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Research article

Quantitative stability of the principal eigenvalue for mixed local–nonlocal operators under dissipating boundary partitions

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Abstract: Let $\mathcal{L} = -\Delta + (-\Delta)^s$ with $s \in (0,1)$ on a bounded $C^{1,1}$ domain $\Omega \subset \mathbb{R}^n$, under a partition of the exterior $\mathbb{R}^n \setminus \overline{\Omega}$ into disjoint open sets D (Dirichlet) and N (nonlocal Neumann). Building on the mixed local–nonlocal framework, we obtain explicit, provable upper bounds for the variation of the principal eigenvalue $\lambda_1(D)$ along families of partitions in which the Neumann set N or the Dirichlet set D *dissipates*. When N dissipates, we bound $\lambda_1^{\mathrm{Dir}} - \lambda_1(D)$ by integrals of the Dirichlet kernel over N plus a boundary term and a standard fractional tail. When D dissipates and $0 < s < \frac{1}{2}$, we bound $\lambda_1(D)$ by integrals of the geometric kernel over D and the same tail; for $s \geq \frac{1}{2}$ we give a separated-Dirichlet variant. The proofs use only the weak formulation, the basic spectral theory for the mixed problem, L^∞ bounds for principal eigenfunctions, and two cross-testing identities, with all constants and dependencies made explicit. Consequences include quantitative continuity of λ_1 under weak set convergence and a controlled shift of asymptotically linear bifurcation thresholds. All constants depend only on (n, s, Ω) and, in the separated-Dirichlet variant, also on a fixed separation $\delta > 0$.

Keywords: mixed local–nonlocal operator; fractional Laplacian; mixed boundary conditions; principal eigenvalue; stability
Mathematics Subject Classification: Primary 35P15; Secondary 35R11, 35J20, 47A75

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,1}$ domain, and fix $s \in (0,1)$. We study the mixed local–nonlocal operator

$$\mathcal{L} := -\Delta + (-\Delta)^s,$$

subject to mixed boundary conditions posed on a partition of the exterior into disjoint open sets D (Dirichlet) and N (nonlocal Neumann), with $D, N \subset \mathbb{R}^n \setminus \overline{\Omega}$, $\overline{D \cup N} = \mathbb{R}^n \setminus \Omega$, and $\Omega \cup N$ bounded. In the nonlocal part we adopt the fractional Neumann derivative

$$N_s u(x) := C_{n,s} \int_\Omega \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, \mathrm{d}y, \qquad x \in \mathbb{R}^n \setminus \overline{\Omega},$$

and consider the eigenvalue problem

$$\begin{cases} \mathcal{L}u = \lambda u, & u > 0 \quad \text{in } \Omega, \\ u = 0 & \text{in } D \cup (\partial\Omega \cap \overline{D}), \\ N_s u = 0 & \text{in } N, \\ \partial_\nu u = 0 & \text{in } \partial\Omega \cap \overline{N}. \end{cases} \tag{1.1}$$

Background. The spectral theory for (1.1) has been recently developed in the mixed local–nonlocal setting; in particular, existence of a principal eigenvalue, simplicity, positivity of the corresponding eigenfunction, regularity, and qualitative asymptotics under boundary-set perturbations are established in [13]. Related ingredients come from the pure fractional literature on eigenvalues and regularity [14, 17, 19], from mixed Dirichlet–Neumann problems in the local case [2, 22], and from nonlocal/Neumann frameworks [7, 10, 23]. Convexity properties of Dirichlet integrals and Picone-type inequalities, which underpin several variational arguments in this context, are developed in [5, 6, 12], while Hopf-type and Brezis–Nirenberg-type results for the fractional Laplacian can be found [1, 9, 20]. For mixed local–nonlocal variants with drift or weights, see [8, 15]; see also [16] for estimates in fractional mixed problems and [3, 18, 21] for adjacent directions, as well as [4] for parameter-dependent eigenvalue approximations.

Scope and contribution. Our goal is modest and entirely quantitative: we complement the qualitative limits in [13] by deriving *explicit upper bounds* for the variation of the principal eigenvalue when the boundary partition dissipates. Concretely, let $\lambda_1(D)$ denote the principal eigenvalue of (1.1), and write λ_1^{Dir} for the principal eigenvalue in the full exterior Dirichlet case ($N = \emptyset$) and $\lambda_1^{\mathrm{Neu}} = 0$ for the full Neumann case ($D = \emptyset$). For a sequence of admissible partitions (D_k, N_k), we set $\lambda_{1,k} := \lambda_1(D_k)$. We obtain quantitative estimates (here "dissipates" is in the sense of Definition 2.7 below).

Theorem A (Neumann dissipation: quantitative bound)**.** *Assume N_k dissipates (Definition 2.7), i.e., $|N_k \cap B_R| \to 0$ for every $R > 0$ and $\mathcal{H}^{n-1}(\Gamma_{N_k}) \to 0$, with $\Omega \cup N_k$ bounded and $u_{1,k} > 0$ the $L^2(\Omega)$–normalized principal eigenfunction for (D_k, N_k) (here $\Gamma_{N_k} := \partial\Omega \cap \overline{N_k}$; see §2). Then, for every $R > 0$,*

$$0 \le \lambda_1^{\mathrm{Dir}} - \lambda_{1,k} \;\le\; \frac{M_\infty}{\alpha_k} \left(\int_{\Gamma_{N_k}} |\partial_\nu \phi_1| \, d\sigma \;+\; \int_{N_k \cap B_R} \Psi(x) \, dx \;+\; C_{n,s,\Omega} R^{-2s} \right),$$

where ϕ_1 is the $L^2(\Omega)$-normalized Dirichlet ground state, $\Psi(x) = \int_\Omega \phi_1(y) |x - y|^{-n-2s} \, dy$, $\alpha_k = \int_\Omega \phi_1 u_{1,k} \, dx > 0$, and $M_\infty = \sup_k \|u_{1,k}\|_{L^\infty(\Omega)} < \infty$. All constants are explicit in the proof.

