Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(12): 28182–28206.
DOI: 10.3934/math.20251239
Received: 17 August 2025
Revised: 20 November 2025
Accepted: 25 November 2025
Published: 01 December 2025

Research article

Generating functions for circular sums of binomial products

Marta Na Chen[1] **and Wenchang Chu**[2,*]

[1] School of Mathematics and Statistics, Zhoukou Normal University, Zhoukou, China; chennamarta@zknu.edu.cn

[2] Via Dalmazio Birago 9/E, Lecce 73100, Italy

* **Correspondence:** Email: hypergeometricx@outlook.com, chu.wenchang@unisalento.it.

Abstract: By means of the recursive construction method, we examine a class of multiple sums (with a free variable "x") for circular products of binomial coefficients. They are first expressed as coefficients of bivariate rational functions. Then the rational generating functions are determined by deftly employing Knuth's bracket calculus, resultants of polynomials, and Hadamard products of rational functions.

Keywords: generating function; binomial coefficient; circular sum; Knuth's bracket calculus; resultant of polynomials; Hadamard products of rational functions
Mathematics Subject Classification: Primary 11B39, Secondary 05A15, 11B65

1. Introduction and motivation

By examining the characteristic polynomial of a certain binomial matrix, Carlitz [4] discovered, in 1965, the following beautiful identity for the multiple sum of binomial circular products:

$$\frac{F_{mn+m}}{F_m} = \sum_{0\le k_1,k_2,\cdots,k_m\le n} \prod_{\iota=1}^{m} \binom{n-k_\iota}{k_{\iota+1}}, \qquad \boxed{k_{m+1} := k_1}$$

where F_m denotes the usual Fibonacci number. Due to the cyclic property, it seems that this is the unique deep result among the known multiple binomial sums (see for example [3]). Up to now, there are three alternative proofs. The first one is due to Benjianmin and Rouse [2], who offered a combinatorial proof by counting domino–tillings. Recently, Mikic [13] furnished an analytic proof based on the induction principle. By introducing "the recursive construction" method, the second author [6,7] found a more transparent proof for Carlitz' formula, and established analogous identities for the alternating multiple sums.

Motivated by these works, this paper will investigate the following product sums of circular binomial coefficients with free variable "x":

$$\Omega_m(n) := \Omega_m(n|x) = \sum_{0\le k_1,k_2,\cdots,k_m\le n} \prod_{\imath=1}^{m} \binom{1+n+k_\imath}{1+2k_{\imath+1}} x^{k_\imath}. \qquad \boxed{k_{m+1} := k_1}$$

In Section 2 (see Theorem 1), we shall show, by means of "the recursive construction" method, that this multiple sum can be expressed as the coefficient of $T^{n(2^m-1)}$ extracted from a bivariate rational function in T and x. By combining artfully Knuth's bracket calculus, the resultant of two polynomials, and the Hadamard product of two formal power series, the generating function of $\Omega_m(n)$ for any specific m is theoretically determined (see Theorem 5) to be a bivariate rational function in x and y (and proper in y). Two special numerical sums corresponding to $x = \pm 1$ will be examined in Sections 4 and 5, respectively. Finally, the paper will end with Section 6, where conclusive comments will be presented briefly.

2. $\Omega_m(n)$ as coefficients of rational functions

By means of the recursive construction method, we are going to show that the multiple sums $\Omega_m(n)$ can be expressed as coefficients of rational functions.

Theorem 1. *Letting Λ_m and Θ_m be defined as below, then*

$$\Omega_m(n) := \Omega_m(n|x) = [T^{(2^m-1)n}] \frac{(1+T)^{1+n}}{\Theta_m(T,x)} \frac{\prod_{\jmath=1}^{m-2} \Lambda_\jmath^{2+2n}}{x^{n(2^m-m-1)}}.$$

Throughout the paper, two sequences of bivariate polynomials $\Lambda_k := \Lambda_k(T,x)$ and $\Theta_k := \Theta_k(T,x)$ are introduced that will play a crucial role in realizing the "recursive construction" process. First, the Λ_m-polynomials are defined by the quadratic recurrence relation of the first order

$$\Lambda_m = \Lambda_{m-1}^2 - (Tx)^{2^m}\Big/x \quad \text{with} \quad \Lambda_0 = 1.$$

The initial terms read explicitly as

$$\begin{aligned}
\Lambda_1 &= 1 - T^2x,\\
\Lambda_2 &= 1 - 2T^2x + T^4x^2 - T^4x^3,\\
\Lambda_3 &= 1 - 4T^2x + 6T^4x^2 - 2T^4x^3 - 4T^6x^3 + 4T^6x^4 + T^8x^4 - 2T^8x^5 + T^8x^6 - T^8x^7.
\end{aligned}$$

It is almost obvious that polynomial $\Lambda_m(T,x)$ has degree 2^m in variable "T" and degree 2^m-1 in variable "x". Then, we have another polynomial sequence $\Theta_m := \Theta_m(T,x)$ defined by

$$\Theta_m = \Lambda_m - (Tx)^{2^m-1} \quad \text{with} \quad \Theta_0 = 0,$$

which satisfies also the quadratic relation as below

$$\Theta_m = \left(\Theta_{m-1} + (Tx)^{2^{m-1}-1}\right)^2 - (Tx)^{2^m-1}(1+T).$$

The initial terms are recorded as follows:

$$\begin{aligned}
\Theta_1 &= 1 - Tx - T^2x,\\
\Theta_2 &= \Theta_1(1 + Tx - T^2x + T^2x^2),\\
\Theta_3 &= \Theta_1\Big(1 + Tx - 3T^2x + T^2x^2 - 2T^3x^2 + T^3x^3 + 3T^4x^2 - 3T^4x^3\\
&\quad + T^4x^4 + T^5x^3 - 2T^5x^4 + T^5x^5 - T^6x^3 + 2T^6x^4 - T^6x^5 + T^6x^6\Big).
\end{aligned}$$

The polynomial $\Theta_m(T, x)$ has the same degrees as $\Lambda_m(T, x)$ in both "T and x". As will be shown in the Appendix, this polynomial sequence is highly decomposable, i.e., for the two natural numbers m and d with $d|m$, we always have that $\Theta_d(T, x)$ divides $\Theta_m(T, x)$: $\boxed{\Theta_d(T, x)|\Theta_m(T, x)}$. This property will be useful subsequently in reducing computations for evaluating the resultant of two polynomials.

Proof of Theorem 1. Observing that

$$\begin{aligned}
x^{k_1}\binom{1+n+k_1}{1+2k_2} &= [T^{2n+2k_1}]\frac{T^{4k_2}x^{2k_2-n}}{(1-T^2x)^{2+2k_2}},\\
\binom{1+n+k_m}{1+2k_1} &= [T^{n+k_m-2k_1}](1+T)^{1+n+k_m},
\end{aligned}$$

we can express the sum with respect to k_1 in $\Omega_m(n)$ as

$$\Omega_m^1 = \sum_{k_1=0}^{n} x^{k_1}\binom{1+n+k_m}{1+2k_1}\binom{1+n+k_1}{1+2k_2} = [T^{3n+k_m}]\frac{(1+T)^{1+n+k_m}}{x^n\Lambda_1^2}\frac{T^{4k_2}x^{2k_2}}{\Lambda_1^{2k_2}},$$

from which we deduce further

$$\begin{aligned}
\Omega_m^2 &= \sum_{k_2=0}^{n}\Omega_m^1\binom{1+n+k_2}{1+2k_3}x^{k_2} \qquad \boxed{y = \frac{T^4x^3}{\Lambda_1^2}}\\
&= [T^{3n+k_m}]\frac{(1+T)^{1+n+k_m}}{x^n\Lambda_1^2}\sum_{k_2=0}^{\infty}\binom{1+n+k_2}{1+2k_3}y^{k_2}\\
&= [T^{3n+k_m}]\frac{(1+T)^{1+n+k_m}}{x^n\Lambda_1^2}\frac{y^{2k_3-n}}{(1-y)^{2+2k_3}}\\
&= [T^{7n+k_m}]\frac{(1+T)^{1+n+k_m}\Lambda_1^{2+2n}}{x^{4n}\Lambda_2^2}\frac{T^{8k_3}x^{6k_3}}{\Lambda_2^{2k_3}},
\end{aligned}$$

where for the summation with respect to k_2 in the second line, the upper limit "n" is replaced by "∞". This is justified by the fact that

$$[T^{3n+k_m}]y^{k_2} = [T^{3n+k_m}]\Big(\frac{T^4x^3}{\Lambda_1^2}\Big)^{k_2} = 0 \quad \text{for} \quad k_2 > n \geq k_m.$$

The same change for the upper limit of subsequent sums will be applied without further explanation.

Then, we can deal with the sum with respect to k_3 in $\Omega_m(n)$

$$\begin{aligned}
\Omega_m^3 &= \sum_{k_3=0}^{n} \Omega_m^2 \binom{1+n+k_3}{1+2k_4} x^{k_3} \qquad \boxed{y = \frac{T^8 x^7}{\Lambda_2^2}} \\
&= [T^{7n+k_m}] \frac{(1+T)^{1+n+k_m} \Lambda_1^{2+2n}}{x^{4n} \Lambda_2^2} \sum_{k_3=0}^{\infty} \binom{1+n+k_3}{1+2k_4} y^{k_3} \\
&= [T^{7n+k_m}] \frac{(1+T)^{1+n+k_m} \Lambda_1^{2+2n}}{x^{4n} \Lambda_2^2} \frac{y^{2k_4-n}}{(1-y)^{2+2k_4}} \\
&= [T^{15n+k_m}] \frac{(1+T)^{1+n+k_m} (\Lambda_1 \Lambda_2)^{2+2n}}{x^{11n} \Lambda_3^2} \frac{T^{16k_4} x^{14k_4}}{\Lambda_3^{2k_4}},
\end{aligned}$$

and the next one with respect to k_4 in $\Omega_m(n)$

$$\begin{aligned}
\Omega_m^4 &= \sum_{k_4=0}^{n} \Omega_m^3 \binom{1+n+k_4}{1+2k_5} x^{k_4} \qquad \boxed{y = \frac{T^{16} x^{15}}{\Lambda_3^2}} \\
&= [T^{15n+k_m}] \frac{(1+T)^{1+n+k_m} (\Lambda_1 \Lambda_2)^{2+2n}}{x^{11n} \Lambda_3^2} \sum_{k_4=0}^{\infty} \binom{1+n+k_4}{1+2k_5} y^{k_4} \\
&= [T^{15n+k_m}] \frac{(1+T)^{1+n+k_m} (\Lambda_1 \Lambda_2)^{2+2n}}{x^{11n} \Lambda_3^2} \frac{y^{2k_5-n}}{(1-y)^{2+2k_5}} \\
&= [T^{31n+k_m}] \frac{(1+T)^{1+n+k_m} (\Lambda_1 \Lambda_2 \Lambda_3)^{2+2n}}{x^{26n} \Lambda_4^2} \frac{T^{32k_5} x^{30k_5}}{\Lambda_4^{2k_5}}.
\end{aligned}$$

