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Research article

New developments in fixed point theorems for θ-Branciari contractions on strong-controlled Branciari b distance spaces

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Abstract: In this paper, we introduced the strong-controlled Branciari b-distance, a generalized metric structure designed to consolidate disparate fixed point theorems scattered throughout existing literature. Through illustrative examples, we demonstrated how this framework subsumes and extends previous results. We further established the applicability of our theoretical developments by presenting a concrete problem whose solution leverages the properties of strong-controlled Branciari b-distance spaces.

Keywords: metric space $\left(M_{sp}\right)$; Branciari distance (B_d); strong-controlled Branciari b-distance (SCB_{bd}); θ-Branciari contraction (θ_{BC}); Ciric-Reich-Rus-type θ-Branciari contraction $(CRRT - \theta_{BC})$
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1. Introduction

The Banach contraction principle, introduced in 1922 [1], established the cornerstone methodology for proving fixed point theorems in metric spaces M_{sp}. This seminal work catalyzed extensive research in fixed point theory, yielding profound implications across diverse mathematical disciplines. This has led to significant developments and interests in the field of fixed point theory, which has broad implications in numerous branches of mathematics. As an extension of M_{sp}, Kamran and Samreen introduced the notion of extended $b{-}M_{sp}$ [2]. Abdeljawad then established the concept of controlled $b{-}M_{sp}$ [3]. This concept was elaborated upon by Santina et al. who progressed it into strong-controlled $b{-}M_{sp}$ [4]. Nevertheless, Branciari introduced the Branciari distance as a generalization of M_{sp} [5]. It is often referred to as the Branciari metric or the Branciari distance function. Those generalizations open up new avenues and possibilities, forming a vibrant and developing field marked by ongoing research projects [6, 7]. The resulting spaces offer fresh and fascinating interpretations on M_{sp} ideas, indicating potential for a wide range of uses. Moreover, existence and uniqueness issues are fundamentally resolved by fixed point theory, especially when dealing with differential and integral equations. It provides a basic structure for coping with a number of issues, such as integrodifferential equations. Under diverse contraction conditions, such as the θ-contraction presented by Jleli and Samet [8] in the framework of Branciari M_{sp}, $CRRT-\theta_{BC}$, and interpolative-θ_{BC} [9], many researchers have investigated a variety of $b - M_{sp}$. When determining the presence and uniqueness of fixed points for mappings in M_{sp}, the Banach contraction principle—a foundational finding in nonlinear analysis—is essential. Motivated by the need to unify and extend existing distance notions, we propose the *strong-controlled Branciari b-metric spaces* (SCB_{bd}), which combine the flexibility of the strong-controlled b-metric with the quadrilateral structure of Branciari distances. Building on this framework, we develop new fixed point results of θ-type and demonstrate their utility through an application to boundary value problems.

The Banach contraction principle, introduced in 1922 [1], established the cornerstone methodology for proving fixed point theorems in metric spaces M_{sp}. This seminal work catalyzed extensive research in fixed point theory, yielding profound implications across diverse mathematical disciplines.

The main contributions of this paper are as follows:

(1) **Definition of the** (SCB_{bd}) **(Section 2).** We introduce a new distance structure (SCB_{bd}) controlled by a function $\omega(x, y) \geq 1$. This structure generalizes Branciari distance ($\omega \equiv 1$) and we provide explicit examples to illustrate its validity.

(2) **θ-Branciari contraction and fixed point theorem (Section 3, Theorem 3.1).** We establish a θ-type contractive condition in (SCB_{bd}) spaces, proving that every θ-Branciari contraction admits a *unique fixed point*. Moreover, we quantify the convergence of the Picard sequence by showing

$$\vartheta\left(SCB_{bd}\left(g_p, g_{p+1}\right)\right) \leq \left[\vartheta\left(SCB_{bd}\left(g_0, g_1\right)\right)\right]^{t^p}.$$

(3) **Ciric-Reich-Rus-type θ-Branciari contraction** ($CRRT - \theta_{BC}$) **(Section 3, Theorem 3.2).** We generalize classical Ciric-Reich-Rus-type conditions to the (SCB_{bd}) setting and prove that such mappings also possess a *unique fixed point*, thereby extending known results from rectangular and Branciari contexts.

(4) **Interpolative θ-Branciari contractions (Section 3, Theorem 3.3).** We formulate a three-term interpolative contractive condition with exponents t_1, t_2, t_3 satisfying $t_1 + t_2 + t_3 < 1$.

(5) **Application to a fourth-order boundary value problem (Section 4, Theorem 4.1).** We apply the abstract results to a nonlinear cantilever beam problem, reformulating it via a Green function into an operator that satisfies our θ-contractive condition. Under suitable growth restrictions on the nonlinear term g, we prove the existence and uniqueness of a solution, showing the practical impact of our theoretical framework.

2. Preliminary

We start by defining the Branciari distance (B_d) [5], which is one of the ideas put out to broaden and generalize the scope of the metric.

Definition 2.1. *Let $\mathbb{X}$ be a nonempty set and let $Br : \mathbb{X} \times \mathbb{X} \longrightarrow [0, \infty)$ such that for all $g, h \in \mathbb{X}$ and all $k \neq l \in \mathbb{X} \backslash \{g, h\}$,*

$$\begin{aligned} &(Br1) \quad Br(g,h) = 0 \text{ if and only if } g = h,\\ &(Br2) \quad Br(g,h) = Br(h,g)(\text{symmetry}),\\ &(Br3) \quad Br(g,h) \le Br(g,k) + Br(k,l) + Br(l,h). \end{aligned} \tag{2.1}$$

Then, Branciari defines Br as B_d. Then, the pair $(\mathbb{X}, Br)$ is referred to as B_d space.

Now we revisit the notion of θ-contraction that is established by Jleli and Samet [8]. Consider the set θ that contains all the functions that are continuous and non-decreasing $\vartheta : (0,\infty) \to (1,\infty)$ such that it fulfills the conditions below:
($\square$) for each sequence $\{g_n\} \subset (0,\infty)$, $\lim_{n\to\infty} \vartheta(g_n) = 1 \Leftrightarrow \lim_{n\to\infty} g_n = 0^+$;
($\triangle$) there exist $e \in (0,1)$ and $L \in (0,\infty)$ such that $\lim_{g\to 0^+} \frac{\vartheta(g)-1}{g^e} = L$.
Several fixed-point results have been improved by using this notion (see, e.g., [10–12]).

Recall the notion of controlled strong $b - M_{sp}$ introduced by Santina et al. [4].

