Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(11): 28077–28099.
DOI:10.3934/math.20251234
Received: 16 September 2025
Revised: 10 November 2025
Accepted: 24 November 2025
Published: 28 November 2025

Research article

On a second-order system of difference equations: expressions and behavior of the solutions

Nouressadat Touafek[1,*] **and Jawharah Ghuwayzi AL-Juaid**[2]

[1] LMAM Laboratory, Faculty of Exact Sciences and Informatics, University of Jijel, Jijel 18000, Algeria

[2] Department of Mathematics and Statistics, College of Science, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia

* **Correspondence:** Email: touafek@univ-jijel.dz.

Abstract: We explicitly solve the following second-order system of difference equations:

$$x_{n+1} = \frac{a x_n y_{n-1}}{y_n - \beta y_{n-1} - \gamma} + b x_n + c,\ y_{n+1} = \frac{\alpha y_n x_{n-1}}{x_n - b x_{n-1} - c} + \beta y_n + \gamma,$$

where $n \in \mathbb{N}_0$, a, α, and the initial conditions x_{-1}, y_{-1}, x_0, and y_0 are nonzero real numbers, while the remaining parameters b, c, β, and γ are real numbers. A detailed analysis of the solutions of our system when $\alpha = a$ with respect to the form, the periodicity and the limiting behavior is presented. To support and illustrate our theoretical results, numerical examples are provided. Our study, considerably generalizes some existing results in the literature.

Keywords: systems of difference equations; well-defined solutions; explicit formulas of solutions; periodic solutions; limiting behavior of solutions
Mathematics Subject Classification: 39A05, 39A10, 39A20, 39A30

1. Introduction and some preliminary results

The search for closed formulas of the solutions of difference equations and systems, especially for those which are nonlinear, is a topic of a major interest for many researchers, and this can be justified by the fact that such type of equations and systems have the advantage of providing explicit formulas of the solutions, and this permits us to understand their behavior; for example, one can consult references [1–3].

When the difference equation or the system under study models a real phenomena, then knowing the form of the solutions, allow us, among other to achieve the following:

- Determine the conditions under which the phenomena presents repeated patterns so we can predict its behavior;
- Study the limiting behavior of the solutions; this is crucial because it indicates if the phenomena grow with oscillation, divergence, or convergence to the equilibrium point.

In the literature, we find many studies that use difference equations to conduct in-depth analyzes of real models from different branches of modern science. For example, in [4], using the forward Euler scheme, the authors studied a discrete-time Leslie-Gower type predator-prey model of ratio-dependent functional response and Michalis-Menten function prey harvesting. In [5], using a non-standard finite difference scheme, the authors investigated the stability and carried out a bifurcation analysis of a chemical reaction system. For readers interested in more applications of difference equations in studying concrete discrete models, we refer to the contributions of [6, 7].

Consider the following second-order rational system of difference equations:

$$x_{n+1} = \frac{a x_n y_{n-1}}{y_n - \beta y_{n-1} - \gamma} + b x_n + c,\ y_{n+1} = \frac{\alpha y_n x_{n-1}}{x_n - b x_{n-1} - c} + \beta y_n + \gamma, \tag{1.1}$$

where $n \in \mathbb{N}_0$, a, α, and the initial values x_{-1}, y_{-1}, x_0, and y_0 are nonzero real numbers, while the remaining parameters b, c, β, and γ are real numbers.

Our aim is to solve system (1.1) explicitly, and then, by using the obtained formulas, we derive explicit forms for the solutions of the system

$$x_{n+1} = \frac{a x_n y_{n-1}}{y_n - \beta y_{n-1} - \gamma} + b x_n + c,\ y_{n+1} = \frac{a y_n x_{n-1}}{x_n - b x_{n-1} - c} + \beta y_n + \gamma, \tag{1.2}$$

when $\alpha = a$. We then provide detailed results on the periodicity and limiting behavior for the solutions of (1.2).

Taking $b = \beta = 0$ in (1.1), we get the following system:

$$x_{n+1} = \frac{a x_n y_{n-1}}{y_n - \gamma} + c,\ y_{n+1} = \frac{\alpha y_n x_{n-1}}{x_n - c} + \gamma,\ n \in \mathbb{N}_0. \tag{1.3}$$

Haddad et al. [8] showed the solvability of (1.3) and investigated the periodic nature and the limiting behavior of its solutions. Therefore, (1.1) is a generalization of (1.3). The study in [8] was motivated by the following difference equation:

$$x_{n+1} = \frac{x_n x_{n-1}}{x_n - 1} + 1,\ n \in \mathbb{N}_0,$$

which was solved in [9]. Later, following the methods in [8], Yazlik et al. [10] extended (1.3) to

$$x_{n+1} = \frac{a_n x_{n-k+1} y_{n-k}}{y_n - \gamma_n} + c_{n+1},\ y_{n+1} = \frac{\alpha_n y_{n-k+1} x_{n-k}}{x_n - c_n} + \gamma_{n+1},\ n, \in \mathbb{N}_0, \tag{1.4}$$

where $k \in \mathbb{N}$, $(a_n)_{n\in\mathbb{N}_0}$, $(\alpha_n)_{n\in\mathbb{N}_0}$, $(c_n)_{n\in\mathbb{N}_0}$, $(\gamma_n)_{n\in\mathbb{N}_0}$ are two periodic sequences of real numbers. More works related to the above-mentioned contributions can be found in [11, 12].

We note that if an $n_0 \in \mathbb{N}_0$ exists such that the denominator

$$y_{n_0} - \beta y_{n_0-1} - \gamma = 0,$$

and,

$$x_{n_0} - bx_{n_0-1} - c = 0,$$

then x_{n_0+1} is not defined, and respectively, y_{n_0+1} is not defined. Based on this, we introduce the following definition:

Definition 1.1. *We say that a solution* $\{(x_n, y_n)\}_{n=-1}^{+\infty}$ *of the system* (1.1) *is well-defined if*

$$(x_n - bx_{n-1} - c)(y_n - \beta y_{n-1} - \gamma) \neq 0,\ n \in \mathbb{N}_0.$$

From here on, we admit that a solution of (1.1) is a well-defined solution. In the following result, we prove $x_n.y_n \neq 0,\ n = -1, 0, ...$, and this explains our choice for the initial values to be nonzero.

Lemma 1.1. *Every solution* $\{(x_n, y_n)\}_{n=-1}^{+\infty}$ *of system* (1.1)*, satisfies*

$$x_n.y_n \neq 0,\ n \geq -1.$$

Proof. Assume that $n_0 \geq -1$ exists such that $x_{n_0} = 0$. From (1.1), we get

$$y_{n_0+2} = \frac{\alpha y_{n_{0+1}} x_{n_0}}{x_{n_0+1} - bx_{n_0} - c} + \beta y_{n_0+1} + \gamma = \beta y_{n_0+1} + \gamma.$$

We have

$$x_{n_0+3} = \frac{ax_{n_0+2}y_{n_0+1}}{y_{n_0+2} - \beta y_{n_0+1} - \gamma} + bx_{n_0+2} + c = \frac{ax_{n_0+2}y_{n_0+1}}{\beta y_{n_0+1} + \gamma - \beta y_{n_0+1} - \gamma} + bx_{n_0+2} + c = \frac{ax_{n_0+2}y_{n_0+1}}{0} + bx_{n_0+2} + c.$$

Therefore, x_{n_0+3} is not defined. Similarly, $n_0 \geq -1$ exists such that $y_{n_0} = 0$, we get y_{n_0+3} is not defined. □

Let us recall the following known result.

Lemma 1.2. *[13] Consider the following first-order difference equation:*

$$s_{n+1} = A_n s_n + B_n,\ n \in \mathbb{N}_0,$$

where s_0 *and the terms of the sequences* $(A_n)_{n\in\mathbb{N}_0}$ *and* $(B_n)_{n\in\mathbb{N}_0}$ *are real numbers. In this case*

$$s_n = \left(\prod_{i=0}^{n-1} A_i\right) s_0 + \sum_{j=0}^{n-1} \left(\prod_{i=j+1}^{n-1} A_i\right) B_j,\ n \in \mathbb{N}_0,$$

with the convention

$$\prod_{i=k}^{l} (e_i) = 1,\ \sum_{i=k}^{l} (e_i) = 0,\ l < k.$$

If $A_n = A,\ B_n = B$*, then for* $n \in \mathbb{N}_0$*, we have*

$$s_n = s_0 A^n + B\left(\frac{A^n - 1}{A - 1}\right),\ A \neq 1,\ s_n = s_0 + Bn,\ A = 1.$$

In the following, we justify the choice of the parameters a and α to be non-zero.

Lemma 1.3. *(1) If $a = \alpha = 0$, the system* (1.1) *will be*

$$x_{n+1} = bx_n + c,\ y_{n+1} = \beta y_n + \gamma,\ n \in \mathbb{N}_0,$$

and we have

$$x_n = x_0 b^n + c\left(\frac{b^n - 1}{b - 1}\right),\ b \neq 1, x_n = x_0 + cn,\ b = 1,\ n \in \mathbb{N}_0,$$

$$y_n = y_0 \beta^n + \gamma\left(\frac{\beta^n - 1}{\beta - 1}\right),\ \beta \neq 1, y_n = y_0 + \gamma n,\ b = 1,\ n \in \mathbb{N}_0.$$

(2) If $a = 0$, $\alpha \neq 0$, the system (1.1) *will be*

$$x_{n+1} = bx_n + c,\ y_{n+1} = \frac{\alpha y_n x_{n-1}}{x_n - bx_{n-1} - c} + \beta y_n + \gamma,\ n \in \mathbb{N}_0,$$

and we get

$$x_n = x_0 b^n + c\left(\frac{b^n - 1}{b - 1}\right),\ b \neq 1, x_n = x_0 + cn,\ b = 1,\ n \in \mathbb{N}_0.$$

However, the y_n-component of the solution is not defined for $n \geq 2$.