Iterating this process, we can show that the sum with respect to k_{m-1} in $\Omega_m(n)$ results in the following expression:

$$\begin{aligned}
\Omega_m^{m-1} &= \sum_{k_{m-1}=0}^{n} \Omega_m^{m-2} \binom{1+n+k_{m-1}}{1+2k_m} x^{k_{m-1}} \\
&= [T^{(2^m-1)n}] \frac{(1+T)^{1+n}}{\Lambda_{m-1}^{4+2n}} \frac{\prod_{J=1}^{m-1} \Lambda_J^{2+2n}}{x^{n(2^m-m-1)}} \frac{(1+T)^{k_m}}{\Lambda_{m-1}^{2k_m}} \frac{(Tx)^{(2^m-1)k_m}}{x^{k_m}}.
\end{aligned}$$

Finally, by summing over k_m, we arrive at the formula for $\Omega_m(n)$:

$$\begin{aligned}
\Omega_m(n) = \Omega_m^m &= \sum_{k_m=0}^{n} \Omega_m^{m-1} x^{k_m} \qquad \boxed{y = \frac{1+T}{\Lambda_{m-1}^2} (Tx)^{2^m-1}} \\
&= [T^{(2^m-1)n}] \frac{(1+T)^{1+n}}{\Lambda_{m-1}^{4+2n}} \frac{\prod_{J=1}^{m-1} \Lambda_J^{2+2n}}{x^{n(2^m-m-1)}} \sum_{k_m=0}^{\infty} y^{k_m} \\
&= [T^{(2^m-1)n}] \frac{(1+T)^{1+n}}{\Lambda_{m-1}^{4+2n}} \frac{\prod_{J=1}^{m-1} \Lambda_J^{2+2n}}{x^{n(2^m-m-1)}} \Bigg/ \left\{1 - \frac{1+T}{\Lambda_{m-1}^2} (Tx)^{2^m-1}\right\} \\
&= [T^{(2^m-1)n}] \frac{(1+T)^{1+n}}{\Lambda_m - (Tx)^{2^m-1}} \frac{\prod_{J=1}^{m-2} \Lambda_J^{2+2n}}{x^{n(2^m-m-1)}}.
\end{aligned}$$

□

3. Rational generating functions for $\Omega_m(n)$

In order to determine the generating functions for $\Omega_m(n)$, we have to review basic facts about Knuth's bracket calculus, the resultant of two polynomials, and the Hadamard product of two formal power series.

3.1. Knuth's bracket calculus

Let $F(x)$ and $G(x)$ be two formal power series:

$$F(x) = \sum_{n=0}^{\infty} f_n x^n, \qquad G(x) = \sum_{n=0}^{\infty} g_n x^n.$$

Write $[x^n]G(x)$ for the coefficient of x^n in $G(x)$. As an alternative generalization, Knuth [12] introduced $[F(x)]G(x)$, which is a linear function of both F and G

$$[F(x)]G(x) = \sum_{n=0}^{\infty} f_n g_n.$$

Suppose that $H(x)$ is a third formal power series. Knuth's bracket calculus satisfies the following important properties:

$$\begin{aligned}
&[F(x)]G(x) = [G(x)]F(x),\\
&[F(x)]G(x)H(x) = [F(x)G(x^{-1})]H(x),\\
&[F(x)G(x)]H(x) = [F(x)]G(x^{-1})H(x).
\end{aligned}$$

3.2. The resultant of two polynomials

Given two polynomials

$$\begin{aligned}
P(T) &= \sum_{i=0}^{m} a_i T^i = a_m \prod_{i=1}^{m} (T - \alpha_i), \quad a_m \neq 0\\
Q(T) &= \sum_{j=0}^{n} b_j T^j = b_n \prod_{j=1}^{n} (T - \beta_j), \quad b_n \neq 0;
\end{aligned}$$

their resultant with respect to the variable T may be defined by

$$\mathrm{RS}\{P(T), Q(T) : T\} = a_m^n b_n^m \prod_{i=1}^{m} \prod_{j=1}^{n} (\alpha_i - \beta_j).$$

It is well known (cf. [9] and [14, §1.3]) that the resultant can alternatively be expressed by the determinant of the Sylvester matrix of order $n + m$:

$$
\mathrm{RS}\{P(T), Q(T) : T\} = \det \begin{bmatrix}
a_m & a_{m-1} & \cdots & a_1 & a_0 & & & \\
 & a_m & a_{m-1} & \cdots & a_1 & a_0 & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & & a_m & a_{m-1} & \cdots & a_1 & a_0 \\
b_n & b_{n-1} & \cdots & b_1 & b_0 & & & \\
 & b_n & b_{n-1} & \cdots & b_1 & b_0 & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & & b_n & b_{n-1} & \cdots & b_1 & b_0
\end{bmatrix},
$$

which consists of n-shifted rows formed by the coefficients of $P(T)$ and m-shifted rows by the coefficients of $Q(T)$.

- Symmetry:
$$\mathrm{RS}\{P(T), Q(T) : T\} = (-1)^{mn}\mathrm{RS}\{Q(T), P(T) : T\}.$$
- Reciprocity:
$$\mathrm{RS}\{Q(T), P(T) : T\} = \mathrm{RS}\{T^m \times P(T^{-1}), T^n \times Q(T^{-1}) : T\}.$$
- Factorization:
$$\begin{aligned}
\mathrm{RS}\{P(T) \times \mathcal{P}(T), Q(T) : T\} &= \mathrm{RS}\{P(T), Q(T) : T\} \times \mathrm{RS}\{\mathcal{P}(T), Q(T) : T\}, \\
\mathrm{RS}\{P(T), Q(T) \times \mathcal{Q}(T) : T\} &= \mathrm{RS}\{P(T), Q(T) : T\} \times \mathrm{RS}\{P(T), \mathcal{Q}(T) : T\}.
\end{aligned}$$
- Bezout's formula: When $P(0) = a_0 = 1$ and $\dfrac{Q(T)}{P(T)} = \sum_{k \geq 0} d_k T^k$, we have
$$
\mathrm{RS}\{P(T), Q(T) : T\} = \det_{m \times m} \begin{bmatrix}
d_n & d_{n+1} & \cdots & d_{n+m-2} & d_{n+m-1} \\
d_{n-1} & d_n & \cdots & d_{n+m-3} & d_{n+m-2} \\
\vdots & \vdots & \cdots & \vdots & \vdots \\
d_{n-m+1} & d_{n-m+2} & \cdots & d_{n-1} & d_n
\end{bmatrix}.
$$

3.3. Hadamard product of rational power series

The Hadamard product of the power series

$$F(T) = \sum_{n=0}^{\infty} f_n T^n \quad \text{and} \quad G(T) = \sum_{n=0}^{\infty} g_n T^n$$

is defined by

$$F \star G(T) = \sum_{n=0}^{\infty} f_n g_n T^n = [F(x)]G(Tx) = [F(Tx)]G(x).$$

There is an integral representation

$$F \star G(T) = \frac{1}{2\pi\mathbf{i}} \oint_{\mathcal{L}} \frac{F(z)}{z} G\Big(\frac{T}{z}\Big) dz = \sum_{\tau} \operatorname*{Res}_{z=\tau} \Big\{ \frac{F(z)}{z} G\Big(\frac{T}{z}\Big) \Big\}.$$

The residue sum runs over all the poles τ inside $\mathcal{L}$, where the integral path $\mathcal{L}$ is the circle $|z| = (\alpha + |T|/\beta)/2$ with α and β being convergence radii of $F(T)$ and $G(T)$, respectively.

When $F(T)$ and $G(T)$ are rational power series with their respective denominators $Q_f(T) = \prod_{i=1}^m (1 - T\alpha_i)$ and $Q_g(T) = \prod_{j=1}^n (1 - T\beta_j)$, we have the following useful properties (cf. [1, 8, 10]):

- If $F(T)$ and $G(T)$ are proper, then so is $F \star G(T)$.
- The Hadamard product $F \star G(T)$ is also a rational power series that may be written with denominator $\prod_{i=1}^m \prod_{j=1}^n (1 - \alpha_i\beta_j T)$.
- There is an explicit expression in terms of resultant

$$\prod_{i=1}^{m} \prod_{j=1}^{n} (1 - \lambda\alpha_i\beta_j) = \mathrm{RS}\Big\{\overline{Q}_f(T), Q_g(T\lambda) : T\Big\} = \mathrm{RS}\Big\{\overline{Q}_g(T), Q_f(T\lambda) : T\Big\};$$

  where the two reciprocal polynomials are defined by

$$\overline{Q}_f(T) = T^m \times Q_f(T^{-1}) \quad \text{and} \quad \overline{Q}_g(T) = T^n \times Q_g(T^{-1}).$$

Summing up, for two parametric rational functions

$$F(T) := F(T|x, y) = \frac{P_f(T)}{Q_f(T)}, \quad G(T) := G(T|x, y) = \frac{P_g(T)}{Q_g(T)};$$

their bracket $[F(T)]G(T)$ equals Hadamard product $[F(T)]G(T) = F \star G(T)$ at $T = 1$, and hence is a bivariate rational function in "x and y", whose denominator divides the resultant $\mathrm{RS}\Big\{\overline{Q}_f(T), Q_g(T) : T\Big\}$.

Armed with these algebraic machineries, we are now ready to examine (ordinary) generating functions for the multiple sums $\Omega_m(n)$ in accordance with Theorem 1.

$\boxed{m = 1}$ By means of the bracket calculus, we can proceed with

$$\begin{aligned}
\sum_{n=0}^{\infty} \Omega_1(n) y^n &= \sum_{n=0}^{\infty} y^n [T^n] \frac{(1+T)^{n+1}}{\Lambda_1 - Tx} \\
&= \sum_{n=0}^{\infty} \big[y^n (1+T)^n\big] \frac{1+T}{1 - Tx - T^2 x} \\
&= \Big[\frac{1}{1 - y - yT}\Big] \frac{1+T}{1 - Tx - T^2 x}.
\end{aligned}$$

Then, the denominator of the diagonal series results in

$$\overline{Q}_f^1(T) = T - y - Ty \quad \text{and} \quad Q_g^1(T) = 1 - Tx - T^2 x,$$

$$\mathbf{Q}_1(y) := \mathrm{RS}\Big\{\overline{\mathrm{Q}}_{\mathrm{f}}^1(T), \mathrm{Q}_{\mathrm{g}}^1(T) : T\Big\} = 1 - (2+x)y + y^2.$$

Consequently, we can formally write the generating function as

$$\sum_{n=0}^{\infty} \Omega_1(n) y^n = \frac{\mathbf{P}_1(y)}{\mathbf{Q}_1(y)} \iff \mathbf{P}_1(y) = \mathbf{Q}_1(y) \times \sum_{n=0}^{\infty} \Omega_1(n) y^n,$$

where $\mathbf{P}_1(y)$ is a linear function in y, and can be shown to be $\mathbf{P}_1(y) = 1$. In fact, this can be done by comparing the coefficients of y^0 and y in the rightmost equation and keeping in mind the initial values $\Omega_1(0) = 1$ and $\Omega_1(1) = 2 + x$.