Definition 2.2. *Let $\mathbb{X}$ be a non-empty set and let $\omega : \mathbb{X}\times\mathbb{X} \to [1,\infty)$. The following function $SC : \mathbb{X}\times\mathbb{X} \to [0,\infty)$ is called a strong-controlled $b - M_{sp}$ if*
(1) $SC(g,h) = 0$ iff $g = h$,
(2) $SC(g,h) = SC(h,g)$,
(3) $SC(g,h) \le SC(g,l) + \omega(l,h)SC(l,h)$,
for all $g,h,l \in \mathbb{X}$. The pair $(\mathbb{X}, SC)$ is called a strong-controlled $b - M_{sp}$.

We will merge these two concepts, strong-controlled $b - M_{sp}$ and B_d, under the designation of a SCB_{bd} space according to the following.

Definition 2.3. *Consider the set $\mathbb{X}$ that contains at least one element and $\omega : \mathbb{X}\times\mathbb{X} \to [1,\infty)$ is a mapping. Hence, the function $SCB_{bd} : \mathbb{X}\times\mathbb{X} \to [0,\infty)$ is a strong-controlled Branciari b-distance if it fulfills:*
(i) $SCB_{bd}(g,h) = SCB_{bd}(h,g)$,
(ii) $SCB_{bd}(g,h) = 0$ if and only if $g = h$,
(iii) $SCB_{bd}(g,h) \le SCB_{bd}(g,l) + SCB_{bd}(l,k) + \omega(k,h)SCB_{bd}(k,h)$,
for all $g,h \in \mathbb{X}$ and all distinct $k,l \in \mathbb{X}\backslash\{g,h\}$. The couple of the symbols $(\mathbb{X}, SCB_{bd})$ denotes $SCB_{bd} - M_{sp}$.

Example 2.1. *Let $\mathbb{X} = \{10,11,12,13\}$. Define $SCB_{bd} : \mathbb{X}\times\mathbb{X} \to [0,\infty)$ as follows:*

$$\begin{aligned} &SCB_{bd}(t,t) = 0, \forall t \in \mathbb{X}, SCB_{bd}(10,t) = SCB_{bd}(t,10) = 40, \forall t \in \mathbb{X} - \{10\},\\ &SCB_{bd}(11,12) = SCB_{bd}(12,11) = SCB_{bd}(11,13) = SCB_{bd}(13,11) = 211,\\ &SCB_{bd}(13,12) = SCB_{bd}(12,13) = 999. \end{aligned}$$

Consider the symmetric function $\omega : \mathbb{X}\times\mathbb{X} \to [1,\infty)$ with the following characteristics:

$$\begin{aligned} &\omega(t,t) = 10, \forall t \in \mathbb{X},\\ &\omega(10,11) = 3, \omega(10,12) = 4, \omega(10,13) = \omega(11,12) = 2, \omega(11,13) = 9, \omega(12,13) = 3. \end{aligned}$$

Therefore, $(\mathbb{X}, SCB_{bd})$ is a $SCB_{bd} - M_{sp}$. In spite of that, we shall observe that
(1) $SCB_{bd}(12,13) = 999 > SCB_{bd}(12,10) + \omega(10,13)SCB_{bd}(10,13) = 120$.
(2) $SCB_{bd}(12,13)=999>\omega(12,11)SCB_{bd}(12,11)+\omega(11,10)SCB_{bd}(11,10)+\omega(10,13)SCB_{bd}(10,13)=622$.
Thus $(\mathbb{X}, SCB_{bd})$ is neither a strong-controlled metric-type space nor a controlled Branciari b-distance space.

Example 2.2. *Let* $\mathbb{X} = \{1, 2, 3, 4\}$. *Define* $SCB_{bd} : \mathbb{X}\times\mathbb{X} \to \mathbb{R}^+$ *by* $\omega(g, h) = 10g + 10h + 100$ *and then* $(\mathbb{X}, SCB_{bd})$ *is a strong-controlled Branciari b-distance space.*

Once all other conditions are met, we shall demonstrate the amended quadrilateral inequality.

$$\begin{aligned}
&SCB_{bd}(g, h)\\
=&|g - h|^2\\
=&|g - l + l - k + k - h|^2, \text{ where } g \neq h \neq l \neq k\\
\leq&|g - l|^2 + |l - k|^2 + |k - h|^2 + 2|g - l||l - k| + 2|l - k||k - h| + 2|k - h||g - l|\\
\leq&|g - l|^2 + |l - k|^2 + (10g + 10h + 100)|k - h|^2\\
=&SCB_{bd}(g, l) + SCB_{bd}(l, k) + \omega(g, h)SCB_{bd}(k, h).
\end{aligned}$$

Therefore, $SCB_{bd}(g, h) \leq SCB_{bd}(g, l) + SCB_{bd}(l, k) + \omega(g, h)SCB_{bd}(k, h)$.

Remark 2.1. *If* $\omega(g, h) = s = 1$, *then it is the standard* B_d. *It is widely recognized that the b-metric does not require continuity. As a result,* SCB_{bd} *is also not always continuous. We assume* SCB_{bd} *is continuous.*

Now, we present the topological properties of strong-controlled Branciari b-distance (SCB_{bd}).

Definition 2.4. *Let* $\mathbb{X}$ *be a set that includes at least one element and endowed with* SCB_{bd}, *and then a sequence* $\{g_n\}$ *in* $\mathbb{X}$ *is*
(a) Convergent to g if for every $\epsilon > 0$ *there exists* $N = N(\epsilon) \in \mathbb{N}$ *such that* $SCB_{bd}(g_n, g) < \epsilon$, *for all* $n \geq N$. *Particularly, for this instance, we define* $\lim_{n\to\infty} g_n = g$.
(b) Cauchy if for every $\epsilon > 0$ *there exists* $N = N(\epsilon) \in \mathbb{N}$ *such that* $SCB_{bd}(g_m, g_n) < \epsilon$, *for all* $m, n \geq N$.
(c) An SCB_{bd}*-metric space* $(\mathbb{X}, SCB_{bd})$ *is complete if every Cauchy sequence in* $\mathbb{X}$ *is convergent.*

3. Primary findings

We shall commence this portion by providing an introduction to the notion of a θ-Branciari contraction.

Definition 3.1. *Let* $(\mathbb{X}, SCB_{bd})$ *be a* $(SCB_{bd}) - M_{sp}$ *and consider the self-mapping* $L : \mathbb{X} \to \mathbb{X}$ *where* $\mathbb{X}$ *is a non-empty set. Then, L is called a* θ_{BC} *if* $\exists\, \vartheta \in \theta$ *satisfying*

$$\vartheta\left(SCB_{bd}(Lg, Lh)\right) \leq \left[\vartheta\left(SCB_{bd}(g, h)\right)\right]^t \text{ if } SCB_{bd}(Lg, Lh) \neq 0 \text{ for } g, h \in \mathbb{X},$$

where $t \in (0, 1)$ *and* $g_p = L^p g_0$ *(p=0,1,2...), for some* $g_0 \in \mathbb{X}$. *Here,* g_p *denotes the sequence (called the orbit or Picard iteration) generated by repeated application of L.*

Therefore, $g_1 = L(g_0), g_2 = L(g_1) = L^2(g_0), ..., g_p = L(g_{p-1}) = L^p(g_0)$.