(3) If $a \neq 0$, $\alpha = 0$, the system (1.1) *will be*

$$x_{n+1} = \frac{ax_n y_{n-1}}{y_n - \beta y_{n-1} - \gamma} + bx_n + c,\ y_{n+1} = \beta y_n + \gamma,\ n \in \mathbb{N}_0,$$

and we get

$$y_n = y_0 \beta^n + \gamma\left(\frac{\beta^n - 1}{\beta - 1}\right),\ \beta \neq 1, y_n = y_0 + \gamma n,\ \beta = 1,\ n \in \mathbb{N}_0.$$

However, the x_n-component of the solution is not defined for $n \geq 2$.

Proof. (1) If $a = \alpha = 0$, then from (1.1), we get

$$x_{n+1} = bx_n + c,\ y_{n+1} = \beta y_n + \gamma,\ n \in \mathbb{N}_0.$$

Using Lemma 1.2, for $n \in \mathbb{N}_0$,we get

$$x_n = x_0 b^n + c\left(\frac{b^n - 1}{b - 1}\right),\ b \neq 1, x_n = x_0 + cn,\ b = 1,$$

$$y_n = y_0 \beta^n + \gamma\left(\frac{\beta^n - 1}{\beta - 1}\right),\ \beta \neq 1, y_n = y_0 + \gamma n,\ b = 1.$$

(2) If $a = 0$, $\alpha \neq 0$, then from (1.1), we have

$$x_{n+1} = bx_n + c,\ y_{n+1} = \frac{\alpha y_n x_{n-1}}{x_n - bx_{n-1} - c} + \beta y_n + \gamma,\ n \in \mathbb{N}_0.$$

From Lemma 1.2, for $n \in \mathbb{N}_0$, we get

$$x_n = x_0 b^n + c\left(\frac{b^n - 1}{b - 1}\right),\ b \neq 1, x_n = x_0 + cn,\ b = 1.$$

Thus the x_n-component of the solution is defined. We will show that the y_n-component of the solution is not defined for $n \geq 2$.
If we let $n_0 \geq 1$, we find the denominator

$$x_{n_0} - bx_{n_0-1} - c = 0,$$

so

$$y_{n_0+1} = \frac{ay_{n_0}x_{n_0-1}}{0} + \beta y_{n_0} + \gamma$$

will be not defined. In fact, we not the following.

- If $b = 1$

$$x_{n_0} - bx_{n_0-1} - c = x_0 + cn_0 - (x_0 + c(n_0 - 1)) - c = 0.$$

- If $b \neq 1$

$$x_{n_0} - bx_{n_0-1} - c = x_0 b^{n_0} + c\left(\frac{b^{n_0} - 1}{b - 1}\right) - b\left(x_0 b^{n_0-1} + c\left(\frac{b^{n_0-1} - 1}{b - 1}\right)\right) - c = 0.$$

(3) If $a \neq 0$ and $\alpha = 0$, then from (1.1), we get

$$x_{n+1} = \frac{ax_n y_{n-1}}{y_n - \beta y_{n-1} - \gamma} + bx_n + c,\ y_{n+1} = \beta y_n + \gamma,\ n \in \mathbb{N}_0.$$

From Lemma 1.2, we get, for $n \in \mathbb{N}_0$

$$y_n = y_0 \beta^n + \gamma\left(\frac{\beta^n - 1}{\beta - 1}\right), \beta \neq 1, y_n = y_0 + \gamma n, \beta = 1.$$

Thus the y_n-component of the solution is defined. We will show that the x_n-component of the solution is not defined for $n \geq 2$.
Let $n_0 \geq 1$, we find the denominator

$$y_{n_0} - \beta y_{n_0-1} - \gamma = 0,$$

so

$$x_{n_0+1} = \frac{\alpha x_{n_0} y_{n_0-1}}{0} + bx_{n_0} + c$$

will be not defined. In fact, we note the following

- If $\beta = 1$

$$y_{n_0} - \beta y_{n_0-1} - \gamma = y_0 + \gamma n_0 - (y_0 + \gamma(n_0 - 1)) - \gamma = 0.$$

- If $\beta \neq 1$

$$y_{n_0} - \beta y_{n_0-1} - \gamma = y_0 \beta^{n_0} + \gamma\left(\frac{\beta^{n_0} - 1}{\beta - 1}\right) - \beta\left(y_0 \beta^{n_0-1} + \gamma\left(\frac{\beta^{n_0-1} - 1}{\beta - 1}\right)\right) - \gamma = 0.$$

□

In the rest of this work, we consider only the case $a\alpha \neq 0$.

2. Expressions of the solutions for the system (1.1)

In this part, we show how to solve explicitly the system (1.1). For this, we need first to transform our system to an equivalent solvable system, and then we use the formulas of the solutions the equivalent system to obtain some first-order linear difference equations which are solvable using Lemma 1.2.

From (1.1), we obtain

$$\frac{x_{n+1}-bx_n-c}{x_n}=\frac{ay_{n-1}}{y_n-\beta y_{n-1}-\gamma},\ \frac{y_{n+1}-\beta y_n-\gamma}{y_n}=\frac{\alpha x_{n-1}}{x_n-bx_{n-1}-c}. \tag{2.1}$$

Let

$$u_n=\frac{x_n-bx_{n-1}-c}{x_{n-1}},\ v_n=\frac{y_n-\beta y_{n-1}-\gamma}{y_{n-1}},\ n\in\mathbb{N}_0. \tag{2.2}$$

From (2.1), we obtain

$$u_{n+1}=\frac{a}{v_n},\ v_{n+1}=\frac{\alpha}{u_n},\ n\in\mathbb{N}_0. \tag{2.3}$$

It is not hard to see from (2.3) that

$$u_{2n}=u_0\left(\frac{a}{\alpha}\right)^n,\ u_{2n+1}=\frac{a}{v_0}\left(\frac{a}{\alpha}\right)^n,\ v_{2n}=v_0\left(\frac{\alpha}{a}\right)^n,\ v_{2n+1}=\frac{\alpha}{u_0}\left(\frac{\alpha}{a}\right)^n,\ n\in\mathbb{N}_0. \tag{2.4}$$

Using (2.2), we get

$$x_n=(b+u_n)x_{n-1}+c,\ n\in\mathbb{N}_0, \tag{2.5}$$

$$y_n=(\beta+v_n)y_{n-1}+\gamma,\ n\in\mathbb{N}_0. \tag{2.6}$$

Depending on the parity of n, from (2.5) and (2.6), we obtain

$$x_{2n}=(b+u_{2n})x_{2n-1}+c,\ x_{2n+1}=(b+u_{2n+1})x_{2n}+c,\ n\in\mathbb{N}_0, \tag{2.7}$$

$$y_{2n}=(\beta+v_{2n})y_{2n-1}+\gamma,\ y_{2n+1}=(\beta+v_{2n+1})y_{2n}+\gamma,\ n\in\mathbb{N}_0. \tag{2.8}$$

Now replacing by (2.4) in (2.7) and (2.8), we get

$$x_{2n}=(b+u_0\left(\frac{a}{\alpha}\right)^n)x_{2n-1}+c,\ n\in\mathbb{N}_0, \tag{2.9}$$

$$x_{2n+1}=(b+\frac{a}{v_0}\left(\frac{a}{\alpha}\right)^n)x_{2n}+c,\ n\in\mathbb{N}_0, \tag{2.10}$$

$$y_{2n}=(\beta+v_0\left(\frac{\alpha}{a}\right)^n)y_{2n-1}+\gamma,\ n\in\mathbb{N}_0, \tag{2.11}$$

$$y_{2n+1}=(\beta+\frac{\alpha}{u_0}\left(\frac{\alpha}{a}\right)^n)y_{2n}+\gamma,\ n\in\mathbb{N}_0. \tag{2.12}$$

It follows from (2.9)–(2.12) that

$$x_{2n+2}=\left(b+\frac{a}{v_0}\left(\frac{a}{\alpha}\right)^n\right)\left(b+u_0\left(\frac{a}{\alpha}\right)^{n+1}\right)x_{2n}+c\left(b+u_0\left(\frac{a}{\alpha}\right)^{n+1}+1\right),\ n\in\mathbb{N}_0, \tag{2.13}$$

$$x_{2n+1}=\left(b+\frac{a}{v_0}\left(\frac{a}{\alpha}\right)^n\right)\left(b+u_0\left(\frac{a}{\alpha}\right)^n\right)x_{2n-1}+c\left(b+\frac{a}{v_0}\left(\frac{a}{\alpha}\right)^n+1\right),\ n\in\mathbb{N}_0, \tag{2.14}$$

$$y_{2n+2}=\left(\beta+\frac{\alpha}{u_0}\left(\frac{\alpha}{a}\right)^n\right)\left(\beta+v_0\left(\frac{\alpha}{a}\right)^{n+1}\right)y_{2n}+\gamma\left(\beta+v_0\left(\frac{\alpha}{a}\right)^{n+1}+1\right),\ n\in\mathbb{N}_0,\tag{2.15}$$

$$y_{2n+1}=\left(\beta+\frac{\alpha}{u_0}\left(\frac{\alpha}{a}\right)^n\right)\left(\beta+v_0\left(\frac{\alpha}{a}\right)^{n}\right)y_{2n-1}+\gamma\left(\beta+\frac{\alpha}{u_0}\left(\frac{\alpha}{a}\right)^{n}+1\right),\ n\in\mathbb{N}_0.\tag{2.16}$$