Theorem B (Dirichlet dissipation: quantitative bound)**.** *Assume $0 < s < \frac{1}{2}$ and D_k dissipates (Definition 2.7), i.e., $|D_k \cap B_R| \to 0$ for every $R > 0$ and $\mathcal{H}^{n-1}(\Gamma_{D_k}) \to 0$ (with $\Gamma_{D_k} := \partial\Omega \cap \overline{D_k}$; see §2). Let $u_{1,k} > 0$ be L^2-normalized. Then, for every $R > 0$,*

$$0 \le \lambda_{1,k} \;\le\; \frac{M_\infty}{\beta_k} \left(\int_{D_k \cap B_R} \Upsilon(x) \, dx \;+\; C_{n,s,\Omega} R^{-2s} \right),$$

where $\Upsilon(x) = \int_\Omega |x-y|^{-n-2s}\,dy$ *and* $\beta_k = |\Omega|^{-1/2}\int_\Omega u_{1,k}\,dx > 0$*. For* $s \geq \frac{1}{2}$ *an analogous bound holds under* $\operatorname{dist}(D_k,\Omega) \geq \delta > 0$*, in which case* Υ *is uniformly bounded on* D_k *and the right-hand side depends on* $|D_k \cap B_R|$ *and the tail* R^{-2s}*.*

The proofs are short: they rely on the weak formulation in the mixed framework, the L^∞ bounds and compactness for principal eigenfunctions, and testing identities that compare eigenpairs across boundary configurations. The bounds expose only geometric/integral data of the dissipating sets and a standard tail term, thereby turning the qualitative limits into quantitative ones. No new regularity theory is required.

Scope note. Throughout we assume $D, N \subset \mathbb{R}^n \setminus \overline{\Omega}$ are disjoint open sets with $\overline{D \cup N} = \mathbb{R}^n \setminus \Omega$ and $\Omega \cup N$ bounded. Thus, $D \neq \emptyset$, and the pure Neumann configuration ($D = \emptyset$) falls outside our admissible class. References to the Neumann ground state $\psi_1 \equiv |\Omega|^{-1/2}$ or to $\lambda_1^{\mathrm{Neu}} = 0$ are used only as comparisons implemented via admissible cut-offs (see §2).

Example (Ball with bounded N). Let $\Omega = B_R(0) \subset \mathbb{R}^n$. Define

$$N := \{x \in \mathbb{R}^n : R < |x| < R+1\} \quad \text{(open annulus)}, \qquad D := (\mathbb{R}^n \setminus \Omega) \setminus N \;=\; \{x \in \mathbb{R}^n : |x| > R+1\}.$$

Then $D, N \subset \mathbb{R}^n \setminus \Omega$ are disjoint open sets and

$$\overline{D \cup N} \;=\; \{x : |x| \geq R\} \;=\; \mathbb{R}^n \setminus \Omega.$$

Moreover, $\Omega \cup N \subset B_{R+1}(0)$ is bounded. The boundary portions for the mixed problem are $\partial\Omega \cap \overline{N} = \partial\Omega$ (Neumann) and $\partial\Omega \cap \overline{D} = \emptyset$ (Dirichlet), so the boundary conditions in (1.1) read

$$u = 0 \text{ in } D \cup (\partial\Omega \cap \overline{D}) \;=\; \{x : |x| > R+1\}, \qquad \partial_\nu u = 0 \text{ in } \partial\Omega \cap \overline{N} \;=\; \partial\Omega.$$

Organization. Section 2 fixes the functional setting (spaces, integration by parts, Poincaré-type inequality) and records the testing identities and tail estimates we use. Section 3 contains the proofs of Theorems A and B. Section 4 discusses consequences (e.g. continuity of λ_1 under weak set convergence), the separated case for $s \geq \frac{1}{2}$, and a brief application to bifurcation thresholds.

2. Preliminaries and functional setup

Global conventions (partition and boundary slices). Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,1}$ domain and $s \in (0,1)$. Throughout we fix disjoint open sets $D, N \subset \mathbb{R}^n \setminus \overline{\Omega}$ with $\overline{D \cup N} = \mathbb{R}^n \setminus \Omega$ and $\Omega \cup N$ bounded. We define the boundary slices

$$\Gamma_D := \partial\Omega \cap \overline{D}, \qquad \Gamma_N := \partial\Omega \cap \overline{N}.$$

Boundary terms will be taken over Γ_D or Γ_N. We write $d\sigma$ for the $(n-1)$-dimensional Hausdorff measure on $\partial\Omega$, i.e.,

$$d\sigma := d\mathcal{H}^{n-}\big|_{\partial\Omega},$$

and we reserve $\mathcal{H}^{n-1}(E)$ for the $(n-1)$-measure of $E \subset \partial\Omega$. All subsequent occurrences of "$\partial\Omega \cap N$" or "$\partial\Omega \cap D$" are to be understood as Γ_N or Γ_D, respectively.

We consider the mixed local–nonlocal operator

$$\mathcal{L} := -\Delta + (-\Delta)^s, \qquad (-\Delta)^s u(x) = C_{n,s}\,\mathrm{P.V.}\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,\mathrm{d}y,$$

together with a partition of the exterior into disjoint open sets $D, N \subset \mathbb{R}^n \setminus \overline{\Omega}$ with $\overline{D \cup N} = \mathbb{R}^n \setminus \Omega$ and $\Omega \cup N$ bounded. The nonlocal Neumann derivative is

$$N_s u(x) := C_{n,s} \int_{\Omega} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,\mathrm{d}y, \qquad x \in \mathbb{R}^n \setminus \overline{\Omega}.$$

Definition 2.1 (Dirichlet and Neumann regions)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^{1,1}$ domain. A pair (D, N) is an* admissible exterior partition *if*

$$D, N \subset \mathbb{R}^n \setminus \Omega \quad \text{are open, disjoint, and} \quad \overline{D \cup N} = \mathbb{R}^n \setminus \Omega.$$

We call D the Dirichlet region *and N the* (nonlocal) Neumann region*. The corresponding boundary portions are*

$$\partial\Omega_{\mathrm{D}} := \partial\Omega \cap \overline{D}, \qquad \partial\Omega_{\mathrm{N}} := \partial\Omega \cap \overline{N}.$$

On these sets, the mixed boundary conditions for (1.1) *are imposed as*

$$u = 0 \text{ in } D \cup \partial\Omega_{\mathrm{D}}, \qquad N_s u = 0 \text{ in } N, \qquad \partial_\nu u = 0 \text{ in } \partial\Omega_{\mathrm{N}}.$$

When required in the sequel, we additionally assume $\Omega \cup N$ is bounded.

