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Research article

On the generalized coupled Hadamard-Gronwall-Bellman-type inequalities with applications to fractional delay systems

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Abstract: The Gronwall-Bellman inequality is a primary tool for proving various types of stability. For this importance, the present paper focuses on the generalized forms of the well-known Gronwall-Bellman inequality in the context of the Hadamard fractional calculus. We prove and generalize the coupled version of the Hadamard-Gronwall-Bellman inequality and then, generalize its extended form with the sum of two non-decreasing functions. In the sequel, the applicability of these inequalities is established in proving the existence and Ulam-Hyers stability of a Caputo-Hadamard coupled delay system and a Caputo-Hadamard damped initial value problem, which are appeared in population dynamic problems.

Keywords: Hadamard fractional integral; Gronwall-Bellman inequality; coupled system; stability of solutions; existence results
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1. Introduction

In mathematics, inequalities play a fundamental role in establishing bounds, comparing quantities, and analyzing the behavior of functions and operators. They are essential tools in analysis, optimization, probability, and many other fields. In fractional calculus, inequalities are particularly important because fractional derivatives and integrals often lack straightforward closed-form expressions, making inequalities crucial for estimating their behavior. If we want to state such applications precisely, we can mention the approximation theory in which inequalities measure how well one function approximates another. In differential equations and dynamical systems, inequalities (e.g., Gronwall's inequality) help establish stability and convergence.

For example, the Lyapunov-type inequalities for fractional equations help determine stability conditions. In optimization, some conditions in constrained optimization rely on inequalities such as the Karush-Kuhn-Tucker conditions.

As we mentioned above, since fractional derivatives are non-local and often defined via integral operators, direct computation is difficult, and so, the inequalities provide needed bounds for us. In recent years, many published articles have worked on fractional generalizations of some well-known inequalities. In between, some inequalities are famous and most applicable. For instance, to find a bound for integral mean of a convex function, the Hermite-Hadamard inequality is used, because the exact integration is difficult in some cases. Its ability to provide two-sided bounds makes it valuable in both theoretical and practical contexts [1–6]. The generalized forms of the Milne-type inequality reinforce the interplay between sums and reciprocals, often leading to useful bounds [7,8]. To estimate for the deviation of a function's value from its integral average, the Ostrowski inequality plays the vital role. In fact, this inequality quantifies how much a function's value $\omega(t)$ can differ from its average value on a closed interval [9–11]. To provide a weighted estimate for integrals involving averages of functions, we use the Hardy's inequality, and it has two continuous and discrete forms. Some applications of this inequality can be found in Sobolev embeddings, martingale theory in probabilty, or interpolation theory in functional spaces [12]. For more studies on inequalities, see [13–15].

The Gronwall (Gronwall-Bellman) inequality is considered as an important tool in mathematical analysis, differential equations, and control theory. It provides an essential tool for bounding the solutions of integral and differential inequalities, particularly in the study of stability, uniqueness, and long-term behavior of dynamical systems. Originally introduced by Thomas Hakon Gronwall [16] in 1919 and later refined by Richard Bellman [17] in 1943, this inequality has become a cornerstone in the analysis of ordinary and partial differential equations (ODEs and PDEs), stochastic processes, and mathematical modeling.

The classical form of the Gronwall inequality [16] states that if $\omega(t)$ is a non-negative continuous function such that for each constants $\lambda, \gamma \in \mathbb{R}^+$,

$$\omega(t) \leq \int_{t_1}^{t} (\lambda\omega(r) + \gamma)\, \mathrm{d}r,$$

then

$$\omega(t) \leq \gamma(t_2 - t_1) \exp\left(\lambda(t_2 - t_1)\right),$$

for all $t \in [t_1, t_2]$. In 1943, Bellman [17] presented the following structure for this inequality and called it as the Gronwall-Bellman inequality. Indeed, If two non-negative continuous functions like $\omega(t)$ and $v(t)$ are defined on $[t_1, t_2]$ such that

$$\omega(t) \leq \rho + \int_{t_1}^{t} v(r)\omega(r)\,\mathrm{d}r, \quad \rho \in \mathbb{R}^+,$$

then

$$\omega(t) \leq \rho \exp\left(\int_{t_1}^{t} v(r)\mathrm{d}r\right),$$

for all $t \in [t_1, t_2]$.

Later, in a general version of the Gronwall-Bellman inequality, Pachpatte [18] proved, in 2001, that if we have two non-negative continuous functions like $\omega(t)$ and $v(t)$ on $[t_1, t_2]$, and the function $h(t)$ is non-decreasing, then

$$\omega(t) \leq h(t) \exp\left(\int_{t_1}^{t} v(r)\mathrm{d}r\right),$$

whenever

$$\omega(t) \leq h(t) + \int_{t_1}^{t} v(r)\omega(r)\,\mathrm{d}r$$

is satisfied for all $t \in [t_1, t_2]$. Subsequently, we see different types of generalizations for this inequality under some interesting conditions. For instances, Pachpatte [19] considered three non-negative continuous functions like $\omega(t)$, $k(t)$, and $v(t)$ on $[t_1, t_2]$, and assumed that

$$\omega(t) \leq h(t) + k(t) \int_{t_1}^{t} v(r)\omega(r)\,\mathrm{d}r.$$

Then, he proved that the inequality

$$\omega(t) \leq h(t)\left[1 + k(t) \int_{t_1}^{t} v(r) \cdot \exp\left(\int_{r}^{t} v(s)k(s)\mathrm{d}s\right) \mathrm{d}r\right],$$

holds for all $t \in [t_1, t_2]$.

In 2017, Adjabi et al. [20] used the generalized fractional integrals unifying the Hadamard and Reimann-Liouville integrals to obtain a new version of the Gronwall-Bellman-type inequality. In 2018, Alzabut and Abdeljawad [21] presented a discrete version of the Gronwall-Bellman inequality under the discrete form of the Mittag-Leffler function. In 2019, Butt et al. [22] extended this structure on time scales and obtained the quantum-type of the Gronwall-Bellman inequality to prove the stability results. In 2021, Alzabut et al. [23] applied the ψ-fractional integrals for proving the ψ-type Gronwall-Bellman inequality and then, they completed the stability results by using this type of the inequality. Other types of the Gronwall-Bellman inequalities can be found in [24–27].

In this paper, we also extend new forms of the Gronwall-Bellman inequalities in the context of the Hadamard fractional integrals. In fact, we extract two different structures under the name of the Hadamard-Gronwall-Bellman inequalities. In the first generalization, we investigate and prove the coupled Hadamard-Gronwall-Bellman-type inequality. In the second generalization, we establish an extended form of the Hadamard-Gronwall-Bellman-type inequality under the sum of two non-decreasing functions. These inequalities are new and are used to prove the existence and stability theorems. In fact, to do this purpose, we consider a Caputo-Hadamard coupled delay system and a Caputo-Hadamard damped initial value problem. The illustrative example will confirm the theoretical results.

2. Preliminaries

This section is began by recalling some needed concepts.

Definition 2.1. *[28] Let $p \geq 0$ and ω be a function which is continuous integrable on $[t_1, t_2]$. The Hadamard fractional integral of $\omega : [t_1, t_2] \to \mathbb{R}$ of order p is given by ${}^{H}I_{t_1}^{0}\omega(t) = \omega(t)$ and*

$$ {}^{H}I_{t_1}^{p}\omega(t) = \frac{1}{\Gamma(p)} \int_{t_1}^{t} \left(\ln \frac{t}{r}\right)^{p-1} \omega(r)\, \frac{\mathrm{d}r}{r}. $$

This operator has a unique structure in its kernel. In fact, this kernel makes it the natural choice for problems with certain geometric and scaling properties like like in finance or population dynamics.

Lemma 2.2. *[28] For each $p_1, p_2 \geq 0$ and $t > t_1$,*

(1H) ${}^{H}I_{t_1}^{p_1}\, {}^{H}I_{t_1}^{p_2}\omega(t) = {}^{H}I_{t_1}^{p_1+p_2}\omega(t)$;

(2H) ${}^{H}I_{t_1}^{p_1}\left(\ln \frac{t}{t_1}\right)^{p_2} = \frac{\Gamma(p_2+1)}{\Gamma(p_1+p_2+1)}\left(\ln \frac{t}{t_1}\right)^{p_1+p_2}$;

(3H) ${}^{H}I_{t_1}^{p_1} 1 = \frac{1}{\Gamma(p_1+1)}\left(\ln \frac{t}{t_1}\right)^{p_1}$ *by setting $p_2 = 0$ in (2H).*

Definition 2.3. *[28] Let $m = [p]+1$. The Hadamard fractional derivative of order p for $\omega : [t_1, t_2] \to \mathbb{R}$ is defined by*

$$ {}^{H}D_{t_1}^{p}\omega(t) = \frac{1}{\Gamma(m-p)} \left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^{m} \int_{t_1}^{t} \left(\ln \frac{t}{r}\right)^{m-p-1} \omega(r)\, \frac{\mathrm{d}r}{r}. $$

We define the space $AC_{\mathbb{R}}^{m}([t_1, t_2])$, for $0 < t_1 < t_2 < \infty$ and $m = [p] + 1$, as

$$ AC_{\mathbb{R}}^{m}([t_1, t_2]) = \left\{ \omega : [t_1, t_2] \to \mathbb{R} : \ \left(t\, \frac{\mathrm{d}}{\mathrm{d}t}\right)^{m-1} \omega(t) \in AC_{\mathbb{R}}([t_1, t_2]) \right\}. $$

Here, the space $AC_{\mathbb{R}}([t_1, t_2])$ includes all absolutely continuous functions on $[t_1, t_2]$.

Definition 2.4. *[29] The Caputo-Hadamard fractional derivative of order p for $\omega \in AC_{\mathbb{R}}^{m}([t_1, t_2])$ is formulated by*

$$ {}^{CH}D_{t_1}^{p}\omega(t) = \frac{1}{\Gamma(m-p)} \int_{t_1}^{t} \left(\ln\left(\frac{t}{r}\right)\right)^{m-p-1} \left(r\, \frac{\mathrm{d}}{\mathrm{d}r}\right)^{m} \omega(r)\, \frac{\mathrm{d}r}{r}. $$

Here, $m - 1 < p \leq m$.

Lemma 2.5. *[29] Let $m - 1 < p \leq m$ and $\omega \in AC_{\mathbb{R}}^{m}([t_1, t_2])$. Then*

$$ {}^{H}I_{t_1}^{p}\left({}^{CH}D_{t_1}^{p}\omega(t)\right) = \omega(t) - \sum_{i=0}^{m-1} \frac{1}{i!} \left(t\, \frac{\mathrm{d}}{\mathrm{d}t}\right)^{i} \omega(t_1) \left(\ln \frac{t}{t_1}\right)^{i}. $$

3. Coupled Hadamard-Gronwall-Bellman-type inequality

This section introduces and proves a generalized form of the coupled Hadamard-Gronwall-Bellman-type inequality. We state the following theorem in this direction.

Theorem 3.1. *Let $p, q > 0$. Suppose that:*

(G1) Four functions $\omega_1(t)$, $\omega_2(t)$, $h_1(t)$ and $h_2(t)$ are integrable with non-negative values on the domain $[t_1, t_2]$.

(G2) Two non-negative-valued functions $f_1(t)$ and $f_2(t)$ are non-decreasing provided that $f_i(t) \leq M$ $(i = 1, 2)$ for all $t \in [t_1, t_2]$ and for some constant $M > 0$.

