4+W_1+W_2$	Ordinary
40	Ξ_{56}	$2+\sqrt{2}$	$4Q_5+4Q_6$	Perfect
41	Ξ_{76}	u_3	$4Q_6+2Q_7+M_1+N_1$	Semiperfect
42	Ξ_{86}	$1+i+\sqrt[4]{-1}$	$4Q_6+2Q_8+M_2+N_2$	Semiperfect
43	Ξ_{17}	$3+2\sqrt{2}$	$4Q_7+Q_1+Q_2+W_1+W_2$	Ordinary
44	Ξ_{27}	$3+2\sqrt{2}$	$4Q_7+Q_1+Q_2+W_1+W_2$	Ordinary
45	Ξ_{37}	1	$4Q_7+Q_3+Q_4+W_3+W_\infty$	Ordinary
46	Ξ_{47}	1	$4Q_7+Q_3+Q_4+W_3+W_\infty$	Ordinary
47	Ξ_{57}	u_3	$4Q_7+2Q_5+M_3+N_3$	Semiperfect
48	Ξ_{67}	$1+i+\sqrt[4]{-1}$	$4Q_7+2Q_6+M_4+N_4$	Semiperfect
49	Ξ_{87}	$2+\sqrt{2}$	$4Q_7+4Q_8$	Perfect
50	Ξ_{18}	$3+2\sqrt{2}$	$4Q_8+Q_1+Q_2+W_1+W_2$	Ordinary
51	Ξ_{28}	$3+2\sqrt{2}$	$4Q_8+Q_1+Q_2+W_1+W_2$	Ordinary
52	Ξ_{38}	1	$4Q_8+Q_3+Q_4+W_3+W_\infty$	Ordinary
53	Ξ_{48}	1	$4Q_8+Q_3+Q_4+W_3+W_\infty$	Ordinary
54	Ξ_{58}	$1+i+\sqrt[4]{-1}$	$4Q_7+2Q_5+M_3+N_3$	Semiperfect
55	Ξ_{68}	u_3	$4Q_8+2Q_6+M_4+N_4$	Semiperfect
56	Ξ_{78}	$2+\sqrt{2}$	$4Q_7+4Q_8$	Perfect

where

$$
\begin{array}{llll}
W_1=\left[-2+2\sqrt{2}:0:1\right], & W_2=[0:0:1], & W_3=[1:0:1], & W_\infty=\left[\frac{1}{\sqrt{2}}:0:0\right], \\
R_1=[\alpha:-i\beta:1], & S_1=[\gamma:-i\delta:1], & M_1=[\zeta:-i\eta:1], & N_1=[\lambda:-i\mu:1], \\
R_2=[\alpha:i\beta:1], & S_2=[\gamma:i\delta:1], & M_2=[\zeta:i\eta:1], & N_2=[\lambda:i\mu:1], \\
R_3=[\alpha:-\beta:1], & S_3=[\gamma:-\delta:1], & M_3=[\zeta:-\eta:1], & N_3=[\lambda:-\mu:1], \\
R_4=[\alpha:\beta:1], & S_4=[\gamma:\delta:1], & M_4=[\zeta:\eta:1], & N_4=[\lambda:\mu:1].
\end{array}
$$

The explicit values of these coordinates are

$$u_1 = \tfrac{1}{2}(1+i)\left(2-\sqrt{2}\right), \qquad \gamma = -11\sqrt{2}+16-2\sqrt{116-82\sqrt{2}},$$
$$u_2 = \tfrac{1}{2}(1-i)\left(2-\sqrt{2}\right), \qquad \delta = \left(-3+2\sqrt{2}\right)\left(2+\sqrt{2-\sqrt{2}}\right),$$
$$u_3 = \tfrac{1}{2}(1-i)\left(2+\sqrt{2}\right), \qquad \zeta = 3\sqrt{2}+2-2\sqrt{4+2\sqrt{2}},$$
$$u_4 = 1+\tfrac{1}{\sqrt{2}}-\tfrac{i}{\sqrt{2}}\left(\sqrt{3+2\sqrt{2}}\right), \qquad \eta = \sqrt{8+7\sqrt{2}-4\sqrt{10+7\sqrt{2}}},$$
$$\alpha = -11\sqrt{2}+16+2\sqrt{116-82\sqrt{2}}, \qquad \lambda = 3\sqrt{2}+2+2\sqrt{4+2\sqrt{2}},$$
$$\beta = \left(-3+2\sqrt{2}\right)\left(-2+\sqrt{2-\sqrt{2}}\right), \qquad \mu = -2\sqrt{\sqrt{2}+1}-\sqrt{4+3\sqrt{2}}.$$

7. Future work

The following lemma will be useful for any future work studying groups generated by four total sextactic points of smooth quartic curves.