Theorem overview. We show that a θ_{BC} L on a complete strong-controlled Branciari b-distance space $(\mathbb{X}, SCB_{bd})$ has a unique fixed point. Moreover, for every $g_0 \in \mathbb{X}$, the Picard orbit (g_p) with $g_p = L^p g_0$ converges to that point.

Theorem 3.1 (Fixed point for θ_{BC})**.** *Let* $(\mathbb{X}, SCB_{bd})$ *be a complete* $SCB_{bd} - M_{sp}$, $L : \mathbb{X} \to \mathbb{X}$ *be a* θ_{BC}, *and* $\omega : \mathbb{X} \times \mathbb{X} \to [1, \infty)$ *be a control function in a strong-controlled Branciari b-distance. If* $\limsup_{p,q\to\infty} \omega(g_p, g_q) = K,\ K \geq 1$, *and for each* $g \in \mathbb{X}, \lim_{p\to+\infty} \omega\left(g, g_p\right)$ *exists and is finite*, *then L has only one fixed point in* $\mathbb{X}$.

Proof (outline). (1) Define $g_p = L^p g_0$; (2) apply (□,△) to get a decay of $\vartheta\left(SCB_{bd}\left(g_{p+1}, g_p\right)\right)$; (3) deduce Cauchy via the strong-controlled Branciari b-distance inequality; (4) completeness ⇒ limit g^*; (5) show $Lg^* = g^*$ and uniqueness. □

Proof (details). For any point $g_0 \in \mathbb{X}$ we generate the following iterative sequence $\left\{g_p\right\}$ where $g_p = L^p g_0$ for all $p \in \mathbb{N}$. Assume $L^{p_*} g = L^{p_*+1} g$ for some $p_* \in \mathbb{N}$, and then $L^{p_*} g$ is certainly a fixed point of L. Thus, without losing generality, we can presume that $SCB_{bd}\left(L^p g, L^{p+1} g\right) > 0$, $\forall\ p \in \mathbb{N}$. From Definition 3.1, we have

$$\vartheta\left(SCB_{bd}\left(g_p, g_{p+1}\right)\right) = \vartheta\left(SCB_{bd}\left(Lg_{p-1}, Lg_p\right)\right) \le \left[\vartheta\left(SCB_{bd}\left(g_{p-1}, g_p\right)\right)\right]^t \le \left[\vartheta\left(SCB_{bd}\left(g_{p-2}, g_{p-1}\right)\right)\right]^{t^2}.$$

Recursively, we find that

$$\vartheta\left(SCB_{bd}\left(g_p, g_{p+1}\right)\right) \le \left[\vartheta\left(SCB_{bd}\left(g_0, g_1\right)\right)\right]^{t^p}. \tag{3.1}$$

Accordingly, we obtain that

$$1 < \vartheta\left(SCB_{bd}\left(g_p, g_{p+1}\right)\right) \le \left[\vartheta\left(SCB_{bd}\left(g_0, g_1\right)\right)\right]^{t^p} \text{ for all } p \in \mathbb{N}. \tag{3.2}$$

Letting $p \to \infty$ in (3.2), we get $\vartheta\left(SCB_{bd}\left(g_p, g_{p+1}\right)\right) \to 1$ as $p \to \infty$.
From (□) , we have

$$\lim_{p\to\infty} SCB_{bd}\left(g_p, g_{p+1}\right) = 0. \tag{3.3}$$

Similarly, we can easily deduce that

$$\lim_{p\to\infty} SCB_{bd}\left(g_p, g_{p+2}\right) = 0. \tag{3.4}$$

From (△), there exist $e \in (0, 1)$ and $F \in (0, \infty)$ such that

$$\lim_{p\to\infty} \frac{\vartheta\left(SCB_{bd}\left(g_p, g_{p+1}\right)\right) - 1}{\left[SCB_{bd}\left(g_p, g_{p+1}\right)\right]^e} = F.$$

Suppose that $F < \infty$. In this case, let $C = \frac{F}{2} > 0$. Utilizing the definition of a limit, choose $p_0 \in \mathbb{N}$ such that

$$\left|\frac{\vartheta\left(SCB_{bd}\left(g_p, g_{p+1}\right)\right) - 1}{\left[SCB_{bd}\left(g_p, g_{p+1}\right)\right]^e} - F\right| \le C,$$

for all $p \ge p_0$. This implies that $\frac{\vartheta\left(SCB_{bd}\left(g_p, g_{p+1}\right)\right) - 1}{\left[SCB_{bd}\left(g_p, g_{p+1}\right)\right]^e} \ge F - C = C$ for all $p \ge p_0$.

Then, we derive that

$$p\left[SCB_{bd}\left(g_p, g_{p+1}\right)\right]^e \le p\left[\frac{\vartheta\left(SCB_{bd}\left(g_p, g_{p+1}\right)\right) - 1}{C}\right] \text{ for all } p \ge p_0.$$

Suppose that $F = \infty$. Let $C > 0$ be an arbitrary positive number. Using the limit definition, we find $p_0 \in \mathbb{N}$ such that $\frac{\vartheta(SCB_{bd}(g_p,g_{p+1}))-1}{[SCB_{bd}(g_p,g_{p+1})]^e} \geq C$ for all $p \geq p_0$. This implies that $p\left[SCB_{bd}\left(g_p, g_{p+1}\right)\right]^e \leq p\left[\frac{\vartheta(SCB_{bd}(g_p,g_{p+1}))-1}{C}\right]$, for all $p \geq p_0$. Thus, in all cases, there exist $\frac{1}{C} > 0$ and $p_0 \in \mathbb{N}$ such that

$$p\left[SCB_{bd}\left(g_p, g_{p+1}\right)\right]^e \leq p\left[\frac{\vartheta\left(SCB_{bd}\left(g_p, g_{p+1}\right)\right) - 1}{C}\right],$$

for all $p \geq p_0$.

Using Eq (3.2), we obtain $p\left[SCB_{bd}\left(g_p, g_{p+1}\right)\right]^e \leq \frac{p}{C}\left[\left[\vartheta\left(SCB_{bd}\left(g_0, g_1\right)\right)\right]^{t^p} - 1\right]$ for all $p \geq p_0$. Letting $p \to \infty$, we have $\lim_{p\to\infty} p\left[SCB_{bd}\left(g_p, g_{p+1}\right)\right]^e = 0$. Thus, there exists $p_1 \in \mathbb{N}$ such that

$$SCB_{bd}\left(g_p, g_{p+1}\right) \leq \frac{1}{p^{\frac{1}{e}}} \text{ for all } p \geq p_1. \tag{3.5}$$

Let $N = \max\{p_0, p_1\}$. Due to the modified triangle inequality, we have two cases.