Then

$$\omega_1(t) \leq h_1(t) + f_1(t) \int_{t_1}^{t} \left(\ln \frac{t}{r}\right)^{p-1} h_2(r) \frac{\mathrm{d}r}{r} \tag{3.1}$$

$$+ \sum_{m=1}^{\infty} \frac{(f_1(t) f_2(t) \Gamma(p)\Gamma(q))^m}{\Gamma(m(p+q))} \int_{t_1}^{t} \left(\ln \frac{t}{r}\right)^{m(p+q)-1} \left(h_1(r) + f_1(r) \int_{t_1}^{r} \left(\ln \frac{r}{s}\right)^{p-1} h_2(s) \frac{\mathrm{d}s}{s}\right) \frac{\mathrm{d}r}{r},$$

and

$$\omega_2(t) \leq h_2(t) + f_2(t) \int_{t_1}^{t} \left(\ln \frac{t}{r}\right)^{q-1} h_1(r) \frac{\mathrm{d}r}{r} \tag{3.2}$$

$$+ \sum_{m=1}^{\infty} \frac{(f_1(t) f_2(t) \Gamma(p)\Gamma(q))^m}{\Gamma(m(p+q))} \int_{t_1}^{t} \left(\ln \frac{t}{r}\right)^{m(p+q)-1} \left(h_2(r) + f_2(r) \int_{t_1}^{r} \left(\ln \frac{r}{s}\right)^{q-1} h_1(s) \frac{\mathrm{d}s}{s}\right) \frac{\mathrm{d}r}{r},$$

if

$$\begin{cases} \omega_1(t) \leq h_1(t) + A_1 \omega_2(t), \\ \omega_2(t) \leq h_2(t) + A_2 \omega_1(t), \end{cases} \tag{3.3}$$

where

$$A_1 \omega_2(t) := f_1(t) \int_{t_1}^{t} \left(\ln \frac{t}{r}\right)^{p-1} \omega_2(r) \frac{\mathrm{d}r}{r},$$

$$A_2 \omega_1(t) := f_2(t) \int_{t_1}^{t} \left(\ln \frac{t}{r}\right)^{q-1} \omega_1(r) \frac{\mathrm{d}r}{r}. \tag{3.4}$$

Proof. In view of the coupled system of the inequalities (3.3) and by considering the operators A_1 and A_2 defined in (3.4), we get

$$\omega_1(t) \leq h_1(t) + A_1 \omega_2(t), \quad \omega_2(t) \leq h_2(t) + A_2 \omega_1(t). \tag{3.5}$$

The monotonicity property of two operators A_1 and A_2, along with the inequalities (3.5), give

$$\omega_1(t) \leq h_1(t) + A_1 \omega_2(t) \leq h_1(t) + A_1 \left(h_2(t) + A_2 \omega_1(t)\right)$$

$$= h_1(t) + A_1 h_2(t) + A_1 A_2 \omega_1(t).$$

Now, by above inequality, since $\omega_1(t) \leq h_1(t) + A_1 h_2(t) + A_1 A_2 \omega_1(t)$, we can continue this process for term $A_1 A_2 \omega_1(t)$ as follows

$$\begin{aligned}\omega_1(t) &\leq h_1(t) + A_1 h_2(t) + A_1 A_2 \left(h_1(t) + A_1 h_2(t) + A_1 A_2 \omega_1(t)\right)\\ &= h_1(t) + A_1 h_2(t) + A_1 A_2 h_1(t) + A_1 A_2 A_1 h_2(t) + (A_1 A_2)^2 \omega_1(t).\end{aligned}$$

Again, this technique is used for term $(A_1 A_2)^2 \omega_1(t)$ and it gives

$$\begin{aligned}\omega_1(t) &\leq h_1(t) + A_1 A_2 h_1(t) + A_1 h_2(t) + A_1 A_2 A_1 h_2(t)\\ &\quad + (A_1 A_2)^2 \Big(h_1(t) + A_1 h_2(t) + A_1 A_2 h_1(t) + A_1 A_2 A_1 h_2(t) + (A_1 A_2)^2 \omega_1(t)\Big)\\ &= h_1(t) + A_1 A_2 h_1(t) + (A_1 A_2)^2 h_1(t) + (A_1 A_2)^3 h_1(t)\\ &\quad + A_1 h_2(t) + A_1 A_2 A_1 h_2(t) + (A_1 A_2)^2 A_1 h_2(t) + (A_1 A_2)^3 A_1 h_2(t) + (A_1 A_2)^4 \omega_1(t).\end{aligned}$$

If we continue this iterative scheme, then for $n \in \mathbb{N}$, we have

$$\omega_1(t) \leq \sum_{m=0}^{n-1} (A_1 A_2)^m h_1(t) + \sum_{m=0}^{n-1} (A_1 A_2)^m A_1 h_2(t) + (A_1 A_2)^n \omega_1(t), \quad t \in [t_1, t_2], \tag{3.6}$$

where $(A_1 A_2)^0 h_1(t) = h_1(t)$. In a similar manner, we may write

$$\omega_2(t) \leq \sum_{m=0}^{n-1} (A_2 A_1)^m h_2(t) + \sum_{m=0}^{n-1} (A_2 A_1)^m A_2 h_1(t) + (A_2 A_1)^n \omega_2(t), \quad t \in [t_1, t_2], \tag{3.7}$$

where $(A_2 A_1)^0 h_2(t) = h_2(t)$.

In the following, we will prove that

$$\begin{cases} (A_1 A_2)^n \omega_1(t) \leq \dfrac{(f_1(t) f_2(t) \Gamma(p) \Gamma(q))^n}{\Gamma(n(p+q))} \displaystyle\int_{t_1}^{t} \left(\ln \frac{t}{r}\right)^{n(p+q)-1} \omega_1(r) \frac{\mathrm{d}r}{r}, \\ (A_2 A_1)^n \omega_2(t) \leq \dfrac{(f_1(t) f_2(t) \Gamma(p) \Gamma(q))^n}{\Gamma(n(p+q))} \displaystyle\int_{t_1}^{t} \left(\ln \frac{t}{r}\right)^{n(p+q)-1} \omega_2(r) \frac{\mathrm{d}r}{r}, \end{cases} \tag{3.8}$$

for $t \in [t_1, t_2]$, and

$$\lim_{n \to \infty} (A_1 A_2)^n \omega_1(t) = 0, \qquad \lim_{n \to \infty} (A_2 A_1)^n \omega_2(t) = 0. \tag{3.9}$$

Clearly, the inequalities in the coupled system (3.8) are to be held if $n = 1$. In other words, we have

$$(A_1 A_2) \omega_1(t) = A_1 (A_2 \omega_1(t)) = f_1(t) \int_{t_1}^{t} \left(\ln \frac{t}{r}\right)^{p-1} (A_2 \omega_1)(r) \frac{\mathrm{d}r}{r}$$

$$
\begin{aligned}
&= f_1(t)\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{p-1}\left(f_2(r)\int_{t_1}^{r}\left(\ln\frac{r}{s}\right)^{q-1}\omega_1(s)\,\frac{\mathrm{d}s}{s}\right)\frac{\mathrm{d}r}{r}\\
&\leq f_1(t)f_2(t)\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{p-1}\int_{t_1}^{r}\left(\ln\frac{r}{s}\right)^{q-1}\omega_1(s)\,\frac{\mathrm{d}s}{s}\frac{\mathrm{d}r}{r}\\
&= f_1(t)f_2(t)\int_{t_1}^{t}\int_{s}^{t}\left(\ln\frac{t}{r}\right)^{p-1}\left(\ln\frac{r}{s}\right)^{q-1}\omega_1(s)\,\frac{\mathrm{d}r}{r}\frac{\mathrm{d}s}{s}\\
&= f_1(t)f_2(t)\int_{t_1}^{t}\omega_1(s)\left(\int_{s}^{t}\left(\ln\frac{t}{r}\right)^{p-1}\left(\ln\frac{r}{s}\right)^{q-1}\frac{\mathrm{d}r}{r}\right)\frac{\mathrm{d}s}{s}.
\end{aligned}
$$

Define a new variable $v = \dfrac{\ln(r)-\ln(s)}{\ln(t)-\ln(s)}$. In this case, we know that if $r \in [s,t]$, then $v \in [0,1]$. Now, the rule of the change of the variables and definition of the Beta function $B(p,q) = \displaystyle\int_0^1 (1-r)^{p-1}r^{q-1}\mathrm{d}r$ follow that

$$
\begin{aligned}
\int_{s}^{t}\left(\ln\frac{t}{r}\right)^{p-1}\left(\ln\frac{r}{s}\right)^{q-1}\frac{\mathrm{d}r}{r} &= \int_{s}^{t}\left(\ln\frac{t}{s}\right)^{p-1}\left(1-\frac{\ln(r)-\ln(s)}{\ln(t)-\ln(s)}\right)^{p-1}\left(\ln\frac{r}{s}\right)^{q-1}\frac{\mathrm{d}r}{r}\\
&= \int_{s}^{t}\left(\ln\frac{t}{s}\right)^{p-1}\left(1-\frac{\ln(r)-\ln(s)}{\ln(t)-\ln(s)}\right)^{p-1}v^{q-1}\left(\ln\frac{t}{s}\right)^{q-1}\frac{\mathrm{d}r}{r}\\
&= \left(\ln\frac{t}{s}\right)^{(p+q)-1}\int_0^1 (1-v)^{p-1}v^{q-1}\,\mathrm{d}v\\
&= \left(\ln\frac{t}{s}\right)^{(p+q)-1}B(p,q)\\
&= \left(\ln\frac{t}{s}\right)^{(p+q)-1}\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)},
\end{aligned}
$$

where B stands for the Beta function.

Hence,

$$
(A_1A_2)\omega_1(t) \leq \frac{f_1(t)f_2(t)\Gamma(p)\Gamma(q)}{\Gamma(p+q)}\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{(p+q)-1}\omega_1(r)\,\frac{\mathrm{d}r}{r}.
$$

Again, in a similar manner, we get

$$
(A_2A_1)\omega_2(t) \leq \frac{f_1(t)f_2(t)\Gamma(p)\Gamma(q)}{\Gamma(p+q)}\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{(p+q)-1}\omega_2(r)\,\frac{\mathrm{d}r}{r}.
$$

Now, we continue the proof by using the mathematical induction. Let, for $n = j \in \mathbb{N}$ and $t \in [t_1, t_2]$, the coupled inequalities

$$\begin{cases} (A_1A_2)^j\omega_1(t) \le \dfrac{(f_1(t)f_2(t)\Gamma(p)\Gamma(q))^j}{\Gamma(j(p+q))} \displaystyle\int_{t_1}^{t} \left(\ln\frac{t}{r}\right)^{j(p+q)-1} \omega_1(r)\,\frac{\mathrm{d}r}{r}, \\ (A_2A_1)^j\omega_2(t) \le \dfrac{(f_1(t)f_2(t)\Gamma(p)\Gamma(q))^j}{\Gamma(j(p+q))} \displaystyle\int_{t_1}^{t} \left(\ln\frac{t}{r}\right)^{j(p+q)-1} \omega_2(r)\,\frac{\mathrm{d}r}{r}, \end{cases} \tag{3.10}$$

be satisfied. We show that (3.8) holds for $n = j+1$. The functions $f_1(t)$ and $f_2(t)$ are non-decreasing. So the induction hypothesis implies that

$$\begin{aligned}
(A_1A_2)^{j+1}\omega_1(t) &= A_1A_2((A_1A_2)^j\omega_1(t)) \\
&\le \frac{f_1(t)f_2(t)\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \int_{t_1}^{t} \left(\ln\frac{t}{r}\right)^{(p+q)-1} (A_1A_2)^j\omega_1(r)\,\frac{\mathrm{d}r}{r} \\
&\le \frac{f_1(t)f_2(t)\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \int_{t_1}^{t} \left(\ln\frac{t}{r}\right)^{(p+q)-1} \left(\frac{(f_1(r)f_2(r)\Gamma(p)\Gamma(q))^j}{\Gamma(j(p+q))} \int_{t_1}^{r} \left(\ln\frac{r}{s}\right)^{j(p+q)-1} \omega_1(s)\,\frac{\mathrm{d}s}{s}\right) \frac{\mathrm{d}r}{r} \\
&\le \frac{(f_1(t)f_2(t)\Gamma(p)\Gamma(q))^{j+1}}{\Gamma(p+q)\Gamma(j(p+q))} \int_{t_1}^{t} \left(\ln\frac{t}{r}\right)^{(p+q)-1} \left(\int_{t_1}^{r} \left(\ln\frac{r}{s}\right)^{j(p+q)-1} \omega_1(s)\,\frac{\mathrm{d}s}{s}\right) \frac{\mathrm{d}r}{r} \\
&\le \frac{(f_1(t)f_2(t)\Gamma(p)\Gamma(q))^{j+1}}{\Gamma(p+q)\Gamma(j(p+q))} \int_{t_1}^{t} \left(\int_{s}^{t} \left(\ln\frac{t}{r}\right)^{(p+q)-1} \left(\ln\frac{r}{s}\right)^{j(p+q)-1} \omega_1(s)\,\frac{\mathrm{d}r}{r}\right) \frac{\mathrm{d}s}{s} \\
&\le \frac{(f_1(t)f_2(t)\Gamma(p)\Gamma(q))^{j+1}}{\Gamma(p+q)\Gamma(j(p+q))} \int_{t_1}^{t} \omega_1(s) \left(\int_{s}^{t} \left(\ln\frac{t}{r}\right)^{(p+q)-1} \left(\ln\frac{r}{s}\right)^{j(p+q)-1} \frac{\mathrm{d}r}{r}\right) \frac{\mathrm{d}s}{s}.
\end{aligned}$$

We again define a new variable $v = \dfrac{\ln(r) - \ln(s)}{\ln(t) - \ln(s)}$. According to the rule of the change of the variables and definition of the Beta function $B(p, q)$, we obtain

$$\begin{aligned}
&\int_{s}^{t} \left(\ln\frac{t}{r}\right)^{(p+q)-1} \left(\ln\frac{r}{s}\right)^{j(p+q)-1} \frac{\mathrm{d}r}{r} \\
&\qquad = \int_{s}^{t} \left(\ln\frac{t}{s}\right)^{(p+q)-1} \left(1 - \frac{\ln(r)-\ln(s)}{\ln(t)-\ln(s)}\right)^{(p+q)-1} \left(\ln\frac{r}{s}\right)^{j(p+q)-1} \frac{\mathrm{d}r}{r} \\
&\qquad = \int_{s}^{t} \left(\ln\frac{t}{s}\right)^{(p+q)-1} \left(1 - \frac{\ln(r)-\ln(s)}{\ln(t)-\ln(s)}\right)^{(p+q)-1} v^{(p+q)-1} \left(\ln\frac{t}{s}\right)^{j(p+q)-1} \frac{\mathrm{d}r}{r} \\
&\qquad \le \left(\ln\frac{t}{s}\right)^{(j+1)(p+q)-1} \int_{0}^{1} (1-v)^{(p+q)-1} v^{j(p+q)-1}\,\mathrm{d}v \\
&\qquad = \left(\ln\frac{t}{s}\right)^{(j+1)(p+q)-1} B(p+q, j(p+q))
\end{aligned}$$

$$= \left(\ln\frac{t}{s}\right)^{(j+1)(p+q)-1} \frac{\Gamma(p+q)\Gamma(j(p+q))}{\Gamma((j+1)(p+q))}.$$