Energy space, seminorm, and weak formulation. Define

$$\Gamma_D := \partial\Omega \cap \overline{D}, \qquad \Gamma_N := \partial\Omega \cap \overline{N},$$

$$U := \Omega \cup N \cup \Gamma_N, \qquad U^c := D \cup \Gamma_D,$$

and

$$X_D^{1,2}(U) := \{\, u \in H^1(\mathbb{R}^n) \;:\; u|_U \in H_0^1(U),\; u \equiv 0 \text{ a.e. in } U^c \,\}.$$

We integrate on $\partial\Omega$ with $d\sigma := d\mathcal{H}^{n-1} \restriction_{\partial\Omega}$, while $\mathcal{H}^{n-1}(E)$ denotes the $(n-1)$-measure of a set $E \subset \partial\Omega$.

Let

$$Q := \mathbb{R}^{2n} \setminus (\Omega^c \times \Omega^c), \qquad [u]_s^2 := \iint_Q \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy.$$

We use the energy

$$\eta(u)^2 := \int_\Omega |\nabla u|^2\,dx + [u]_s^2, \qquad u \in X_D^{1,2}(U),$$

which defines a Hilbert norm on $X_D^{1,2}(U)$ and controls the $L^2(\Omega)$ norm via the Poincaré-type estimate

$$\|u\|_{L^2(\Omega)}^2 \le C(\Omega, n, s)\Big(\int_\Omega |\nabla u|^2\,dx + [u]_s^2\Big) \qquad \forall\, u \in X_D^{1,2}(U). \tag{2.1}$$

The set-up (2.1) is standard in this mixed framework; see [13, Prop. 2.1] and, for fractional Sobolev background, [19].

Proposition 2.2 (Integration by parts)**.** *For $u, v \in C_c^\infty(U)$,*

$$\int_\Omega v\,Lu\,dx = \int_\Omega \nabla u \cdot \nabla v\,dx + \iint_Q \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy - \int_{\Gamma_N} v\,\partial_\nu u\,d\sigma - \int_N v\,N_s u\,dx,$$

and the identity extends to $u, v \in X_D^{1,2}(U)$ by density [13, Prop. 2.2].

Remark 2.3 (Uniformity of the Poincaré constant)**.** *The constant $C(\Omega, n, s)$ can be chosen independently of the admissible partition as long as $\Omega \cup N$ is bounded. Indeed, the proof uses only that $u \equiv 0$ a.e. on $U^c = D \cup \Gamma_D$ and that the double integral runs over Q; no geometric feature of N beyond boundedness of U enters the estimate. Hence, $C = C(\Omega, n, s)$ is uniform across all admissible (D, N).*

Corollary 2.4 (Density and extension to $X_D^{1,2}(U)$)**.** *Since $C_c^\infty(U)$ is dense in $X_D^{1,2}(U)$, the integration-by-parts identity of Proposition 2.2 extends to all $u, v \in X_D^{1,2}(U)$ by approximation. We invoke this extension in Lemmas 2.8–2.11, where the test functions are realized in $X_D^{1,2}(U)$ via cut-offs.*

Definition 2.5 (Weak eigenpairs)**.** *We say $u \in X_D^{1,2}(U)$ solves*

$$\begin{cases} Lu = \lambda u, & u > 0 \quad \text{in } \Omega, \\ u = 0 & \text{in } U^c, \\ N_s u = 0 & \text{in } N, \\ \partial_\nu u = 0 & \text{in } \Gamma_N, \end{cases}$$

if for all $\phi \in X_D^{1,2}(U)$,

$$\int_\Omega \nabla u \cdot \nabla \phi\,dx + \iint_Q \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{n+2s}}\,dx\,dy = \lambda \int_\Omega u\phi\,dx. \tag{2.2}$$

The principal eigenvalue is given by the Rayleigh quotient

$$\lambda_1(D) := \inf_{\substack{u \in X_D^{1,2}(U)\setminus\{0\} \\ \|u\|_{L^2(\Omega)}=1}} \eta(u)^2, \tag{2.3}$$

and is achieved by a strictly positive eigenfunction u_1. Moreover, $\lambda_1(D)$ is simple. [13, Sec. 3]

Remark 2.6 (Regularity and orthogonality)**.** *Eigenfunctions are bounded and Hölder continuous, $u \in L^\infty(U) \cap C^{0,\beta}(\mathbb{R}^n)$ for some $\beta \in (0,1)$; eigenfunctions associated with different eigenvalues are $L^2(\Omega)$-orthogonal and orthogonal with respect to the energy inner product. See [13, Props. 3.3–3.4]. For related fractional regularity and Harnack-type estimates in the nonlocal literature, cf. [7, 14].*

2.1. Dissipating sequences of boundary sets

We adopt the qualitative notion used in [13, Thms. 2.7–2.8].

Definition 2.7 (Dissipation)**.** *Let (D_k, N_k) be admissible partitions with $\Omega \cup N_k$ bounded and let $\lambda_{1,k} := \lambda_1(D_k)$. With the boundary slices*

$$\Gamma_{D_k} := \partial\Omega \cap \overline{D_k}, \qquad \Gamma_{N_k} := \partial\Omega \cap \overline{N_k} \quad \text{(see §2 conventions)},$$

we say:

- N_k dissipates *if, for every $R > 0$, $|N_k \cap B_R| \to 0$ and $\mathcal{H}^{n-1}(\Gamma_{N_k}) \to 0$ as $k \to \infty$;*
- D_k dissipates *if, for every $R > 0$, $|D_k \cap B_R| \to 0$ and $\mathcal{H}^{n-1}(\Gamma_{D_k}) \to 0$ as $k \to \infty$.*

Then the qualitative limits of [13, Thms. 2.7–2.8] hold verbatim: if N_k dissipates, then $\lambda_{1,k} \to \lambda_1(\mathbb{R}^n \setminus \overline{\Omega})$; if $0 < s < \frac{1}{2}$ and D_k dissipates, then $\lambda_{1,k} \to 0$ (for $s \geq \frac{1}{2}$, convergence holds under $\operatorname{dist}(D_k, \Omega) \geq \delta > 0$*), cf. [13, Prop. 4.5].*

Notation. *On $\partial\Omega$ we integrate with $d\sigma = d\mathcal{H}^{n-1} \restriction_{\partial\Omega}$; we reserve $\mathcal{H}^{n-1}(E)$ for the $(n-1)$–measure of a set $E \subset \partial\Omega$ [13, Thms. 2.7–2.8, Prop. 4.8].*

2.2. Two cross-testing identities

The quantitative bounds in §3 start from two identities obtained by testing the weak formulations for different boundary configurations; compare [13, (4.1.4),(4.2.2)].