Case 1. Let $g_p = g_q$ where $p \neq q$. In the case where $q > p$, we have $L^{q-p}\left(g_p\right) = g_p$. Choose $h = g_p$ and $s = q - p$. Then $L^s h = h$. Consequently, L has h as a periodic point. Hence, $SCB_{bd}(h, Lh) = SCB_{bd}\left(L^s h, L^{s+1}h\right) = SCB_{bd}\left(L^{ks}h, L^{ks+1}h\right)$ for all $k \in \mathbb{N}$. Therefore, it is clear from the above reasoning that $SCB_{bd}(h, Lh) = 0$, so $h = Lh$, that is, h is a fixed point of L.

Case 2. Suppose that $L^p g \neq L^q g$ for all integers $p \neq q$. Let $p < q$ be two natural numbers; to show that $\left\{g_p\right\}$ is a Cauchy sequence, we need to consider two subcases:

Subcase 1. We claim that if $p - q$ is odd, then $SCB_{bd}\left(g_p, g_q\right)$ converges to 0 as $p, q \to \infty$. To prove this, we may assume that $q = p + 2s + 1$.

Thus,

$$SCB_{bd}(g_p, g_{p+2s+1}) \leq SCB_{bd}(g_p, g_{p+2s-1}) + SCB_{bd}(g_{p+2s-1}, g_{p+2s}) + \omega(g_{p+2s}, g_{p+2s+1})SCB_{bd}(g_{p+2s}, g_{p+2s+1}).$$

Using $\omega(g_{p+2s}, g_{p+2s+1}) \leq K$ and $SCB_{bd}(g_p, g_{p+1}) \leq \frac{1}{(p)^{1/e}}$, we have

$$SCB_{bd}(g_p, g_{p+2s+1}) \leq SCB_{bd}(g_p, g_{p+2s-1}) + \frac{1}{(p + 2s - 1)^{1/e}} + K\frac{1}{(p + 2s)^{1/e}}.$$

Since $K \geq 1$, we have

$$SCB_{bd}(g_p, g_{p+2s+1}) \leq SCB_{bd}(g_p, g_{p+2s-1}) + K\left(\frac{1}{(p + 2s - 1)^{1/e}} + \frac{1}{(p + 2s)^{1/e}}\right).$$

Doing this recursively, we have

$$SCB_{bd}(g_p, g_{p+2s+1}) \leq SCB_{bd}(g_p, g_{p+2s-3}) + K\left(\frac{1}{(p+2s-3)^{1/e}} + \frac{1}{(p+2s-2)^{1/e}} + \frac{1}{(p+2s-1)^{1/e}} + \frac{1}{(p+2s)^{1/e}}\right).$$

Eventually, we obtain

$$SCB_{bd}(g_p, g_{p+2s+1}) \leq SCB_{bd}(g_p, g_{p+1}) + K\sum_{i=p+1}^{p+2s}\frac{1}{(i)^{1/e}} \leq \frac{1}{(p)^{1/e}} + K\sum_{i=p+1}^{p+2s}\frac{1}{(i)^{1/e}}.$$

Now, using $K \geq 1$, we have

$$SCB_{bd}(g_p, g_{p+2s+1}) \leq K \sum_{i=p}^{p+2s} \frac{1}{(i)^{1/e}}.$$

Now, the series $\sum_{i=1}^{\infty} \frac{1}{(i)^{1/e}}$ is convergent, so

$$\lim_{p,s\to\infty} SCB_{bd}(g_p, g_{p+2s+1}) = 0.$$

Subcase 2. We may assume that $q = p + 2s$. Thus, we start with the recursive inequality:

$$SCB_{bd}(g_p, g_{p+2s}) \leq SCB_{bd}(g_p, g_{p+2s-2}) + SCB_{bd}(g_{p+2s-2}, g_{p+2s-1}) + \omega(g_{p+2s-1}, g_{p+2s}) SCB_{bd}(g_{p+2s-1}, g_{p+2s}).$$

Using $\omega(g_{p+2s-1}, g_{p+2s}) \leq K$ and $SCB_{bd}(g_p, g_{p+1}) \leq \frac{1}{(p)^{1/e}}$, we have

$$SCB_{bd}(g_p, g_{p+2s}) \leq SCB_{bd}(g_p, g_{p+2s-2}) + \frac{1}{(p+2s-2)^{1/e}} + K \frac{1}{(p+2s-1)^{1/e}}.$$

Since $K \geq 1$, we have

$$SCB_{bd}(g_p, g_{p+2s}) \leq SCB_{bd}(g_p, g_{p+2s-2}) + K\left(\frac{1}{(p+2s-2)^{1/e}} + \frac{1}{(p+2s-1)^{1/e}}\right).$$

Doing this recursively, we have

$$SCB_{bd}(g_p, g_{p+2s}) \leq SCB_{bd}(g_p, g_{p+2s-4}) + K\left(\frac{1}{(p+2s-4)^{1/e}} + \frac{1}{(p+2s-3)^{1/e}} + \frac{1}{(p+2s-2)^{1/e}} + \frac{1}{(p+2s-1)^{1/e}}\right).$$

Thus, we obtain

$$SCB_{bd}(g_p, g_{p+2s}) \leq SCB_{bd}(g_p, g_{p+2}) + K \sum_{i=p+2}^{p+2s-1} \frac{1}{(i)^{1/e}}.$$

Using the fact that the series $\displaystyle\sum_{i=1}^{\infty} \frac{1}{(i)^{1/e}}$ is convergent (since $1/e > 1$), and $\lim_{p\to\infty} SCB_{bd}(g_p, g_{p+2}) = 0$, we can deduce that

$$\lim_{p,s\to\infty} SCB_{bd}(g_p, g_{p+2s}) = 0.$$

Therefore, $\{g_p\}$ is a Cauchy sequence in $\mathbb{X}$. Given $(\mathbb{X}, SCB_{bd})$ is a complete SCB_{bd}, it implies that the sequence $\{g_p\}$ converges to point μ in $\mathbb{X}$. Next, we show that μ is a fixed point of L.

Note that if $Lg \neq Lh$ and by employing (3.1), we have

$$\ln\left[\vartheta SCB_{bd}(Lg, Lh)\right] \leq t \ln\left[\vartheta SCB_{bd}(g, h)\right] \leq \ln\left[\vartheta SCB_{bd}(g, h)\right].$$

Given that ϑ is non-decreasing, the aforementioned observance leads to the conclusion that $SCB_{bd}(Lg, Lh) \leq SCB_{bd}(g, h)$ for all distinct $g, h \in \mathbb{X}$.