Thus,

$$(A_1A_2)^{j+1}\omega_1(t) \leq \frac{(f_1(t)f_2(t)\Gamma(p)\Gamma(q))^{j+1}}{\Gamma((j+1)(p+q))} \int_{t_1}^{t} \left(\ln\frac{t}{r}\right)^{(j+1)(p+q)-1} \omega_1(r)\,\frac{\mathrm{d}r}{r}. \tag{3.11}$$

If we repeat the proof of (3.11) for $(A_2A_1)^{j+1}\omega_2(t)$, we obtain

$$(A_2A_1)^{j+1}\omega_2(t) \leq \frac{(f_1(t)f_2(t)\Gamma(p)\Gamma(q))^{j+1}}{\Gamma((j+1)(p+q))} \int_{t_1}^{t} \left(\ln\frac{t}{r}\right)^{(j+1)(p+q)-1} \omega_2(r)\,\frac{\mathrm{d}r}{r}. \tag{3.12}$$

This means that (3.8) has been proven. Finally, we prove (3.9) as follows. By the hypothesis of theorem, we know that there is a positive constant M such that $f_1(t) \leq M$ and $f_2(t) \leq M$ for each $t \in [t_1, t_2]$. In this case, from (3.8), one can write

$$(A_1A_2)^n\omega_1(t) \leq \frac{\left(M^2\Gamma(p)\Gamma(q)\right)^n}{\Gamma(n(p+q))} \int_{t_1}^{t} \left(\ln\frac{t}{r}\right)^{n(p+q)-1} \omega_1(r)\,\frac{\mathrm{d}r}{r}.$$

By considering

$$a_n := \frac{\left(M^2\Gamma(p)\Gamma(q)\right)^n}{\Gamma(n(p+q))},$$

and using the ratio test and an application of the asymptotic approximation property, the series $\sum_{n=1}^{\infty} a_n$ is convergent because

$$\lim_{n\to\infty} \frac{\Gamma(n(p+q))}{\Gamma((n+1)(p+q))} = 0.$$

Therefore, it is found that $\lim_{n\to\infty}(A_1A_2)^n\omega_1(t) = 0$. Similarly, $\lim_{n\to\infty}(A_2A_1)^n\omega_2(t) = 0$. This means that (3.9) is valid.

In the last step, in the basis of the inequality (3.6), we have

$$\omega_1(t) \leq \sum_{m=0}^{n-1}(A_1A_2)^m h_1(t) + \sum_{m=0}^{n-1}(A_1A_2)^m A_1h_2(t) + (A_1A_2)^n\omega_1(t)$$

$$= h_1(t) + A_1h_2(t) + \sum_{m=1}^{n-1}(A_1A_2)^m h_1(t) + \sum_{m=1}^{n-1}(A_1A_2)^m A_1h_2(t) + (A_1A_2)^n\omega_1(t).$$

Then, as n goes to infinite, we obtain

$$\begin{aligned}
\omega_1(t) &\leq h_1(t) + A_1 h_2(t) + \sum_{m=1}^{\infty} (A_1 A_2)^m h_1(t) + \sum_{m=1}^{\infty} (A_1 A_2)^m A_1 h_2(t) \\
&\leq h_1(t) + f_1(t) \int_{t_1}^{t} \left(\ln \frac{t}{r}\right)^{p-1} h_2(r) \frac{\mathrm{d}r}{r} + \sum_{m=1}^{\infty} (A_1 A_2)^m \left(h_1(t) + A_1 h_2(t)\right) \\
&\leq h_1(t) + f_1(t) \int_{t_1}^{t} \left(\ln \frac{t}{r}\right)^{p-1} h_2(r) \frac{\mathrm{d}r}{r} \\
&+ \sum_{m=1}^{\infty} \frac{(f_1(t) f_2(t) \Gamma(p) \Gamma(q))^m}{\Gamma(m(p+q))} \int_{t_1}^{t} \left(\ln \frac{t}{r}\right)^{m(p+q)-1} \left(h_1(r) + f_1(r) \int_{t_1}^{r} \left(\ln \frac{r}{s}\right)^{p-1} h_2(s) \frac{\mathrm{d}s}{s}\right) \frac{\mathrm{d}r}{r},
\end{aligned}$$

proving the inequality (3.1). We obtain (3.2) in a similar way and so, the proof is completed. □

In view of the above theorem, we now prove the coupled Hadamard-Gronwall-Bellman-type inequality as follows.

Theorem 3.2. *Assume that all hypotheses of the previous theorem are satisfied and also, $h_1(t)$ and $h_2(t)$ are non-decreasing on $[t_1, t_2]$. Then*

$$\omega_1(t) \leq \left(h_1(t) + h_2(t) f_1(t) \frac{\left(\ln \frac{t}{t_1}\right)^p}{p}\right) \mathbb{E}_{p+q}\left((f_1(t) f_2(t) \Gamma(p) \Gamma(q) \left(\ln \tfrac{t}{t_1}\right)^{p+q}\right), \tag{3.13}$$

where $\mathbb{E}_{p+q}$ stands for the Mittag-Leffler function of index $p+q$, and

$$\omega_2(t) \leq \left(h_2(t) + h_1(t) f_2(t) \frac{\left(\ln \frac{t}{t_1}\right)^q}{q}\right) \mathbb{E}_{p+q}\left(f_1(t) f_2(t) \Gamma(p) \Gamma(q) \left(\ln \tfrac{t}{t_1}\right)^{p+q}\right). \tag{3.14}$$

Proof. Since $r < t$ and $h_2(t)$ is non-decreasing, so $h_2(r) < h_2(t)$ and we have

$$\int_{t_1}^{t} \left(\ln \frac{t}{r}\right)^{p-1} h_2(r) \frac{\mathrm{d}r}{r} \leq h_2(t) \int_{t_1}^{t} \left(\ln \frac{t}{r}\right)^{p-1} \frac{\mathrm{d}r}{r} = h_2(t) \frac{\left(\ln \frac{t}{t_1}\right)^p}{p}. \tag{3.15}$$

Therefore, (3.1) and (3.15) imply that

$$\begin{aligned}
\omega_1(t) &\leq \left(h_1(t) + h_2(t) f_1(t) \frac{\left(\ln \frac{t}{t_1}\right)^p}{p}\right) \left(1 + \sum_{m=1}^{\infty} \frac{(f_1(t) f_2(t) \Gamma(p) \Gamma(q))^m}{\Gamma(m(p+q))} \int_{t_1}^{t} \left(\ln \frac{t}{r}\right)^{m(p+q)-1} \frac{\mathrm{d}r}{r}\right) \\
&= \left(h_1(t) + h_2(t) f_1(t) \frac{\left(\ln \frac{t}{t_1}\right)^p}{p}\right) \left(1 + \sum_{m=1}^{\infty} \frac{(f_1(t) f_2(t) \Gamma(p) \Gamma(q))^m}{\Gamma(m(p+q)+1)} \left(\ln \tfrac{t}{t_1}\right)^{m(p+q)}\right)
\end{aligned}$$

$$= \left(h_1(t) + h_2(t)f_1(t)\frac{\left(\ln\frac{t}{t_1}\right)^p}{p}\right)\sum_{m=0}^{\infty}\frac{(f_1(t)f_2(t)\Gamma(p)\Gamma(q))^m}{\Gamma(m(p+q)+1)}\left(\ln\frac{t}{t_1}\right)^{m(p+q)}$$

$$= \left(h_1(t) + h_2(t)f_1(t)\frac{\left(\ln\frac{t}{t_1}\right)^p}{p}\right)\mathbb{E}_{p+q}\left(f_1(t)f_2(t)\Gamma(p)\Gamma(q)\left(\ln\frac{t}{t_1}\right)^{p+q}\right).$$

This proves (3.13). To prove (3.14), we proceed it similar to above. Therefore, the proof of the coupled Hadamard-Gronwall-Bellman-type inequality is completed. □

In the following, we provide a new extended form of the Hadamard-Gronwall-Bellman-type inequality.

Theorem 3.3. *Let $p, q > 0$. Suppose that:*

(G1) Two functions $\omega(t)$ and $h(t)$ are integrable with non-negative values on the domain $[t_1, t_2]$.
(G2) Two non-negative-valued functions $f_1(t)$ and $f_2(t)$ are non-decreasing provided that $f_1(t) \le M_1$ and $f_2(t) \le M_2$ for all $t \in [t_1, t_2]$ and for some constants $M_1, M_2 > 0$.

Then

$$\omega(t) \le h(t) + \int_{t_1}^{t}\sum_{m=1}^{\infty} f^m(t)\sum_{i=0}^{m}\frac{\binom{m}{i}(\Gamma(p))^{m-i}(\Gamma(q))^i}{\Gamma((m-i)p+iq)}\left(\ln\frac{t}{r}\right)^{((m-i)p+iq)-1} h(r)\,\frac{\mathrm{d}r}{r}, \tag{3.16}$$

where $\binom{m}{i} = \dfrac{m(m-1)(m-2)\dots(m-i+1)}{i!}$ *and $f(t) = f_1(t) + f_2(t)$, if*

$$\omega(t) \le h(t) + f_1(t)\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{p-1}\omega(r)\,\frac{\mathrm{d}r}{r} + f_2(t)\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{q-1}\omega(r)\,\frac{\mathrm{d}r}{r}. \tag{3.17}$$

Proof. By the hypothesis, since $f(t) = f_1(t) + f_2(t)$, so $f_1(t) \le f(t)$ and $f_2(t) \le f(t)$. Hence,

$$\omega(t) \le h(t) + f(t)\int_{t_1}^{t}\left(\left(\ln\frac{t}{r}\right)^{p-1} + \left(\ln\frac{t}{r}\right)^{q-1}\right)\omega(r)\,\frac{\mathrm{d}r}{r}.$$

Now, put

$$A\omega(t) = f(t)\int_{t_1}^{t}\left(\left(\ln\frac{t}{r}\right)^{p-1} + \left(\ln\frac{t}{r}\right)^{q-1}\right)\omega(r)\,\frac{\mathrm{d}r}{r}.$$

Naturally, we have

$$\omega(t) \le h(t) + A\omega(t). \tag{3.18}$$

The monotonicity property of the operator A and the inequality (3.18) imply that

$$\begin{aligned}\omega(t) &\le h(t) + A\omega(t) \le h(t) + A\left(h(t) + A\omega(t)\right)\\ &= h(t) + Ah(t) + A^2\omega(t)\end{aligned}$$

$$\leq h(t) + Ah(t) + A^2\left(h(t) + Ah(t) + A^2\omega(t)\right)$$

$$= h(t) + Ah(t) + A^2h(t) + A^3h(t) + A^4\omega(t).$$

If we continue this iterative scheme, then for $n \in \mathbb{N}$, we have

$$\omega(t) \leq \Big(\sum_{m=0}^{n-1} A^m h(t)\Big) + A^n\omega(t), \quad t \in [t_1, t_2], \tag{3.19}$$

where $A^0h(t) = h(t)$. In the following, we will prove that

$$A^n\omega(t) \leq f^n(t) \int_{t_1}^{t} \sum_{i=0}^{n} \frac{\binom{n}{i}(\Gamma(p))^{n-i}(\Gamma(q))^i}{\Gamma((n-i)p+iq)} \left(\ln\frac{t}{r}\right)^{((n-i)p+iq)-1} \omega(r)\,\frac{\mathrm{d}r}{r}, \tag{3.20}$$

for $t \in [t_1, t_2]$, and

$$\lim_{n\to\infty} A^n\omega(t) = 0. \tag{3.21}$$

Based on the mathematical induction, we know that (3.20) is obviously valid for $n = 1$. As an induction hypothesis, let (3.20) be also true for $n = j$; that is,

$$A^j\omega(t) \leq f^j(t) \int_{t_1}^{t} \sum_{i=0}^{j} \frac{\binom{j}{i}(\Gamma(p))^{j-i}(\Gamma(q))^i}{\Gamma((j-i)p+iq)} \left(\ln\frac{t}{r}\right)^{((j-i)p+iq)-1} \omega(r)\,\frac{\mathrm{d}r}{r}.$$