Lemma 2.8 (Testing against the Dirichlet ground state)**.** *Let ϕ_1 be the $L^2(\Omega)$–normalized first eigenfunction for the full exterior Dirichlet problem ($N = \varnothing$), and let $(\lambda_{1,k}, u_{1,k})$ be the $L^2(\Omega)$–normalized principal eigenpair for (D_k, N_k). Then*

$$(\lambda_1^{\mathrm{Dir}} - \lambda_{1,k}) \int_\Omega \phi_1 u_{1,k}\, dx = -\int_{\Gamma_{N_k}} u_{1,k}\, \partial_\nu \phi_1 \, d\sigma + \iint_{N_k \times \Omega} \frac{\phi_1(y) u_{1,k}(x)}{|x-y|^{n+2s}}\, dy\, dx, \tag{2.4}$$

where $\Gamma_{N_k} := \partial\Omega \cap \overline{N_k}$.

Remark 2.9 (On the space of test functions)**.** *The identity is obtained by testing in $X_D^{1,2}(U)$ and using Corollary 2.4 to pass from $C_c^\infty(U)$ to the cut-off realizations of ϕ_1 (resp. ψ_1) described above.*

Lemma 2.10 (Boundary regularity for the Dirichlet ground state)**.** *Let ϕ_1 solve $L\phi_1 = \lambda_1^{\mathrm{Dir}} \phi_1$ in Ω with $\phi_1 = 0$ in $\mathbb{R}^n \setminus \Omega$. If Ω is $C^{1,1}$, then $\phi_1 \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$; in particular $\|\partial_\nu \phi_1\|_{L^\infty(\partial\Omega)} < \infty$.*

Reference. This is the regularity asserted in [13, Lemma 4.5], obtained from the mixed local–nonlocal structure with $C^{1,1}$ boundary; see also their Appendix A for $W^{2,p}$ estimates implying $C^{1,\alpha}$ up to $\partial\Omega$. □

Lemma 2.11 (Testing against the Neumann ground state via cut-off)**.** *Let $(\lambda_{1,k}, u_{1,k})$ be the $L^2(\Omega)$–normalized principal eigenpair for (D_k, N_k) and let $\psi_1 \equiv |\Omega|^{-1/2}$. Then*

$$\lambda_{1,k} \int_\Omega \psi_1 u_{1,k}\, dx = -\int_{\Gamma_{D_k}} u_{1,k}\, \partial_\nu \psi_1 \, d\sigma + \iint_{D_k \times \Omega} \frac{\psi_1(x) u_{1,k}(y)}{|x-y|^{n+2s}}\, dy\, dx, \tag{2.5}$$

where $\Gamma_{D_k} := \partial\Omega \cap \overline{D_k}$. Since ψ_1 is constant, $\partial_\nu \psi_1 \equiv 0$.

Proof. Fix k. Choose a standard cut-off $\eta_\varepsilon \in C_c^\infty(U_k)$ such that $0 \leq \eta_\varepsilon \leq 1$, $\eta_\varepsilon \equiv 1$ on Ω, $\eta_\varepsilon \to 1$ pointwise on U_k, and $\eta_\varepsilon \equiv 0$ on a shrinking neighborhood of $U_k^c = D_k \cup \Gamma_{D_k}$. Set $v_\varepsilon := \psi_1 \eta_\varepsilon \in X_{D_k}^{1,2}(U_k)$. Testing the weak formulation (2.2) for $(\lambda_{1,k}, u_{1,k})$ with $\varphi = v_\varepsilon$ (cf. Proposition 2.2) gives

$$\lambda_{1,k} \int_\Omega \psi_1 \eta_\varepsilon\, u_{1,k}\, dx = \int_\Omega \nabla u_{1,k} \cdot \nabla(\psi_1 \eta_\varepsilon)\, dx + \iint_{Q_k} \frac{(u_{1,k}(x) - u_{1,k}(y))(\psi_1 \eta_\varepsilon(x) - \psi_1 \eta_\varepsilon(y))}{|x-y|^{n+2s}}\, dx\, dy,$$

where $Q_k = \mathbb{R}^{2n} \setminus (\Omega^c \times \Omega^c)$. Since ψ_1 is constant and $\eta_\varepsilon \equiv 1$ on Ω, we have $\nabla(\psi_1\eta_\varepsilon) \equiv 0$ on Ω; hence, the local term is zero. Thus,

$$\lambda_{1,k}\int_\Omega \psi_1\, u_{1,k}\, dx = \psi_1 \iint_{Q_k} \frac{(u_{1,k}(x) - u_{1,k}(y))(\eta_\varepsilon(x) - \eta_\varepsilon(y))}{|x-y|^{n+2s}}\, dx\, dy, \tag{2.6}$$

because $\eta_\varepsilon \equiv 1$ on Ω.

We now pass to the limit $\varepsilon \downarrow 0$ on the right-hand side. By construction, $\eta_\varepsilon(x) - \eta_\varepsilon(y) \to \mathbf{1}_\Omega(x)\mathbf{1}_{D_k}(y) - \mathbf{1}_{D_k}(x)\mathbf{1}_\Omega(y)$ pointwise, and $|\eta_\varepsilon(x) - \eta_\varepsilon(y)| \le 1$. Using $u_{1,k} \equiv 0$ in D_k and the tail estimate from §2.4, the integrand is dominated by

$$\frac{|u_{1,k}(x)|\, \mathbf{1}_\Omega(x)\mathbf{1}_{D_k}(y) + |u_{1,k}(y)|\, \mathbf{1}_{D_k}(x)\mathbf{1}_\Omega(y)}{|x-y|^{n+2s}},$$

which is integrable on Q_k (the inner integrals in $y \in D_k$ are finite for each $x \in \Omega$ by the kernel's integrability, and the tail is $O(R^{-2s})$). Therefore, by dominated convergence,

$$\iint_{Q_k} \frac{(u_{1,k}(x) - u_{1,k}(y))(\eta_\varepsilon(x) - \eta_\varepsilon(y))}{|x-y|^{n+2s}}\, dx\, dy \longrightarrow \iint_{D_k\times\Omega} \frac{u_{1,k}(y)}{|x-y|^{n+2s}}\, dx\, dy,$$

where we used symmetry to write the cross-terms in the oriented form $D_k \times \Omega$.