On the other hand, $SCB_{bd}\left(g_{p+1}, L\mu\right) = SCB_{bd}\left(Lg_p, L\mu\right) \leq SCB_{bd}\left(g_p, \mu\right)$, $\forall\ p \in \mathbb{N}$. If we take $p \to \infty$ in the preceding inequality, we obtain $g_{p+1} \to L\mu$. By the uniqueness of the limit we deduce that $L\mu = \mu$.

Assume that $\xi \in L$ is a fixed point different than μ. Accordingly, $SCB_{bd}(\mu, \xi) = SCB_{bd}(L\mu, L\xi) \neq 0$. Now using the Definition 3.1, we get

$$\vartheta\left(SCB_{bd}(\mu, \xi)\right) = \vartheta\left(SCB_{bd}(L\mu, L\xi)\right) \leq \left[\vartheta\left(SCB_{bd}(\mu, \xi)\right)\right]^t < \vartheta\left(SCB_{bd}(\mu, \xi)\right), \text{which is a contradiction.}$$

Therefore, $\mu = \xi$. Consequently, L is asserted to possess a unique fixed point in $\mathbb{X}$. □

Now, let us consider the following example that validates our findings.

Example 3.1. *Construct the following sequence:*

$$\begin{aligned}
\sigma_1 &= 1 \times 2,\\
\sigma_2 &= 1 \times 2 + 2 \times 5,\\
\sigma_3 &= 1 \times 2 + 2 \times 5 + 3 \times 10,\\
\sigma_4 &= 1 \times 2 + 2 \times 5 + 3 \times 10 + 4 \times 17,\\
\sigma_p &= 1 \times 2 + 2 \times 5 + 3 \times 10 + 4 \times 17 + \ldots. + p\left(p^2 + 1\right) = \sum_{i=1}^{p}(i^3 + i)\\
&= \left[\frac{p(p+1)}{2}\right]^2 + \frac{p(p+1)}{2} = \frac{p^4 + 2p^3 + 3p^2 + 2p}{4}.
\end{aligned} \tag{3.6}$$

Let $\mathbb{X} = \left\{\sigma_p : p \geq 0\right\}$. *Define* $SCB_{bd} : \mathbb{X} \times \mathbb{X} \to [0, \infty)$ *as* $SCB_{bd}(g, h) = |g - h|^2$. *Consider* $\omega : \mathbb{X} \times \mathbb{X} \to [1, \infty)$ *as* $\omega(g, h) = 5g + 3h + 20$. *Subsequently,* $(\mathbb{X}, SCB_{bd})$ *is a complete* SCB_{bd} *space. We take p to be a non-negative real number. For the last two lines, we define*

$$L(\sigma_p) = \sigma_{p/2}, \text{ for all } p \geq 0.$$

We will now demonstrate that L is a θ_{BC} *where* $\vartheta(x) = e^x$. *Since* $\vartheta\left(SCB_{bd}(Lg, Lh)\right) \leq \left[\vartheta\left(SCB_{bd}(g, h)\right)\right]^t$, *this yields* $e^{(SCB_{bd}(Lg,Lh))} \leq \left[e^{(SCB_{bd}(g,h))}\right]^t$. *Applying* log *on both sides, we get*

$$SCB_{bd}(Lg, Lh) \leq tSCB_{bd}(g, h).$$

Therefore, proving the preceding equation is sufficient to demonstrate that L is a θ_{BC}.

Consider $q > p \geq 0$. *We have*

$$\begin{aligned}
SCB_{bd}(L\sigma_p, L\sigma_q) &= SCB_{bd}(\sigma_{p/2}, \sigma_{q/2})\\
&= \left|\frac{\left(\frac{q}{2}\right)^4 + 2\left(\frac{q}{2}\right)^3 + 3\left(\frac{q}{2}\right)^2 + 2\left(\frac{q}{2}\right) - \left(\frac{p}{2}\right)^4 - 2\left(\frac{p}{2}\right)^3 - 3\left(\frac{p}{2}\right)^2 - 2\left(\frac{p}{2}\right)}{4}\right|^2\\
&= \left|\frac{\left(\left(\frac{q}{2}\right)^4 - \left(\frac{p}{2}\right)^4\right) + 2\left(\left(\frac{q}{2}\right)^3 - \left(\frac{p}{2}\right)^3\right) + 3\left(\left(\frac{q}{2}\right)^2 - \left(\frac{p}{2}\right)^2\right) + 2\left(\left(\frac{q}{2}\right) - \left(\frac{p}{2}\right)\right)}{4}\right|^2,
\end{aligned}$$

and

$$SCB_{bd}(\sigma_p,\sigma_q)=\left|\frac{(q^4-p^4)+2(q^3-p^3)+3(q^2-p^2)+2(q-p)}{4}\right|^2.$$

Consider

$$\frac{SCB_{bd}(L\sigma_p,L\sigma_q)}{SCB_{bd}(\sigma_p,\sigma_q)}=\left|\frac{\left(\left(\frac{q}{2}\right)^4-\left(\frac{p}{2}\right)^4\right)+2\left(\left(\frac{q}{2}\right)^3-\left(\frac{p}{2}\right)^3\right)+3\left(\left(\frac{q}{2}\right)^2-\left(\frac{p}{2}\right)^2\right)+2\left(\left(\frac{q}{2}\right)-\left(\frac{p}{2}\right)\right)}{(q^4-p^4)+2(q^3-p^3)+3(q^2-p^2)+2(q-p)}\right|^2.$$

Clearly,

$$\left(\frac{q}{2}\right)^n-\left(\frac{p}{2}\right)^n\leq\frac{1}{2}(q^n-p^n)\, for\ all\ n\geq 1.$$

Therefore,

$$\frac{\left(\left(\frac{q}{2}\right)^4-\left(\frac{p}{2}\right)^4\right)+2\left(\left(\frac{q}{2}\right)^3-\left(\frac{p}{2}\right)^3\right)+3\left(\left(\frac{q}{2}\right)^2-\left(\frac{p}{2}\right)^2\right)+2\left(\left(\frac{q}{2}\right)-\left(\frac{p}{2}\right)\right)}{(q^4-p^4)+2(q^3-p^3)+3(q^2-p^2)+2(q-p)}\leq\frac{1}{2},$$

and this implies that

$$\frac{SCB_{bd}(L\sigma_p,L\sigma_q)}{SCB_{bd}(\sigma_p,\sigma_q)}\leq\frac{1}{2}.$$