Putting

$$w_n=x_{2n},\ z_n=x_{2n-1},\ r_n=y_{2n},\ t_n=y_{2n-1},\ n\in\mathbb{N}_0.\tag{2.17}$$

Then Eqs (2.13) and (2.16) become the following non-autonomous first-order linear difference equations:

$$w_{n+1}=\left(b+\frac{a}{v_0}\left(\frac{a}{\alpha}\right)^n\right)\left(b+u_0\left(\frac{a}{\alpha}\right)^{n+1}\right)w_{n}+c\left(b+u_0\left(\frac{a}{\alpha}\right)^{n+1}+1\right),\ n\in\mathbb{N}_0,\tag{2.18}$$

$$z_{n+1}=\left(b+\frac{a}{v_0}\left(\frac{a}{\alpha}\right)^n\right)\left(b+u_0\left(\frac{a}{\alpha}\right)^{n}\right)z_{n-1}+c\left(b+\frac{a}{v_0}\left(\frac{a}{\alpha}\right)^{n}+1\right),\ n\in\mathbb{N}_0,\tag{2.19}$$

$$r_{n+1}=\left(\beta+\frac{\alpha}{u_0}\left(\frac{\alpha}{a}\right)^n\right)\left(\beta+v_0\left(\frac{\alpha}{a}\right)^{n+1}\right)r_{n}+\gamma\left(\beta+v_0\left(\frac{\alpha}{a}\right)^{n+1}+1\right),\ n\in\mathbb{N}_0,\tag{2.20}$$

$$t_{n+1}=\left(\beta+\frac{\alpha}{u_0}\left(\frac{\alpha}{a}\right)^n\right)\left(\beta+v_0\left(\frac{\alpha}{a}\right)^{n}\right)t_{n-1}+\gamma\left(\beta+\frac{\alpha}{u_0}\left(\frac{\alpha}{a}\right)^{n}+1\right),\ n\in\mathbb{N}_0.\tag{2.21}$$

Using Lemma 1.2, we get the solutions of (2.18) and (2.21) as follows:

$$w_n=w_0\prod_{i=0}^{n-1}\left(b+\frac{a}{v_0}\left(\frac{a}{\alpha}\right)^i\right)\left(b+u_0\left(\frac{a}{\alpha}\right)^{i+1}\right)+c\sum_{j=0}^{n-1}\left(\prod_{i=j+1}^{n-1}\left(b+\frac{a}{v_0}\left(\frac{a}{\alpha}\right)^i\right)\left(b+u_0\left(\frac{a}{\alpha}\right)^{i+1}\right)\right)\left(b+u_0\left(\frac{a}{\alpha}\right)^{j+1}+1\right),$$

$$z_n=z_0\prod_{i=0}^{n-1}\left(b+\frac{a}{v_0}\left(\frac{a}{\alpha}\right)^i\right)\left(b+u_0\left(\frac{a}{\alpha}\right)^{i}\right)+c\sum_{j=0}^{n-1}\left(\prod_{i=j+1}^{n-1}\left(b+\frac{a}{v_0}\left(\frac{a}{\alpha}\right)^i\right)\left(b+u_0\left(\frac{a}{\alpha}\right)^{i}\right)\right)\left(b+\frac{a}{v_0}\left(\frac{a}{\alpha}\right)^{j}+1\right),$$

$$r_n=r_0\prod_{i=0}^{n-1}\left(\beta+\frac{\alpha}{u_0}\left(\frac{\alpha}{a}\right)^i\right)\left(\beta+v_0\left(\frac{\alpha}{a}\right)^{i+1}\right)+\gamma\sum_{j=0}^{n-1}\left(\prod_{i=j+1}^{n-1}\left(\beta+\frac{\alpha}{u_0}\left(\frac{\alpha}{a}\right)^i\right)\left(\beta+v_0\left(\frac{\alpha}{a}\right)^{i+1}\right)\right)\left(\beta+v_0\left(\frac{\alpha}{a}\right)^{j+1}+1\right),$$

$$t_n=t_0\prod_{i=0}^{n-1}\left(\beta+\frac{\alpha}{u_0}\left(\frac{\alpha}{a}\right)^i\right)\left(\beta+v_0\left(\frac{\alpha}{a}\right)^{i}\right)+\gamma\sum_{j=0}^{n-1}\left(\prod_{i=j+1}^{n-1}\left(\beta+\frac{\alpha}{u_0}\left(\frac{\alpha}{a}\right)^i\right)\left(\beta+v_0\left(\frac{\alpha}{a}\right)^{i}\right)\right)\left(\beta+\frac{\alpha}{u_0}\left(\frac{\alpha}{a}\right)^{j}+1\right).$$

Now, using the fact that $w_n=x_{2n}$, $z_n=x_{2n-1}$, $r_n=y_{2n}$, and $t_n=y_{2n-1}$, we get

$$x_{2n}=x_0\prod_{i=0}^{n-1}\left(b+\frac{a}{v_0}\left(\frac{a}{\alpha}\right)^i\right)\left(b+u_0\left(\frac{a}{\alpha}\right)^{i+1}\right)+c\sum_{j=0}^{n-1}\left(\prod_{i=j+1}^{n-1}\left(b+\frac{a}{v_0}\left(\frac{a}{\alpha}\right)^i\right)\left(b+u_0\left(\frac{a}{\alpha}\right)^{i+1}\right)\right)\left(b+u_0\left(\frac{a}{\alpha}\right)^{j+1}+1\right),$$

$$x_{2n-1}=x_{-1}\prod_{i=0}^{n-1}\left(b+\frac{a}{v_0}\left(\frac{a}{\alpha}\right)^i\right)\left(b+u_0\left(\frac{a}{\alpha}\right)^i\right)+c\sum_{j=0}^{n-1}\left(\prod_{i=j+1}^{n-1}\left(b+\frac{a}{v_0}\left(\frac{a}{\alpha}\right)^i\right)\left(b+u_0\left(\frac{a}{\alpha}\right)^i\right)\right)\left(b+\frac{a}{v_0}\left(\frac{a}{\alpha}\right)^j+1\right),$$

$$y_{2n}=y_0\prod_{i=0}^{n-1}\left(\beta+\frac{\alpha}{u_0}\left(\frac{\alpha}{a}\right)^i\right)\left(\beta+v_0\left(\frac{\alpha}{a}\right)^{i+1}\right)+\gamma\sum_{j=0}^{n-1}\left(\prod_{i=j+1}^{n-1}\left(\beta+\frac{\alpha}{u_0}\left(\frac{\alpha}{a}\right)^i\right)\left(\beta+v_0\left(\frac{\alpha}{a}\right)^{i+1}\right)\right)\left(\beta+v_0\left(\frac{\alpha}{a}\right)^{j+1}+1\right),$$

$$y_{2n-1}=y_{-1}\prod_{i=0}^{n-1}\left(\beta+\frac{\alpha}{u_0}\left(\frac{\alpha}{a}\right)^i\right)\left(\beta+v_0\left(\frac{\alpha}{a}\right)^i\right)+\gamma\sum_{j=0}^{n-1}\left(\prod_{i=j+1}^{n-1}\left(\beta+\frac{\alpha}{u_0}\left(\frac{\alpha}{a}\right)^i\right)\left(\beta+v_0\left(\frac{\alpha}{a}\right)^i\right)\right)\left(\beta+\frac{\alpha}{u_0}\left(\frac{\alpha}{a}\right)^j+1\right).$$

From the calculations above, we have the following result.

Theorem 2.1. *The form of every solution* $\{(x_n,y_n)\}_{n=-1}^{+\infty}$ *of* (1.1) *is given, for all* $n\in\mathbb{N}_0$, *by the following formulas:*

$$x_{2n}=x_0\prod_{i=0}^{n-1}\left(b+\frac{a}{v_0}\left(\frac{a}{\alpha}\right)^i\right)\left(b+u_0\left(\frac{a}{\alpha}\right)^{i+1}\right)+c\sum_{j=0}^{n-1}\left(\prod_{i=j+1}^{n-1}\left(b+\frac{a}{v_0}\left(\frac{a}{\alpha}\right)^i\right)\left(b+u_0\left(\frac{a}{\alpha}\right)^{i+1}\right)\right)\left(b+u_0\left(\frac{a}{\alpha}\right)^{j+1}+1\right),$$

$$x_{2n-1}=x_{-1}\prod_{i=0}^{n-1}\left(b+\frac{a}{v_0}\left(\frac{a}{\alpha}\right)^i\right)\left(b+u_0\left(\frac{a}{\alpha}\right)^i\right)+c\sum_{j=0}^{n-1}\left(\prod_{i=j+1}^{n-1}\left(b+\frac{a}{v_0}\left(\frac{a}{\alpha}\right)^i\right)\left(b+u_0\left(\frac{a}{\alpha}\right)^i\right)\right)\left(b+\frac{a}{v_0}\left(\frac{a}{\alpha}\right)^j+1\right),$$

$$y_{2n}=y_0\prod_{i=0}^{n-1}\left(\beta+\frac{\alpha}{u_0}\left(\frac{\alpha}{a}\right)^i\right)\left(\beta+v_0\left(\frac{\alpha}{a}\right)^{i+1}\right)+\gamma\sum_{j=0}^{n-1}\left(\prod_{i=j+1}^{n-1}\left(\beta+\frac{\alpha}{u_0}\left(\frac{\alpha}{a}\right)^i\right)\left(\beta+v_0\left(\frac{\alpha}{a}\right)^{i+1}\right)\right)\left(\beta+v_0\left(\frac{\alpha}{a}\right)^{j+1}+1\right),$$

$$y_{2n-1}=y_{-1}\prod_{i=0}^{n-1}\left(\beta+\frac{\alpha}{u_0}\left(\frac{\alpha}{a}\right)^i\right)\left(\beta+v_0\left(\frac{\alpha}{a}\right)^i\right)+\gamma\sum_{j=0}^{n-1}\left(\prod_{i=j+1}^{n-1}\left(\beta+\frac{\alpha}{u_0}\left(\frac{\alpha}{a}\right)^i\right)\left(\beta+v_0\left(\frac{\alpha}{a}\right)^i\right)\right)\left(\beta+\frac{\alpha}{u_0}\left(\frac{\alpha}{a}\right)^j+1\right),$$

where

$$u_0=\frac{x_0-bx_{-1}-c}{x_{-1}},\ v_0=\frac{y_0-\beta y_{-1}-\gamma}{y_{-1}}.$$