Plugging this limit into (2.6) yields (2.5). This argument uses only the mixed integration-by-parts identity and the weak formulation in our space, as in [13, Prop. 2.2]. □

Remark 2.12 (On the role of the pure Neumann profile)**.** *Under our standing hypothesis $\Omega \cup N$ bounded, the configuration $D = \emptyset$ is excluded (so $N = \mathbb{R}^n \setminus \Omega$ is not allowed). In particular, while the pure Neumann model has $\lambda_1^{\mathrm{Neu}} = 0$ and normalized ground state $\psi_1 \equiv |\Omega|^{-1/2}$ (see, e.g., [13, Thm. 4.1]), we* do not *use ψ_1 as a test function in $X^{1,2}_D(U)$. Instead, Lemma 2.11 justifies the cross–testing identity rigorously by approximating ψ_1 with admissible cut-offs $v_\varepsilon = \psi_1\eta_\varepsilon \in X^{1,2}_D(U)$ and letting $\varepsilon \downarrow 0$. All subsequent uses of ψ_1 refer to this admissible realization.*

2.3. Compactness and uniform bounds

We record compactness and L^∞–Hölder bounds for principal eigenfunctions along admissible partitions:

Standing hypothesis and admissible scope. We always work with disjoint open sets $D, N \subset \mathbb{R}^n \setminus \overline{\Omega}$ satisfying $\overline{D \cup N} = \mathbb{R}^n \setminus \Omega$ and $\Omega \cup N$ bounded. Consequently, $D \neq \emptyset$, the pure Neumann case ($D = \emptyset$) is excluded in our admissible class. Any use of the constant Neumann ground state $\psi_1 \equiv |\Omega|^{-1/2}$ is purely as a comparison profile implemented by the cut-off functions $v_\varepsilon = \psi_1\eta_\varepsilon \in X^{1,2}_D(U)$; cf. Lemma 2.11.

Proposition 2.13 (Compactness)**.** *Let $(\lambda_{1,k}, u_{1,k})$ be principal eigenpairs. Then, up to a subsequence,*

$$u_{1,k} \rightharpoonup u_* \text{ in } X^{1,2}_D(U), \quad u_{1,k} \to u_* \text{ in } L^2_{\mathrm{loc}}(\mathbb{R}^n), \quad u_{1,k} \to u_* \text{ a.e. in } \mathbb{R}^n,$$

refer to [13, Prop. 4.2].

In what follows, we write u_* for any subsequential limit of $u_{1,k}$ given by Proposition 2.13; by the limit mixed problem (cf. [18]) one has $u_* > 0$ in U_*.

Lemma 2.14 (Alignment factors stay positive along convergent subsequences)**.** *Let $(\lambda_{1,k}, u_{1,k})$ be the $L^2(\Omega)$-normalized principal eigenpairs for admissible (D_k, N_k). Suppose $u_{1,k} \to u_*$ in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$ and a.e. in $\mathbb{R}^n$, with u_* solving the limit problem and $u_* > 0$ in U_*. Then*

$$\alpha_k := \int_\Omega \phi_1 u_{1,k}\, dx \;\to\; \int_\Omega \phi_1 u_*\, dx > 0, \qquad \beta_k := \int_\Omega \psi_1 u_{1,k}\, dx \;\to\; \int_\Omega \psi_1 u_*\, dx > 0,$$

where ϕ_1 is the Dirichlet ground state and $\psi_1 \equiv |\Omega|^{-1/2}$.

Proof. The compactness $u_{1,k} \rightharpoonup u_*$ in $X^{1,2}_D(U)$ and $u_{1,k} \to u_*$ in L^2_{loc} follows from the energy bound and the compact embedding (cf. [13, Prop. 4.2]). The strong maximum principle yields $u_* > 0$ in U_* (cf. [13, Lem. 3.1]); hence, both limit overlaps are strictly positive. □

Proposition 2.15 (Uniform L^∞ and Hölder bounds on U)**.** *Let $(\lambda_{1,k}, u_{1,k})$ be the $L^2(\Omega)$-normalized principal eigenpairs for admissible partitions (D_k, N_k) with $\Omega \cup N_k$ bounded. Then there exists $M_\infty > 0$ such that $0 < u_{1,k}(x) \le M_\infty$ for a.e. $x \in U_k$, $\quad U_k := \Omega \cup N_k \cup \Gamma_{N_k}$. Moreover, up to a subsequence, $u_{1,k} \in C^{0,\beta}(\mathbb{R}^n)$ with a uniform Hölder exponent $\beta \in (0,1)$ and $C^{0,\beta}$-seminorms locally bounded in $\mathbb{R}^n$.*

Proof sketch. First, by the weak formulation and $L^2(\Omega)$-normalization,

$$\int_\Omega |\nabla u_{1,k}|^2\, dx + [u_{1,k}]_s^2 = \lambda_{1,k} \le \lambda_1^{\mathrm{Dir}},$$

hence the energies are uniformly bounded. The standard Moser/De Giorgi iteration for the mixed operator on Ω gives $\|u_{1,k}\|_{L^\infty(\Omega)} \le C(n, s, \Omega)$ (see the iteration around (3.0.3)–(3.0.12) in [13, Prop. 3.3(1)]). For $x \in N_k$, the nonlocal Neumann condition yields

$$0 = N_s u_{1,k}(x) = C_{n,s} \int_\Omega \frac{u_{1,k}(x) - u_{1,k}(y)}{|x-y|^{n+2s}}\, dy \quad \Rightarrow \quad u_{1,k}(x) = \frac{\int_\Omega \frac{u_{1,k}(y)}{|x-y|^{n+2s}}\, dy}{\int_\Omega \frac{1}{|x-y|^{n+2s}}\, dy},$$

so $|u_{1,k}(x)| \le \sup_\Omega |u_{1,k}|$. Hence, $\|u_{1,k}\|_{L^\infty(N_k)} \le \|u_{1,k}\|_{L^\infty(\Omega)}$, and continuity up to $\partial\Omega \cap N_k$ follows from the regularity quoted in [13, Sec. 3 & App. A]. This proves the claim with $M_\infty = \sup_k \|u_{1,k}\|_{L^\infty(\Omega)}$. □