Proposition 2 (Generating function for $\Omega_1(n)$)**.**

$$\sum_{n=0}^{\infty} \Omega_1(n) y^n = \frac{1}{1 - (2+x)y + y^2}.$$

Alternatively, by making use of the partial fraction decomposition

$$\frac{1}{1 - Tx - T^2 x} = \frac{1}{\alpha - \gamma}\Big\{\frac{\alpha}{1 - T\alpha} - \frac{\gamma}{1 - T\gamma}\Big\} : \ \alpha, \gamma = \frac{x \pm \sqrt{x^2 + 4x}}{2},$$

we can also compute the generating function directly as follows:

$$\begin{aligned}
\sum_{n=0}^{\infty} \Omega_1(n) y^n &= \sum_{k=0}^{\infty} \frac{y^k}{(1-y)^{k+1}} \big[T^k\big] \frac{1+T}{1 - Tx - T^2 x} \\
&= \frac{1}{\alpha - \gamma} \sum_{k=0}^{\infty} \frac{y^k}{(1-y)^{k+1}} \big[T^k\big] \Big\{\frac{\alpha(1+T)}{1 - T\alpha} - \frac{\gamma(1+T)}{1 - T\gamma}\Big\} \\
&= \sum_{k=0}^{\infty} \frac{y^k}{(1-y)^{k+1}} \frac{(1+\alpha)\alpha^k - (1+\gamma)\gamma^k}{\alpha - \gamma} \\
&= \frac{1}{(1-y)(\alpha - \gamma)} \Big\{\frac{1+\alpha}{1 - \frac{y\alpha}{1-y}} - \frac{1+\gamma}{1 - \frac{y\gamma}{1-y}}\Big\} \\
&= \frac{1}{1 - (2+x)y + y^2}.
\end{aligned}$$

□

$\boxed{m = 2}$ Express the generating function in terms of the bracket calculus

$$\begin{aligned}
\sum_{n=0}^{\infty} \Omega_2(n) y^n &= \sum_{n=0}^{\infty} y^n [T^{3n}] \frac{(1+T)^{n+1} x^{-n}}{\Lambda_2 - (Tx)^3} \\
&= \sum_{n=0}^{\infty} \big[(y/x)^n T^{2n} (1+T)^n\big] \frac{1+T}{(1 - Tx - T^2 x)(1 + Tx - T^2 x + T^2 x^2)} \\
&= \Big[\frac{1}{1 - T^2(1+T)y/x}\Big] \frac{1+T}{(1 - Tx - T^2 x)(1 + Tx - T^2 x + T^2 x^2)}.
\end{aligned}$$

Then, the denominator of the generating function can be determined by

$$\begin{aligned}
&\overline{\mathrm{Q}}_{\mathrm{f}}^{2}(T) = T^3 - (1+T)y/x,\\
&\mathrm{Q}_{\mathrm{g}}^{2}(T) = (1-Tx-T^2x)(1+Tx-T^2x+T^2x^2),\\
&\mathbf{Q}_2(y) := \mathrm{RS}\Big\{\overline{\mathrm{Q}}_{\mathrm{f}}^{2}(T), \mathrm{Q}_{\mathrm{g}}^{2}(T) : T\Big\} = \Big\{1-(1-x)^2y\Big\}^2\Big(1-(2+4x+x^2)y+y^2\Big).
\end{aligned}$$

Thus, the generating function can be written as

$$\sum_{n=0}^{\infty}\Omega_2(n)y^n = \frac{\mathbf{P}_2(y)}{\mathbf{Q}_2(y)} \Longleftrightarrow \mathbf{P}_2(y) = \mathbf{Q}_2(y)\times\sum_{n=0}^{\infty}\Omega_2(n)y^n;$$

where $\mathbf{P}_2(y)$ is a polynomial of degree ≤ 3 in y, and can be shown to be

$$\begin{aligned}
\mathbf{P}_2(y) &= 1-2x^2y-y^2+4xy^2-4x^2y^2+x^4y^2\\
&= \{1-(1-x)^2y\}(1+(1-2x-x^2)y).
\end{aligned}$$

In fact, this can be done by comparing the coefficients of $\{y^n\}_{0\leq n\leq 3}$ and taking into account the initial values $\{\Omega_2(n)\}_{0\leq n\leq 3}$:

$$\Big\{1, 4+x^2, 9+8x+16x^2+x^4, 16+40x+100x^2+40x^3+36x^4+x^6\Big\}.$$

Proposition 3 (Generating function for $\Omega_2(n)$)**.**

$$\sum_{n=0}^{\infty}\Omega_2(n)y^n = \frac{1+(1-2x-x^2)y}{\{1-(1-x)^2y\}(1-(2+4x+x^2)y+y^2)}.$$

$\boxed{m=3}$ The generating function can be manipulated by the bracket calculus

$$\begin{aligned}
\sum_{n=0}^{\infty}\Omega_3(n)y^n &= \sum_{n=0}^{\infty}y^n[T^{7n}]\frac{(1+T)^{n+1}x^{-4n}\Lambda_1^{2+2n}}{\Lambda_3-(Tx)^7}\\
&= \sum_{n=0}^{\infty}y^n[(T/x^2)^{2n}(1+T)^n(T^2-x)^{2n}]\frac{(1+T)(1-T^2x)^2}{(1-Tx-T^2x)\mathrm{P}_3(T,x)}\\
&= \Big[\frac{1}{1-T^2(1+T)(T^2-x)^2y/x^4}\Big]\frac{(1+T)(1-T^2x)^2}{(1-Tx-T^2x)\mathrm{P}_3(T,x)},
\end{aligned}$$

where $\mathrm{P}_3(T,x)$ is a bivariate polynomial

$$\begin{aligned}
\mathrm{P}_3(T,x) = 1 &+ Tx - 3T^2x + T^2x^2 - 2T^3x^2 + T^3x^3 + 3T^4x^2 - 3T^4x^3\\
&+ T^4x^4 + T^5x^3 - 2T^5x^4 + T^5x^5 - T^6x^3 + 2T^6x^4 - T^6x^5 + T^6x^6,
\end{aligned}$$

$$\begin{aligned}
\Theta_3(T,x) &= \Lambda_3 - (Tx)^7 = \Lambda_2^2 - T^8x^7 - T^7x^7\\
&= (1-2T^2x+T^4x^2-T^4x^3)^2 - T^8x^7 - T^7x^7\\
&= (1-Tx-T^2x)\mathrm{P}_3(T,x).
\end{aligned}$$

Then, we can compute the denominator of the generating function

$$\begin{aligned}
&\overline{\mathrm{Q}}_{\mathrm{f}}^{3}(T) = T^7 - (1+T)(1-T^2x)^2 y/x^4,\\
&\mathrm{Q}_{\mathrm{g}}^{3}(T) = (1 - Tx - T^2x)\mathrm{P}_3(T,x),\\
&\mathbf{Q}_3(y) := \mathrm{RS}\Big\{\overline{\mathrm{Q}}_{\mathrm{f}}^{3}(T), \mathrm{Q}_{\mathrm{g}}^{3}(T) : T\Big\} = \Big(1 - (2+x)(1+4x+x^2)y + y^2\Big)\\
&\qquad \times \Big(1 - (2-3x-2x^2+2x^3)y + (1-2x+x^2-x^3)^2y^2\Big)^3.
\end{aligned}$$

Thus, the generating function can formally be written as

$$\sum_{n=0}^{\infty}\Omega_3(n)y^n = \frac{\mathbf{P}_3(y)}{\mathbf{Q}_3(y)} \Longleftrightarrow \mathbf{P}_3(y) = \mathbf{Q}_3(y) \times \sum_{n=0}^{\infty}\Omega_3(n)y^n;$$

where $\mathbf{P}_3(y)$ is a polynomial of degree ≤ 7 in y, and can be shown to be

$$\begin{aligned}
\mathbf{P}_3(y) &= 1 - 6x^3y - 3\Big(3 - 8x - 4x^2 + 16x^3 - 5x^6\Big)y^2\\
&+ 4\Big(4 - 18x + 18x^2 + 15x^3 - 12x^4 - 36x^5 + 32x^6 - 5x^9\Big)y^3\\
&- 3\big(3 - 20x + 51x^2 - 60x^3 + 19x^4 + 48x^5 - 94x^6 + 96x^7 - 72x^8 + 32x^9 - 5x^{12}\big)y^4\\
&+ 6\Big(2x^2 - 16x^3 + 54x^4 - 107x^5 + 152x^6 - 169x^7 + 140x^8\\
&\quad - 86x^9 + 40x^{10} - 8x^{11} - x^{15}\Big)y^5\\
&+ \Big(1 - 12x + 63x^2 - 196x^3 + 420x^4 - 684x^5 + 885x^6 - 912x^7 + 747x^8\\
&\quad - 468x^9 + 183x^{10} - 73x^{12} + 72x^{13} - 36x^{14} + 16x^{15} + x^{18}\Big)y^6\\
&= \Big(1 - (2-3x-2x^2+2x^3)y + (1-2x+x^2-x^3)^2y^2\Big)^2\\
&\times \Big\{1 + 2(2-3x-2x^2-x^3)y + (1-4x+3x^2+2x^4+4x^5+x^6)y^2\Big\}.
\end{aligned}$$

Proposition 4 (Generating function for $\Omega_3(n)$)**.**

$$\sum_{n=0}^{\infty}\Omega_3(n)y^n = \frac{1 + 2\Big(2-3x-2x^2-x^3\Big)y + \Big(1-4x+3x^2+2x^4+4x^5+x^6\Big)y^2}{(1-(2+x)(1+4x+x^2)y+y^2)\Big(1-(2-3x-2x^2+2x^3)y+(1-2x+x^2-x^3)^2y^2\Big)}.$$

$\boxed{m > 3}$ The generating function corresponding to $m > 3$ can be reformulated via Theorem 1 as

$$\begin{aligned}
\sum_{n=0}^{\infty}\Omega_m(n)y^n &= \sum_{n=0}^{\infty} y^n[T^{(2^m-1)n}]\frac{(1+T)^{1+n}}{\Lambda_m - (Tx)^{2^m-1}}\frac{\prod_{J=1}^{m-2}\Lambda_J^{2+2n}}{x^{n(2^m-1-m)}}\\
&= \sum_{n=0}^{\infty} y^n\left[\left(\frac{(1+T)\prod_{J=1}^{m-2}\Lambda_J^2}{(Tx)^{2^m-1}x^{-m}}\right)^n\Big|_{T\to T^{-1}}\right]\frac{(1+T)\prod_{J=1}^{m-2}\Lambda_J^2}{\Lambda_m - (Tx)^{2^m-1}}\\
&= \left[\frac{1}{1 - y\Big(\frac{(1+T)\prod_{J=1}^{m-2}\Lambda_J^2}{(Tx)^{2^m-1}x^{-m}}\Big)\Big|_{T\to T^{-1}}}\right]\frac{(1+T)\prod_{J=1}^{m-2}\Lambda_J^2}{\Lambda_m - (Tx)^{2^m-1}}.
\end{aligned}$$

Hence, we can express the denominator of the generating function.