3. The case $\alpha = a$

In this case the system (1.1) takes the form

$$x_{n+1}=\frac{ax_ny_{n-1}}{y_n-\beta y_{n-1}-\gamma}+bx_n+c,\ y_{n+1}=\frac{ay_nx_{n-1}}{x_n-bx_{n-1}-c}+\beta y_n+\gamma,\ n\in\mathbb{N}_0. \tag{3.1}$$

We use the formulas of the solutions of (3.1) to analyze their periodicity and limiting behavior.

Theorem 3.1. *For every solution* $\{(x_n,y_n)\}_{n=-1}^{+\infty}$ *of* (3.1), *we have the following.*

- *If* $p \neq 1$

$$x_{2n-1} = x_{-1}p^n + c\left(b + \frac{a}{v_0} + 1\right)\left(\frac{p^n - 1}{p - 1}\right),\ x_{2n} = x_0 p^n + c\,(b + u_0 + 1)\left(\frac{p^n - 1}{p - 1}\right),\ n \in \mathbb{N}.$$

- *If* $p = 1$

$$x_{2n-1} = x_{-1} + c\left(b + \frac{a}{v_0} + 1\right)n,\ x_{2n} = x_0 + c\,(b + u_0 + 1)\,n,\ n \in \mathbb{N}_0.$$

- *If* $q \neq 1$

$$y_{2n-1} = y_{-1}q^n + c\left(\beta + \frac{a}{u_0} + 1\right)\left(\frac{q^n - 1}{q - 1}\right),\ y_{2n} = y_0 q^n + \gamma\,(\beta + v_0 + 1)\left(\frac{q^n - 1}{q - 1}\right),\ n \in \mathbb{N}.$$

- *If* $q = 1$

$$y_{2n-1} = y_{-1} + \gamma\left(\beta + \frac{a}{u_0} + 1\right)n,\ y_{2n} = y_0 + \gamma\,(\beta + v_0 + 1)\,n,\ n \in \mathbb{N}_0.$$

where

$$p = (b + u_0)\left(b + \frac{a}{v_0}\right),\ q = (\beta + v_0)\left(\beta + \frac{a}{u_0}\right),$$

$$u_0 = \frac{x_0 - bx_{-1} - c}{x_{-1}},\ v_0 = \frac{y_0 - \beta y_{-1} - \gamma}{y_{-1}}.$$

Proof. Taking $\alpha = a$ in Theorem 2.1, we get

$$x_{2n} = x_0 \prod_{i=0}^{n-1}\left(b + \frac{a}{v_0}\right)(b + u_0) + c\sum_{j=0}^{n-1}\left(\prod_{i=j+1}^{n-1}\left(b + \frac{a}{v_0}\right)(b + u_0)\right)(b + u_0 + 1)\,, \tag{3.2}$$

$$x_{2n-1} = x_{-1} \prod_{i=0}^{n-1}\left(b + \frac{a}{v_0}\right)(b + u_0) + c\sum_{j=0}^{n-1}\left(\prod_{i=j+1}^{n-1}\left(b + \frac{a}{v_0}\right)(b + u_0)\right)\left(b + \frac{a}{v_0} + 1\right), \tag{3.3}$$

$$y_{2n} = y_0 \prod_{i=0}^{n-1}\left(\beta + \frac{a}{u_0}\right)(\beta + v_0) + \gamma\sum_{j=0}^{n-1}\left(\prod_{i=j+1}^{n-1}\left(\beta + \frac{a}{u_0}\right)(\beta + v_0)\right)(\beta + v_0 + 1)\,, \tag{3.4}$$

$$y_{2n-1} = y_{-1} \prod_{i=0}^{n-1}\left(\beta + \frac{\alpha}{u_0}\right)(\beta + v_0) + \gamma\sum_{j=0}^{n-1}\left(\prod_{i=j+1}^{n-1}\left(\beta + \frac{a}{u_0}\right)(\beta + v_0)\right)\left(\beta + \frac{a}{u_0} + 1\right). \tag{3.5}$$

Let

$$p = (b + u_0)\left(b + \frac{a}{v_0}\right),\ q = (\beta + v_0)\left(\beta + \frac{a}{u_0}\right).$$

From (3.2)–(3.5), for $n \in \mathbb{N}$, we obtain

$$x_{2n-1} = x_{-1}p^n + c\left(b + \frac{a}{v_0} + 1\right)\left(1 + p + ... + p^{n-1}\right), \tag{3.6}$$

$$x_{2n} = x_0 p^n + c\,(b + u_0 + 1)\left(1 + p + ... + p^{n-1}\right), \tag{3.7}$$

$$y_{2n-1} = y_{-1}q^n + \gamma\left(\beta + \frac{a}{u_0} + 1\right)\left(1 + q + ... + q^{n-1}\right), \tag{3.8}$$

$$y_{2n} = y_0 q^n + \gamma\left(\beta + v_0 + 1\right)\left(1 + q + ... + q^{n-1}\right). \tag{3.9}$$

From (3.6) and (3.7), we get

$$x_{2n-1} = x_{-1}p^n + c\left(b + \frac{a}{v_0} + 1\right)\left(\frac{p^n - 1}{p - 1}\right),\ x_{2n} = x_0 p^n + c\left(b + u_0 + 1\right)\left(\frac{p^n - 1}{p - 1}\right),\ n \in \mathbb{N},$$

if $p \neq 1$, and

$$x_{2n} = x_0 + c\left(b + u_0 + 1\right)n,\ x_{2n-1} = x_{-1} + c\left(b + \frac{a}{v_0} + 1\right)n,\ n \in \mathbb{N}, \tag{3.10}$$

if $p = 1$. The formulas in (3.10) are also correct for $n = 0$. Similarly, from (3.8) and (3.9), we obtain

$$y_{2n-1} = y_{-1}q^n + c\left(\beta + \frac{a}{u_0} + 1\right)\left(\frac{q^n - 1}{q - 1}\right),\ y_{2n} = y_0 q^n + \gamma\left(\beta + v_0 + 1\right)\left(\frac{q^n - 1}{q - 1}\right),\ n \in \mathbb{N},$$

if $q \neq 1$, and

$$y_{2n-1} = y_{-1} + \gamma\left(\beta + \frac{a}{u_0} + 1\right)n,\ y_{2n} = y_0 + \gamma\left(\beta + v_0 + 1\right)n,\ n \in \mathbb{N}_0,$$

if $q = 1$. □

The subsequent statements follow directly from Theorem 3.1, so their proofs will be omitted.

Lemma 3.1. *Let* $\{(x_n, y_n)\}_{n=-1}^{+\infty}$ *be a solution of* (3.1).

(1) Assume that $(b + u_0)\left(b + \frac{a}{v_0}\right) = 1$ *and* $(\beta + v_0)\left(\beta + \frac{a}{u_0}\right) = 1$. *The following then hold.*

- *If* $c\gamma\left(b + \frac{a}{v_0} + 1\right)(b + u_0 + 1)\left(\beta + \frac{a}{u_0} + 1\right)(\beta + v_0 + 1) \neq 0$, *we get*

$$\lim_{n\to+\infty} |x_{2n-1}| = \lim_{n\to+\infty} |x_{2n}| = \lim_{n\to+\infty} |y_{2n-1}| = \lim_{n\to+\infty} |y_{2n}| = +\infty,$$

  that is, the solution is unbounded.
- *If* $c\gamma = 0$ *or* $\left(b + \frac{a}{v_0} + 1\right)(b + u_0 + 1)\left(\beta + \frac{a}{u_0} + 1\right)(\beta + v_0 + 1) = 0$, *we get*

$$x_{2n-1} = x_{-1},\ x_{2n} = x_0,\ y_{2n-1} = y_{-1},\ y_{2n} = y_0,\ n \in \mathbb{N}_0,$$

  and the solution will be periodic of period two, provided that $(x_{-1} - x_0)(y_{-1} - y_0) \neq 0$.

(2) Assume that $(b + u_0)\left(b + \frac{a}{v_0}\right) \neq 1$ *and* $(\beta + v_0)\left(\beta + \frac{a}{u_0}\right) \neq 1$. *In this case, if* $x_0 + \frac{c(b+u_0+1)}{p-1}$, $x_{-1} + \frac{c\left(b+\frac{a}{v_0}+1\right)}{p-1}$, $y_0 + \frac{\gamma(\beta+v_0+1)}{q-1}$, $y_{-1} + \frac{\gamma\left(\beta+\frac{a}{u_0}+1\right)}{q-1}$ *are zero, we get*

$$x_{2n-1} = \frac{c\left(b + \frac{a}{v_0} + 1\right)}{1 - p},\ x_{2n} = \frac{c\left(b + u_0 + 1\right)}{1 - p}, n \in \mathbb{N},$$

$$y_{2n-1} = \frac{\gamma\left(\beta + \frac{a}{u_0} + 1\right)}{1 - q},\ y_{2n} = \frac{\gamma\left(\beta + v_0 + 1\right)}{1 - q}, n \in \mathbb{N},$$

that is, the solution is eventually periodic of period two provided that $u_0 v_0 \neq a$.