Now, we put $n = j + 1$. Moreover, $f(t)$ is non-decreasing since $f_1(t)$ and $f_2(t)$ are non-decreasing. Then, we have

$$\begin{aligned}
A^{j+1}\omega(t) &= A(A^j\omega(t)) \\
&\leq f(t) \int_{t_1}^{t} \left(\left(\ln\frac{t}{r}\right)^{p-1} + \left(\ln\frac{t}{r}\right)^{q-1}\right)(A^j\omega(r))\,\frac{\mathrm{d}r}{r} \\
&\leq f(t) \int_{t_1}^{t} \left(\left(\ln\frac{t}{r}\right)^{p-1} + \left(\ln\frac{t}{r}\right)^{q-1}\right)\left(f^j(r)\int_{t_1}^{r} \sum_{i=0}^{j} \frac{\binom{j}{i}(\Gamma(p))^{j-i}(\Gamma(q))^i}{\Gamma((j-i)p+iq)} \left(\ln\frac{r}{s}\right)^{((j-i)p+iq)-1} \omega(s)\,\frac{\mathrm{d}s}{s}\right)\frac{\mathrm{d}r}{r} \\
&\leq f^{j+1}(t) \int_{t_1}^{t} \left(\left(\ln\frac{t}{r}\right)^{p-1} + \left(\ln\frac{t}{r}\right)^{q-1}\right)\left(\int_{t_1}^{r} \sum_{i=0}^{j} \frac{\binom{j}{i}(\Gamma(p))^{j-i}(\Gamma(q))^i}{\Gamma((j-i)p+iq)} \left(\ln\frac{r}{s}\right)^{((j-i)p+iq)-1} \omega(s)\,\frac{\mathrm{d}s}{s}\right)\frac{\mathrm{d}r}{r} \\
&\leq f^{j+1}(t) \int_{t_1}^{t} \Bigg[\int_{s}^{t} \sum_{i=0}^{j} \frac{\binom{j}{i}(\Gamma(p))^{j-i}(\Gamma(q))^i}{\Gamma((j-i)p+iq)} \left(\ln\frac{t}{r}\right)^{p-1}\left(\ln\frac{r}{s}\right)^{((j-i)p+iq)-1} \\
&\quad + \sum_{i=0}^{j} \frac{\binom{j}{i}(\Gamma(p))^{j-i}(\Gamma(q))^i}{\Gamma((j-i)p+iq)} \left(\ln\frac{t}{r}\right)^{q-1}\left(\ln\frac{r}{s}\right)^{((j-i)p+iq)-1} \frac{\mathrm{d}r}{r}\Bigg]\omega(s)\,\frac{\mathrm{d}s}{s}
\end{aligned}$$

$$
\begin{aligned}
&= f^{j+1}(t)\int_{t_1}^{t}\Bigg[\sum_{i=0}^{j}\frac{\binom{j}{i}(\Gamma(p))^{j+1-i}(\Gamma(q))^{i}}{\Gamma((j+1-i)p+iq)}\left(\ln\frac{t}{r}\right)^{((j+1-i)p+iq)-1}\\
&\quad+\sum_{i=0}^{j}\frac{\binom{j}{i}(\Gamma(p))^{j-i}(\Gamma(q))^{i+1}}{\Gamma((j-i)p+(i+1)q)}\left(\ln\frac{t}{r}\right)^{((j-i)p+(i+1)q)-1}\Bigg]\omega(r)\,\frac{\mathrm{d}r}{r}\\
&= f^{j+1}(t)\int_{t_1}^{t}\Bigg[\sum_{i=0}^{j}\frac{\binom{j}{i}(\Gamma(p))^{j+1-i}(\Gamma(q))^{i}}{\Gamma((j+1-i)p+iq)}\left(\ln\frac{t}{r}\right)^{((j+1-i)p+iq)-1}\\
&\quad+\sum_{i=1}^{j+1}\frac{\binom{j}{i-1}(\Gamma(p))^{j+1-i}(\Gamma(q))^{i}}{\Gamma((j+1-i)p+iq)}\left(\ln\frac{t}{r}\right)^{((j+1-i)p+iq)-1}\Bigg]\omega(r)\,\frac{\mathrm{d}r}{r}\\
&= f^{j+1}(t)\int_{t_1}^{t}\Bigg[\frac{\binom{j}{0}(\Gamma(p))^{j+1}}{\Gamma((j+1)p)}\left(\ln\frac{t}{r}\right)^{(j+1)p)-1}\\
&\quad+\sum_{i=1}^{j}\frac{\left(\binom{j}{i}+C_j^{i-1}\right)(\Gamma(p))^{j+1-i}(\Gamma(q))^{i}}{\Gamma((j+1-i)p+iq)}\left(\ln\frac{t}{r}\right)^{((j+1-i)p+iq)-1}\\
&\quad+\frac{\binom{j}{j}(\Gamma(q))^{j+1}}{\Gamma((j+1)q)}\left(\ln\frac{t}{r}\right)^{(j+1)q)-1}\Bigg]\omega(r)\,\frac{\mathrm{d}r}{r}\\
&= f^{j+1}(t)\int_{t_1}^{t}\sum_{i=0}^{j+1}\frac{\binom{j+1}{i}(\Gamma(p))^{j+1-i}(\Gamma(q))^{i}}{\Gamma((j+1-i)p+iq)}\left(\ln\frac{t}{r}\right)^{((j+1-i)p+iq)-1}\omega(r)\,\frac{\mathrm{d}r}{r}.
\end{aligned}
$$

In view of the above computations, we follow that (3.20) is valid.

Now, to prove (3.21), by the hypothesis, we know that $f_1(t) \leq M_1$ and $f_2(t) \leq M_2$ for all $t \in [t_1, t_2]$ and for some constants $M_1, M_2 > 0$. In this case, since $f(t)$ is the sum of two functions $f_1(t)$ and $f_2(t)$, so we get $f(t) \leq M_1 + M_2$. Therefore, according to (3.20), one can write

$$
A^n\omega(t) \leq \int_{t_1}^{t}\sum_{i=0}^{n}\frac{\binom{n}{i}(M_1+M_2)^n(\Gamma(p))^{n-i}(\Gamma(q))^{i}}{\Gamma((n-i)p+iq)}\left(\ln\frac{t}{r}\right)^{((n-i)p+iq)-1}\omega(r)\,\frac{\mathrm{d}r}{r}.
$$

For the Gamma function, we know that the Stirling's formula

$$
\Gamma(\alpha+1) \approx (2\pi\alpha)^{\frac{1}{2}}\left(\frac{\alpha}{e}\right)^{\alpha}, \quad \alpha > 0
$$

holds. On the other hand, for the finite integrals, the first mean-value theorem implies the existence of a constant $b \in [t_1, t_2]$ so that

$$
\lim_{n\to\infty} A^n\omega(t)
$$

$$
\begin{aligned}
&\le \lim_{n\to\infty} \omega(b) \sum_{i=0}^{n} \frac{\binom{n}{i}(M_1+M_2)^n(\Gamma(p))^{n-i}(\Gamma(q))^i}{\Gamma((n-i)p+iq)} \int_{t_1}^{t} \left(\ln\frac{t}{r}\right)^{((n-i)p+iq)-1} \frac{\mathrm{d}r}{r} \\
&\le \lim_{n\to\infty} \omega(b) \sum_{i=0}^{n} \frac{\binom{n}{i}(M_1+M_2)^n(\Gamma(p))^{n-i}(\Gamma(q))^i}{\Gamma((n-i)p+iq+1)} \left(\ln\frac{t_2}{t_1}\right)^{(n-i)p+iq} \\
&= \lim_{n\to\infty} \omega(b) \sum_{i=0}^{n} \frac{\binom{n}{i}(M_1+M_2)^n \left(\Gamma(p)\left(\ln(\frac{t_2}{t_1})\right)^p\right)^{n-i} \left(\Gamma(q)\left(\ln\frac{t_2}{t_1}\right)^q\right)^i}{\Gamma((n-i)p+iq+1)} \\
&= \lim_{n\to\infty} \omega(b) \sum_{i=0}^{n} \frac{\binom{n}{i}(M_1+M_2)^n \left(\Gamma(p)\left(\ln(\frac{t_2}{t_1})\right)^p\right)^{n-i} \left(\Gamma(q)\left(\ln\frac{t_2}{t_1}\right)^q\right)^i}{(2\pi((n-i)p+iq))^{\frac{1}{2}}\left(\frac{(n-i)p+iq}{e}\right)^{(n-i)p+iq}} \\
&= \lim_{n\to\infty} \omega(b) \sum_{i=0}^{n} \frac{\binom{n}{i}(M_1+M_2)^n}{(2\pi((n-i)p+iq))^{\frac{1}{2}}} \left(\frac{\Gamma(p)\left(\ln\frac{t_2}{t_1}\right)^p}{\left(\frac{(n-i)p+iq}{e}\right)^p}\right)^{n-i} \left(\frac{\Gamma(q)\left(\ln\frac{t_2}{t_1}\right)^q}{\left(\frac{(n-i)p+iq}{e}\right)^q}\right)^{i}.
\end{aligned}
$$

Therefore, we obtain

$$
\lim_{n\to\infty} A^n\omega(t) \le \lim_{n\to\infty} \omega(b)\frac{((M_1+M_2)(\Delta_1+\Delta_2))^n}{[2n\pi z^*]^{\frac{1}{2}}},
$$

by assuming

$$
\Delta_1 := \left(\frac{\Gamma(p)\left(\ln(\frac{t_2}{t_1})\right)^p}{(\frac{(n-i)p+iq}{e})^p}\right), \qquad \Delta_2 := \left(\frac{\Gamma(q)\left(\ln(\frac{t_2}{t_1})\right)^q}{(\frac{(n-i)p+iq}{e})^q}\right),
$$

and $z^* := \min\{p, q\}$.

By letting $(M_1 + M_2)(\Delta_1 + \Delta_2) < 1$, we get $((M_1 + M_2)(\Delta_1 + \Delta_2))^n \to 0$, as $n \to \infty$. Hence, one can find that $\lim\limits_{n\to\infty} A^n\omega(t) = 0$ and so, (3.21) is proved.

Finally, again from (3.19), we have

$$
\omega(t) \le \Big(\sum_{m=0}^{n-1} A^m h(t)\Big) + A^n\omega(t) = h(t) + \Big(\sum_{m=1}^{n-1} A^m h(t)\Big) + A^n\omega(t).
$$

When n goes to infinite, then

$$
\begin{aligned}
\omega(t) &\le h(t) + \sum_{m=1}^{\infty} A^m h(t) \\
&\le h(t) + \sum_{m=1}^{\infty} f^m(t) \int_{t_1}^{t} \sum_{i=0}^{m} \frac{\binom{m}{i}(\Gamma(p))^{m-i}(\Gamma(q))^i}{\Gamma((m-i)p+iq)} \left(\ln\frac{t}{r}\right)^{((m-i)p+iq)-1} h(r)\frac{\mathrm{d}r}{r}
\end{aligned}
$$

$$\leq h(t) + \int_{t_1}^{t} \sum_{m=1}^{\infty} f^m(t) \sum_{i=0}^{m} \frac{\binom{m}{i}(\Gamma(p))^{m-i}(\Gamma(q))^i}{\Gamma((m-i)p+iq)} \left(\ln \frac{t}{r}\right)^{((m-i)p+iq)-1} h(r) \frac{\mathrm{d}r}{r}.$$

This completes the proof. □

In view of the above theorem, we now prove the extended form of the Hadamard-Gronwall-Bellman-type inequality as follows.

Theorem 3.4. *Assume that all hypotheses of Theorem 3.3 are satisfied and assume that $h(t)$ is non-decreasing function on $[t_1, t_2]$. Also, to find a uniform bound for the terms in the Mittag-Leffler function, we consider $z^* = \min\{p, q\}$. Then, we have*

$$\omega(t) \leq h(t)\mathbb{E}_{z^*}\left[f(t)\left(\Gamma(p)\left(\ln \frac{t}{t_1}\right)^p + \Gamma(q)\left(\ln \frac{t}{t_1}\right)^q\right)\right]. \tag{3.22}$$

Proof. Since $r < t$ and $h(t)$ is non-decreasing, so $h(r) < h(t)$ and we have

$$\omega(t) \leq h(t) + h(t) \int_{t_1}^{t} \sum_{m=1}^{\infty} f^m(t) \sum_{i=0}^{m} \frac{\binom{m}{i}(\Gamma(p))^{m-i}(\Gamma(q))^i}{\Gamma((m-i)p+iq)} \left(\ln \frac{t}{r}\right)^{((m-i)p+iq)-1} \frac{\mathrm{d}r}{r}.$$

Hence,

$$\begin{aligned}
\omega(t) &\leq h(t)\left[1 + \int_{t_1}^{t} \sum_{m=1}^{\infty} f^m(t) \sum_{i=0}^{m} \frac{\binom{m}{i}(\Gamma(p))^{m-i}(\Gamma(q))^i}{\Gamma((m-i)p+iq)} \left(\ln \frac{t}{r}\right)^{((m-i)p+iq)-1} \frac{\mathrm{d}r}{r}\right] \\
&= h(t)\left[1 + \sum_{m=1}^{\infty} f^m(t) \sum_{i=0}^{m} \frac{\binom{m}{i}(\Gamma(p))^{m-i}(\Gamma(q))^i}{\Gamma((m-i)p+iq+1)} \left(\ln \frac{t}{t_1}\right)^{(m-i)p+iq}\right] \\
&\leq h(t)\left[1 + \sum_{m=1}^{\infty} \frac{f^m(t) \sum_{i=0}^{m} \binom{m}{i} \left(\Gamma(p)\left(\ln \frac{t}{t_1}\right)^p\right)^{m-i} \left(\Gamma(q)\left(\ln \frac{t}{t_1}\right)^q\right)^i}{\Gamma(mz^*+1)}\right) \\
&= h(t)\left(1 + \sum_{m=1}^{\infty} \frac{f^m(t)\left(\Gamma(p)\left(\ln \frac{t}{t_1}\right)^p + \Gamma(q)\left(\ln \frac{t}{t_1}\right)^q\right)^m}{\Gamma(mz^*+1)}\right) \\
&= h(t) \sum_{m=0}^{\infty} \frac{f^m(t)\left(\Gamma(p)\left(\ln \frac{t}{t_1}\right)^p + \Gamma(q)\left(\ln \frac{t}{t_1}\right)^q\right)^m}{\Gamma(mz^*+1)} \\
&= h(t)\mathbb{E}_{z^*}\left[f(t)\left(\Gamma(p)\left(\ln \frac{t}{t_1}\right)^p + \Gamma(q)\left(\ln \frac{t}{t_1}\right)^q\right)\right].
\end{aligned}$$

This completes the proof of the extended form of the Hadamard-Gronwall-Bellman-type inequality (3.22). □

4. Application: Existence and stability of solutions

This section aims to show an application of the coupled Hadamard-Gronwall-Bellman-type inequalities (3.13) and (3.14) and also, the extended form of the Hadamard-Gronwall-Bellman-type inequality (3.22) in the existence and stability theories. To do this, we choose a coupled delay system and a fractional damped system, because most of real-world models have delay and damped properties, and this helps us to guarantee our theoretical results on such applied systems.