Theorem 5. *The* (ordinary) *generating function for* $\Omega_m(n)$ *is a rational function with its denominator dividing* $\mathbf{Q}_m(y)$*, defined by*

$$\mathbf{Q}_m(y) := \mathrm{RS}\Big\{\overline{\mathrm{Q}}_{\mathrm{f}}^{\mathrm{m}}(T), \mathrm{Q}_{\mathrm{g}}^{\mathrm{m}}(T) : T\Big\},$$

where

$$\overline{\mathrm{Q}}_{\mathrm{f}}^{\mathrm{m}}(T) = T^{2^m-1} - yx^{1+m-2^m}(1+T)\prod_{j=1}^{m-2}\Lambda_j^2,$$

$$\mathrm{Q}_{\mathrm{g}}^{\mathrm{m}}(T) = \Theta_m(T) = \Lambda_m - (Tx)^{2^m-1}.$$

The related polynomial degrees are determined by

$$\delta(\mathbf{Q}_m(y)) = \delta(\mathrm{Q}_{\mathrm{g}}^{\mathrm{m}}(T)) = 2^m \quad \text{and} \quad \delta\Big(\overline{\mathrm{Q}}_{\mathrm{f}}^{\mathrm{m}}(T)\Big) = 2^m - 1.$$

Thus, sequence $\{\Omega_m(n)\}_{n\geq 0}$ *satisfies a linear recurrence relation of order "* 2^m*" with the coefficients being determined by* $\mathbf{Q}_m(y)$*.*

Since the generating function is a proper rational function

$$\sum_{n=0}^{\infty}\Omega_m(n)y^n = \frac{\mathbf{P}_m(y)}{\mathbf{Q}_m(y)} \Longleftrightarrow \mathbf{P}_m(y) = \mathbf{Q}_m(y) \times \sum_{n=0}^{\infty}\Omega_m(n)y^n.$$

The numerator polynomial $\mathbf{P}_m(y)$ can be determined by comparing the initial coefficients across the rightmost equality. Theoretically, for any given natural number m, we can find the corresponding rational generating function for $\Omega_m(n)$.

However, for $m > 5$ it practically becomes difficult to calculate the related generating functions, since $\Omega_m(n) = \Omega_m(n|x)$ is a large polynomial in "x" of degree "mn". For example, to determine the generating function with $m = 5$, we have to explicitly figure out the polynomials $\Omega_5(n)$ with $0 \leq n \leq 32 = 2^5$. Among them, $\Omega_5(25)$ is shown (by numerical computations according to the definition) to have 125 terms with the maximum coefficient $\approx 55.3822185 \times 10^{50}$. We record three further generating functions for "$m = 4, 5, 6$". They are the best possible results that we are able to obtain by employing *Wolfram Mathematica* (version 11) with our personal computers.

$\boxed{m = 4}$ According to Theorem 5, we first compute the resultant

$$\begin{aligned}
&\overline{\mathrm{Q}}_{\mathrm{f}}^{4}(T) = T^{15} - (1+T)\Lambda_1^2\Lambda_2^2 y/x^{11},\\
&\mathrm{Q}_{\mathrm{g}}^{4}(T) = \Lambda_4 - (Tx)^{15},\\
&\mathbf{Q}_4(y) := \mathrm{RS}\Big\{\overline{\mathrm{Q}}_{\mathrm{f}}^{4}(T), \mathrm{Q}_{\mathrm{g}}^{4}(T) : T\Big\}\\
&\qquad = \Big(1 - (1-x)^4 y\Big)^2\Big(1 - (2 + 16x + 20x^2 + 8x^3 + x^4)y + y^2\Big)\\
&\qquad \times\Big\{1 - (1+x)(3 - 5x - 3x^2 + 3x^3)y + (3 - 8x - x^2 + 14x^3 - 19x^4\\
&\qquad\quad + 24x^5 - 4x^6 + 3x^8)y^2 - (1 - 3x + 3x^2 - 3x^3 + 2x^4 + x^6)^2 y^3\Big\}^4.
\end{aligned}$$

Then, the generating function is given as in the proposition below.

Proposition 6 (Generating function for $\Omega_4(n)$)**.**

$$\sum_{n=0}^{\infty} \Omega_4(n) y^n = \frac{\mathcal{P}_4(y)}{\mathcal{Q}_4(y)},$$

where the rational function is proper and irreducible defined by

$$\begin{aligned}
\mathcal{P}_4(y) = {} & 1 + 2(5 - 5x - 9x^2 - 2x^3 - 2x^4)y + x(14 - 67x + 86x^2 - 23x^3 - 6x^4 \\
& + 34x^5 + 12x^6 + 6x^7)y^2 - (10 - 66x + 154x^2 - 142x^3 + 27x^4 + 54x^5 \\
& - 108x^6 + 16x^7 + 120x^8 - 26x^9 + 14x^{10} + 12x^{11} + 4x^{12})y^3 \\
& - (1 - 10x + 41x^2 - 92x^3 + 132x^4 - 136x^5 + 94x^6 - 22x^7 - 19x^8 \\
& + 24x^9 - 23x^{10} + 6x^{11} - 5x^{12} + 10x^{13} + 2x^{14} - 4x^{15} - x^{16})y^4, \\
\mathcal{Q}_4(y) = {} & \Big(1 - (1 - x)^4 y\Big)\Big(1 - (2 + 16x + 20x^2 + 8x^3 + x^4)y + y^2\Big) \\
& \times \Big\{1 - (1 + x)(3 - 5x - 3x^2 + 3x^3)y + (3 - 8x - x^2 + 14x^3 - 19x^4 \\
& + 24x^5 - 4x^6 + 3x^8)y^2 - (1 - 3x + 3x^2 - 3x^3 + 2x^4 + x^6)^2 y^3\Big\}.
\end{aligned}$$

$\boxed{m = 5}$ The resultant $\mathbf{Q}_5(y)$ is a polynomial of degree $32 = 2^5$ in "y":

$$\begin{aligned}
\mathbf{Q}_5(y) = {} & (1 - (2 + x)(1 + 12x + 19x^2 + 8x^3 + x^4)y + y^2) \\
& \times \Big\{1 - (1 + x)(6 - 11x + x^2 - 8x^3 + 6x^4)y + (15 - 36x - 29x^2 + 80x^3 + 70x^4 - 258x^5 \\
& + 242x^6 + 34x^7 - 23x^8 - 10x^9 + 15x^{10})y^2 - (20 - 94x + 74x^2 + 271x^3 - 602x^4 + 415x^5 \\
& + 124x^6 - 437x^7 + 58x^8 + 527x^9 - 590x^{10} + 94x^{11} + 148x^{12} - 22x^{13} - 20x^{14} + 20x^{15})y^3 \\
& + (15 - 116x + 315x^2 - 200x^3 - 862x^4 + 2698x^5 - 3968x^6 + 2836x^7 + 1366x^8 - 6588x^9 \\
& + 9871x^{10} - 9752x^{11} + 6667x^{12} - 2898x^{13} + 361x^{14} + 670x^{15} - 516x^{16} + 140x^{17} + 2x^{18} \\
& - 20x^{19} + 15x^{20})y^4 - (6 - 69x + 342x^2 - 943x^3 + 1470x^4 - 644x^5 - 3264x^6 + 11151x^7 \\
& - 21578x^8 + 30796x^9 - 34952x^{10} + 32275x^{11} - 24038x^{12} + 13787x^{13} - 4758x^{14} - 1303x^{15} \\
& + 3972x^{16} - 4255x^{17} + 3310x^{18} - 1923x^{19} + 896x^{20} - 321x^{21} + 22x^{22} + 13x^{23} - 10x^{24} \\
& + 6x^{25})y^5 + (1 - 8x + 28x^2 - 60x^3 + 94x^4 - 116x^5 + 114x^6 - 94x^7 \\
& + 69x^8 - 44x^9 + 26x^{10} - 14x^{11} + 5x^{12} - 2x^{13} + x^{14} - x^{15})^2 y^6\Big\}^5.
\end{aligned}$$

Proposition 7 (Generating function for $\Omega_5(n)$)**.**

$$\sum_{n=0}^{\infty} \Omega_5(n) y^n = \frac{\mathcal{P}_5(y)}{\mathcal{Q}_5(y)},$$

where $\mathcal{Q}_5(y)$ is of degree "8" and consists of the two distinct factors in $\mathbf{Q}_5(y)$, and

$$\begin{aligned}
\mathcal{P}_5(y) = {} & 1 + (24 - 20x - 40x^2 - 28x^3 - 8x^4 - 6x^5)y + (15 + 4x - 194x^2 + 160x^3 \\
& + 300x^4 - 248x^5 - 78x^6 + 144x^7 + 132x^8 + 40x^9 + 15x^{10})y^2 - (80 - 344x \\
& + 24x^2 + 1516x^3 - 1752x^4 + 560x^5 - 736x^6 - 572x^7 - 312x^8 + 2332x^9 + 1160x^{10}
\end{aligned}$$

$$- 176x^{11} + 208x^{12} + 248x^{13} + 80x^{14} + 20x^{15})y^3 + (15 - 156x + 765x^2 - 2290x^3 + 4393x^4 - 5262x^5 + 3482x^6 + 26x^7 - 3039x^8 + 3122x^9 - 834x^{10} - 652x^{11} + 637x^{12} + 432x^{13} - 1084x^{14} + 920x^{15} - 36x^{16} + 160x^{17} + 232x^{18} + 80x^{19} + 15x^{20})y^4 + (24 - 276x + 1328x^2 - 3472x^3 + 5340x^4 - 4516x^5 - 236x^6 + 7914x^7 - 13772x^8 + 13074x^9 - 7988x^{10} + 1440x^{11} + 3908x^{12} - 3782x^{13} + 3188x^{14} - 3272x^{15} + 488x^{16} - 550x^{17} + 120x^{18} + 1308x^{19} + 464x^{20} - 44x^{21} - 72x^{22} - 108x^{23} - 40x^{24} - 6x^{25})y^5 + (1 - 16x + 115x^2 - 498x^3 + 1482x^4 - 3296x^5 + 5793x^6 - 8278x^7 + 9707x^8 - 9374x^9 + 7373x^{10} - 4448x^{11} + 1686x^{12} + 230x^{13} - 1302x^{14} + 1572x^{15} - 1354x^{16} + 1006x^{17} - 507x^{18} + 194x^{19} - 101x^{20} - 17x^{22} + 16x^{24} + 2x^{26} + 16x^{27} + 20x^{28} + 8x^{29} + x^{30})y^6.$$

$\boxed{m = 6}$ The resultant $\mathbf{Q}_6(y)$ is a polynomial of degree $64 = 2^6$ in "y":