(3) Assume that $x_0 + \frac{c(b+u_0+1)}{p-1}$, $x_{-1} + \frac{c\left(b+\frac{a}{v_0}+1\right)}{p-1}$, $y_0 + \frac{\gamma(\beta+v_0+1)}{q-1}$, *and* $y_{-1} + \frac{\gamma\left(\beta+\frac{a}{u_0}+1\right)}{q-1}$ *are nonzero. The following then hold.*

- *If* $|\,(b+u_0)\left(b+\frac{a}{v_0}\right)| > 1$, *and* $|\,(\beta+v_0)\left(\beta+\frac{a}{u_0}\right)| > 1$, *we get*

$$\lim_{n\to+\infty}|x_{2n-1}| = \lim_{n\to+\infty}|x_{2n}| = \lim_{n\to+\infty}|y_{2n-1}| = \lim_{n\to+\infty}|y_{2n}| = +\infty,$$

that is, the solution is unbounded.

- *If* $|\,(b+u_0)\left(b+\frac{a}{v_0}\right)| < 1$ *and* $|\,(\beta+v_0)\left(\beta+\frac{a}{u_0}\right)| < 1$, *we get*

$$\lim_{n\to+\infty} x_{2n-1} = \frac{c\left(b+\frac{a}{v_0}+1\right)}{1-p}, \lim_{n\to+\infty} x_{2n} = \frac{c\,(b+u_0+1)}{1-p}$$

$$\lim_{n\to+\infty} y_{2n-1} = \frac{\gamma\left(\beta+\frac{a}{u_0}+1\right)}{1-q}, \lim_{n\to+\infty} y_{2n} = \frac{\gamma\,(\beta+v_0+1)}{1-q},$$

that is, the solution converge to a two-period solution.

In what follows, we look for the form and the periodicity of the solutions of some specific cases of the system (3.1).

Theorem 3.2. *Assume that*

$$(b+u_0)\left(b+\frac{a}{v_0}\right) = (\beta+v_0)\left(\beta+\frac{a}{u_0}\right) = -1.$$

Thus, every solution of (3.1) *takes the form*

$$x_{4n-1} = x_{-1},\ x_{4n} = x_0,\ x_{4n+1} = -x_{-1} + c\left(b+\frac{a}{v_0}+1\right),\ x_{4n+2} = -x_0 + c\,(b+u_0+1)\,,\ n \in \mathbb{N}_0, \quad (3.11)$$

$$y_{4n-1} = y_{-1},\ y_{4n} = y_0,\ y_{4n+1} = -y_{-1} + \gamma\left(\beta+\frac{a}{u_0}+1\right),\ y_{4n+2} = -y_0 + \gamma\,(\beta+v_0+1)\,,\ n \in \mathbb{N}_0. \quad (3.12)$$

Moreover, we have the following statements.

(1) Assume that

$$x_{-1},\ x_0,\ -x_{-1} + c\left(b+\frac{a}{v_0}+1\right),\ -x_0 + c\,(b+u_0+1)$$

are pairwise different, and

$$y_{-1},\ y_0,\ -y_{-1} + \gamma\left(\beta+\frac{a}{u_0}+1\right),\ -y_0 + \gamma\,(\beta+v_0+1)$$

are pairwise different, then the solution is periodic of period four.

(2) Assume that

$$x_{-1} = \frac{c}{2}\left(b+\frac{a}{v_0}+1\right),\ x_0 = \frac{c}{2}\,(b+u_0+1)\,,\ y_{-1} = \frac{\gamma}{2}\left(\beta+\frac{a}{u_0}+1\right),\ y_0 = \frac{\gamma}{2}\,(\beta+v_0+1)\,,$$

then the solution takes the form

$$x_{2n-1} = x_{-1},\ x_{2n} = x_0,\ y_{2n-1} = y_{-1},\ y_{2n} = y_0,\ n \in \mathbb{N}_0. \quad (3.13)$$

and it is periodic of period two, provided that $(x_{-1} - x_0)(y_{-1} - y_0) \neq 0$.

Proof. From the assumption, we have $p = q = -1$, so by replacement in the formulas of of the solutions given in Theorem 3.1, we obtain

$$x_{2n-1} = x_{-1}(-1)^n + c\left(b + \frac{a}{v_0} + 1\right)\left(\frac{(-1)^n - 1}{-1-1}\right), n \in \mathbb{N},$$

$$x_{2n} = x_0(-1)^n + c\left(b + u_0 + 1\right)\left(\frac{(-1)^n - 1}{-1-1}\right), \ n \in \mathbb{N},$$

$$y_{2n-1} = y_{-1}(-1)^n + \gamma\left(\beta + \frac{a}{u_0} + 1\right)\left(\frac{(-1)^n - 1}{-1-1}\right), \ n \in \mathbb{N},$$

$$y_{2n} = y_0(-1)^n + \gamma\left(\beta + v_0 + 1\right)\left(\frac{(-1)^n - 1}{-1-1}\right), \ n \in \mathbb{N}.$$

Therefore, we get

$$x_{4n-1} = x_{-1}, \ x_{4n} = x_0, \ n \in \mathbb{N}, \tag{3.14}$$

$$x_{4n+1} = -x_{-1} + c\left(b + \frac{a}{v_0} + 1\right), \ x_{4n+2} = -x_0 + c\left(b + u_0 + 1\right), \ n \in \mathbb{N}_0,$$

$$y_{4n-1} = y_{-1}, \ y_{4n} = y_0, \ n \in \mathbb{N}, \tag{3.15}$$

$$y_{4n+1} = -y_{-1} + \gamma\left(\beta + \frac{a}{u_0} + 1\right), \ y_{4n+2} = -y_0 + \gamma\left(\beta + v_0 + 1\right), \ n \in \mathbb{N}_0.$$

The formulas in (3.14) and (3.15) are also correct for $n = 0$.

Now, it is clear that if

$$x_{-1}, \ x_0, \ -x_{-1} + c\left(b + \frac{a}{v_0} + 1\right), \ -x_0 + c\left(b + u_0 + 1\right)$$

are pairwise different, and

$$y_{-1}, \ y_0, \ -y_{-1} + \gamma\left(\beta + \frac{a}{u_0} + 1\right), \ -y_0 + \gamma\left(\beta + v_0 + 1\right)$$

are pairwise different, the solution will be periodic of period four and takes the form (x_{-1}, y_{-1}), $(x_0, y_0,)$, $\left(-x_{-1} + c\left(b + \frac{a}{v_0} + 1\right), -y_{-1} + \gamma\left(\beta + \frac{a}{u_0} + 1\right)\right)$, $\left(-x_0 + c\left(b + u_0 + 1\right), -y_0 + \gamma\left(\beta + v_0 + 1\right)\right)$, (x_{-1}, y_{-1}), $(x_0, y_0,), \cdots$.

Moreover, we have

$$x_{-1} = \frac{c}{2}\left(b + \frac{a}{v_0} + 1\right) \Leftrightarrow x_{-1} = -x_{-1} + c\left(b + \frac{a}{v_0} + 1\right),$$

$$x_0 = \frac{c}{2}\left(b + u_0 + 1\right) \Leftrightarrow x_0 = -x_0 + c\left(b + u_0 + 1\right),$$

$$y_{-1} = \frac{\gamma}{2}\left(\beta + \frac{a}{u_0} + 1\right) \Leftrightarrow y_{-1} = -y_{-1} + \gamma\left(\beta + \frac{a}{u_0} + 1\right),$$

and

$$y_0 = \frac{\gamma}{2}\left(\beta + v_0 + 1\right) \Leftrightarrow y_0 = -y_0 + \gamma\left(\beta + v_0 + 1\right).$$

Therefore, if

$$x_{-1} = \frac{c}{2}\left(b + \frac{a}{v_0} + 1\right),\ x_0 = \frac{c}{2}\left(b + u_0 + 1\right),\ y_{-1} = \frac{\gamma}{2}\left(\beta + \frac{a}{u_0} + 1\right),\ y_0 = \frac{\gamma}{2}\left(\beta + v_0 + 1\right),$$

we get

$$x_{4n+1} = x_{4n-1} = x_{-1},\ x_{4n+2} = x_{4n} = x_0,\ y_{4n+1} = y_{4n-1} = y_{-1},\ y_{4n+2} = y_{4n} = y_0,\ n \in \mathbb{N}_0,$$

and the solution takes the form

$$x_{2n-1} = x_{-1},\ x_{2n} = x_0,\ y_{2n-1} = y_{-1},\ y_{2n} = y_0,\ n \in \mathbb{N}_0.$$

and it is periodic of period two, provided that $(x_{-1} - x_0)(y_{-1} - y_0) \neq 0$. □

Theorem 3.3. *Let* $\{(x_n, y_n)\}_{n=-1}^{+\infty}$ *be a solution of* (3.1).

$$(b + u_0)\left(b + \frac{a}{v_0}\right) = (\beta + v_0)\left(\beta + \frac{a}{u_0}\right) = 0.$$

The following statements are true.

(1) If $b + u_0 = 0,\ b + \frac{a}{v_0} = 0,\ \beta + v_0 = 0,$ *and* $\beta + \frac{a}{u_0} = 0,$ *then*

$$x_n = c,\ y_n = \gamma,\ n \in \mathbb{N}, \tag{3.16}$$

with $c = x_0$ *and* $\gamma = y_0$.