4.1. Application to a coupled delay system

Let $p, q \in (0, 1]$ and $\phi_1, \phi_2 \in C([t_1, t_2]\times\mathbb{R}^2, \mathbb{R})$ and $\omega_1^*, \omega_2^* \in C([t_1-\tau, t_1], \mathbb{R})$. As the first application, we consider a Caputo-Hadamard coupled delay system given by

$$\begin{cases} {}^{CH}D_{t_1}^{p}\omega_1(t) = \phi_1(t, \omega_2(t), \omega_2(t-\tau)), & t \in [t_1, t_2], \\ {}^{CH}D_{t_1}^{q}\omega_2(t) = \phi_2(t, \omega_1(t), \omega_1(t-\tau)), & t \in [t_1, t_2], \\ \omega_1(t) = \omega_1^*(t), \qquad \omega_2(t) = \omega_2^*(t), & t \in [t_1-\tau, t_1]. \end{cases} \tag{4.1}$$

Lemma 2.5 easily follows that the coupled $(\omega_1(t), \omega_2(t))$ is a solution of (4.1) if and only if it satisfies the Hadamard integral sysstem

$$\begin{cases} \omega_1(t) = \omega_1^*(t_1) + \dfrac{1}{\Gamma(p)}\displaystyle\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{p-1}\phi_1(r, \omega_2(r), \omega_2(r-\tau))\,\frac{\mathrm{d}r}{r}, & t \in [t_1, t_2], \\ \omega_2(t) = \omega_2^*(t_1) + \dfrac{1}{\Gamma(q)}\displaystyle\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{q-1}\phi_2(r, \omega_1(r), \omega_1(r-\tau))\,\frac{\mathrm{d}r}{r}, & t \in [t_1, t_2], \\ \omega_1(t) = \omega_1^*(t), \qquad \omega_2(t) = \omega_2^*(t), & t \in [t_1-\tau, t_1]. \end{cases} \tag{4.2}$$

In the following, we consider the Banach space $W \times W$ under the norm $\|(\omega_1, \omega_2)\| = \|\omega_1\| + \|\omega_2\|$, where $\|\omega_1\| = \sup_t(|\omega_1(t)| + |\omega_1(t-\tau)|)$. The next result proves the existence of unique solutions for (4.1).

Theorem 4.1. *Let W be a Banach space and suppose that:*

(G1) $\phi_1, \phi_2 \in C([t_1, t_2]\times\mathbb{R}^2, \mathbb{R})$ *and* $\omega_1^*, \omega_2^* \in C([t_1-\tau, t_1], \mathbb{R})$*;*
(G2) There exists $L_1, L_2 > 0$ *such that for all* $\omega_1, \omega_2, \bar{\omega}_1, \bar{\omega}_2 \in W$*, we have*

$$\left|\phi_1(t, \omega_1, \omega_2) - \phi_1(t, \bar{\omega}_1, \bar{\omega}_2)\right| \leq L_1\left(|\omega_1 - \bar{\omega}_1| + |\omega_2 - \bar{\omega}_2|\right),$$

$$\left|\phi_2(t, \omega_1, \omega_2) - \phi_2(t, \bar{\omega}_1, \bar{\omega}_2)\right| \leq L_2\left(|\omega_1 - \bar{\omega}_1| + |\omega_2 - \bar{\omega}_2|\right);$$

(G3) Let $M := \max\left\{\dfrac{L_1\left(\ln(\frac{t_2}{t_1})\right)^p}{\Gamma(p+1)}, \dfrac{L_2\left(\ln(\frac{t_2}{t_1})\right)^q}{\Gamma(q+1)}\right\} < 1.$

Then, the Caputo-Hadamard coupled delay system (4.1) *admits one and only one solution on* $W\times W$*.*

Proof. Firstly, we define the operator

$$T(\omega_1,\omega_2)(t) := (T_1\omega_1(t), T_2\omega_2(t))$$

as follows

$$T_1\omega_1(t) = \begin{cases} \omega_1^*(t); & t \in [t_1-\tau, t_1], \\ \omega_1^*(t_1) + \dfrac{1}{\Gamma(p)} \displaystyle\int_{t_1}^{t} \left(\ln\frac{t}{r}\right)^{p-1} \phi_1(r,\omega_2(r),\omega_2(r-\tau))\,\frac{\mathrm{d}r}{r}; & t \in [t_1, t_2], \end{cases} \tag{4.3}$$

and

$$T_2\omega_2(t) = \begin{cases} \omega_2^*(t); & t \in [t_1-\tau, t_1], \\ \omega_2^*(t_1) + \dfrac{1}{\Gamma(q)} \displaystyle\int_{t_1}^{t} \left(\ln\frac{t}{r}\right)^{q-1} \phi_2(r,\omega_1(r),\omega_1(r-\tau))\,\frac{\mathrm{d}r}{r}; & t \in [t_1, t_2]. \end{cases} \tag{4.4}$$

For $t \in [t_1-\tau, t_1]$, we may write

$$\left|T_1\omega_1(t) - T_1\bar{\omega}_1(t)\right| = 0,$$

and

$$\left|T_2\omega_2(t) - T_2\bar{\omega}_2(t)\right| = 0,$$

if $(\omega_1,\omega_2),(\bar{\omega}_1,\bar{\omega}_2) \in C([t_1-\tau,t_1],\mathbb{R}) \times C([t_1-\tau,t_1],\mathbb{R})$.

Now, for all $t \in [t_1,t_2]$ and for each $(\omega_1,\omega_2),(\bar{\omega}_1,\bar{\omega}_2) \in C([t_1,t_2],\mathbb{R}) \times C([t_1,t_2],\mathbb{R})$, we estimate

$$\begin{aligned} \left|T_1\omega_1(t) - T_1\bar{\omega}_1(t)\right| &= \frac{1}{\Gamma(p)}\left| \int_{t_1}^{t} \left(\ln\frac{t}{r}\right)^{p-1} \phi_1(r,\omega_2(r),\omega_2(r-\tau))\,\frac{\mathrm{d}r}{r} - \int_{t_1}^{t} \left(\ln\frac{t}{r}\right)^{p-1} \phi_1(r,\bar{\omega}_2(r),\bar{\omega}_2(r-\tau))\,\frac{\mathrm{d}r}{r}\right| \\ &\le \frac{1}{\Gamma(p)} \int_{t_1}^{t} \left(\ln\frac{t}{r}\right)^{p-1} \left|\phi_1(r,\omega_2(r),\omega_2(r-\tau)) - \phi_1(r,\bar{\omega}_2(r),\bar{\omega}_2(r-\tau))\right| \frac{\mathrm{d}r}{r} \\ &\le \frac{L_1}{\Gamma(p)} \int_{t_1}^{t} \left(\ln\frac{t}{r}\right)^{p-1} \left(|\omega_2(r)-\bar{\omega}_2(r)| + |\omega_2(r-\tau)-\bar{\omega}_2(r-\tau)|\right) \frac{\mathrm{d}r}{r} \\ &\le \frac{L_1\left(\ln\frac{t}{t_1}\right)^p}{\Gamma(p+1)}\|\omega_2-\bar{\omega}_2\| \\ &\le \frac{L_1\left(\ln\frac{t_2}{t_1}\right)^p}{\Gamma(p+1)}\|\omega_2-\bar{\omega}_2\|. \end{aligned} \tag{4.5}$$

In a similar manner, it yields that

$$\left|T_2\omega_2(t) - T_2\bar{\omega}_2(t)\right| \le \frac{L_2\left(\ln(\frac{t_2}{t_1})\right)^q}{\Gamma(q+1)}\|\omega_1-\bar{\omega}_1\|. \tag{4.6}$$

The inequalities (4.5) and (4.6) imply that

$$
\begin{aligned}
\left\|T(\omega_1,\omega_2)(t)-T(\bar{\omega}_1,\bar{\omega}_2)(t)\right\| &= \left\|T_1\omega_1-T_1\bar{\omega}_1\right\|+\left\|T_2\omega_2-T_2\bar{\omega}_2\right\| \\
&\le \frac{L_1\left(\ln\frac{t_2}{t_1}\right)^p}{\Gamma(p+1)}\|\omega_2-\bar{\omega}_2\|+\frac{L_2\left(\ln\frac{t_2}{t_1}\right)^q}{\Gamma(q+1)}\|\omega_1-\bar{\omega}_1\| \\
&\le \max\left\{\frac{L_1\left(\ln(\frac{t_2}{t_1})\right)^p}{\Gamma(p+1)},\frac{L_2\left(\ln(\frac{t_2}{t_1})\right)^q}{\Gamma(q+1)}\right\}(\|\omega_1-\bar{\omega}_1\|+\|\omega_2-\bar{\omega}_2\|) \\
&= M\left\|(\omega_1-\bar{\omega}_1,\omega_2-\bar{\omega}_2)\right\| \\
&= M\left\|(\omega_1,\omega_2)-(\bar{\omega}_1,\bar{\omega}_2)\right\|.
\end{aligned}
$$

This implies that the operator T is acontraction by (G3). On the basis of the Banach fixed point theorem, T has a unique fixed point. So (4.2) implies that the Caputo-Hadamard coupled delay system (4.1) admits one and only one solution on $W\times W$. □

For proving the next theorem, we need to define the Ulam-Hyers stability: The Caputo-Hadamard coupled delay system (4.1) is Ulam-Hyers stable if there exists $c_1,c_2\in\mathbb{R}$ such that for all $\varepsilon>0$ and every $\bar{\omega}_1(t),\bar{\omega}_2(t)\in W$ with $(\bar{\omega}_1(t),\bar{\omega}_2(t))=(\omega_1^*(t),\omega_2^*(t))$ for $t\in[t_1-\tau,t_1]$, satisfying

$$
\begin{cases}
\left|{}^{CH}D_{t_1}^{p}\bar{\omega}_1(t)-\phi_1(t,\bar{\omega}_2(t),\bar{\omega}_2(t-\tau))\right|\le\varepsilon, & t\in[t_1,t_2],\\
\left|{}^{CH}D_{t_1}^{q}\bar{\omega}_2(t)-\phi_2(t,\bar{\omega}_1(t),\bar{\omega}_1(t-\tau))\right|\le\varepsilon, & t\in[t_1,t_2],
\end{cases}
$$

there exists $(\omega_1,\omega_2)\in W\times W$ as the solution of the delay system (4.1) so that

$$
\left|\bar{\omega}_1(t)-\omega_1(t)\right|\le c_1\varepsilon\quad\text{and}\quad\left|\bar{\omega}_2(t)-\omega_2(t)\right|\le c_2\varepsilon.
$$

Now, everything is ready for the next theorem.

Theorem 4.2. *The Caputo-Hadamard coupled delay system* (4.1) *is Ulam-Hyers stable under the conditions of Theorem 4.1.*

Proof. To prove this theorem, assume that $(\bar{\omega}_1(t),\bar{\omega}_2(t))\in W\times W$ is a solution for two inequalities

$$
\left|{}^{CH}D_{t_1}^{p}\bar{\omega}_1(t)-\phi_1(t,\bar{\omega}_2(t),\bar{\omega}_2(t-\tau))\right|\le\varepsilon, \tag{4.7}
$$

and

$$
\left|{}^{CH}D_{t_1}^{q}\bar{\omega}_2(t)-\phi_2(t,\bar{\omega}_1(t),\bar{\omega}_1(t-\tau))\right|\le\varepsilon, \tag{4.8}
$$

for all $\varepsilon>0$ and all $t\in[t_1,t_2]$, and also, assume that $(\omega_1(t),\omega_2(t))$ is the unique solution for the Caputo-Hadamard coupled delay system (4.1) which satisfies the conditions

$$
\omega_1(t)=\bar{\omega}_1(t)=\omega_1^*(t),\quad\omega_2(t)=\bar{\omega}_2(t)=\omega_2^*(t),
$$

for all $t \in [t_1 - \tau, t_1]$. We have

$$\omega_1(t) = \begin{cases} \bar{\omega}_1(t); & t \in [t_1 - \tau, t_1], \\ \bar{\omega}_1(t_1) + \dfrac{1}{\Gamma(p)} \displaystyle\int_{t_1}^{t} \left(\ln \frac{t}{r}\right)^{p-1} \phi_1(r, \omega_2(r), \omega_2(r-\tau)) \frac{\mathrm{d}r}{r}; & t \in [t_1, t_2], \end{cases}$$

and

$$\omega_2(t) = \begin{cases} \bar{\omega}_2(t); & t \in [t_1 - \tau, t_1], \\ \bar{\omega}_2(t_1) + \dfrac{1}{\Gamma(q)} \displaystyle\int_{t_1}^{t} \left(\ln \frac{t}{r}\right)^{q-1} \phi_2(r, \omega_1(r), \omega_1(r-\tau)) \frac{\mathrm{d}r}{r}; & t \in [t_1, t_2]. \end{cases}$$

These structures are guaranteed by Theorem 4.1.