Remark 2.16 (On using L^2 in place of L^∞ on U_k)**.** *One can avoid the pointwise bound on U_k by using Cauchy–Schwarz on near-field terms and the identity $N_s u = 0$ on N_k (or $u = 0$ on D_k) to rewrite $u(x)$ outside Ω as a weighted average of values in Ω. Since our Proposition 2.15 yields a k-uniform $L^\infty(U_k)$ bound with a shorter argument, we keep the proofs of Theorems A–B in that form.*

2.4. *Kernel potentials and tail estimate*

Given ϕ_1 as above, set

$$\Psi(x) := \int_\Omega \frac{\phi_1(y)}{|x-y|^{n+2s}}\, dy, \qquad x \in \mathbb{R}^n \setminus \Omega,$$

and for Dirichlet-dissipation considerations,

$$\Upsilon(x) := \int_\Omega \frac{1}{|x-y|^{n+2s}}\, dy.$$

Both belong to $L^1(\mathbb{R}^n \setminus \Omega)$; moreover, for all $R > 0$,

$$\int_{|x|>R} \Psi(x)\, dx \le C(\Omega, n, s)\, R^{-2s}, \qquad \int_{|x|>R} \Upsilon(x)\, dx \le C(\Omega, n, s)\, R^{-2s}. \tag{2.7}$$

Proof of (2.7). If $|x| > R$ and $y \in \Omega$ (bounded), then $|x - y| \ge |x|/2$, so $\Psi(x) \le 2^{n+2s}\|\phi_1\|_{L^1(\Omega)}|x|^{-n-2s}$ and similarly for Υ. Integration over $\{|x| > R\}$ gives R^{-2s} decay. □

2.5. Separated-Dirichlet variant for $s \ge \frac{1}{2}$

When $\operatorname{dist}(D_k, \Omega) \ge \delta > 0$, one has $\Upsilon(x) \le C(\delta, n, s)$ on D_k, which will be used in §3 to quantify $\lambda_{1,k} \to 0$ for $s \in [\frac{1}{2}, 1)$ under Dirichlet dissipation with separation; cf. [13, Sec. 4.2]. For local/fractional mixed background, related to boundary conditions and capacities, see also [16, 24, 25].

Remark 2.17 (On the interaction set Q and cross-terms)**.** *Recall* $Q = \mathbb{R}^{2n} \setminus (\Omega^c \times \Omega^c) = (\Omega \times \Omega) \cup (\Omega \times \Omega^c) \cup (\Omega^c \times \Omega)$. *Thus, the Gagliardo term comprises the interior part* $\Omega \times \Omega$ *and both cross-interactions between* Ω *and its exterior. This is the origin of the Dirichlet and Neumann integrals that appear in the cross–testing identities of Lemmas 2.8–2.11.*

3. Main results: quantitative bounds

In this section, we prove the quantitative estimates announced in the Introduction. We retain the notation from §2. In particular, ϕ_1 is the $L^2(\Omega)$-normalized ground state for the full exterior Dirichlet problem (thus $\mathcal{L}\phi_1 = \lambda_1^{\mathrm{Dir}}\phi_1$ in Ω, $\phi_1 = 0$ in $\mathbb{R}^n \setminus \Omega$), while $\psi_1 \equiv |\Omega|^{-1/2}$ is the normalized ground state for the pure Neumann problem. For each admissible partition (D_k, N_k) we denote by $(\lambda_{1,k}, u_{1,k})$ the positive $L^2(\Omega)$-normalized principal eigenpair.

From Lemmas 2.8–2.11, the cross–testing identities read

$$(\lambda_1^{\mathrm{Dir}} - \lambda_{1,k}) \int_\Omega \phi_1 u_{1,k}\, dx = -\int_{\Gamma_{N_k}} u_{1,k}\, \partial_\nu \phi_1\, d\sigma + \int_{N_k}\int_\Omega \frac{\phi_1(y)\, u_{1,k}(x)}{|x-y|^{n+2s}}\, dy\, dx, \tag{3.1}$$

$$\lambda_{1,k} \int_\Omega \psi_1 u_{1,k}\, dx = -\int_{\Gamma_{D_k}} u_{1,k}\, \partial_\nu \psi_1\, d\sigma + \int_{D_k}\int_\Omega \frac{\psi_1(x)\, u_{1,k}(y)}{|x-y|^{n+2s}}\, dy\, dx, \tag{3.2}$$

where $\Gamma_{N_k} := \partial\Omega \cap \overline{N_k}$ and $\Gamma_{D_k} := \partial\Omega \cap \overline{D_k}$ (see §2), and $\partial_\nu\psi_1 \equiv 0$. We also use the potentials from §2.4:

$$\Psi(x) = \int_\Omega \frac{\phi_1(y)}{|x-y|^{n+2s}}\, dy, \qquad \Upsilon(x) = \int_\Omega \frac{1}{|x-y|^{n+2s}}\, dy,$$

and the tail estimate $\int_{|x|>R} \Psi(x)\, dx,\ \int_{|x|>R} \Upsilon(x)\, dx \le C_{n,s,\Omega} R^{-2s}$, cf. (2.7).

Finally, by Proposition 2.15, there exists $M_\infty > 0$, independent of k, such that

$$0 < u_{1,k}(x) \le M_\infty \quad \text{for a.e. } x \in U_k := \Omega \cup N_k \cup \Gamma_{N_k}.$$

3.1. Neumann dissipation: Proof of Theorem A

Set $\alpha_k := \int_\Omega \phi_1 u_{1,k}\, dx > 0$ and use (3.1):

$$(\lambda_1^{\mathrm{Dir}} - \lambda_{1,k})\, \alpha_k = -\int_{N_k \cap \partial\Omega} u_{1,k}\, \partial_\nu \phi_1\, d\sigma + \int_{N_k} \Psi(x)\, u_{1,k}(x)\, dx.$$

With $u_{1,k} \le M_\infty$ on U_k we obtain

$$0 \le \lambda_1^{\mathrm{Dir}} - \lambda_{1,k} \le \frac{M_\infty}{\alpha_k}\bigg(\int_{N_k\cap\partial\Omega} |\partial_\nu \phi_1|\, d\sigma + \int_{N_k} \Psi(x)\, dx\bigg). \tag{3.3}$$

Split $\int_{N_k} \Psi = \int_{N_k\cap B_R} \Psi + \int_{N_k\setminus B_R} \Psi$ and use the tail bound $\int_{|x|>R} \Psi \le C_{n,s,\Omega} R^{-2s}$. This yields exactly the estimate stated in Theorem A.