By choosing $t\in[1/2,1)$*, we have*

$$SCB_{bd}(L\sigma_p,L\sigma_q)\leq t\cdot SCB_{bd}(\sigma_p,\sigma_q).$$

Hence, L meets θ_{BC} *with* $\vartheta(x)=e^x$*. Then, from Theorem 3.1, L has a unique fixed point* σ_1*. Letting* $\omega(g,h)=1$ *in the preceding theorem, the following corollary is obtained.*

Corollary 3.1. *Consider L as a self mapping on a complete* SCB_d-M_{sp} $(\mathbb{X},SCB_d)$*. If* $\exists\,\Theta\in\theta$ *and* $t\in(0,1)$ *satisfying*

$$\Theta(SCB_d(Lg,Lh))\leq[\Theta(SCB_{bd}(g,h))]^t\ when\ SCB_{bd}(Lg,Lh)\neq 0\, for\ g,h\in\mathbb{X},$$

then L possesses a unique fixed point in $\mathbb{X}$*.*

Definition 3.2. *Consider the* $SCB_{bd}-M_{sp}$ $(\mathbb{X},SCB_{bd})$*. According to Reich, the self-mapping* $\mathfrak{L}:\mathbb{X}\to\mathbb{X}$ *is Ćirić-Reich-Rus-type* θ_{BC}*, briefly,* $CRRT-\theta_{BC}$*, if there exists a function* $\vartheta\in\theta$ *and non-negative real number* $t<1$ *such that*

$$\vartheta\left(SCB_{bd}(\mathfrak{L}g,\mathfrak{L}h)\right)\leq\left[M_{\mathfrak{L},\vartheta}(g,h)\right]^t, \tag{3.7}$$

for all $g,h\in\mathbb{X}$*, where*

$$M_{\mathfrak{L},\vartheta}(g,h):=\max\left\{\vartheta\left(SCB_{bd}(g,h)\right),\vartheta\left(SCB_{bd}(h,\mathfrak{L}h)\right),\vartheta\left(SCB_{bd}(g,\mathfrak{L}g)\right)\right\},$$

where $\limsup_{p,q\to\infty}\omega\left(g_p,g_q\right)<K,\ K\geq 1$*, and* $g_p=\mathfrak{L}^p g_0$ *for* $g_0\in\mathbb{X}$ *and* $t\in(0,1)$*.*

Theorem overview. We extend the fixed point result to the $CRRT-\theta_{BC}$ class: every $CRRT-\theta_{BC}$ on a complete SCB_{bd} space admits a unique fixed point.

Theorem 3.2 (Fixed point for $CRRT-\theta_{BC}$)**.** *Consider* $(\mathbb{X}, SCB_{bd})$ *a complete* SCB_{bd} *space and* $\mathfrak{L} : \mathbb{X} \to \mathbb{X}$ *is a* $CRRT-\theta_{BC}$. *Then* $\mathfrak{L}$ *has a unique fixed point.*

Proof (outline). Consider the Picard sequence $g_{p+1} = \mathfrak{L}(g_p)$. The $CRR-\theta$-Branciari contractive condition yields a geometric decay for the successive gaps $\vartheta(SCB_{bd}(g_{p+1}, g_p))$. Using the SCB_{bd}-inequality (with the control function), this implies that (g_p) is Cauchy; completeness gives a limit $g^* \in \mathbb{X}$. Passing to the limit in the contractive inequality shows $\mathfrak{L}(g^*) = g^*$. Uniqueness follows by applying the same inequality to two proposed fixed points. □

Proof (details). As in Theorem 3.1, we establish an iterative sequence $\{g_p\}$. Let $g_0 \in \mathbb{X}$ and define

$$g_p = \mathfrak{L}^p g_0, \ \forall\ p \in \mathbb{N}.$$

Without loss of generality, we presume that $SCB_{bd}\left(\mathfrak{L}^p g, \mathfrak{L}^{p+1} g\right) > 0$ for all $p \in \mathbb{N}$. Certainly, if $\mathfrak{L}^{p_*} g = \mathfrak{L}^{p_*+1} g$ for some $p_* \in \mathbb{N}$, then $\mathfrak{L}^{p_*} g$ will be a fixed point of L. We show that $\lim_{p\to\infty} SCB_{bd}\left(g_p, g_{p+1}\right) = 0$. Applying the condition of contraction (3.7), we obtain

$$\vartheta\left(SCB_{bd}\left(g_{p+1}, g_p\right)\right) \leq \left[M_{\mathfrak{L},\vartheta}\left(g_p, g_{p-1}\right)\right]^t, \tag{3.8}$$

in which

$$\begin{aligned} M_{\mathfrak{L},\vartheta}\left(g_p, g_{p-1}\right) &= \max\left\{\vartheta\left(SCB_{bd}\left(g_p, g_{p-1}\right)\right), \vartheta\left(SCB_{bd}\left(g_p, \mathfrak{L}g_p\right)\right), \vartheta\left(SCB_{bd}\left(g_{p-1}, \mathfrak{L}g_{p-1}\right)\right)\right\} \\ &= \max\left\{\vartheta\left(SCB_{bd}\left(g_p, g_{p-1}\right)\right), \vartheta\left(SCB_{bd}\left(g_p, g_{p+1}\right)\right), \vartheta\left(SCB_{bd}\left(g_{p-1}, g_p\right)\right)\right\} \\ &\leq \max\left\{\vartheta\left(SCB_{bd}\left(g_p, g_{p-1}\right)\right), \vartheta\left(SCB_{bd}\left(g_p, g_{p+1}\right)\right)\right\}. \end{aligned}$$

If $M_{\mathfrak{L},\theta}\left(g_p, g_{p-1}\right) = \vartheta\left(SCB_{bd}\left(g_p, g_{p+1}\right)\right)$, then the inequality (3.8) turns into $\vartheta\left(SCB_{bd}\left(g_{p+1}, g_p\right)\right) \leq \vartheta\left(SCB_{bd}\left(g_p, g_{p+1}\right)\right)^t \Leftrightarrow \ln\left(\vartheta\left(SCB_{bd}\left(g_{p+1}, g_p\right)\right)\right) \leq t\ln\left(\vartheta\left(SCB_{bd}\left(g_{p+1}, g_p\right)\right)\right)$, which is a contradiction (because $t < 1$). Hence, we have $M_{\mathfrak{L},\vartheta}\left(g_p, g_{p-1}\right) = \vartheta\left(SCB_{bd}\left(g_{p-1}, g_p\right)\right)$. From (3.8), it follows that

$$\vartheta\left(SCB_{bd}\left(g_p, g_{p+1}\right)\right) \leq \left[\vartheta\left(SCB_{bd}\left(g_{p-1}, g_p\right)\right)\right]^t.$$