By the definition, we know that $(\bar{\omega}_1(t), \bar{\omega}_2(t))$ satisfies the inequalities (4.7) and (4.8) if and only if there exists two functions $h_1(t), h_2(t) \in C([t_1, t_2], \mathbb{R})$ so that $|h_i(t)| \leq \varepsilon$ $(i = 1, 2)$, and

$$^{CH}D_{t_1}^{p} \bar{\omega}_1(t) - \phi_1(t, \bar{\omega}_2(t), \bar{\omega}_2(t-\tau)) = h_1(t), \tag{4.9}$$

$$^{CH}D_{t_1}^{q} \bar{\omega}_2(t) - \phi_2(t, \bar{\omega}_1(t), \bar{\omega}_1(t-\tau)) = h_2(t), \tag{4.10}$$

for all $t \in [t_1, t_2]$. If we apply the Hadamard fractional integral of order p on (4.9), then Lemma 2.5 implies that

$$\begin{aligned} \left| \bar{\omega}_1(t) - \bar{\omega}_1(t_1) - \frac{1}{\Gamma(p)} \int_{t_1}^{t} \left(\ln \frac{t}{r}\right)^{p-1} \phi_1(r, \bar{\omega}_2(r), \bar{\omega}_2(r-\tau)) \frac{\mathrm{d}r}{r} \right| &= \left| \frac{1}{\Gamma(p)} \int_{t_1}^{t} \left(\ln \frac{t}{r}\right)^{p-1} h_1(r) \frac{\mathrm{d}r}{r} \right| \\ &\leq \frac{1}{\Gamma(p)} \int_{t_1}^{t} \left(\ln \frac{t}{r}\right)^{p-1} |h_1(r)| \frac{\mathrm{d}r}{r} \\ &\leq \frac{\left(\ln \frac{t}{t_1}\right)^{p}}{\Gamma(p+1)} \varepsilon \\ &\leq \frac{\left(\ln \frac{t_2}{t_1}\right)^{p}}{\Gamma(p+1)} \varepsilon. \end{aligned} \tag{4.11}$$

In a similar manner,

$$\left| \bar{\omega}_2(t) - \bar{\omega}_2(t_1) - \frac{1}{\Gamma(q)} \int_{t_1}^{t} \left(\ln \frac{t}{r}\right)^{q-1} \phi_2(r, \bar{\omega}_1(r), \bar{\omega}_1(r-\tau)) \frac{\mathrm{d}r}{r} \right| \leq \frac{\left(\ln \frac{t_2}{t_1}\right)^{q}}{\Gamma(q+1)} \varepsilon. \tag{4.12}$$

On the other side, $|\omega_1(t) - \bar{\omega}_1(t)| = 0$ and $|\omega_2(t) - \bar{\omega}_2(t)| = 0$ clearly, for $t \in [t_1 - \tau, t_1]$. Also, for $t \in [t_1, t_1 + \tau]$, it gives

$$\left| \bar{\omega}_1(t) - \omega_1(t) \right| = \left| \bar{\omega}_1(t) - \bar{\omega}_1(t_1) - \frac{1}{\Gamma(p)} \int_{t_1}^{t} \left(\ln \frac{t}{r}\right)^{p-1} \phi_1(r, \omega_2(r), \omega_2(r-\tau)) \frac{\mathrm{d}r}{r} \right|$$

$$
\begin{aligned}
&\leq \left|\bar{\omega}_1(t)-\bar{\omega}_1(t_1)-\frac{1}{\Gamma(p)}\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{p-1}\phi_1(r,\bar{\omega}_2(r),\bar{\omega}_2(r-\tau))\,\frac{\mathrm{d}r}{r}\right|\\
&+\left|\frac{1}{\Gamma(p)}\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{p-1}\phi_1(r,\bar{\omega}_2(r),\bar{\omega}_2(r-\tau))\,\frac{\mathrm{d}r}{r}\right.\\
&\left.-\frac{1}{\Gamma(p)}\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{p-1}\phi_1(r,\omega_2(r),\omega_2(r-\tau))\,\frac{\mathrm{d}r}{r}\right| \qquad (4.13)\\
&\leq \left|\bar{\omega}_1(t)-\bar{\omega}_1(t_1)-\frac{1}{\Gamma(p)}\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{p-1}\phi_1(r,\bar{\omega}_2(r),\bar{\omega}_2(r-\tau))\,\frac{\mathrm{d}r}{r}\right|\\
&+\frac{1}{\Gamma(p)}\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{p-1}\left|\phi_1(r,\bar{\omega}_2(r),\bar{\omega}_2(r-\tau))-\phi_1(r,\omega_2(r),\omega_2(r-\tau))\right|\frac{\mathrm{d}r}{r}\\
&\leq \left|\bar{\omega}_1(t)-\bar{\omega}_1(t_1)-\frac{1}{\Gamma(p)}\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{p-1}\phi_1(r,\bar{\omega}_2(r),\bar{\omega}_2(r-\tau))\,\frac{\mathrm{d}r}{r}\right|\\
&+\frac{L_1}{\Gamma(p)}\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{p-1}\left(|\bar{\omega}_2(r)-\omega_2(r)|+|\bar{\omega}_2(r-\tau)-\omega_2(r-\tau)|\right)\,\frac{\mathrm{d}r}{r}.
\end{aligned}
$$

In a similar manner, for $t \in [t_1, t_1+\tau]$, we have

$$
\begin{aligned}
|\bar{\omega}_2(t)-\omega_2(t)| &\leq \left|\bar{\omega}_2(t)-\bar{\omega}_2(t_1)-\frac{1}{\Gamma(q)}\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{q-1}\phi_2(r,\bar{\omega}_1(r),\bar{\omega}_1(r-\tau))\,\frac{\mathrm{d}r}{r}\right|\\
&+\frac{L_2}{\Gamma(q)}\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{q-1}\left(|\bar{\omega}_1(r)-\omega_1(r)|+|\bar{\omega}_1(r-\tau)-\omega_1(r-\tau)|\right)\,\frac{\mathrm{d}r}{r}. \qquad (4.14)
\end{aligned}
$$

In this case, (4.11) and (4.13), (4.12) and (4.14), respectively, give

$$
|\bar{\omega}_1(t)-\omega_1(t)| \leq \frac{\left(\ln(\frac{t_2}{t_1})\right)^p}{\Gamma(p+1)}\varepsilon+\frac{L_1}{\Gamma(p)}\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{p-1}\|\bar{\omega}_2(r)-\omega_2(r)\|\,\frac{\mathrm{d}r}{r}, \qquad (4.15)
$$

and

$$
|\bar{\omega}_2(t)-\omega_2(t)| \leq \frac{\left(\ln(\frac{t_2}{t_1})\right)^q}{\Gamma(q+1)}\varepsilon+\frac{L_2}{\Gamma(q)}\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{q-1}\|\bar{\omega}_1(r)-\omega_1(r)\|\,\frac{\mathrm{d}r}{r}. \qquad (4.16)
$$

By the coupled Hadamard-Gronwall-Bellman-type inequality (Theorem 3.2), we find out from (4.15) and (4.16) that

$$
|\bar{\omega}_1(t)-\omega_1(t)| \leq \left[\frac{\left(\ln\frac{t_2}{t_1}\right)^p}{\Gamma(p+1)}\varepsilon+\frac{L_1}{\Gamma(p+1)}\frac{\left(\ln\frac{t_2}{t_1}\right)^q}{\Gamma(q+1)}\varepsilon\right]\mathbb{E}_{p+q}\left[L_1L_2\left(\ln\frac{t}{t_1}\right)^{p+q}\right].
$$

Hence, we obtain

$$|\bar{\omega}_1(t)-\omega_1(t)|\le\left(\frac{\left(\ln\frac{t_2}{t_1}\right)^p}{\Gamma(p+1)}+\frac{L_1}{\Gamma(p+1)}\frac{\left(\ln\frac{t_2}{t_1}\right)^q}{\Gamma(q+1)}\right)\mathbb{E}_{p+q}\left[L_1L_2\left(\ln\frac{t_1+\tau}{t_1}\right)^{p+q}\right]\varepsilon.$$

for all $t\in[t_1,t_1+\tau]$. Again, similarly, and on the same interval $[t_1,t_1+\tau]$, we obtain

$$|\bar{\omega}_2(t)-\omega_2(t)|\le\left(\frac{\left(\ln\frac{t_2}{t_1}\right)^q}{\Gamma(q+1)}+\frac{L_2}{\Gamma(q+1)}\frac{\left(\ln\frac{t_2}{t_1}\right)^p}{\Gamma(p+1)}\right)\mathbb{E}_{p+q}\left[L_1L_2\left(\ln\frac{t_1+\tau}{t_1}\right)^{p+q}\right]\varepsilon.$$

This time, for $t\in[t_1+\tau,t_2]$, we repeat the same proofs and obtain

$$\begin{aligned}|\bar{\omega}_1(t)-\omega_1(t)|\le&\frac{\left(\ln\frac{t_2}{t_1}\right)^p}{\Gamma(p+1)}\varepsilon+\frac{L_1}{\Gamma(p)}\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{p-1}|\bar{\omega}_2(r)-\omega_2(r)|\,\frac{\mathrm{d}r}{r}\\&+\frac{L_1}{\Gamma(p)}\int_{t_1+\tau}^{t}\left(\ln\frac{t}{r}\right)^{p-1}|\bar{\omega}_2(r-\tau)-\omega_2(r-\tau)|\,\frac{\mathrm{d}r}{r},\end{aligned}\tag{4.17}$$

and

$$\begin{aligned}|\bar{\omega}_2(t)-\omega_2(t)|\le&\frac{\left(\ln\frac{t_2}{t_1}\right)^q}{\Gamma(q+1)}\varepsilon+\frac{L_2}{\Gamma(q)}\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{q-1}|\bar{\omega}_1(r)-\omega_1(r)|\,\frac{\mathrm{d}r}{r}\\&+\frac{L_2}{\Gamma(q)}\int_{t_1+\tau}^{t}\left(\ln\frac{t}{r}\right)^{q-1}|\bar{\omega}_1(r-\tau)-\omega_1(r-\tau)|\,\frac{\mathrm{d}r}{r}.\end{aligned}\tag{4.18}$$

We set

$$v_1(t)=\max_{s\in[-\tau,0]}|\bar{\omega}_1(t+s)-\omega_1(t+s)|,$$

and

$$v_2(t)=\max_{s\in[-\tau,0]}|\bar{\omega}_2(t+s)-\omega_2(t+s)|.$$

The inequalities (4.17) and (4.18) imply that

$$\begin{aligned}v_1(t)\le&\frac{\left(\ln\frac{t_2}{t_1}\right)^p}{\Gamma(p+1)}\varepsilon+\frac{L_1}{\Gamma(p)}\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{p-1}v_2(r)\,\frac{\mathrm{d}r}{r}+\frac{L_1}{\Gamma(p)}\int_{t_1+\tau}^{t}\left(\ln\frac{t}{r}\right)^{p-1}v_2(r)\,\frac{\mathrm{d}r}{r}\\\le&\frac{\left(\ln\frac{t_2}{t_1}\right)^p}{\Gamma(p+1)}\varepsilon+\frac{2L_1}{\Gamma(p)}\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{p-1}v_2(r)\,\frac{\mathrm{d}r}{r},\end{aligned}\tag{4.19}$$

and

$$v_2(t)\le\frac{\left(\ln\frac{t_2}{t_1}\right)^q}{\Gamma(q+1)}\varepsilon+\frac{2L_2}{\Gamma(q)}\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{q-1}v_1(r)\,\frac{\mathrm{d}r}{r}.\tag{4.20}$$