Lemma 7.1. *Let C be a smooth projective plane quartic curve with three total sextactic points. If these points are collinear, then the fourth point of intersection to C is either a total sextactic point or a 2-flex point.*

Proof. Suppose that $\mathcal{P}, \mathcal{Q}$, and ∞ are total sextactic points of the curve C, and they lie on a line L. Assume that the intersection divisor of L cut out on C is defined to be $\operatorname{div}(L) = \mathcal{P}+\mathcal{Q}+\infty+R$, where $R \in C \setminus \{\mathcal{P}, \mathcal{Q}, \infty\}$. Taking ∞ as a base point of the Jacobian embedding, let Λ_∞ be the quadratic form defining the sextactic conic to C at ∞. Then, $\operatorname{div}(\Lambda_\infty) = 8\infty$. We have a relation in J_C of the form

$$[\operatorname{div}\frac{L^2}{\Lambda_\infty}] = [2\mathcal{P}+2\mathcal{Q}+2R-6\infty] = 0.$$

Multiplying this relation by 4, we get $[8R-8\infty] = 0$. Note that $\Lambda_\infty \in H^0(C, \mathcal{O}_C(2)) \cong H^0(C, (\Omega^{(1)})^2(C))$. The relation $[8R-8\infty] = 0$ implies the existence of a conic Λ with a contact order of 8 to C at R. It turns out that either $\Lambda = 2\mathcal{T}_R$, and then R is a 2-flex and $\mathcal{T}_R$ is the tangent to C at R, or the conic Λ is irreducible, in which case R is a total sextactic point and Λ is the sextactic conic to C at R. □

In Examples 4.5 and 4.6, it is clear that the points Q_1, Q_2, and Q_3 are collinear. Indeed, they lie on the line $L := X-\sqrt{2}Z$. This line L intersects the curve C once more at the point $Q_4 = \left[\sqrt{2} : -\sqrt{\sqrt{2}-2} : 1\right]$ which is a total sextactic point on C (see Example 5.11). We also have the following interesting example when the fourth intersection point is a 2-flex point.

Example 7.2. *Let $\mho$ be the smooth Picard quartic curve defined by*

$$\mho : Y^3Z = X^4 - bX^2Z^2 - Z^4;$$

$b = i\sqrt{9-3\sqrt{3}}$. It is not hard to show that the points $Q_1 = [0:-1:1]$, $Q_2 = [0:\zeta:1]$, and $Q_3 = \left[0:-\zeta^2:1\right]$, where $\zeta = e^{\frac{\pi i}{3}}$, are total sextactic points on $\mho$. We note that the distinct lines

$$\mathcal{T}_1 = Y+Z,\ \mathcal{T}_2 = 2Y-\left(1+i\sqrt{3}\right)Z,\ \text{and}\ \mathcal{T}_3 = 2Y-\left(1-i\sqrt{3}\right)Z$$

are the tangent lines to $\mho$ *at* Q_1, Q_2, *and* Q_3, *respectively. The equations of the sextactic conics* Λ_i *associated to* Q_i $(i = 1, 2, 3)$ *are*

$$\Lambda_1 = -6X^2 - b\,Y^2 + \left(\sqrt{3}+1\right)bYZ + \left(\sqrt{3}+2\right)b\,Z^2,$$
$$\Lambda_2 = -6X^2 + \zeta b\,Y^2 + \zeta^2\left(1 + \sqrt{3}\right)bYZ - \left(\sqrt{3}+2\right) ibZ^2, \text{ and}$$
$$\Lambda_3 = 6X^2 + \zeta^2 bY^2 + \zeta\left(1 + \sqrt{3}\right)bYZ - \left(\sqrt{3}+2\right)bZ^2.$$