$$\mathbf{Q}_6(y) = \Big\{1 - (2 + 4x + x^2)(1 + 16x + 20x^2 + 8x^3 + x^4)y + y^2\Big\} \times \{1 - (1 - x)^6 y\}^2 \times \Big\{(1 - (2 - 4x + x^2)(1 - 6x^2 + 4x^3 + 2x^4)y + (1 - 2x + x^2 - x^3)^4 y^2\Big\}^3 \times \Big\{1 - (9 - 2x - 17x^2 - 28x^3 - 5x^4 + 2x^5 + 9x^6)y + 2(18 - 20x - 81x^2 + 3x^3 + 226x^4 - 9x^5 - 402x^6 + 478x^7 + 173x^8 - 44x^9 - 16x^{10} + 8x^{11} + 18x^{12})y^2 - (84 - 224x - 434x^2 + 1298x^3 + 986x^4 - 4542x^5 + 4655x^6 - 1214x^7 - 4090x^8 + 5460x^9 - 4758x^{10} + 3950x^{11} - 3298x^{12} + 376x^{13} + 1658x^{14} + 56x^{15} - 84x^{16} + 56x^{17} + 84x^{18})y^3 + (126 - 616x - 10x^2 + 4566x^3 - 7155x^4 - 4800x^5 + 27752x^6 - 43750x^7 + 33817x^8 - 1630x^9 - 4816x^{10} - 33514x^{11} + 83367x^{12} - 105444x^{13} + 71918x^{14} - 17206x^{15} - 11543x^{16} + 12550x^{17} + 1976x^{18} - 6956x^{19} + 2514x^{20} + 616x^{21} - 112x^{22} + 112x^{23} + 126x^{24})y^4 - (126 - 980x + 1980x^2 + 4026x^3 - 25115x^4 + 43762x^5 - 14900x^6 - 82702x^7 + 184551x^8 - 123542x^9 - 225870x^{10} + 738096x^{11} - 1085481x^{12} + 955694x^{13} - 319000x^{14} - 439048x^{15} + 888856x^{16} - 860596x^{17} + 466077x^{18} - 65206x^{19} - 131671x^{20} + 147874x^{21} - 54773x^{22} - 6714x^{23} + 15588x^{24} - 12420x^{25} + 1000x^{26} + 1120x^{27} - 70x^{28} + 140x^{29} + 126x^{30})y^5 + (84 - 952x + 4058x^2 - 5630x^3 - 16714x^4 + 97046x^5 - 221948x^6 + 255604x^7 + 52681x^8 - 888614x^9 + 2076873x^{10} - 2992438x^{11} + 2888589x^{12} - 1423140x^{13} - 967233x^{14} + 3265958x^{15} - 4582858x^{16} + 4616466x^{17} - 3715214x^{18} + 2569282x^{19} - 1638319x^{20} + 1070602x^{21} - 813376x^{22} + 649304x^{23} - 492950x^{24} + 336112x^{25} - 160826x^{26} + 45346x^{27} + 9422x^{28} - 30742x^{29} + 14236x^{30} - 6172x^{31} - 1042x^{32} + 952x^{33} + 112x^{35} + 84x^{36})y^6 - (36 - 560x + 3778x^2 - 13946x^3 + 26400x^4 + 4742x^5 - 196445x^6 + 691290x^7 - 1471297x^8 + 2083964x^9 - 1525918x^{10} - 1420738x^{11} + 7408981x^{12} - 15752078x^{13} + 24274021x^{14} - 30112252x^{15} + 31067449x^{16} - 26640272x^{17} + 18205420x^{18} - 8288062x^{19} - 562978x^{20} + 6616716x^{21} - 9315122x^{22} + 9163668x^{23} - 7242143x^{24} + 4699510x^{25} - 2408957x^{26} + 786280x^{27} + 131542x^{28} - 485354x^{29} + 500409x^{30} - 380058x^{31} + 226884x^{32} - 111984x^{33} + 49184x^{34} - 11158x^{35} + 1274x^{36} + 936x^{37} - 1114x^{38} + 392x^{39} + 28x^{40} + 56x^{41} + 36x^{42})y^7 + (9 - 184x + 1746x^2 - 10206x^3 + 40969x^4$$

$$- 117908x^5 + 238784x^6 - 276794x^7 - 175152x^8 + 1925448x^9 - 6100963x^{10} + 13730528x^{11}$$
$$- 25068946x^{12} + 38992142x^{13} - 52895355x^{14} + 63323458x^{15} - 67133828x^{16} + 62660344x^{17}$$
$$- 50340467x^{18} + 32548908x^{19} - 12786262x^{20} - 5384788x^{21} + 19274176x^{22} - 27582730x^{23}$$
$$+ 30413906x^{24} - 28905292x^{25} + 24667807x^{26} - 19265394x^{27} + 13905121x^{28} - 9321462x^{29}$$
$$+ 5811960x^{30} - 3367500x^{31} + 1804618x^{32} - 881580x^{33} + 381176x^{34} - 136334x^{35} + 30851x^{36}$$
$$+ 6460x^{37} - 14394x^{38} + 12318x^{39} - 8299x^{40} + 4130x^{41} - 1776x^{42} + 1036x^{43} - 282x^{44} + 56x^{45}$$
$$+ 16x^{46} + 16x^{47} + 9x^{48})y^8 - (1 - 13x + 78x^2 - 293x^3 + 792x^4 - 1672x^5 + 2892x^6 - 4219x^7$$
$$+ 5313x^8 - 5892x^9 + 5843x^{10} - 5258x^{11} + 4346x^{12} - 3310x^{13} + 2331x^{14} - 1525x^{15} + 927x^{16}$$
$$- 536x^{17} + 298x^{18} - 155x^{19} + 76x^{20} - 35x^{21} + 17x^{22} - 7x^{23} + 3x^{24} - x^{25} - x^{26} - x^{27})^2 y^9 \Big\}^6.$$

Proposition 8 (Generating function for $\Omega_6(n)$)**.**

$$\sum_{n=0}^{\infty} \Omega_6(n) y^n = \frac{\mathcal{P}_6(y)}{\mathcal{Q}_6(y)}.$$

where $\mathcal{Q}_6(y)$ is of degree "14" and consists of the four distinct factors in $\mathbf{Q}_6(y)$, and

$$\begin{aligned}\mathcal{P}_6(y) =& 1 + (50 - 24x - 92x^2 - 96x^3 - 46x^4 - 4x^5 - 12x^6)y - (76 - 656x + 1284x^2 + 542x^3 - 1742x^4\\
&- 2198x^5 + 2444x^6 + 560x^7 + 17x^8 - 692x^9 - 454x^{10} - 44x^{11} - 66x^{12})y^2 - (650 - 776x\\
&- 10732x^2 + 23868x^3 + 16188x^4 - 70098x^5 + 41855x^6 - 618x^7 - 3480x^8 - 42068x^9\\
&+ 26746x^{10} + 68540x^{11} + 15446x^{12} - 2888x^{13} - 2958x^{14} + 1896x^{15} + 2010x^{16}\\
&+ 220x^{17} + 220x^{18})y^3 + (2325 - 13704x + 7918x^2 + 101902x^3 - 246715x^4 + 134902x^5\\
&+ 142101x^6 - 311984x^7 + 524554x^8 - 362598x^9 - 71967x^{10} - 412858x^{11} + 748133x^{12}\\
&- 267112x^{13} + 319156x^{14} - 102786x^{15} - 336125x^{16} + 26502x^{17} + 100304x^{18} - 2496x^{19}\\
&- 11973x^{20} + 1764x^{21} + 5250x^{22} + 660x^{23} + 495x^{24})y^4 - (2652 - 28416x + 112868x^2\\
&- 144344x^3 - 337294x^4 + 1557056x^5 - 2396574x^6 + 1524332x^7 - 464149x^8 + 2842162x^9\\
&- 9379227x^{10} + 15586276x^{11} - 16348067x^{12} + 13242852x^{13} - 11957290x^{14} + 12513534x^{15}\\
&- 9436046x^{16} + 804816x^{17} + 4626124x^{18} - 1401360x^{19} - 2012410x^{20} + 1936212x^{21}\\
&- 285840x^{22} - 647248x^{23} + 153372x^{24} + 11248x^{25} - 21696x^{26} - 2784x^{27} + 8940x^{28}\\
&+ 1320x^{29} + 792x^{30})y^5 - (8496x - 121876x^2 + 720076x^3 - 2127014x^4 + 2449476x^5\\
&+ 3800510x^6 - 18179436x^7 + 24844972x^8 + 5370958x^9 - 82220121x^{10} + 164074940x^{11}\\
&- 177538677x^{12} + 96335378x^{13} + 15259868x^{14} - 66681992x^{15} + 31677747x^{16}\\
&+ 27103450x^{17} - 33751409x^{18} - 177422x^{19} + 16116272x^{20} + 3806516x^{21}\\
&- 36084124x^{22} + 38128432x^{23} - 9498873x^{24} - 8537580x^{25} + 8535634x^{26}\\
&- 2130288x^{27} - 2188024x^{28} + 1403380x^{29} + 25832x^{30} - 34720x^{31} + 19026x^{32}\\
&+ 10584x^{33} - 10332x^{34} - 1848x^{35} - 924x^{36})y^6 + (2652 - 36096x + 166876x^2\\
&- 7216x^3 - 3201710x^4 + 15256636x^5 - 33449804x^6 + 19853092x^7 + 101373461x^8\\
&- 365333278x^9 + 638955843x^{10} - 607932130x^{11} + 13308676x^{12} + 998644558x^{13}\end{aligned}$$

$$
\begin{aligned}
&- 1819325081x^{14} + 1837093706x^{15} - 1045084001x^{16} + 141536192x^{17} + 82853974x^{18}\\
&+ 515159016x^{19} - 1340770205x^{20} + 1664955776x^{21} - 1278817948x^{22} + 569157474x^{23}\\
&- 31802209x^{24} - 130898048x^{25} + 38705816x^{26} + 64956662x^{27} - 75658298x^{28}\\
&+ 32390540x^{29} + 12903088x^{30} - 18997604x^{31} + 1881578x^{32} + 4476772x^{33} - 4048900x^{34}\\
&+ 901304x^{35} + 297764x^{36} - 44016x^{37} + 3108x^{38} + 15120x^{39} - 8148x^{40} - 1848x^{41}\\
&- 792x^{42})y^7 - (2325 - 45024x + 377384x^2 - 1704492x^3 + 3696668x^4 + 2980636x^5\\
&- 50403706x^6 + 186300736x^7 - 399403626x^8 + 486121126x^9 + 2354687x^{10}\\
&- 1549569394x^{11} + 4223867125x^{12} - 7368475312x^{13} + 9889546994x^{14} - 11072095778x^{15}\\
&+ 11165672434x^{16} - 11057479248x^{17} + 11273541940x^{18} - 11355136130x^{19}\\
&+ 10382226949x^{20} - 8104422620x^{21} + 5363644951x^{22} - 3377919542x^{23}\\
&+ 2698028232x^{24} - 2874668668x^{25} + 3101908881x^{26} - 2969132826x^{27} + 2525470636x^{28}\\
&- 1964270434x^{29} + 1394672289x^{30} - 838507222x^{31} + 378601906x^{32} - 112154980x^{33}\\
&+ 16744298x^{34} - 4970372x^{35} + 3412967x^{36} + 7216108x^{37} - 10153550x^{38} + 6120760x^{39}\\
&- 1714202x^{40} - 209820x^{41} + 311664x^{42} - 29312x^{43} - 9822x^{44} + 12504x^{45} - 4260x^{46}\\
&- 1320x^{47} - 495x^{48})y^8 + (650 - 16296x + 189520x^2 - 1349208x^3 + 6498776x^4 - 21796924x^5\\
&+ 48234882x^6 - 45198292x^7 - 145507165x^8 + 875399722x^9 - 2646541635x^{10}\\
&+ 5822816380x^{11} - 10140421571x^{12} + 14373912340x^{13} - 16666697980x^{14}\\
&+ 15691108854x^{15} - 12030821179x^{16} + 8614213966x^{17} - 9393699612x^{18}\\
&+ 16765086910x^{19} - 29450799825x^{20} + 42544067844x^{21} - 49958248318x^{22}\\
&+ 47754892842x^{23} - 36202777201x^{24} + 19410471038x^{25} - 3029301975x^{26}\\
&- 8410455730x^{27} + 13085324963x^{28} - 11912838320x^{29} + 7441517898x^{30}\\
&- 2397495170x^{31} - 1357824897x^{32} + 3174545270x^{33} - 3368880221x^{34} + 2667613728x^{35}\\
&- 1740314143x^{36} + 983929980x^{37} - 506794438x^{38} + 253785758x^{39} - 132841014x^{40}\\
&+ 71098224x^{41} - 33887648x^{42} + 12818288x^{43} - 3671938x^{44} + 242084x^{45}\\
&+ 727936x^{46} - 428912x^{47} + 119146x^{48} - 9464x^{49} - 9972x^{50} + 6336x^{51} - 1350x^{52}\\
&- 660x^{53} - 220x^{54})y^9 + (76 - 2016x + 23856x^2 - 158742x^3 + 556060x^4 + 170542x^5\\
&- 14843150x^6 + 103553244x^7 - 460887219x^8 + 1556529798x^9 - 4246647079x^{10}\\
&+ 9654708120x^{11} - 18597939017x^{12} + 30576936562x^{13} - 42848208274x^{14}\\
&+ 50513000588x^{15} - 48335064285x^{16} + 33931347164x^{17} - 10631069603x^{18}\\
&- 12199538324x^{19} + 22447976824x^{20} - 10974174596x^{21} - 23268652618x^{22}\\
&+ 71712298596x^{23} - 119278283133x^{24} + 150534313088x^{25} - 155793004677x^{26}\\
&+ 134365308574x^{27} - 93864920490x^{28} + 46413567166x^{29} - 3829525588x^{30}\\
&- 26081276914x^{31} + 40923768742x^{32} - 42847936258x^{33} + 36427479602x^{34}\\
&- 26493837004x^{35} + 16705540865x^{36} - 9052475520x^{37} + 4064538718x^{38}\\
&- 1350341588x^{39} + 162744193x^{40} + 198278034x^{41} - 210867637x^{42} + 130685578x^{43}\\
&- 60760412x^{44} + 23592108x^{45} - 10995092x^{46} + 8301520x^{47} - 6302277x^{48}
\end{aligned}
$$