Repeatedly, we discover that

$$\vartheta\left(SCB_{bd}\left(g_p, g_{p+1}\right)\right) \leq \left[\vartheta\left(SCB_{bd}\left(g_0, g_1\right)\right)\right]^{t^p}.$$

Following this insight, we deduce that $\{g_p\}$ in $\mathbb{X}$ is a Cauchy sequence by tracing the relevant lines in the Theorem 3.1 proof. In conclusion, the sequence $\{g_p\}$ in $\mathbb{X}$ is Cauchy. Because $(\mathbb{X}, SCB_{bd})$ is a complete SCB_{bd}, there is a certain point μ in $\mathbb{X}$ in a way that $\{g_p\}$ converges to μ. Without loss of generality, we presume that $\mathfrak{L}^p g \neq \mu$ for all p (or p tends to infinity). Assume that $SCB_{bd}(\mu, \mathfrak{L}\mu) > 0$. Using (3.7), we obtain

$$\vartheta\left(SCB_{bd}\left(\mathfrak{L}g_p, \mathfrak{L}\mu\right)\right) \leq \left[M_{\mathfrak{L},\vartheta}\left(g_p, \mu\right)\right]^t, \tag{3.9}$$

for all $g, h \in \mathbb{X}$, in which

$$M_{\mathfrak{L},\vartheta}\left(g_p, \mu\right) := \max\left\{\vartheta\left(SCB_{bd}\left(g_p, \mu\right)\right), \vartheta\left(SCB_{bd}\left(g_p, \mathfrak{L}g_p\right)\right), \vartheta\left(SCB_{bd}(\mu, \mathfrak{L}\mu)\right)\right\}.$$

Taking $p \to \infty$ in the preceding inequality, we get

$$\vartheta(SCB_{bd}(\mu, \mathfrak{L}\mu)) \le [\vartheta(SCB_{bd}(\mu, \mathfrak{L}\mu))]^t < \vartheta(SCB_{bd}(\mu, \mathfrak{L}\mu)),$$

which is a contradiction. Therefore, $\mathfrak{L}\mu = \mu$. Hence, $\mathfrak{L}$ has a fixed point in $\mathbb{X}$. Assume that $\mu \neq \xi$ are distinct fixed points of $\mathfrak{L}$. Afterwards, obviously

$$SCB_{bd}(\mu, \xi) = SCB_{bd}(\mathfrak{L}\mu, L\xi) \neq 0.$$

Applying condition (3.10) now, we obtain

$$\begin{aligned} 1 &< \vartheta(SCB_{bd}(\mu, \xi)) \\ &= \vartheta(SCB_{bd}(\mathfrak{L}\mu, \mathfrak{L}\xi)) \\ &\le [\max\{\vartheta(SCB_{bd}(\mu, \xi)), \vartheta(SCB_{bd}(\mu, \mathfrak{L}\mu)), \vartheta(SCB_{bd}(\xi, \mathfrak{L}\xi))\}]^t \\ &< \vartheta(SCB_{bd}(\mu, \xi)), \end{aligned}$$

which is clearly a contradiction. Consequently, we obtain $\xi = \mu$. As a result, $\mathfrak{L}$ has just one fixed point in $\mathbb{X}$. □

Definition 3.3. *Consider the SCB_{bd} space, $(\mathbb{X}, SCB_{bd})$, and a self-mapping $\mathfrak{L} : \mathbb{X} \to \mathbb{X}$. Hence, $\mathfrak{L}$ is considered an interpolative-θ_{BC} when there exists a function $\vartheta \in \theta$ such that $t_1 + t_2 + t_3 < 1$, where t_1, t_2, t_3 are positive real numbers satisfying*

$$\vartheta(SCB_{bd}(\mathfrak{L}g, \mathfrak{L}h)) \le [\vartheta(SCB_{bd}(g, h))]^{t_1} [\vartheta(SCB_{bd}(g, \mathfrak{L}g))]^{t_2} [\vartheta(SCB_{bd}(h, \mathfrak{L}h))]^{t_3}, \tag{3.10}$$

for all $g, h \in \mathbb{X}$, where $\limsup_{p,q\to\infty} \omega\left(g_p, g_q\right) < K$, $K \ge 1$, and $g_p = \mathfrak{L}^p g_0$ for $g_0 \in \mathbb{X}$ and $t \in (0, 1)$.

Theorem 3.3 (Fixed point for interpolative-θ_{BC})**.** *Let $(\mathbb{X}, SCB_{bd})$ be a complete and continuous function SCB_{bd}. If $\mathfrak{L} : \mathbb{X} \to \mathbb{X}$ is an interpolative-θ_{BC}, then in $\mathbb{X}$, $\mathfrak{L}$ has a single fixed point.*

We omit this proof because

$$[\vartheta(SCB_{bd}(g, h))]^{t_1} [\vartheta(SCB_{bd}(g, \mathfrak{L}g))]^{t_2} [\vartheta(SCB_{bd}(h, \mathfrak{L}h))]^{t_3} \le [M_{\vartheta,\mathfrak{L}}(g, h)]^{t_1+t_2+t_3}.$$

Then, selecting $t := t_1 + t_2 + t_3 < 1$ is sufficient in Theorem 3.1 to sum up the preceding theorem.

4. Application to differential equations

Consider the following system of differential equations:

$$\begin{cases} \psi''''(\chi) = g(\chi, \psi(\chi), \psi'(\chi), \psi''(\chi), \psi'''(\chi)), \\ \psi(0) = \psi'(0) = \psi''(1) = \psi'''(1) = 0;\ \chi \in [0, 1], \end{cases} \tag{4.1}$$

so that g is a continuous function defined as $g : [0, 1] \times \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$.

The focus of the study is on the boundary value problem (BVP), a fundamental concept in mathematical analysis, particularly when applied to the modeling of complex physical phenomena. In this case, the problem is situated within the context of elastic beam deformations, with an emphasis on the equilibrium configuration. Specifically, the problem models scenarios where one end of the beam is free to move, while the other is fixed in place. This setup is commonly referred to as the cantilever beam problem in the field of mechanics, underlining its wide applicability and importance in both theoretical and applied mathematics.

The solutions to this type of BVP play a crucial role in numerous applications, ranging from structural engineering to mathematical physics. As such, determining the existence and uniqueness of solutions is of central importance. To achieve this, we apply the fixed point technique, a well-established method in mathematical analysis that is particularly effective for solving BVPs of this nature.

Here we examine if a boundary value problem with a differential equation of fourth order has a solution. On the interval $[0, 1]$, $\mathbb{X} = \mathcal{F}[0, 1]$ is the notation for the space of continuous bounded functions. Within this space, we introduce the SCB_{bd} as a means to measure distances between functions. The metric is denoted by

$$SCB_{bd}(L(v), g(v)) = \max_{v\in\mathbb{X}} |L(v) - g(v)|^2,$$

which provides the necessary structure for analyzing the convergence of sequences of functions in $\mathbb{X}$.