Once again, if we refer to the coupled Hadamard-Gronwall-Bellman-type inequality (Theorem 3.2), then for $t \in [t_1 + \tau, t_2]$, anf by (4.19) and (4.20), we have

$$v_1(t) \leq \left(\frac{\left(\ln \frac{t_2}{t_1}\right)^p}{\Gamma(p+1)} + \frac{2L_1}{\Gamma(p+1)} \frac{\left(\ln \frac{t_2}{t_1}\right)^q}{\Gamma(q+1)} \right) \mathbb{E}_{p+q} \left[2L_1 L_2 \left(\ln \frac{t_2}{t_1} \right)^{p+q} \right] \varepsilon,$$

and

$$v_2(t) \leq \left(\frac{\left(\ln \frac{t_2}{t_1}\right)^q}{\Gamma(q+1)} + \frac{2L_2}{\Gamma(q+1)} \frac{\left(\ln \frac{t_2}{t_1}\right)^p}{\Gamma(p+1)} \right) \mathbb{E}_{p+q} \left[2L_1 L_2 \left(\ln \frac{t_2}{t_1} \right)^{p+q} \right] \varepsilon.$$

For the sake of the inequalities $|\bar{\omega}_1(t) - \omega_1(t)| \leq v_1(t)$ and $|\bar{\omega}_2(t) - \omega_2(t)| \leq v_2(t)$, for $t \in [t_1 + \tau, t_2]$, we may write

$$|\bar{\omega}_1(t) - \omega_1(t)| \leq \left(\frac{\left(\ln \frac{t_2}{t_1}\right)^p}{\Gamma(p+1)} + \frac{2L_1}{\Gamma(p+1)} \frac{\left(\ln \frac{t_2}{t_1}\right)^q}{\Gamma(q+1)} \right) \mathbb{E}_{p+q} \left(2L_1 L_2 \left(\ln \frac{t_2}{t_1} \right)^{p+q} \right) \varepsilon,$$

and

$$|\bar{\omega}_2(t) - \omega_2(t)| \leq \left(\frac{\left(\ln \frac{t_2}{t_1}\right)^q}{\Gamma(q+1)} + \frac{2L_2}{\Gamma(q+1)} \frac{\left(\ln \frac{t_2}{t_1}\right)^p}{\Gamma(p+1)} \right) \mathbb{E}_{p+q} \left[2L_1 L_2 \left(\ln \frac{t_2}{t_1} \right)^{p+q} \right] \varepsilon.$$

Finally, we define the constants

$$c_1 := \left(\frac{\left(\ln \frac{t_2}{t_1}\right)^p}{\Gamma(p+1)} + \frac{2L_1}{\Gamma(p+1)} \frac{\left(\ln \frac{t_2}{t_1}\right)^q}{\Gamma(q+1)} \right) \mathbb{E}_{p+q} \left[L_1 L_2 \left(\ln \frac{t_2}{t_1} \right)^{p+q} \right],$$

and

$$c_2 := \left(\frac{\left(\ln \frac{t_2}{t_1}\right)^q}{\Gamma(q+1)} + \frac{2L_2}{\Gamma(q+1)} \frac{\left(\ln \frac{t_2}{t_1}\right)^p}{\Gamma(p+1)} \right) \mathbb{E}_{p+q} \left[2L_1 L_2 \left(\ln \frac{t_2}{t_1} \right)^{p+q} \right],$$

and obtain the desired inequalities

$$|\bar{\omega}_1(t) - \omega_1(t)| \leq c_1 \varepsilon \quad \text{and} \quad |\bar{\omega}_2(t) - \omega_2(t)| \leq c_2 \varepsilon,$$

which means the Ulam-Hyers stability of the Caputo-Hadamard coupled delay system (4.1), and the proof is ended. □

4.2. Application to a fractional damped system

Let $0 < q \leq 1 < p \leq 2$ and $\phi \in C([t_1, t_2] \times \mathbb{R}, \mathbb{R})$ and $\lambda, \omega^*, \omega_* \in \mathbb{R}^+$. Now, in this step, we investigate an application of the extended form of the Hadamard-Gronwall-Bellman-type inequality (3.22) for a Caputo-Hadamard fractional damped system formulated by

$$\begin{cases} {}^{CH}D_{t_1}^{p} \omega(t) - \lambda \, {}^{CH}D_{t_1}^{q} \omega(t) = \phi(t, \omega(t)), \qquad t \in [t_1, t_2], \\ \omega(t_1) = \omega^*, \qquad \omega'(t_1) = \omega_*. \end{cases} \tag{4.21}$$

The Hadamard fractional integral of order $1 < p \leq 2$ acts on (4.21) and yields the Hadamard integral equation

$$\omega(t) = \omega^* + \left(\ln \frac{t}{t_1}\right)\omega_* - \frac{\lambda\left(\ln \frac{t}{t_1}\right)^{p-q}}{\Gamma(p-q+1)}\omega^* + \frac{\lambda}{\Gamma(p-q)}\int_{t_1}^{t}\left(\ln \frac{t}{r}\right)^{p-q-1}\omega(r)\,\frac{\mathrm{d}r}{r} + \frac{1}{\Gamma(p)}\int_{t_1}^{t}\left(\ln \frac{t}{r}\right)^{p-1}\phi(r,\omega(r))\,\frac{\mathrm{d}r}{r}, \tag{4.22}$$

for all $t \in [t_1, t_2]$.

Before establishing the existence results, some hypotheses are needed to state them below:

(G4) $\phi \in C([t_1, t_2] \times \mathbb{R}, \mathbb{R})$;
(G5) For all $t \in [t_1, t_2], \omega \in \mathbb{R}$, there exists $k_\phi \in C([t_1, t_2], \mathbb{R})$ such that $\left|\phi(t, \omega)\right| \leq k_\phi(t)$ with $k_\phi^* := \sup_{t\in[t_1,t_2]} |k_\phi(t)|$;
(G6) For all $t \in [t_1, t_2], \omega_1, \omega_2 \in \mathbb{R}$, there exists $k \in C([t_1, t_2], \mathbb{R})$ such that $\left|\phi(t, \omega_1) - \phi(t, \omega_2)\right| \leq k(t)\left|\omega_1 - \omega_2\right|$ with $k^* := \sup_{t\in[t_1,t_2]} |k(t)|$;
(G7) For all $t \in [t_1, t_2]\omega \in \mathbb{R}$, there exists $L > 0$ such that $\left|\phi(t, \omega)\right| \leq L(1 + |\omega|)$.

Theorem 4.3. *Under the conditions (G4)–(G6), the Caputo-Hadamard fractional damped system* (4.21) *admits one and only one solution on* $[t_1, t_2]$ *if*

$$\frac{\lambda\left(\ln \frac{t_2}{t_1}\right)^{p-q}}{\Gamma(p-q+1)} + \frac{k^*\left(\ln \frac{t_2}{t_1}\right)^{p}}{\Gamma(p+1)} < 1. \tag{4.23}$$

Proof. Construct a new Banach space as

$$C_\varepsilon := \{\omega \in C([t_1, t_2], \mathbb{R}) : \quad \|\omega\| \leq \varepsilon\},$$

so that the arbitrary constant $\varepsilon > 0$ satisfies

$$\varepsilon \geq \frac{\omega^* + \left(\ln \frac{t_2}{t_1}\right)\omega_* + \dfrac{\lambda\left(\ln \frac{t_2}{t_1}\right)^{p-q}\omega^*}{\Gamma(p-q+1)} + \dfrac{k_\phi^*\left(\ln \frac{t_2}{t_1}\right)^{p}}{\Gamma(p+1)}}{1 - \dfrac{\lambda\left(\ln \frac{t_2}{t_1}\right)^{p-q}}{\Gamma(p-q+1)}}, \tag{4.24}$$

with $\dfrac{\lambda\left(\ln \frac{t_2}{t_1}\right)^{p-q}}{\Gamma(p-q+1)} < 1$.

In the next step, a new operator $Y : C_\varepsilon \to C([t_1, t_2], \mathbb{R})$ is defined, by (4.22), as

$$(Y\omega)(t) = \omega^* + \left(\ln \frac{t}{t_1}\right)\omega_* - \frac{\lambda\left(\ln \frac{t}{t_1}\right)^{p-q}}{\Gamma(p-q+1)}\omega^*$$

$$+\frac{\lambda}{\Gamma(p-q)}\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{p-q-1}\omega(r)\,\frac{\mathrm{d}r}{r}$$

$$+\frac{1}{\Gamma(p)}\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{p-1}\phi(r,\omega(r))\,\frac{\mathrm{d}r}{r}. \tag{4.25}$$

Due to the condition (G4), Y is well-defined. Now, for each $\omega\in C_\varepsilon$, we estimate

$$\begin{aligned}
|(Y\omega)(t)| &\le \omega^* + \left(\ln\frac{t}{t_1}\right)\omega_* + \frac{\lambda\left(\ln\frac{t}{t_1}\right)^{p-q}}{\Gamma(p-q+1)}\omega^* \\
&+ \frac{\lambda}{\Gamma(p-q)}\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{p-q-1}\left|\omega(r)\right|\frac{\mathrm{d}r}{r} \\
&+ \frac{1}{\Gamma(p)}\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{p-1}\left|\phi(r,\omega(r))\right|\frac{\mathrm{d}r}{r} \\
&\le \omega^* + \left(\ln\frac{t}{t_1}\right)\omega_* + \frac{\lambda\left(\ln(\frac{t}{t_1})\right)^{p-q}}{\Gamma(p-q+1)}\omega^* \\
&+ \frac{\varepsilon\lambda}{\Gamma(p-q)}\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{p-q-1}\frac{\mathrm{d}r}{r} \\
&+ \frac{1}{\Gamma(p)}\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{p-1}\left|k_\phi(r)\right|\frac{\mathrm{d}r}{r} \\
&\le \omega^* + \left(\ln(\frac{t_2}{t_1})\right)\omega_* + \frac{\lambda\left(\ln(\frac{t_2}{t_1})\right)^{p-q}}{\Gamma(p-q+1)}\omega^* \\
&+ \frac{\varepsilon\lambda\left(\ln(\frac{t_2}{t_1})\right)^{p-q}}{\Gamma(p-q+1)} + \frac{k_\phi^*\left(\ln(\frac{t_2}{t_1})\right)^{p}}{\Gamma(p+1)} \\
&\le \varepsilon,
\end{aligned}$$

implying $\|Y\omega\|\le\varepsilon$.

In view of (G6), for all $\omega_1,\omega_2\in C_\varepsilon$, we have

$$\begin{aligned}
\left|(Y\omega_1)(t)-(Y\omega_2)(t)\right| &\le \frac{\lambda}{\Gamma(p-q)}\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{p-q-1}\left|\omega_1(r)-\omega_2(r)\right|\frac{\mathrm{d}r}{r} \\
&+ \frac{1}{\Gamma(p)}\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{p-1}\left|\phi(r,\omega_1(r))-\phi(r,\omega_2(r))\right|\frac{\mathrm{d}r}{r} \\
&\le \frac{\lambda}{\Gamma(p-q)}\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{p-q-1}\left|\omega_1(r)-\omega_2(r)\right|\frac{\mathrm{d}r}{r}
\end{aligned}$$

$$+\frac{1}{\Gamma(p)}\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{p-1}k(r)\Big|\omega_1(r)-\omega_2(r)\Big|\frac{\mathrm{d}r}{r}$$

$$\leq\left(\frac{\lambda\left(\ln\frac{t_2}{t_1}\right)^{p-q}}{\Gamma(p-q+1)}+\frac{k^*\left(\ln\frac{t_2}{t_1}\right)^{p}}{\Gamma(p+1)}\right)\|\omega_1-\omega_2\|.$$

From (4.23), we know that Y is a contraction on C_ε. Now, the existence of unique fixed point is confirmed for Y by the Banach contraction mapping principle. Therefore, the Caputo-Hadamard fractional damped system (4.21) admits one and only one solution which completes the proof. □

Theorem 4.4. *Under the conditions (G4), (G6) and (G7), the Caputo-Hadamard fractional damped system* (4.21) *has at least one solution on* $[t_1, t_2]$.

Proof. We again use the same operator $Y : C_\varepsilon \to C([t_1, t_2], \mathbb{R})$ introduced in (4.25) and define a new set as

$$\hat{A}(Y) := \{\omega \in C_\varepsilon : \quad \omega = cY(\omega)\,\text{for some}\,c \in [0, 1]\,\}.$$

In order to apply the Schauder fixed point theorem, it is needed to establish that Y is completely continuous and $\hat{A}(Y)$ is a bounded set. For these, we first consider a sequence $\{\omega_n\}$ in $C([t_1, t_2], \mathbb{R})$ such that ω_n tends to ω. By (G6) and for all $t \in [t_1, t_2]$, we gaet

$$\Big|(Y\omega_n)(t)-(Y\omega)(t)\Big|\leq\frac{\lambda}{\Gamma(p-q)}\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{p-q-1}\Big|\omega_n(r)-\omega(r)\Big|\frac{\mathrm{d}r}{r}$$

$$+\frac{1}{\Gamma(p)}\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{p-1}\Big|\phi(r,\omega_n(r))-\phi(r,\omega(r))\Big|\frac{\mathrm{d}r}{r}$$

$$\leq\frac{\lambda}{\Gamma(p-q)}\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{p-q-1}\Big|\omega_n(r)-\omega(r)\Big|\frac{\mathrm{d}r}{r}$$

$$+\frac{k^*}{\Gamma(p)}\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{p-1}\Big|\omega_n(r)-\omega(r)\Big|\frac{\mathrm{d}r}{r}$$

$$\leq\left(\frac{\lambda\left(\ln\frac{t_2}{t_1}\right)^{p-q}}{\Gamma(p-q+1)}+\frac{k^*\left(\ln\frac{t_2}{t_1}\right)^{p}}{\Gamma(p+1)}\right)\|\omega_n-\omega\|.$$

Hence, $\|Y\omega_n - Y\omega\| \to 0$ as $n \to \infty$, and Y is continuous on C_ε.