$$
\begin{aligned}
&+ 3891236x^{49} - 2289538x^{50} + 1324670x^{51} - 507054x^{52} + 99742x^{53} - 2028x^{54} + 528x^{55}\\
&+ 4259x^{56} - 1884x^{57} + 190x^{58} + 220x^{59} + 66x^{60})y^{10} - (50 - 1720x + 28376x^{2} - 299340x^{3}\\
&+ 2271694x^{4} - 13229874x^{5} + 61579817x^{6} - 235464866x^{7} + 753723901x^{8}\\
&- 2044365354x^{9} + 4724099757x^{10} - 9268513286x^{11} + 15141574834x^{12}\\
&- 19370622298x^{13} + 15071878951x^{14} + 8961061770x^{15} - 66311163224x^{16}\\
&+ 168094187678x^{17} - 316356596418x^{18} + 498620364616x^{19} - 687019868911x^{20}\\
&+ 843950086408x^{21} - 933027152667x^{22} + 931175360844x^{23} - 836678136657x^{24}\\
&+ 669757213794x^{25} - 465679091739x^{26} + 263578909256x^{27} - 95545344535x^{28}\\
&- 20320375634x^{29} + 81343554186x^{30} - 96976138604x^{31} + 82971827683x^{32}\\
&- 55525390454x^{33} + 27180672087x^{34} - 5178899442x^{35} - 8115789095x^{36}\\
&+ 13637596484x^{37} - 13831897958x^{38} + 11278434780x^{39} - 7938633980x^{40}\\
&+ 4947973000x^{41} - 2747312666x^{42} + 1343943026x^{43} - 554798849x^{44}\\
&+ 165501540x^{45} - 3584158x^{46} - 45346994x^{47} + 47084217x^{48} - 34301376x^{49}\\
&+ 20600776x^{50} - 10383590x^{51} + 4190178x^{52} - 1144706x^{53} + 40805x^{54}\\
&+ 169478x^{55} - 128006x^{56} + 55560x^{57} - 1262x^{58} - 13860x^{59} + 6478x^{60}\\
&- 424x^{61} + 802x^{62} - 280x^{63} - 14x^{64} + 44x^{65} + 12x^{66})y^{11} - (1 - 40x + 774x^{2}\\
&- 9670x^{3} + 87865x^{4} - 620262x^{5} + 3549271x^{6} - 16962408x^{7} + 69234732x^{8}\\
&- 245586888x^{9} + 767680238x^{10} - 2138868344x^{11} + 5361591638x^{12} - 12187378412x^{13}\\
&+ 25286587778x^{14} - 48155413716x^{15} + 84570182427x^{16} - 137510754872x^{17}\\
&+ 207712381436x^{18} - 292287226982x^{19} + 384028914465x^{20} - 471928779246x^{21}\\
&+ 543055182415x^{22} - 585407131836x^{23} + 590898738347x^{24} - 557494949408x^{25}\\
&+ 489784493342x^{26} - 397827167146x^{27} + 294702383922x^{28} - 193570372050x^{29}\\
&+ 105111314945x^{30} - 35961138820x^{31} - 11641948295x^{32} + 39176151474x^{33}\\
&- 50542987667x^{34} + 50641988574x^{35} - 44190551968x^{36} + 34983642454x^{37}\\
&- 25608304866x^{38} + 17506805168x^{39} - 11235959880x^{40} + 6783776580x^{41}\\
&- 3849567592x^{42} + 2044080644x^{43} - 1005403705x^{44} + 448488480x^{45}\\
&- 172811200x^{46} + 49487862x^{47} - 2102501x^{48} - 11197718x^{49} + 11440087x^{50}\\
&- 8111556x^{51} + 4838951x^{52} - 2576544x^{53} + 1259792x^{54} - 574818x^{55} + 246508x^{56}\\
&- 99658x^{57} + 38271x^{58} - 13846x^{59} + 4166x^{60} - 564x^{61} - 214x^{62} + 22x^{63} + 181x^{64}\\
&- 158x^{65} + 64x^{67} - 39x^{68} + 12x^{69} + 6x^{70} - 4x^{71} - x^{72})y^{12}.
\end{aligned}
$$

Observe that these generating functions $\dfrac{\mathcal{P}_m(y)}{\mathcal{Q}_m(y)}$ are irreducible fractions. The degrees of numerator and denominator polynomials in "y" are tabulated as below (see Table 1):

Table 1. Degrees of $\mathcal{P}_m(y)$ and $\mathcal{Q}_m(y)$.

m	1	2	3	4	5	6
$\delta(\mathcal{P}_m(y))$	0	1	2	4	6	12
$\delta(\mathcal{Q}_m(y))$	2	3	4	6	8	14

- The degrees of numerator polynomials $\mathcal{P}_m(y)$ result in the Möbius transform of Fibonacci numbers (cf. [15, A007436]):

$$\mathcal{M}_f(m) = \sum_{d|m} \mu\Big(\frac{m}{d}\Big) \times F_d : \ \{\mathcal{M}_f(m)\}_{m=1}^{9} = \{1, 0, 1, 2, 4, 6, 12, 18, 32\}.$$

- The degrees of denominator polynomials $\mathcal{Q}_m(y)$ coincide with the number of m-bead necklaces (cf. [15, A000031]) with two colors when turning over is not allowed:

$$\mathcal{L}_g(m) = \sum_{d|m} \phi\Big(\frac{m}{d}\Big) \times \frac{2^d}{m} : \ \{\mathcal{L}_g(m)\}_{m=1}^{9} = \{2, 3, 4, 6, 8, 14, 20, 36, 60\}.$$

4. Reduced generating functions $\boxed{x_\iota = 1 : \ 1 \le \iota \le n}$

The corresponding positive sum becomes

$$\mathrm{U}_m(n) := \Omega_m(n|x = 1) = \sum_{0\le k_1,k_2,\cdots,k_m\le n} \prod_{\iota=1}^{m} \binom{1 + n + k_\iota}{1 + 2k_{\iota+1}}. \qquad \boxed{k_{m+1} := k_1}$$

The initial values are illustrated in the Table 2:

Table 2. Initial values of $\Omega_m(n|x = 1)$.

$m\backslash n$	1	2	3	4	5	6	7
1	3	8	21	55	144	377	987
2	5	34	233	1597	10946	75025	514229
3	9	176	3153	56569	1015104	18215297	326860233
4	17	962	44833	2105649	98927362	4647459713	218331680913
5	33	5328	646721	79519585	9781368384	1203017770497	147961426640417

Their rational generating functions are given explicitly as follows (where F_n stands for the usual Fibonacci number):

- Generating function for $\mathrm{U}_1(n)$:

$$\sum_{n=0}^{\infty} \mathrm{U}_1(n) y^n = \frac{1}{1 - 3y + y^2} : \ \mathrm{U}_1(n) = F_{2n+2}.$$

- Generating function for $\mathrm{U}_2(n)$:

$$\sum_{n=0}^{\infty} \mathrm{U}_2(n)y^n = \frac{1-2y}{1-7y+y^2} : \ \mathrm{U}_2(n) = F_{4n+1}.$$

- Generating function for $\mathrm{U}_3(n)$:

$$\sum_{n=0}^{\infty} \mathrm{U}_3(n)y^n = \frac{1-8y+7y^2}{(1-18y+y^2)(1+y+y^2)}.$$

- Generating function for $\mathrm{U}_4(n)$:

$$\sum_{n=0}^{\infty} \mathrm{U}_4(n)y^n = \frac{1-26y+56y^2-69y^3+2y^4}{(1-47y+y^2)(1+4y+12y^2-y^3)}.$$

- Generating function for $\mathrm{U}_5(n)$:

$$\sum_{n=0}^{\infty} \mathrm{U}_5(n)y^n = \frac{1-78y+290y^2-2336y^3+926y^4+462y^5+31y^6}{\big(1-123y+y^2\big)\big(1+12y+100y^2+14y^3+36y^4+12y^5+y^6\big)}.$$

- Generating function for $\mathrm{U}_6(n)$: $\sum_{n=0}^{\infty} \mathrm{U}_6(n)y^n = \dfrac{\mathcal{P}_6^u(y)}{\mathcal{Q}_6^u(y)}$, where

$$\begin{aligned}
\mathcal{P}_6^u(y) = {} & 1-224y+929y^2-64021y^3-24352y^4-61811y^5-406887y^6 \\
& -567648y^7-352045y^8-7943y^9+28512y^{10}-641y^{11}+2y^{12}, \\
\mathcal{Q}_6^u(y) = {} & (1-322y+y^2)(1+y+y^2)\{1+32y+704y^2-19y^3 \\
& +1888y^4-928y^5-5357y^6-2560y^7+96y^8-y^9\}.
\end{aligned}$$

Denote by L_{2m} and F_{2m} the bisection Lucas and Fibonacci numbers (cf. [11]):

$$\begin{aligned}
\{L_{2m}\}_{m=1}^{9} &= \{3, 7, 18, 47, 123, 322, 843, 2207, 5778\}, \\
\{F_{2m}\}_{m=1}^{9} &= \{1, 3, 8, \ 21, \ 55, \ 144, 377, \ 987, 2584\}.
\end{aligned}$$

These generating functions contain quadratic polynomial factors in denominators $\boxed{1-yL_{2m}+y^2}$ characterized by bisection Lucas numbers.

Observe that these generating functions $\dfrac{\mathcal{P}_m^u(y)}{\mathcal{Q}_m^u(y)}$ are irreducible fractions. The degrees of numerator and denominator polynomials in "y" are tabulated as below (see Table 3):

Table 3. Degrees of $\mathcal{P}_m^u(y)$ and $\mathcal{Q}_m^u(y)$.

m	1	2	3	4	5	6
$\delta(\mathcal{P}_m^u(y))$	0	1	2	4	6	12
$\delta(\mathcal{Q}_m^u(y))$	2	2	4	5	8	13

- The degrees of numerator polynomials $\mathcal{P}^u_m(y)$ are the same as $\mathcal{P}_m(y)$, and correspond to the Möbius transform of Fibonacci numbers, as described previously.