Using any of computer algebra systems (like Maple), one can show that the conics Λ_i *on the curve* $\mho$ *satisfy that* $\mathrm{div}(\Lambda_i) = 8Q_i$. *Let* $\Xi_{21} = \mathcal{T}_1^2 - g_1(Q_2)\Lambda_1$, *where* $g_1(Q_2) := \frac{\mathcal{T}_1^2}{\Lambda_1}(Q_2) = -\sqrt[4]{\frac{\sqrt{3}-2}{54}}$ *is the mutual conic for* Q_2 *with respect to* Q_1. *Let* $\Xi_{31} = \mathcal{T}_1^2 - g_1(Q_3)\Lambda_1$, *where* $g_1(Q_3) := \frac{\mathcal{T}_1^2}{\Lambda_1}(Q_3) = i\sqrt[4]{\frac{\sqrt{3}-2}{54}}$ *is the mutual conic for* Q_3 *with respect to* Q_1. *Let* $\Xi_{32} = \mathcal{T}_2^2 - g_2(Q_3)\Lambda_2$, *where* $g_2(Q_3) := \frac{\mathcal{T}_2^2}{\Lambda_2}(Q_3) = -\frac{4i}{b\left(2+i+\sqrt{3}\right)}$ *is the mutual conic for* Q_3 *with respect to* Q_2. *The intersection divisors of* Ξ_{21}, Ξ_{31}, *and* Ξ_{32} *on* $\mho$ *are given by* $\mathrm{div}(\Xi_{21}) = 4Q_1 + 2Q_2 + N_1 + N_2$, $\mathrm{div}(\Xi_{31}) = 4Q_1 + 2Q_3 + N_3 + N_4$, *and* $\mathrm{div}(\Xi_{32}) = 4Q_2 + 2Q_3 + N_5 + N_6$, *where* $N_i \in \mho \setminus \{Q_1, Q_2, Q_3\}$ *for* $i = 1, 2, \ldots, 6$. *Hence, the mutual conics are all imperfect. Therefore, the subgroup* $G := \langle [Q_2 - Q_1], [Q_3 - Q_1] \rangle$ *in the Jacobian* $J_\mho$, *under the Abel Jacobi map* $A_{Q_1} : \mho \to J_\mho$, *satisfies* $G \cong (\mathbb{Z}/8\mathbb{Z})^2$. *This example supports the correctness of Theorem 5.3, but also we note that the three total sextactic points* Q_1, Q_2, *and* Q_3 *on the curve* $\mho$ *are colinear. Indeed, they lie on the line* $X = 0$ *and it is not hard to see that its fourth intersection point with the curve* $\mho$, *namely* $Q_4 = [0 : 1 : 0]$, *is a 2-flex point on* $\mho$.

Remark 7.3. *As a promising direction for future work, the authors note that the Jacobian matrix also plays a role in the study of chaotic systems (see [12]).*

8. Conclusions

In this paper, we introduced the concept of the mutual conic as a geometric tool to understand how total sextactic points relate to each other on smooth plane quartic curves. This tool made it possible to give a complete classification of the groups generated by two or three such points in the Jacobian. What we found is that the structure of these groups is fully determined by simple geometric features—mainly the behavior of the tangent lines and the nature of the mutual conics between the points.

Our classifications show exactly which subgroup appears in each geometric situation, and the examples throughout the paper illustrate these possibilities in a concrete and transparent way. Overall, the results indicate that the mutual conic is a natural and powerful invariant for studying higher-order contact phenomena on quartic curves. We expect that this perspective will be useful in further investigations—particularly those involving larger collections of total sextactic points or related higher-order Weierstrass loci—opening the door to new classifications and deeper structural understanding.

Author contributions

Alwaleed Kamel: Methodology, writing—review and editing, supervision; Eman Alluqmani: Validation, data curation, resources; Mohammed A. Saleem: Conceptualization, formal analysis, review and editing, supervision; Waleed Khaled Elshareef: Investigation, validation, creating software, writing—original draft. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used generative-Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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