(2) If $b + u_0 = 0,\ b + \frac{a}{v_0} \neq 0,\ \beta + v_0 = 0,$ *and* $\beta + \frac{a}{u_0} = 0,$ *then*

$$x_{2n} = c,\ x_{2n-1} = c\left(b + \frac{a}{v_0} + 1\right),\ y_n = \gamma,\ n \in \mathbb{N}, \tag{3.17}$$

with $c = x_0$ *and* $\gamma = y_0$.

(3) If $b + u_0 = 0,\ b + \frac{a}{v_0} = 0,\ \beta + v_0 \neq 0,$ *and* $\beta + \frac{a}{u_0} = 0,$ *then*

$$x_n = c,\ y_{2n} = \gamma\left(\beta + v_0 + 1\right),\ y_{2n-1} = \gamma,\ n \in \mathbb{N}, \tag{3.18}$$

with $c = x_0$.

(4) If $b + u_0 = 0,\ b + \frac{a}{v_0} \neq 0,\ \beta + v_0 \neq 0,$ *and* $\beta + \frac{a}{u_0} = 0,$ *then*

$$x_{2n} = c, x_{2n-1} = c\left(b + \frac{a}{v_0} + 1\right),\ y_{2n} = \gamma\left(\beta + v_0 + 1\right),\ y_{2n-1} = \gamma,\ n \in \mathbb{N}, \tag{3.19}$$

with $c = x_0$, *and the solution is eventually periodic of period two.*

(5) If $b + u_0 = 0,\ b + \frac{a}{v_0} = 0,\ \beta + v_0 = 0,$ *and* $\beta + \frac{a}{u_0} \neq 0,$ *then*

$$x_n = c,\ y_{2n-1} = \gamma\left(\beta + \frac{a}{u_0} + 1\right),\ y_{2n} = \gamma,\ n \in \mathbb{N}, \tag{3.20}$$

with $c = x_0$ *and* $\gamma = y_0$.

(6) If $b + u_0 = 0,\ b + \frac{a}{v_0} \neq 0,\ \beta + v_0 = 0,$ *and* $\beta + \frac{a}{u_0} \neq 0,$ *then*

$$x_{2n} = c, x_{2n-1} = c\left(b + \frac{a}{v_0} + 1\right),\ y_{2n-1} = \gamma\left(\beta + \frac{a}{u_0} + 1\right),\ y_{2n} = \gamma,\ n \in \mathbb{N}, \tag{3.21}$$

with $c = x_0$, *and* $\gamma = y_0$, *and the solution is eventually periodic of period two.*

(7) If $b + u_0 \neq 0,\ b + \frac{a}{v_0} = 0,\ \beta + v_0 = 0,$ *and* $\beta + \frac{a}{u_0} = 0,$ *then*

$$x_{2n-1} = c, x_{2n} = c\left(b + u_0 + 1\right),\ y_n = \gamma,\ n \in \mathbb{N}, \tag{3.22}$$

with $\gamma = y_0$.

(8) If $b + u_0 \neq 0,\ b + \frac{a}{v_0} = 0,\ \beta + v_0 = 0,$ *and* $\beta + \frac{a}{u_0} \neq 0,$ *then*

$$x_{2n-1} = c, x_{2n} = c\left(b + u_0 + 1\right),\ y_{2n-1} = \gamma\left(\beta + \frac{a}{u_0} + 1\right),\ y_{2n} = \gamma,\ n \in \mathbb{N}, \tag{3.23}$$

with $\gamma = y_0$, *and the solution is eventually periodic of period two.*

(9) If $b + u_0 \neq 0,\ b + \frac{a}{v_0} = 0,\ \beta + v_0 \neq 0,$ *and* $\beta + \frac{a}{u_0} = 0,$ *then*

$$x_{2n-1} = c, x_{2n} = c\left(b + u_0 + 1\right),\ y_{2n} = \gamma\left(\beta + v_0 + 1\right),\ y_{2n-1} = \gamma,\ n \in \mathbb{N}, \tag{3.24}$$

and the solution is eventually periodic of period two.

Proof. From the assumption, we have $p = q = 0$, so by replacment in the formulas of of the solutions given in Theorem 3.1, we obtain

$$x_{2n} = c\left(b + u_0 + 1\right),\ x_{2n-1} = c\left(b + \frac{a}{v_0} + 1\right),\ n \in \mathbb{N}, \tag{3.25}$$

$$y_{2n} = \gamma\left(\beta + v_0 + 1\right),\ y_{2n-1} = \gamma\left(\beta + \frac{a}{u_0} + 1\right),\ n \in \mathbb{N}. \tag{3.26}$$

We have the following situations.

- Case 1: If we assume that $b + u_0 = 0,\ b + \frac{a}{v_0} \neq 0$, from (3.25), we obtain

$$x_{2n} = c,\ x_{2n-1} = c\left(b + \frac{a}{v_0} + 1\right),\ n \in \mathbb{N}.$$

- Case 2: If we assume that $b + u_0 \neq 0,\ b + \frac{a}{v_0} = 0$, from (3.25), we obtain

$$x_{2n} = c(b + u_0 + 1),\ x_{2n-1} = c,\ n \in \mathbb{N}.$$

- Case 3: If we assume that $b + u_0 = b + \frac{a}{v_0} = 0$, from (3.25), we get

$$x_{2n} = c,\ x_{2n-1} = c,\ n \in \mathbb{N}.$$

  That is

$$x_n = c,\ n \in \mathbb{N}.$$

- Case 4: If we assume that $\beta + v_0 = 0$, and $\beta + \frac{a}{u_0} \neq 0$, from (3.26), we obtain

$$y_{2n} = \gamma,\ y_{2n-1} = \gamma\left(\beta + \frac{a}{u_0} + 1\right),\ n \in \mathbb{N}.$$

- Case 5: If we assume that $\beta + v_0 \neq 0$, and $\beta + \frac{a}{u_0} = 0$, from (3.26), we obtain

$$y_{2n} = \gamma(\beta + v_0 + 1),\ y_{2n-1} = \gamma,\ n \in \mathbb{N}.$$

- Case 6: If we assume that $\beta + v_0 = \beta + \frac{a}{u_0} = 0$, from (3.26), we obtain

$$y_{2n} = \gamma,\ y_{2n-1} = \gamma,\ n \in \mathbb{N}_0.$$

That is

$$y_n = \gamma,\ n \in \mathbb{N}.$$

Combining all possible situations in these cases and accounting for the fact that $b + u_0 = 0 \Leftrightarrow c = x_0$ and $\beta + v_0 = 0 \Leftrightarrow \gamma = y_0$, we get the formulas given in (3.16)–(3.24). □

4. Numerical examples

Here, we provide some numerical examples with their graphical representations to support and illustrate our results on the periodicity, convergence, and divergence behavior of the solutions of (3.1).

Example 4.1. *Consider the system* (3.1) *with the parameters* $a = \frac{3}{2}, b = 1, \beta = \frac{1}{2}, c = 3,$ *and* $\gamma = 1$, *we have*

$$x_{n+1} = \frac{\frac{3}{2}x_n y_{n-1}}{y_n - \frac{1}{2}y_{n-1} - 1} + x_n + 3,\ y_{n+1} = \frac{\frac{3}{2}y_n x_{n-1}}{x_n - x_{n-1} - 3} + \frac{1}{2}y_n + 1. \tag{4.1}$$

Let the initial values be as follows: $x_{-1} = 1, y_{-1} = -4, x_0 = 5,$ *and* $y_0 = 3$. *It is not hard to see that*

$$(b + u_0)\left(b + \frac{a}{v_0}\right) = (\beta + v_0)\left(\beta + \frac{a}{u_0}\right) = 1,$$

and the solution takes the form

$$x_{4n-1} = 1,\ x_{4n} = 5,\ x_{4n+1} = \frac{1}{2},\ x_{4n+2} = 4,$$

$$y_{4n-1} = -4,\ y_{4n} = 3,\ y_{4n+1} = 7,\ y_{4n+2} = -1.$$

That is the solution is periodic of period four, as stated in Case 1 of Theorem 3.2. The graphical representations of the solution are given in Figures 1 and 2.

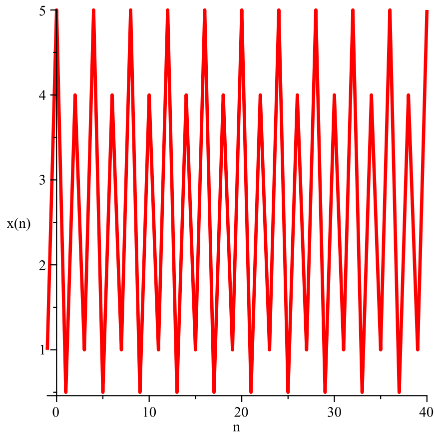

Figure 1. The graphic of the x_n-component of the solution of the system (4.1).

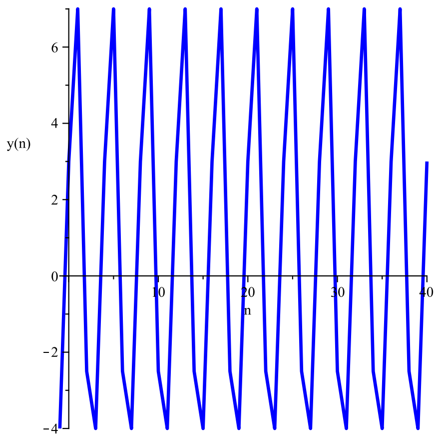

Figure 2. The graphic of the y_n-component of the solution of the system (4.1).