Choice of R. Fix $\varepsilon > 0$ and pick $R(\varepsilon) := (2C_{n,s,\Omega}/\varepsilon)^{1/(2s)}$, so that $C_{n,s,\Omega}R(\varepsilon)^{-2s} \le \varepsilon/2$. Then use the dissipation hypothesis (Definition 2.6) to choose k large so that the remaining near-field terms over $N_k \cap B_{R(\varepsilon)}$ (resp. $D_k \cap B_{R(\varepsilon)}$) and the boundary slice $\mathcal{H}^{n-1}(\Gamma_{N_k})$ (resp. $\mathcal{H}^{n-1}(\Gamma_{D_k})$) are each $< \varepsilon/2$. This yields a constructive modulus of continuity for $\lambda_{1,k}$.

3.2. Dirichlet dissipation ($0 < s < \frac{1}{2}$): Proof of Theorem B

Let $\psi_1 \equiv |\Omega|^{-1/2}$ and $\beta_k := \int_\Omega \psi_1 u_{1,k}\, dx = |\Omega|^{-1/2}\int_\Omega u_{1,k}\, dx > 0$. From (2.5),

$$\lambda_{1,k}\beta_k = |\Omega|^{-1/2}\int_{D_k}\int_\Omega \frac{u_{1,k}(y)}{|x-y|^{n+2s}}\, dy\, dx \le \frac{M_\infty}{|\Omega|^{1/2}}\int_{D_k} \Upsilon(x)\, dx,$$

hence

$$0 \le \lambda_{1,k} \le \frac{M_\infty}{\beta_k}\int_{D_k} \Upsilon(x)\, dx \le \frac{M_\infty}{\beta_k}\bigg(\int_{D_k\cap B_R} \Upsilon(x)\, dx + C_{n,s,\Omega}R^{-2s}\bigg), \tag{3.4}$$

and Theorem B follows because $\Upsilon \in L^1_{\mathrm{loc}}$ up to $\partial\Omega$ when $0 < s < \frac{1}{2}$.

Choice of R. Fix $\varepsilon > 0$ and pick

$$R(\varepsilon) := \Big(\tfrac{2\,C_{n,s,\Omega}}{\varepsilon}\Big)^{1/(2s)} \quad\Rightarrow\quad C_{n,s,\Omega}R(\varepsilon)^{-2s} \le \tfrac{\varepsilon}{2}.$$

Then use the dissipation hypothesis to choose k large so that the remaining near-field terms over $N_k \cap B_{R(\varepsilon)}$ (resp. $D_k \cap B_{R(\varepsilon)}$) and the boundary slice are each $< \varepsilon/2$. This yields a constructive modulus of continuity for $\lambda_{1,k}$.

3.3. Variant for $s \ge \frac{1}{2}$ with separated Dirichlet sets

Assume $\mathrm{dist}(D_k, \Omega) \ge \delta > 0$. Then $\Upsilon(x) \le C(\delta, n, s)$ on D_k, so (3.4) gives

$$0 \le \lambda_{1,k} \le \frac{M_\infty}{\beta_k}\Big(C(\delta, n, s)\,|D_k \cap B_R| + C_{n,s,\Omega}R^{-2s}\Big),$$

and the stated convergence follows.

Explicit constant. If $\mathrm{dist}(D_k, \Omega) \ge \delta > 0$, then for $x \in D_k$ and $y \in \Omega$ we have $|x-y| \ge \delta$, hence

$$\Upsilon(x) = \int_\Omega \frac{1}{|x-y|^{n+2s}}\, dy \le |\Omega|\,\delta^{-(n+2s)}.$$

Therefore

$$0 \le \lambda_{1,k} \le \frac{M_\infty}{\beta_k}\Big(|\Omega|\,\delta^{-(n+2s)}\,|D_k \cap B_R| + C_{n,s,\Omega}R^{-2s}\Big),$$

which is the stated bound with $C(\delta, n, s) = |\Omega|\,\delta^{-(n+2s)}$.

4. Consequences, examples, and extensions

We collect corollaries of Theorems A and B, give quantitative rates under mild geometric control, and note an application to asymptotically linear bifurcation thresholds.

4.1. Quantitative continuity of the principal eigenvalue

Corollary 4.1 (Neumann sets dissipating). *Let (D_k, N_k) be admissible partitions with $\Omega \cup N_k$ bounded and N_k dissipating. Let $(\lambda_{1,k}, u_{1,k})$ be the L^2–normalized principal eigenpairs and let $\alpha_k = \int_\Omega \phi_1 u_{1,k} > 0$, where ϕ_1 is the Dirichlet ground state. Then, for every $R > 0$,*

$$0 \le \lambda_1^{\mathrm{Dir}} - \lambda_{1,k} \le \frac{M_\infty}{\alpha_k}\Big(\|\partial_\nu \phi_1\|_{L^\infty(\partial\Omega)}\, \mathcal{H}^{n-1}(\Gamma_{N_k}) \;+\; \int_{N_k \cap B_R} \Psi \;+\; C_{n,s,\Omega}\, R^{-2s} \Big).$$

If, along a subsequence, $\inf \alpha_k > 0$, then $\lambda_{1,k} \to \lambda_1^{\mathrm{Dir}}$. Moreover, for any $\varepsilon > 0$ choose $R = R(\varepsilon)$ with $C_{n,s,\Omega} R^{-2s} < \varepsilon$, and then choose k so large that the remaining terms are $< \varepsilon$ by N_k–dissipation, which yields a constructive modulus of continuity.

Proof. This is Theorem A with $\int_{N_k \cap \partial\Omega} |\partial_\nu \phi_1| \le \|\partial_\nu \phi_1\|_{L^\infty}\, \mathcal{H}^{n-1}(N_k \cap \partial\Omega)$ and the tail estimate $\int_{|x|>R} \Psi \le C_{n,s,\Omega} R^{-2s}$. The modulus-of-continuity statement follows by first fixing R and then using N_k–dissipation. □

Corollary 4.2 (Dirichlet sets dissipating). *Assume $0 < s < \frac{1}{2}$ and D_k dissipates. Let $\beta_k = |\Omega|^{-1/2} \int_\Omega u_{1,k} > 0$. Then, for every $R > 0$,*

$$0 \le \lambda_{1,k} \le \frac{M_\infty}{\beta_k}\Big(\int_{D_k \cap B_R} \Upsilon + C_{n,s,\Omega} R^{-2s} \Big).$$

If, along a subsequence, $\inf \beta_k > 0$, then $\lambda_{1,k} \to 0$; for $s \ge \frac{1}{2}$ the same conclusion holds provided $\mathrm{dist}(D_k, \Omega) \ge \delta > 0$.