For all $t \in [t_1, t_2]$ and by (G7), we estimate

$$|(Y\omega)(t)|\leq\omega^*+\left(\ln\frac{t}{t_1}\right)\omega_*+\frac{\lambda\left(\ln\frac{t}{t_1}\right)^{p-q}}{\Gamma(p-q+1)}\omega^*+\frac{\lambda}{\Gamma(p-q)}\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{p-q-1}|\omega(r)|\frac{\mathrm{d}r}{r}$$

$$+\frac{1}{\Gamma(p)}\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{p-1}|\phi(r,\omega(r))|\frac{\mathrm{d}r}{r}$$

$$\leq \omega^* + \left(\ln \frac{t_2}{t_1}\right)\omega_* + \frac{\lambda\left(\ln \frac{t_2}{t_1}\right)^{p-q}}{\Gamma(p-q+1)}\omega^* + \frac{\varepsilon\lambda\left(\ln \frac{t_2}{t_1}\right)^{p-q}}{\Gamma(p-q+1)} + \frac{L(1+\varepsilon)\left(\ln \frac{t_2}{t_1}\right)^{p}}{\Gamma(p+1)} := \hat{\varepsilon}.$$

We see that for each $\varepsilon > 0$, one can find a constant $\hat{\varepsilon} > 0$ so that for every $\omega \in C_\varepsilon$, we get $\|Y\omega\| \leq \hat{\varepsilon}$; i.e., Y maps a bounded set into a bounded set in C_ε, and also, for each $t_*, t^* \in [t_1, t_2], t_1 \leq t_* < t^* \leq t_2$ and every $\omega \in C_\varepsilon$, we have by (G7) that

$$\begin{aligned}
|(Y\omega)(t^*) - (Y\omega)(t_*)| &\leq \left(\ln \frac{t^*}{t_*}\right)\omega_* + \frac{\lambda\left(\ln \frac{t^*}{t_*}\right)^{p-q}}{\Gamma(p-q+1)}\omega^* + \frac{\lambda}{\Gamma(p-q)}\int_{t_*}^{t^*}\left(\ln \frac{t^*}{r}\right)^{p-q-1}|\omega(r)|\frac{\mathrm{d}r}{r} \\
&+ \frac{\lambda}{\Gamma(p-q)}\int_{t_1}^{t_*}\left(\left(\ln \frac{t^*}{r}\right)^{p-q-1} - \left(\ln \frac{t_*}{r}\right)^{p-q-1}\right)|\omega(r)|\frac{\mathrm{d}r}{r} \\
&+ \frac{1}{\Gamma(p)}\int_{t_*}^{t^*}\left(\ln \frac{t^*}{r}\right)^{p-1}|\phi(r,\omega(r))|\frac{\mathrm{d}r}{r} + \frac{1}{\Gamma(p)}\int_{t_1}^{t^*}\left(\left(\ln \frac{t^*}{r}\right)^{p-1} - \left(\ln \frac{t_*}{r}\right)^{p-1}\right)|\phi(r,\omega(r))|\frac{\mathrm{d}r}{r} \\
&\leq \left(\ln \frac{t^*}{t_*}\right)\omega_* + \frac{\lambda\left(\ln \frac{t^*}{t_*}\right)^{p-q}}{\Gamma(p-q+1)}\omega^* \\
&+ \frac{\lambda\varepsilon}{\Gamma(p-q)}\left[\int_{t_*}^{t^*}\left(\ln \frac{t^*}{r}\right)^{p-q-1}\frac{\mathrm{d}r}{r} + \int_{t_1}^{t_*}\left(\left(\ln \frac{t^*}{r}\right)^{p-q-1} - \left(\ln \frac{t_*}{r}\right)^{p-q-1}\right)\frac{\mathrm{d}r}{r}\right] \\
&+ \frac{L(1+\varepsilon)}{\Gamma(p)}\left[\int_{t_*}^{t^*}\left(\ln \frac{t^*}{r}\right)^{p-1}\frac{\mathrm{d}r}{r} + \int_{t_1}^{t^*}\left(\left(\ln \frac{t^*}{r}\right)^{p-1} - \left(\ln \frac{t_*}{r}\right)^{p-1}\right)\frac{\mathrm{d}r}{r}\right];
\end{aligned}$$

that is, independently (from ω), $|(Y\omega)(t^*) - (Y\omega)(t_*)$ tends to 0 as t^* tends to t_*, which states that Y is equicontinuous and finally, it is completely continuous by the conclusion of the Arzela-Ascoli theorem. Now, we take $\omega \in \hat{A}(Y)$. Then $\omega = cY\omega$ for some $c \in [0, 1]$. For $t \in [t_1, t_2]$,

$$\begin{aligned}
|\omega(t)| = |c||(Y\omega)(t)| &\leq \omega^* + \left(\ln \frac{t}{t_1}\right)\omega_* + \frac{\lambda\left(\ln \frac{t}{t_1}\right)^{p-q}}{\Gamma(p-q+1)}\omega^* \\
&+ \frac{\lambda}{\Gamma(p-q)}\int_{t_1}^{t}\left(\ln \frac{t}{r}\right)^{p-q-1}|\omega(r)|\frac{\mathrm{d}r}{r} \\
&+ \frac{1}{\Gamma(p)}\int_{t_1}^{t}\left(\ln \frac{t}{r}\right)^{p-1}|\phi(r,\omega(r))|\frac{\mathrm{d}r}{r} \\
&\leq \omega^* + \left(\ln \frac{t}{t_1}\right)\omega_* + \frac{\lambda\left(\ln \frac{t}{t_1}\right)^{p-q}}{\Gamma(p-q+1)}\omega^* + \frac{L\left(\ln \frac{t}{t_1}\right)^{p}}{\Gamma(p+1)} \\
&+ \frac{\lambda}{\Gamma(p-q)}\int_{t_1}^{t}\left(\ln \frac{t}{r}\right)^{p-q-1}|\omega(r)|\frac{\mathrm{d}r}{r}
\end{aligned}$$

$$+\frac{L}{\Gamma(p)}\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{p-1}|\omega(r)|\,\frac{\mathrm{d}r}{r}. \tag{4.26}$$

We define three continuous functions as

$$h(t)=\omega^*+\left(\ln\frac{t}{t_1}\right)\omega_*+\frac{\lambda\left(\ln\frac{t}{t_1}\right)^{p-q}}{\Gamma(p-q+1)}\omega^*+\frac{L\left(\ln\frac{t}{t_1}\right)^{p}}{\Gamma(p+1)},$$

$$f_1(t)=\frac{\lambda}{\Gamma(p-q)},\qquad f_2(t)=\frac{L}{\Gamma(p)}.$$

By the extended form of the Hadamard-Gronwall-Bellman-type inequality (3.22) (in Theorem 3.4) for a Caputo-Hadamard fractional damped system, and by (4.26), we have

$$\begin{aligned}|\omega(t)|&\le h(t)+f_1(t)\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{p-q-1}|\omega(r)|\,\frac{\mathrm{d}r}{r}+f_2(t)\int_{t_1}^{t}\left(\ln\frac{t}{r}\right)^{p-1}|\omega(r)|\,\frac{\mathrm{d}r}{r}\\
&\le h(t)\mathbb{E}_{p-q}\left((f_1(t)+f_2(t))\left(\Gamma(p-q)\left(\ln\frac{t}{t_1}\right)^{p-q}+\Gamma(p)\left(\ln\frac{t}{t_1}\right)^{p}\right)\right)\\
&\le h(t_2)\mathbb{E}_{p-q}\left((f_1(t_2)+f_2(t_2))\left(\Gamma(p-q)\left(\ln\frac{t_2}{t_1}\right)^{p-q}+\Gamma(p)\left(\ln\frac{t_2}{t_1}\right)^{p}\right)\right)=\varepsilon^*.\end{aligned}$$

This shows that there is some $\varepsilon^*>0$ so that $|\omega(t)|\le\varepsilon^*$. Then, one can find out that the set $\hat{A}(Y)$ is bounded. Therefore, by the conclusion of the Schauder fixed point theorem and by above results, we deduce that Y admits a fixed point. Hence, the Caputo-Hadamard fractional damped system (4.21) has at least one solution. □

5. Example

We investigate our theoretical results by giving a numerical example.

Example 5.1. *Let $p,q\in(0,1]$ with $p=0.25$ and $q=0.5$. We define a Caputo-Hadamard coupled delay system given by*

$$\begin{cases}{}^{CH}D_1^{0.25}\omega_1(t)=\dfrac{\sqrt{t+2}}{14}\left(\sin(\omega_2(t))+\arcsin(\omega_2(t-1))\right), & t\in[1,7],\\ {}^{CH}D_1^{0.5}\omega_2(t)=(2t+5)\left(\dfrac{\omega_1(t)+\omega_1(t-1)}{110}\right), & t\in[1,7],\\ \omega_1(t)=1-\cos(\pi t),\qquad \omega_2(t)=\sin\left(\dfrac{\pi}{2}t\right), & t\in[0,1],\end{cases} \tag{5.1}$$

for which $\phi_1,\phi_2\in C([1,7]\times\mathbb{R}^2,\mathbb{R})$ are defined by

$$\phi_1(t,\omega_2(t),\omega_2(t-\tau))=\frac{\sqrt{t+2}}{14}\left(\sin(\omega_2(t))+\arcsin(\omega_2(t-1))\right),$$

$$\phi_2(t, \omega_1(t), \omega_1(t-\tau)) = (2t+5)\left(\frac{\omega_1(t) + \omega_1(t-1)}{110}\right),$$

and $\omega_1^*, \omega_2^* \in C([0,1], \mathbb{R})$ *are defined by*

$$\omega_1^*(t) = 1 - \cos(\pi t), \qquad \omega_2^*(t) = \sin\left(\frac{\pi}{2}t\right).$$

By some computations, it is known that both functions ϕ_1 *and* ϕ_2 *have the Lipschitz constants* $L_1 = \frac{3}{14} \simeq 0.2142$ *and* $L_2 = \frac{19}{110} \simeq 0.1727$*, respectively; since*

$$|\phi_1(t, \omega_1, \omega_2) - \phi_1(t, \bar{\omega}_1, \bar{\omega}_2)| \leq \frac{3}{14}\left(|\omega_1 - \bar{\omega}_1| + |\omega_2 - \bar{\omega}_2|\right),$$

$$|\phi_2(t, \omega_1, \omega_2) - \phi_2(t, \bar{\omega}_1, \bar{\omega}_2)| \leq \frac{19}{110}\left(|\omega_1 - \bar{\omega}_1| + |\omega_2 - \bar{\omega}_2|\right),$$

for each $\omega_1, \omega_2, \bar{\omega}_1, \bar{\omega}_2 \in \mathbb{R}$ *and* $t \in [1,7]$. *On the other hand,*

$$\begin{aligned} M &:= \max\left\{\frac{L_1\left(\ln(\frac{t_2}{t_1})\right)^p}{\Gamma(p+1)}, \frac{L_2\left(\ln(\frac{t_2}{t_1})\right)^q}{\Gamma(q+1)}\right\} \simeq \max\left\{\frac{0.2142\ln(7)^{0.25}}{\Gamma(1.25)}, \frac{0.1727\ln(7)^{0.5}}{\Gamma(1.5)}\right\} \\ &\simeq \max\{0.2793, 0.2716\} = 0.2793 < 1. \end{aligned}$$

In view of numerical results obtained above, Theorem 4.1 implies that the Caputo-Hadamard coupled delay system (5.1) *admits one and only one solution and its solution is Ulam-Hyers stable by Theorem 4.2.*

6. Conclusions

In this paper, we aimed to work on the new versions of the Gronwall-Bellman inequalities. In fact, we proved two new forms of such inequalities by using the Hadamard fractional integrals. In the first version, we generalized the Gronwall-Bellman inequality to a coupled Hadamard-Gronwall-Bellman inequality which is applied to establish the Ulam-Hyers stability of the solutions of the Caputo-Hadamard fractional coupled systems with finite delay. The second version was related to the extended form of the Hadamard-Gronwall-Bellman inequality under the sum of two non-decreasing functions. The application of this inequality could be seen in the existence theory of a Caputo-Hadamard fractional initial value problem with damping. These results were new and the applicability of our generalizations were validated by giving an example. In the next studies, we will try to generalize our studies for extending the Gronwall-Bellman inequality under the non-singular kernels by using the fractional proportional operators in two Caputo-Fabrizio and Atangana-Baleanu settings. Also, we aim to investigate our research on fractional operators defined in (p,q)-calculus to analyze these inequalities on time scales.

Authors' contributions

S.E.: Conceptualization, methodology, software, validation, formal analysis, writing-original draft preparation; A.A.: Conceptualization, methodology, formal analysis, investigation, writing-original draft preparation, project administration; J.A.C.: Methodology, validation, formal analysis, investigation, writing-review and editing, supervision, project administration. All authors have read and agreed to the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

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