- The degrees of denominator polynomials $Q^u_m(y)$ coincide with the first difference of p_n (cf. [15, A116084 and A116085]), where p_n counts partitions "1" into distinct reduced fractions i/j with $j \leq n$. For example,

$$p_6 = 6 = \left\{\frac{1}{2} + \frac{1}{3} + \frac{1}{6}, \frac{1}{3} + \frac{2}{3}, \frac{1}{4} + \frac{3}{4}, \frac{1}{5} + \frac{4}{5}, \frac{2}{5} + \frac{3}{5}, \frac{1}{6} + \frac{5}{6}\right\}.$$

  The initial terms are illustrated by

$$\begin{aligned}\{p_n\}_{n=3}^{12} &= \{1, 2, 4, 6, 10, 15, 23, 36, 47, 70\},\\ \{\Delta p_n\}_{n=3}^{12} &= \{1,\ 2, 2, 4, 5, 8, 13,\ 11,\ 23,\ 17\}.\end{aligned}$$

- Observing that for $1 \leq m \leq 6$, the denominator $Q^u_m(y)$ can be decomposed into linear factors of form $(1 - y\alpha)$. Among these α's, there is a unique (real)

$$\alpha_m := \rho^{2m} = \frac{1}{2}(L_{2m} + F_{2m}\sqrt{5}) \text{ with } \rho = \frac{1+\sqrt{5}}{2},$$

  which takes the maximum of $|\alpha|$'s. According to partial fractions, we have the following asymptotic expansion:

$$\frac{\mathcal{P}^u_m(y)}{Q^u_m(y)} \approx \frac{1}{1 - y\alpha_m} \times \lim_{y \to 1/\alpha_m} \frac{(1 - y\alpha_m)\mathcal{P}^u_m(y)}{Q^u_m(y)}. \qquad \boxed{\begin{matrix} n \to \infty \\ 1 \leq m \leq 6 \end{matrix}}$$

  From this, we deduce, as $n \to \infty$, the asymptotic formulae below

$$\begin{aligned}
\mathrm{U}_1(n) &\approx \frac{\alpha_1^n}{10}\left(5 + 3\sqrt{5}\right),\\
\mathrm{U}_2(n) &\approx \frac{\alpha_2^n}{10}\left(5 + \sqrt{5}\right),\\
\mathrm{U}_3(n) &\approx \frac{\alpha_3^n}{95}\left(25 + 12\sqrt{5}\right),\\
\mathrm{U}_4(n) &\approx \frac{\alpha_4^n}{290}\left(65 + 27\sqrt{5}\right),\\
\mathrm{U}_5(n) &\approx \frac{\alpha_5^n}{550}\left(95 + 43\sqrt{5}\right).
\end{aligned}$$

In a recent paper by the authors [5], triple sums of similar binomial products were examined, where the generating functions were determined by first detecting "recurrence relations" through computational experiments and then verifying the "presumably corresponding (guessed) generating functions". Instead, the approach via "Resultant/Hadamard product of polynomials" is more advantageous, both in theory and practice, since it affirms that the generating functions of the circular sums treated in this paper are rational ones prior to numerical tests.

5. Reduced generating functions $\boxed{x_\iota = -1 :\ 1 \leq \iota \leq n}$

The corresponding alternating sum is denoted by

$$\mathrm{V}_m(n) := \Omega_m(n|x=-1) = \sum_{0\leq k_1,k_2,\cdots,k_m\leq n}\prod_{\iota=1}^{m}(-1)^{k_\iota}\binom{1+n+k_\iota}{1+2k_{\iota+1}}. \qquad \boxed{k_{m+1} := k_1}$$

The initial values are illustrated in Table 4:

Table 4. Initial values of $\Omega_m(n|x=-1)$.

$m\backslash n$	1	2	3	4	5	6	7
1	1	0	-1	-1	0	1	1
2	5	18	73	293	1170	4681	18725
3	7	-24	-193	401	5232	-4799	-135593
4	17	162	2593	36305	562626	8753185	138298769
5	31	-360	-11521	110849	5106960	-30977279	-2276827169

Their rational generating functions are displayed explicitly as follows:

- Generating function for $\mathrm{V}_1(n)$:

$$\sum_{n=0}^{\infty}\mathrm{V}_1(n)y^n = \frac{1}{1-y+y^2}.$$

- Generating function for $\mathrm{V}_2(n)$:

$$\sum_{n=0}^{\infty}\mathrm{V}_2(n)y^n = \frac{1+2y}{(1-4y)(1+y+y^2)}.$$

- Generating function for $\mathrm{V}_3(n)$:

$$\sum_{n=0}^{\infty}\mathrm{V}_3(n)y^n = \frac{1+8y+7y^2}{(1+2y+y^2)(1-y+25y^2)}.$$

- Generating function for $\mathrm{V}_4(n)$:

$$\sum_{n=0}^{\infty}\mathrm{V}_4(n)y^n = \frac{1+2y-156y^2-373y^3-446y^4}{(1-16y)(1+y+y^2)(1-48y^2-169y^3)}.$$

- Generating function for $\mathrm{V}_5(n)$:

$$\sum_{n=0}^{\infty}\mathrm{V}_5(n)y^n = \frac{1+30y+90y^2+4832y^3+8166y^4-14958y^5+45799y^6}{(1-y+y^2)(1+480y^2+1562y^3+26496y^4+157152y^5+458329y^6)}.$$

- Generating function for $\mathrm{V}_6(n)$:

$$\sum_{n=0}^{\infty} \mathrm{V}_6(n) y^n = \frac{\mathcal{P}_6^v(y)}{\mathcal{Q}_6^v(y)},$$

where

$$\begin{aligned}\mathcal{P}_6^v(y) = {} & 1 + 25y - 4022y^2 - 111891y^3 + 2279373y^4 + 97808874y^5 + 584738787y^6 - 8898122085y^7 \\ & - 132822555378y^8 - 445275404497y^9 - 1246731704449y^{10} + 5166241027262y^{11}, \\ \mathcal{Q}_6^v(y) = {} & (1-y)(1-64y)(1+49y+625y^2)\{1 - 24y - 960y^2 + 10413y^3 + 394032y^4 \\ & + 598080y^5 - 15123549y^6 - 118340376y^7 - 235909248y^8 - 2100663889y^9\}.\end{aligned}$$

Observe that these generating functions $\frac{\mathcal{P}_m^v(y)}{\mathcal{Q}_m^v(y)}$ are irreducible fractions. The degrees of numerator and denominator polynomials in "y" are tabulated as below (see Table 5):

Table 5. Degrees of $\mathcal{P}_m^v(y)$ and $\mathcal{Q}_m^v(y)$.

m	1	2	3	4	5	6
$\delta(\mathcal{P}_m^v(y))$	0	1	2	4	6	11
$\delta(\mathcal{Q}_m^v(y))$	2	3	4	6	8	13

- The degrees of denominator polynomials $\mathcal{Q}_m^v(y)$ coincide with the known sequence "**A000029**" recorded in [15]: the number of m-bead necklaces with two colors allowing turning over.

$$\delta\big(\mathcal{Q}_m^v(y)\big) = \sum_{d|m} \phi\Big(\frac{m}{d}\Big) \times \frac{2^d}{2m} + \begin{cases} 2^{\frac{m-1}{2}}, & m \equiv_2 1, \\ \frac{3}{4} \times 2^{\frac{m}{2}}, & m \equiv_2 0. \end{cases}$$

- The degrees of numerator polynomials $\mathcal{P}_m^v(y)$ coincide with sequence "**A056342**" recorded in [15]: the number of m-bead necklaces with exactly two different colors.

$$\delta\big(\mathcal{P}_m^v(y)\big) = \delta\big(\mathcal{Q}_m^v(y)\big) - 2.$$

6. Conclusions and further comments

By means of recursive construction, the preliminary Theorem 1 is established that expresses the multiple sums $\Omega_m(n|x)$ as coefficients of bivariate rational functions in T and x. Then, algebraic machinery (Knuth's bracket calculus, resultants of polynomials, and Hadamard products of rational formal power series) is deliberately utilized to determine theoretically ordinary generating functions for $\Omega_m(n|x)$ in Theorem 5. However, the problem is not resolved in practice for larger $m \geq 7$, since to work out explicitly the related generating functions involves a huge quantity of computations. It would be desirable to have some efficient algorithm to handle this problem.

Apart from the observations (made previously) about polynomial degrees, there remains another intriguing question concerns the resultant $\mathbf{Q}_m(y)$ (which is a multiple of the denominator of the reduced generating function $\frac{\mathcal{P}_m(y)}{\mathcal{Q}_m(y)}$), containing the mth power of a big polynomial factor. Instead, the denominator $\mathcal{Q}_m(y)$ is reduced drastically from $\mathbf{Q}_m(y)$ and turns to be the product of only distinct factors appearing in $\mathbf{Q}_m(y)$. It seems that there are more mysteries hidden behind these nontrivial facts. The interested reader is encouraged to make further exploration.

Author contributions

Marta Na Chen: Computation, Writing, and Editing; Wenchang Chu: Original draft, Review, and Supervision. Both authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors express their sincere gratitude to the three reviewers for their careful reading, positive comments, and constructive suggestions, which contributed to improving the manuscript during revision.

Conflict of interest

Prof. Wenchang Chu is the Guest Editor of special issue "Special Functions and q-Series" for AIMS Mathematics. Prof. Wenchang Chu was not involved in the editorial review and the decision to publish this article. The authors declare no conflicts of interest.

References

1. J. P. Allouche, M. Mendé France, Hadamard grade of power series, *J. Number Theory*, **131** (2011), 2013–2022. https://doi.org/10.1016/j.jnt.2011.04.011
2. A. T. Benjamin, J. A. Rouse, In: F. T. Howard, Recounting binomial Fibonacci identities, *Applications of Fibonacci numbers*, Springer, 2004, 25–28. https://doi.org/10.1007/978-0-306-48517-6_4
3. A. Bostan, P. Flajolet, B. Salvy, É. Schost, Fast computation of special resultants, *J. Symbolic Comput.*, **41** (2006), 1–29. https://doi.org/10.1016/j.jsc.2005.07.001
4. L. Carlitz, The characteristic polynomial of a certain matrix of binomial coefficients, *Fibonacci Quart.*, **3** (1965), 81–89. https://doi.org/10.1080/00150517.1965.12431433
5. M. N. Chen, W. Chu, Triple symmetric sums of circular binomial products, *Mathematics*, **12** (2024), 2303. https://doi.org/10.3390/math12152303
6. W. Chu, Circular sums of binomial coefficients, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.*, **115** (2021), 92. https://doi.org/10.1007/s13398-021-01039-x

7. W. Chu, Alternating circular sums of binomial coefficients, *Bull. Aust. Math. Soc.*, **106** (2022), 385–395. https://doi.org/10.1017/S0004972722000351
8. I. M. Gessel, I. Kar, Binomial convolutions for rational power series, *arXiv*, 2023. https://doi.org/10.48550/arXiv.2304.10426
9. S. Janson, Resultant and discriminant of polynomials, *Preprint*, 2007. Available from: `https://www2.math.uu.se/~svantejs/papers/sjN5.pdf`.
10. I. Kar, A new method to compute the Hadamard product of two rational functions, *Rose-Hulman Undergrad. Math. J.*, **23** (2022), 3.
11. T. Koshy, *Fibonacci and Lucas numbers with applications*, John Wiley & Sons, 2001. https://doi.org/10.1002/9781118033067
12. D. E. Knuth, Bracket notation for the "coefficient of operator", In: A. W. Roscoe, *A classical mind: essays in honour of C. A. R. Hoare*, Hertfordshire: Prentice Hall International Ltd., 1994, 247–258.
13. J. Mikic, A proof of the curious binomial coefficient identity which is connected with the Fibonacci numbers, *Open Access J. Math. Theor. Phys.*, **1** (2017), 1–7.
14. V. V. Prasalov, *Polynomials*, Algorithms and Computation in Mathematics, Vol. 11, Springer, 2004. https://doi.org/10.1007/978-3-642-03980-5
15. The On-Line Encyclopedia of Integer Sequences (OEIS), accessed on 12 May 2025. Available from: `http://oeis.org/`.