Furthermore, we define a mapping $\omega : \mathbb{X}^4 \to [1, \infty)$, where $\omega(u_1, u_2, u_3, u_4) = 5u_1 + 5u_2 + 3$. This mapping introduces an additional layer of complexity to our exploration, enabling us to further study the behavior of solutions within the context of the BVP.

Through this approach, we aim to demonstrate the existence of a solution to the boundary value problem, using the fixed point theorem and the introduced metrics to rigorously establish the conditions under which solutions can be found. Having established the foundational framework, we now proceed to reformulate the fourth-order ordinary differential equation (BVP) as an integral equation. The integral form in which the BVP is presented is below:

$$\psi(\chi) = \int_0^1 \mathcal{G}(\chi, v) g\left(v, \psi(v), \psi'(v), \psi''(v), \psi'''(v)\right) dv, \quad \psi \in \mathcal{F}[0, 1].$$

In this expression, $\mathcal{G}(\chi, v)$ represents Green's function associated with the homogeneous linear problem:

$$\psi''''(\chi) = 0, \quad \psi(0) = \psi'(0) = \psi''(1) = \psi'''(1) = 0.$$

This function provides a key component for solving the boundary value problem and offers valuable insight into the underlying structure of the equation. The use of Green's function allows for a detailed understanding of the interactions between the boundary conditions and the solution, facilitating the analysis of the problem's characteristics.

$$\mathcal{G}(\chi, v) = \begin{cases} \frac{1}{6}\chi^2(3v - \chi), & 0 \le \chi \le v \le 1, \\ \frac{1}{6}v^2(3\chi - v), & 0 \le v \le \chi \le 1. \end{cases} \tag{4.2}$$

The subsequently properties of $\mathcal{G}(\chi, v)$ could be simply verified from (3.2).

$$\frac{1}{3}\chi^2 v^2 \le \mathcal{G}(\chi, v) \le \frac{1}{2}\chi^2 \left(\text{or } \frac{1}{2}v^2 \right), \ \chi, v \in [0, 1].$$

Theorem 4.1. *Assume that the subsequent constraints are satisfied.*
(1) The mapping $g : [0,1] \times \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ *is continuous.*
(2) For every $\psi, y \in \mathbb{X} = \mathcal{F}[0,1]$*, there is a* $\sigma \in [1,\infty)$ *such that the below condition is satisfied.*

$$\left|g\left(v,\psi,\psi',\psi'',\psi'''\right) - g\left(v,y,y',y'',y'''\right)\right| \leq \sqrt{20}e^{\frac{-\sigma}{2}}|\psi(v) - y(v)|, \quad v \in [0,1].$$

(3) There exists $\psi_0 \in \mathbb{X}$ *where for every* $\chi \in [0,1]$*, we deduce*

$$\psi_0(\chi) \leq \int_0^1 \mathcal{G}(\chi, v) g\left(v, \psi_0(v), \psi_0'(v), \psi_0''(v), \psi_0'''(v)\right) dv.$$

Hence, the BVP problem contains a solution in $\mathbb{X}$*.*

Proof. Assume the self-mapping $L : \mathbb{X} \to \mathbb{X}$ is defined as

$$L(\psi(\chi)) = \int_0^1 \mathcal{G}(\chi, v) g\left(v, \psi(v), \psi'(v), \psi''(v), \psi'''(v)\right) dv.$$

Then $\psi(\chi) = L(\psi(\chi))$, which implies that the BVP has only one solution.

Consider,

$$\begin{aligned}
&|L(\psi)(\chi) - L(y)(\chi)|^2 \\
=&\left|\int_0^1 \mathcal{G}(\chi, v) g\left(v, \psi(v), \psi'(v), \psi''(v), \psi'''\right) dv - \int_0^1 \mathcal{G}(\chi, v) g\left(v, y(v), y'(v), y''(v), y'''(v)\right) dv\right|^2 \\
\leq&\int_0^1 (\mathcal{G}(\chi, v))^2 \left|g\left(v, \psi(v), \psi'(v), \psi''(v), \psi'''\right) - g\left(v, y(v), y'(v), y''(v), y'''(v)\right)\right|^2 dv \\
\leq&\int_0^1 \frac{1}{4} v^4 20 e^{-\sigma} |\psi(v) - y(v)|^2 dv \\
\leq&20 e^{-\sigma} SCB_{bd}(\psi, y) \int_0^1 \frac{1}{4} v^4 dv \\
\leq&20 e^{-\sigma} SCB_{bd}(\psi, y) \frac{1}{20} \\
=&e^{-\sigma} SCB_{bd}(\psi, y),
\end{aligned}$$

where this deduces to

$$\begin{aligned}
SCB_{bd}(L(\psi), L(y)) &\leq e^{-\sigma} SCB_{bd}(\psi, y), \\
\sqrt{SCB_{bd}(L(\psi), L(y))} &\leq \sqrt{e^{-\sigma} SCB_{bd}(\psi, y)}, \\
e^{\sqrt{SCB_{bd}(L(\psi), L(y))}} &\leq \left(e^{\sqrt{SCB_{bd}(\psi, y)}}\right)^{\sqrt{e^{-\sigma}}},
\end{aligned}$$

where

$$e^{-\sigma} < 1 \text{ as } \sigma \geq 1.$$

Thus, $e^{\sqrt{SCB_{bd}(L(\psi), L(y))}} \leq \left(e^{\sqrt{SCB_{bd}(\psi, y)}}\right)^{\sqrt{r}}$ with $r = \sqrt{e^{-\sigma}}$ which gives

$$\vartheta\left(SCB_{bd}(L\psi, Ly)\right) \leq \left[\vartheta\left(SCB_{bd}(\psi, y)\right)\right]^r \text{ where } \vartheta(t) = e^{\sqrt{\chi}}.$$

L has a fixed point as all conditions of Theorem 3.1 are met. Therefore, the solution to the BVP is found in $\mathbb{X}$. □

5. Conclusions

As an extension of strong-controlled metric-type spaces and Branciari spaces, we introduced the notion of strong controlled b-Branciari metric type space in this work. Next, in the framework of strong-controlled-b-Branciari metric-type space, we proved several fixed point theorems concerning the θ-contraction, Ciric-Reich-Rus-type θ-Brainciari contraction, and interpolative-θ-Branciari contraction. Additionally, we provided numerous examples to highlight our conclusions.

Author contributions

Dania Santina, Wan Ainun Mior Othman, Kok Bin Wong, and Nabil Mlaiki: Conceptualization, Methodology, Validation, Writing-original draft, Writing-review and editing. All authors of this article have contributed equally. All authors read and approved the final manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

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