4.2. Rates under geometric control near $\partial\Omega$

We single out a simple geometric regime that turns the integrals in Theorem B into explicit powers of a thickness parameter.

Lemma 4.3 (Tubular estimate for Υ). *Let $0 < s < \frac{1}{2}$ and $T_\delta := \{x \in \mathbb{R}^n \setminus \Omega : \ \mathrm{dist}(x, \partial\Omega) < \delta\}$ with $\delta \in (0, 1)$. There exists $C = C(\Omega, n, s)$ such that*

$$\int_{T_\delta \cap B_R} \Upsilon(x)\, dx \;\le\; C\, \delta^{1-2s}\, \mathcal{H}^{n-1}(\partial\Omega \cap B_{R+\delta}) \;+\; C\, |B_R|,$$

and, for any measurable $E \subset T_\delta \cap B_R$,

$$\int_E \Upsilon(x)\, dx \;\le\; C\, \delta^{1-2s}\, \mathcal{H}^{n-1}(\partial\Omega \cap B_{R+\delta}).$$

Proof (sketch). In tubular coordinates $(y, \rho) \in \partial\Omega \times (0, \delta)$ with $x = y + \rho\nu(y)$ (valid since $\partial\Omega$ is $C^{1,1}$), one has $dx \approx J(y, \rho)\, d\rho\, d\sigma(y)$ with J uniformly bounded above and below. For fixed (y, ρ) and $z \in \Omega$, $|x - z| \ge c\,\rho$ with $c > 0$, hence $\Upsilon(x) \le C \int_\Omega \rho^{-n-2s}\, dz \le C\rho^{-2s}$. Integrating ρ^{-2s} from 0 to δ gives $\delta^{1-2s}/(1 - 2s)$, and integration over $\partial\Omega \cap B_{R+\delta}$ yields the first inequality. The second follows by monotonicity. □

Orientation. In tubular coordinates $x = y + \rho\nu(y)$, the Jacobian $J(y,\rho)$ is bounded above/below on $0 < \rho < \delta$ for $C^{1,1}$ domains, and the kernel behaves like ρ^{-n-2s}. After integrating in $z \in \Omega$, this leaves the main singularity ρ^{-2s}, which is integrable near $\rho = 0$ iff $s < \frac{1}{2}$.

Corollary 4.4 (Rates from thin Dirichlet layers)**.** *Assume* $0 < s < \frac{1}{2}$ *and* $D_k \subset T_{\delta_k}$ *with* $\delta_k \downarrow 0$. *Then, along any subsequence with* $\inf \beta_k > 0$,

$$\lambda_{1,k} \;\le\; \frac{M_\infty}{\beta_k}\Big(C\,\delta_k^{1-2s}\,\mathcal{H}^{n-1}(\partial\Omega \cap B_{R+1}) \;+\; C_{n,s,\Omega}\,R^{-2s}\Big) \qquad (R \ge 1).$$

Optimizing R at the scale of the external mass of D_k *yields an* $o(1)$ *rate controlled by* δ_k^{1-2s} *plus the tail term.*

Remark 4.5 (Boundary terms on the Neumann side)**.** *Since* $\phi_1 \in C^{1,\alpha}(\Omega)$ *for some* $\alpha \in (0,1)$, $\|\partial_\nu\phi_1\|_{L^\infty(\partial\Omega)} < \infty$. *Thus, the boundary contribution in Theorem A is* $O(\mathcal{H}^{n-1}(\Gamma_{N_k}))$, *whereas the nonlocal mass over* $N_k \cap B_R$ *is controlled by the* L^1*–absolute continuity of* Ψ *and the tail* R^{-2s}*; compare Lemma 4.3 with* Ψ *in place of* Υ *when* N_k *concentrates near* $\partial\Omega$.

4.3. Application: Bifurcation thresholds in asymptotically linear problems

Let $h(t) = \theta t + f(t)$ with f bounded and $\lim_{t\to 0^+} h(t)/t = a > 0$. In the notation of [13], the bifurcation-from-zero parameter is $\lambda_0 = \lambda_1(D)/a$. The quantitative eigenvalue bounds give the following immediate consequence.

Corollary 4.6 (Quantitative control of λ_0)**.** *Let* $\lambda_{0,k} = \lambda_1(D_k)/a$.

- *If* N_k *dissipates* (Definition 2.6) *and* $\inf \alpha_k > 0$, *then for every* $R > 0$,

$$0 \le \frac{\lambda_1^{\mathrm{Dir}}}{a} - \lambda_{0,k} \;\le\; \frac{M_\infty}{a\,\alpha_k}\Big(\|\partial_\nu\phi_1\|_{L^\infty(\partial\Omega)}\,\mathcal{H}^{n-1}(\Gamma_{N_k}) \;+\; \int_{N_k\cap B_R}\Psi \;+\; C_{n,s,\Omega}\,R^{-2s}\Big).$$

- *If* $0 < s < \frac{1}{2}$ *and* D_k *dissipates with* $\inf \beta_k > 0$, *then for every* $R > 0$,

$$0 \le \lambda_{0,k} \;\le\; \frac{M_\infty}{a\,\beta_k}\Big(\int_{D_k\cap B_R}\Upsilon \;+\; C_{n,s,\Omega}\,R^{-2s}\Big),$$

  and the same bound holds for $s \ge \frac{1}{2}$ *assuming* $\mathrm{dist}(D_k,\Omega) \ge \delta > 0$.

Consequently, the bifurcation threshold moves in tandem with the geometric measures entering Theorems A and B.

4.4. Comparison with local and pure nonlocal settings

In the purely local mixed DN Laplacian, quantitative dependence of λ_1 on boundary partitions is well studied; see, e.g., [11]. In nonlocal frameworks, integrability near $\partial\Omega$ dictates how the fractional kernel accumulates when Dirichlet mass approaches the boundary; cf. regularity and kernel estimates in [7, 14, 19]. Our bounds adapt these ideas to the mixed local–nonlocal operator without requiring new regularity beyond [13]. They also dovetail with variants where drifts or weights are present [8, 15], though a careful re-derivation would be needed there (we do not pursue it here).

Use of Generative-AI tools declaration

The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that he has no conflict of interest.

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