Example 4.2. *Consider the system* (3.1) *with the parameters* $a = 9$, $b = 1$, $\beta = 2$, $c = 4$, *and* $\gamma = 6$*; that is*

$$x_{n+1} = \frac{9x_n y_{n-1}}{y_n - 2y_{n-1} - 6} + x_n + 4,\ y_{n+1} = \frac{9y_n x_{n-1}}{x_n - x_{n-1} - 4} + 2y_n + 6, \tag{4.2}$$

with the initial values $x_{-1} = 8$, $y_{-1} = 5$, $x_0 = 4$, *and* $y_0 = 6$. *We have*

$$b + u_0 = 0,\ \beta + v_0 = 0,\ b + \frac{a}{v_0} = -\frac{7}{2},\ \beta + \frac{a}{u_0} = -7,$$

and the solution takes the form

$$x_{2n} = 4,\ y_{2n} = 6,\ n \in \mathbb{N}_0,\ x_{2n-1} = -10,\ y_{2n-1} = -36,\ n \in \mathbb{N}.$$

That is the solution is eventually periodic of period two, as stated in Case 6 of Theorem 3.3. The plots of the solution are given in Figures 3 and 4.

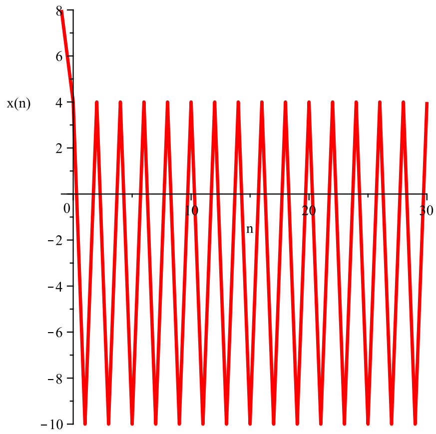

Figure 3. The graphic of the x_n-component of the solution of the system (4.2).

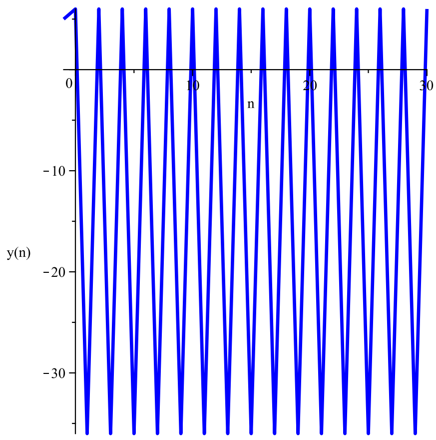

Figure 4. The graphic of the y_n-component of the solution of the system (4.2).

Example 4.3. *Consider the system* (3.1) *with the parameters* $a = 3$, $b = \frac{3}{11}$, $\beta = 11$, $c = 5$, *and* $\gamma = 1$*; that is*

$$x_{n+1} = \frac{3x_n y_{n-1}}{y_n - 11y_{n-1} - 1} + \frac{3}{11}x_n + 5, \; y_{n+1} = \frac{3y_n x_{n-1}}{x_n - \frac{3}{11}x_{n-1} - 5} + 11y_n + 1, \tag{4.3}$$

with the initial values $x_{-1} = 7$, $y_{-1} = 2$, $x_0 = 2$, *and* $y_0 = 1$. *We have*

$$b + u_0 = -\frac{3}{7}, \; b + \frac{a}{v_0} = 0, \; \beta + v_0 = 0, \; \beta + \frac{a}{u_0} = \frac{121}{18},$$

and the solution takes the form

$$y_{2n} = 1, \; n \in \mathbb{N}_0, \; x_{2n} = \frac{20}{7}, \; x_{2n-1} = 5, \; y_{2n-1} = \frac{139}{18}, \; n \in \mathbb{N}.$$

That is, the solution is eventually periodic of period two, as stated in Case 8 of Theorem 3.3. The plots of the solution are given in Figures 5 and 6.

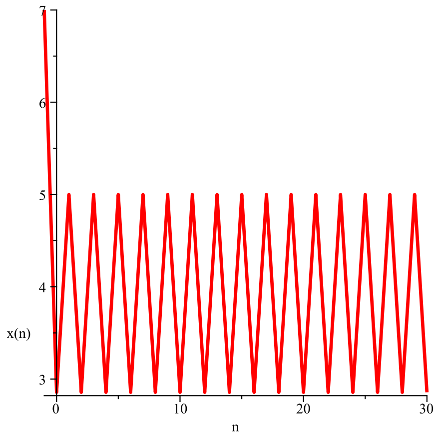

Figure 5. The graphic of the x_n-component of the solution of the system (4.3).

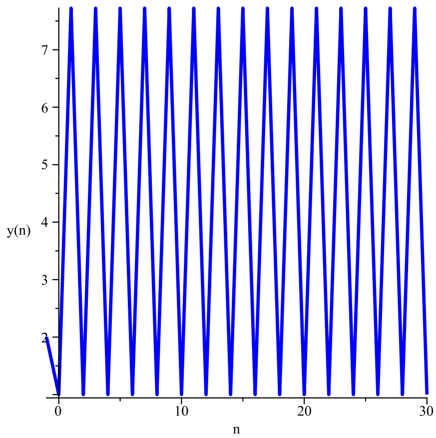

Figure 6. The graphic of the y_n-component of the solution of the system (4.3).

Example 4.4. *Consider the system* (3.1) *with the parameters* $a = 7$, $b = \frac{3}{11}$, $\beta = \frac{3}{4}$, $c = 2$, *and* $\gamma = \frac{1}{2}$; *that is*

$$x_{n+1} = \frac{7x_n y_{n-1}}{y_n - \frac{3}{4}y_{n-1} - \frac{1}{2}} + \frac{3}{11}x_n + 2,\ y_{n+1} = \frac{7y_n x_{n-1}}{x_n - \frac{3}{11}x_{n-1} - 2} + \frac{3}{4}y_n + \frac{1}{2}, \tag{4.4}$$

and let the initial values be as follows: $x_{-1} = 15$, $y_{-1} = 19$, $x_0 = \frac{5}{2}$, *and* $y_0 = 1$. *It is not hard to see that*

$$(b + u_0)\left(b + \frac{a}{v_0}\right) = -\frac{47}{150},\ (\beta + v_0)\left(\beta + \frac{a}{u_0}\right) = -\frac{9003}{12008},$$

and the solution converge to a two-period solution as follows:

$$\lim_{n\to+\infty} x_{2n+1} = -12.57360,\ \lim_{n\to+\infty} x_{2n} = 1.57360,\ \lim_{n\to+\infty} y_{2n+1} = -7.85555,\ \lim_{n\to+\infty} y_{2n} = 0.29327,$$

as stated in Case 3 of Lemma 3.1. The graphical representations of the solution are given in Figures 7–10.

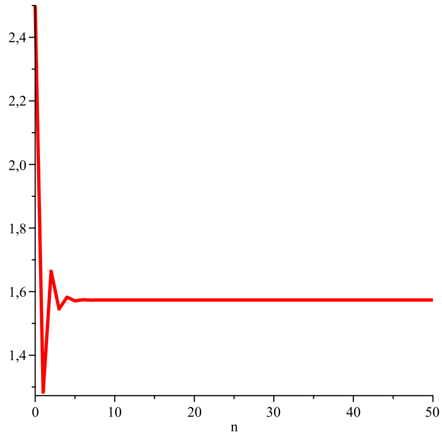

Figure 7. The plot of x_{2n} of the solution of the system (4.4).

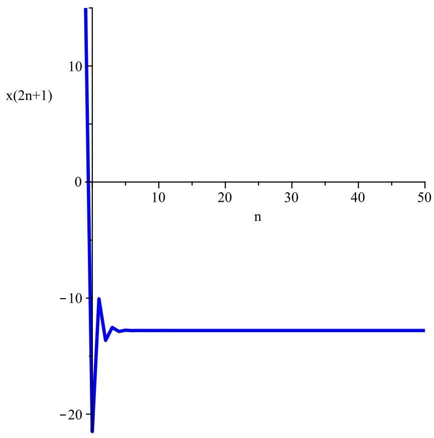

Figure 8. The plot of x_{2n+1} of the solution of the system (4.4).

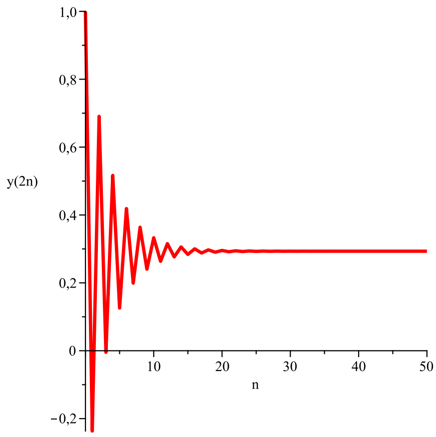

Figure 9. The plot of y_{2n} of the solution of the system (4.4).

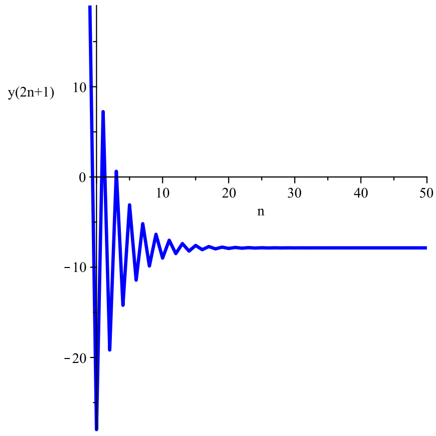

Figure 10. The plot of y_{2n+1} of the solution of the system (4.4).