Appendix. Divisibility of Θ_m

Let $\Lambda_m := \Lambda_m(T,x)$ and $\Theta_m := \Theta_m(T,x)$ be defined as in Section 2. Then, we can express Θ_m in terms of Θ_{m-1} as follows:

$$\begin{aligned}\Theta_m &= \Lambda_m - (Tx)^{2^m-1} = \Lambda_{m-1}^2 - (Tx)^{2^m}\Big\{\frac{1}{x}+\frac{1}{Tx}\Big\}\\ &= \Big(\Theta_{m-1}+(Tx)^{2^{m-1}-1}\Big)^2 - (Tx)^{2^m}\Big\{\frac{1}{x}+\frac{1}{Tx}\Big\}\\ &= \Theta_{m-1}^2 + 2\Theta_{m-1}(Tx)^{2^{m-1}-1} + (Tx)^{2^m}\Big\{\frac{1}{T^2x^2}-\frac{1}{x}-\frac{1}{Tx}\Big\}.\end{aligned}$$

- $\boxed{\Theta_1|\Theta_m}$ This is justified by the recurrence relation

$$\Theta_m = \frac{\Theta_1}{(Tx)^2}(Tx)^{2^m} + \frac{\Theta_{m-1}}{Tx}\Big\{Tx\Theta_{m-1} + 2(Tx)^{2^{m-1}}\Big\} = \frac{\Theta_1}{(Tx)^2}(Tx)^{2^m} + \frac{\Theta_{m-1}}{Tx}\theta_1(m-1);$$

where $\quad \theta_1(m) = \Big\{Tx\Theta_m + 2(Tx)^{2^m}\Big\}.$

- $\boxed{2|m \Longrightarrow \Theta_2|\Theta_m}$ This is justified by the recurrence relation

$$\begin{aligned}\Theta_m = \frac{\Theta_2}{(Tx)^4}(Tx)^{2^m} + \frac{\Theta_{m-2}}{(Tx)^3}\Big\{Tx\Theta_{m-2} + 2(Tx)^{2^{m-2}}\Big\}\\ \times\Big\{(Tx)^{2^{m-1}}(1-2T^2x) + \big(Tx\Theta_{m-2} + (Tx)^{2^{m-2}}\big)^2\Big\}\end{aligned}$$

$$= \frac{\Theta_2}{(Tx)^4}(Tx)^{2^m} + \frac{\Theta_{m-2}}{(Tx)^3}\theta_1(m-2)\theta_2(m-2);$$
$$\text{where} \quad \theta_2(m) = \left\{(Tx)^{2^{m+1}}(1-2T^2x) + (Tx\Theta_m + (Tx)^{2^m})^2\right\}.$$

- $\boxed{3|m \Longrightarrow \Theta_3|\Theta_m}$ This is justified by the recurrence relation

$$\begin{aligned}
\Theta_m = &\frac{\Theta_3}{(Tx)^8}(Tx)^{2^m} + \frac{\Theta_{m-3}}{(Tx)^7}\left\{Tx\Theta_{m-3} + 2(Tx)^{2^{m-3}}\right\} \\
&\times \left\{(Tx)^{2^{m-2}}(1-2T^2x) + (Tx\Theta_{m-3} + (Tx)^{2^{m-3}})^2\right\} \\
&\times \left\{(Tx)^{2^{m-1}}(1-2T^2x+2T^4x^2-2T^4x^3)\right. \\
&- 2T^2x(Tx)^{2^{m-2}}(Tx\Theta_{m-3} + (Tx)^{2^{m-3}})^2 \\
&\left.+ (Tx\Theta_{m-3} + (Tx)^{2^{m-3}})^4\right\} \\
= &\frac{\Theta_3}{(Tx)^8}(Tx)^{2^m} + \frac{\Theta_{m-3}}{(Tx)^7}\theta_1(m-3)\theta_2(m-3)\theta_3(m-3);
\end{aligned}$$
$$\begin{aligned}
\text{where} \quad \theta_3(m) = &\left\{(Tx)^{2^{m+2}}(1-2T^2x+2T^4x^2-2T^4x^3)\right. \\
&\left.- 2T^2x(Tx)^{2^{m+1}}(Tx\Theta_m + (Tx)^{2^m})^2 + (Tx\Theta_m + (Tx)^{2^m})^4\right\}.
\end{aligned}$$

- $\boxed{4|m \Longrightarrow \Theta_4|\Theta_m}$ This is justified by the recurrence relation

$$\Theta_m = \frac{\Theta_4}{(Tx)^{16}}(Tx)^{2^m} + \frac{\Theta_{m-4}}{(Tx)^{15}}\theta_1(m-4)\theta_2(m-4)\theta_3(m-4)\theta_4(m-4);$$
$$\begin{aligned}
\text{where} \qquad \theta_4(m-4) = &(Tx)^{2^{m+3}}(1-4T^2x+6T^4x^2-2T^4x^3 \\
&- 4T^6x^3 + 4T^6x^4 + 2T^8x^4 - 4T^8x^5 + 2T^8x^6 - 2T^8x^7) \\
&- 4T^6x^3(1-x)(Tx)^{3\cdot 2^{m+1}}(Tx\Theta_m + (Tx)^{2^m})^2 \\
&+ 2T^4x^2(3-x)(Tx)^{2^{m+2}}(Tx\Theta_m + (Tx)^{2^m})^4 \\
&- 4T^2x(Tx)^{2^{m+1}}(Tx\Theta_m + (Tx)^{2^m})^6 \\
&+ (Tx\Theta_m + (Tx)^{2^m})^8.
\end{aligned}$$

- $\boxed{d|m \Longrightarrow \Theta_d|\Theta_m}$ Numerical experiments suggest that for $m \leq 6$, we have $\Theta_d|\Theta_m$ when $d|m$. The recurrence relation below provides a possible explanation for this phenomenon:

$$\Theta_m = \frac{\Theta_d}{(Tx)^{2^d}}(Tx)^{2^m} + \frac{\Theta_{m-d}}{(Tx)^{2^d-1}}\prod_{k=1}^{d}\theta_k(m-d).$$

In fact, we have shown that

$$\begin{aligned}
\theta_1(m) &= Tx\Theta_m + 2(Tx)^{2^m}, \\
\theta_2(m) &= (Tx\Theta_m + (Tx)^{2^m})^2 + (Tx)^{2^{1+m}}(1-2T^2x),
\end{aligned}$$

$$\theta_3(m) = \Big((Tx\Theta_m + (Tx)^{2^m})^2 - T^2x(Tx)^{2^{1+m}}\Big)^2 + (Tx)^{2^{2+m}}\Big((1-T^2x)^2 - 2T^4x^3\Big).$$

In general, we can express

$$\theta_k(m) = \phi_k(m) + \psi_k(m) \times (Tx)^{2^{m+k-1}},$$

where the two sequences $\phi_k(m)$ and $\psi_k(m)$ are defined by

$$\phi_1(m) = Tx\Theta_m + (Tx)^{2^m}, \qquad \psi_1(m) = 1,$$
$$\phi_{k+1}(m) = \phi_k^2(m) - \frac{1}{x}(Tx)^{2^k(1+2^m)}, \qquad \psi_{k+1}(m) = \psi_k^2(m) - \frac{1}{x}(Tx)^{2^k}.$$

Then, we can construct recursively

$$\begin{aligned}
\theta_1(m) &= Tx\Theta_m + 2(Tx)^{2^m},\\
\theta_2(m) &= \left(Tx\Theta_m + (Tx)^{2^m}\right)^2 + (Tx)^{2^{1+m}}\left(1 - 2T^2x\right),\\
\theta_3(m) &= \left(\left(Tx\Theta_m + (Tx)^{2^m}\right)^2 - T^2x(Tx)^{2^{1+m}}\right)^2\\
&\quad + (Tx)^{2^{2+m}}\left(\left(1-T^2x\right)^2 - 2T^4x^3\right),\\
\theta_4(m) &= \left(\left(\left(Tx\Theta_m + (Tx)^{2^m}\right)^2 - T^2x(Tx)^{2^{1+m}}\right)^2 - T^4x^3(Tx)^{2^{2+m}}\right)^2\\
&\quad + (Tx)^{2^{3+m}}\left(\left(\left(1-T^2x\right)^2 - T^4x^3\right)^2 - 2T^8x^7\right),\\
\theta_5(m) &= \left(\left(\left(\left(Tx\Theta_m + (Tx)^{2^m}\right)^2 - T^2x(Tx)^{2^{1+m}}\right)^2 - T^4x^3(Tx)^{2^{2+m}}\right)^2 - T^8x^7(Tx)^{2^{3+m}}\right)^2\\
&\quad + (Tx)^{2^{4+m}}\left(\left(\left(\left(1-T^2x\right)^2 - T^4x^3\right)^2 - T^8x^7\right)^2 - 2T^{16}x^{15}\right),\\
\theta_6(m) &= \left(\left(\left(\left(\left(Tx\Theta_m + (Tx)^{2^m}\right)^2 - T^2x(Tx)^{2^{1+m}}\right)^2 - T^4x^3(Tx)^{2^{2+m}}\right)^2 - T^8x^7(Tx)^{2^{3+m}}\right)^2 - T^{16}x^{15}(Tx)^{2^{4+m}}\right)^2\\
&\quad + (Tx)^{2^{5+m}}\left(\left(\left(\left(\left(1-T^2x\right)^2 - T^4x^3\right)^2 - T^8x^7\right)^2 - T^{16}x^{15}\right)^2 - 2T^{32}x^{31}\right),\\
\theta_7(m) &= \left(\left(\left(\left(\left(\left(Tx\Theta_m + (Tx)^{2^m}\right)^2 - T^2x(Tx)^{2^{1+m}}\right)^2 - T^4x^3(Tx)^{2^{2+m}}\right)^2 - T^8x^7(Tx)^{2^{3+m}}\right)^2 - T^{16}x^{15}(Tx)^{2^{4+m}}\right)^2\right.\\
&\quad \left. - T^{32}x^{31}(Tx)^{2^{5+m}}\right)^2 + (Tx)^{2^{6+m}}\left(\left(\left(\left(\left(\left(1-T^2x\right)^2 - T^4x^3\right)^2 - T^8x^7\right)^2 - T^{16}x^{15}\right)^2 - T^{32}x^{31}\right)^2 - 2T^{64}x^{63}\right).
\end{aligned}$$