Example 4.5. *Consider the system* (3.1) *with the parameters* $a = 3$, $b = \frac{7}{3}$, $\beta = \frac{3}{2}$, $c = \frac{3}{4}$, *and* $\gamma = 2$; *that is*

$$x_{n+1} = \frac{3x_n y_{n-1}}{y_n - \frac{3}{2}y_{n-1} - 2} + \frac{7}{3}x_n + \frac{3}{4},\ y_{n+1} = \frac{3y_n x_{n-1}}{x_n - \frac{7}{3}x_{n-1} - \frac{3}{4}} + \frac{3}{2}y_n + 2, \tag{4.5}$$

and let the initial values be as follows: $x_{-1} = \frac{21}{4}$, $y_{-1} = \frac{5}{2}$, $x_0 = 7$, *and* $y_0 = 5$. *We have*

$$(b + u_0)\left(b + \frac{a}{v_0}\right) = -\frac{575}{63},\ (\beta + v_0)\left(\beta + \frac{a}{u_0}\right) = -\frac{27}{20},$$

and

$$\lim_{n\to+\infty} |x_{2n}| = \lim_{n\to+\infty} |x_{2n+1}| = \lim_{n\to+\infty} |y_{2n}| = \lim_{n\to+\infty} |y_{2n+1}| = +\infty$$

as stated in Case 3 of Lemma 3.1. The graphical representations of the solution are given in Figures 11–14.

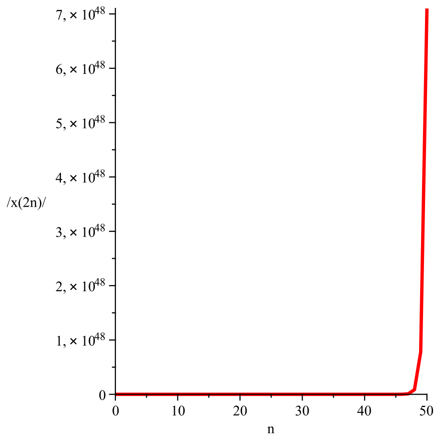

Figure 11. The plot of $|x_{2n}|$ of the solution of the system (4.5).

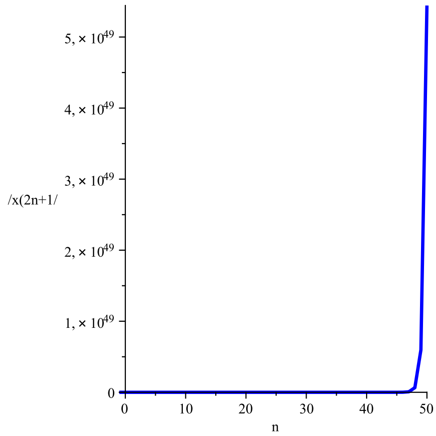

Figure 12. The plot of $|x_{2n+1}|$ of the solution of the system (4.5).

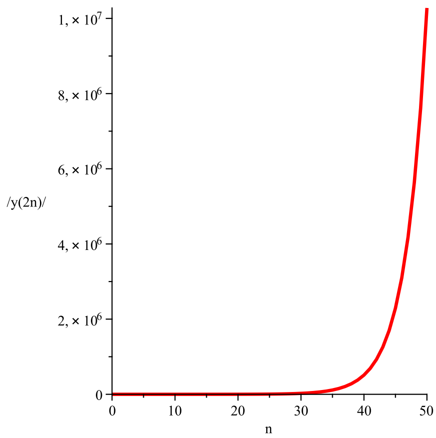

Figure 13. The plot of $|y_{2n}|$ of the solution of the system (4.5).

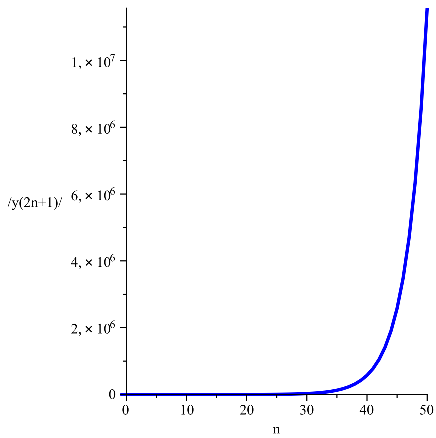

Figure 14. The plot of $|y_{2n+1}|$ of the solution of the system (4.5).

5. Conclusions and remarks

We solved system (1.1) explicitly, and from the formulas of its solutions, we deduce those of (3.1). Moreover, we analyzed in details the periodicity and limiting behavior of the solutions of some particular cases of (3.1). To support our results on the system (3.1), numerical examples are provided with their graphical representations to illustrate the periodicity, convergence, or divergence of the solutions. We note that if we choose

$$\alpha = a, \beta = b, \gamma = c, y_{-1} = x_{-1},\ y_0 = x_0, \tag{5.1}$$

we get

$$y_n = x_n,\ n = -1, 0, ... \tag{5.2}$$

that is, the systems (1.1) and (3.1) will be the following difference equations:

$$x_{n+1} = \frac{a x_n x_{n-1}}{x_n - b x_{n-1} - c} + b x_n + c. \tag{5.3}$$

Consequently, the results on the form, the periodicity, and the limiting behavior of the solutions of (5.3) can be deduced from Theorems 3.1–3.3 and Lemma 3.1. For example, it follows from Theorem 3.1 that every solution $\{x_n\}_{n=-1}^{+\infty}$ of (5.3) takes the form

$$x_{2n} = x_0 p^n + c\,(b + u_0 + 1)\left(\frac{p^n - 1}{p - 1}\right),\ x_{2n-1} = x_{-1} p^n + c\left(b + \frac{a}{u_0} + 1\right)\left(\frac{p^n - 1}{p - 1}\right),$$

if $p \neq 1$, and

$$x_{2n} = x_0 + c\,(b + u_0 + 1)\,n,\ x_{2n-1} = x_{-1} + c\left(b + \frac{a}{u_0} + 1\right)n,$$

if $p = 1$, where $p = (b + u_0)\left(b + \frac{a}{u_0}\right)$ and $u_0 = \frac{x_0 - b x_{-1} - c}{x_{-1}}$.

Author contributions

Nouressadat Touafek: Methodology, investigation, software, writing—review and editing, visualization, supervision; Jawharah Ghuwayzi Al-Juaid: Writing—review and editing, project administration. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors would like to acknowledge the Deanship of Graduate Studies and Scientific Research of Taif University for funding this work.

Conflict of interest

There are no conflicts of interest.

References

1. M. Gümüş, R. Abo-Zeid, An explicit formula and forbidden set for a higher order difference equation, *J. Appl. Math. Comput.*, **63** (2020), 133–142. https://doi.org/10.1007/s12190-019-01311-9
2. Y. Halim, A. Khelifa, M. Berkal, A. Bouchair, On a solvable system of p difference equations of higher order, *Period. Math. Hung.*, **85** (2022), 109–127. https://doi.org/10.1007/s10998-021-00421-x
3. J. Kubayi, M. Folly-Gbetoula, A detailed study of a class of recurrence equations with a generalized order, *J. Math. Computer Sci.*, **32** (2024), 318–331. https://doi.org/10.22436/jmcs.032.04.03
4. Y. Long, X. Pang, Q. Zhang, Codimension-one and codimension-two bifurcations of a discrete Leslie-Gower type predator-prey model, *Discrete Cont. Dyn.-B*, **30** (2025), 1357–1389. https://doi.org/10.3934/dcdsb.2024132
5. Q. Din, U. Saeed, Stability, discretization, and bifurcation analysis for a chemical reaction system, *MATCH Commun. Math. Comput. Chem.*, **90** (2023), 151–174. https://doi.org/10.46793/match.90-1.151D
6. F. Alsharif, R. Ahmed, I. Alraddadi, M. Alsubhi, Md. Uddin, A study of stability and bifurcation in a discretized predator-prey model with holling type III response and prey refuge via piecewise constant argument method, *Complexity*, **2025** (2025), 4542190. https://doi.org/10.1155/cplx/4542190
7. M. Almatrafi, M. Berkal, M. Hamada, Qualitative behavior of a discrete-time predator-prey model with Holling-type III functional response and Gompertz growth of prey, *Math. Method. Appl. Sci.*, **48** (2025), 13100–13112. https://doi.org/10.1002/mma.11087
8. N. Haddad, N. Touafek, J. Rabago, Well-defined solutions of a system of difference equations, *J. Appl. Math. Comput.*, **56** (2018), 439–458. https://doi.org/10.1007/s12190-017-1081-8
9. E. Elabbasy, H. Elmatwally, E. Elsayed, Qualitative behavior of higher order difference equation, *Soochow Journal of Mathematics*, **33** (2007), 861–873.
10. Y. Yazlik, M. Kara, On a solvable system of difference equations of higher-order with periodic coefficients, *Commun. Fac. Sci. Univ.*, **68** (2019), 1675–1693. https://doi.org/10.31801/cfsuasmas.548262
11. M. Kara, N. Touafek, Y. Yazlik, Well-defined solutions of a three-dimensional system of difference equations, *Gazi Univ. J. Sci.*, **33** (2020), 767–778. https://doi.org/10.35378/gujs.641441
12. K. Al-Basyouni, E. Elsayed, On some solvable systems of some rational difference equations of third order, *Mathematics*, **11** (2023), 1047. https://doi.org/10.3390/math11041047
13. S. Elaydi, *An introduction to difference equations*, New York: Springer, 2005. https://doi.org/10.1007/0-387-27602-5

AIMS Press