Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(11): 27934–27953.
DOI: 10.3934/math.20251227
Received: 28 October 2025
Revised: 19 November 2025
Accepted: 24 November 2025
Published: 28 November 2025

Research article

Nonlocal effects in a coupled road-field population system

You Zhou* **and Zhi Ling**

School of Mathematics, Yangzhou University, Yangzhou 225002, China

* **Correspondence:** Email: DX120220050@stu.yzu.edu.cn.

Abstract: Understanding when a population persists or dies out in environments intersected by transportation corridors requires capturing the influence of fast movement along the road and the resulting nonlocal interactions. We investigate a road-field reaction-diffusion framework that links a rapidly diffusing one-dimensional road to a two-dimensional half-plane, where the population follows a Kolmogorov-Petrovsky-Piskunov (KPP) type growth law subject to both local competition and a nonlocal crowding effect. By constructing appropriately coupled lower and upper solutions in this mixed-dimensional domain and applying a comparison principle tailored to the exchange between road and field, we obtain boundedness as well as global existence and uniqueness. To analyze long-term behavior under nonlocal competition, we extend classical persistence techniques and demonstrate that positive solutions remain uniformly away from extinction under suitable hypotheses. Concerning spatial invasion, we determine the propagation speed along the road: When the diffusion rate on the road is at most twice that in the field, the front travels more slowly than the classical KPP wave; once this ratio is exceeded, the spreading speed increases like the square root of the road diffusion parameter. Numerical simulations are consistent with these analytical conclusions and further show that intensifying nonlocal competition tends to reduce the eventual size of the positive equilibrium, thereby increasing the likelihood of extinction. These results highlight the joint impact of road-mediated transport and nonlocal effects on invasion outcomes in coupled habitats.

Keywords: nonlocal interactions; coupled population model; road diffusion; global stability; invasion speed
Mathematics Subject Classification: 35A02, 35B40, 35K57, 92D25

1. Introduction

The present study aims to investigate the following road-field coupled population system, incorporating the spatial nonlocality of reaction in the field, which describes the dynamics of two subpopulations within the same species:

$$\begin{cases} \partial_t u - D\partial_{xx}u = \nu v|_{y=0} - \mu u, & x \in \mathbb{R},\ t > 0, \\ \partial_t v - d\Delta v = v(a - bv - \alpha\phi * v), & (x,y) \in \mathbb{R}\times\mathbb{R}^+,\ t > 0, \\ -d\partial_y v|_{y=0} = \mu u - \nu v|_{y=0}, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \\ v(x,y,0) = v_0(x,y), & (x,y) \in \mathbb{R}\times\mathbb{R}^+. \end{cases} \tag{1.1}$$

Here, $u(x,t)$ and $v(x,y,t)$ are the densities of the two subpopulations on the road and in the field, respectively. The *field* is denoted by the upper half-plane $\mathbb{R}\times\mathbb{R}^+$, and the *road* $\mathbb{R}\times\{0\}$ is simplified as $\mathbb{R}$. The first equation of (1.1) (the road equation) is coupled with the second one (the field equation) through the third, which imposes a Robin boundary condition at $y = 0$. This condition models the population exchanges between individuals residing on the road and those in the field. Specifically, a ratio μ of the subpopulation u on the road disperses into the field, while a ratio ν of the field population v at the boundary $y = 0$ moves onto the road. The term av represents the average growth of the subpopulation v in the field, where parameter a is the intrinsic growth rate. The terms bv^2 and $\alpha v\phi * v$ are, respectively, the local and nonlocal intraspecific competition for space and food resources, etc., among the individuals in the field. Here, b and α represent the local and nonlocal competition strength. We denote the function $a - bv$ by $f(v)$, and define the convolution operator by

$$\phi * v = \int_{\mathbb{R}^2} \phi(x - z_1, y - z_2)v(z_1, |z_2|, t)\mathrm{d}z,$$

where $z = (z_1, z_2) \in \mathbb{R}^2$. The kernel function ϕ adheres to the criteria outlined in [1]. The symbols D and d represent diffusion rates of subpopulations on the road and in the field, respectively. Since the rapid diffusion characteristics were observed on the road, we assume that $D > d$ unless otherwise specified below. The nonnegative initial functions u_0 and v_0 are supposed to be bounded and possess continuous derivatives up to second order. All parameters are taken to be positive constants.

In 2013, Berestycki et al. introduced a road-field model [2] to explore the influence of directional diffusion along transportation networks on biological invasion dynamics. It was motivated by the empirical observation that the transmission of both various ecological species [3] and diseases [4] tends to be more accelerated along transportation lines such as roads and river stream networks than in the field. In [2], the authors posited that the field equation is of the KPP type without nonlocal interaction. They showed that the road diffusion has no effect on the asymptotic spreading speed along the road $c^*(D)$ for $D \le 2d$, but accelerates the speed (exceeding the classic KPP speed c_{KPP}) when $D > 2d$. Extending the foundational model, Berestycki et al. [5, 6] further investigated the qualitative behaviors of road diffusion, and later developed a generalized eigenvalue theory for heterogeneous setting [7], deepening the analytical understanding of anisotropic propagation in biological invasion contexts. Combining the theory with the impact of climate change, they revealed the propagation behavior of the road diffusion model with shifting ecological niches [8]. Such population models with the road diffusion have also garnered attention from other scholars, thereby further enriching the theoretical framework of the road diffusion system. Dietrich [9, 10] and Tellini [11] were concerned about the travelling waves for the road-field model in a bounded strip. The road-field model with spatially periodic exchange terms was investigated by Giletti et al. [12] and that with nonlocal exchange terms was considered by Pauthier [13]. Alfaro et al. [14] derived the fundamental solution and asymptotic behavior of the road-field model. More analysis of road-field coupled systems can be found in [15–17] and related literature.

In the aforementioned models, the authors focused solely on local intraspecific competition in the field. It was emphasized that the population can compete internally not only within local areas but also across distant regions due to the existence of fast diffusion “line”. Motivated by this, we consider nonlocal interactions among individuals in the field, and are interested in the role of nonlocal spatial effects in biological invasion.

The presence of nonlocal term presents certain processing challenges, which was attempted by Deng and Wu [18]. A key challenge arises from the absence of positivity in the nonlocal term (as highlighted in [19, 20]) within the road equation of system (1.1), a feature that is crucial for the comparison framework. To address this, we first define coupled upper and lower solutions, which allows us to rigorously formulate and prove the comparison principle by a new methodology in Section 2. This serves as the foundation for the primary analytical results, including the existence, uniqueness, long-time dynamics, and asymptotic spreading speed. Then, we complement the theoretical analysis with numerical simulations and offer a brief discussion in Section 3.

2. Main results

This section is devoted primarily to analyzing the long-term dynamics of solutions to system (1.1). For the sake of completeness and to ensure the logical coherence of the exposition, we also include a comprehensive derivation of the existence and uniqueness of solutions. These foundational analyses provide a necessary groundwork for studying the asymptotic behavior.

2.1. Existence and uniqueness

We begin by introducing coupled upper and lower solutions, which serves as a foundation for formulating and proving the crucial comparison principle.

Definition 2.1. *Two functions pairs* $\tilde{\mathbf{u}} = (\tilde{u}, \tilde{v})$ *and* $\hat{\mathbf{u}} = (\hat{u}, \hat{v})$ *to form coupled upper and lower solutions of (1.1) if they are bounded and satisfy*

$$
\begin{cases}
\partial_t \tilde{u} - D\partial_{xx}\tilde{u} \geq \nu\tilde{v}|_{y=0} - \mu\tilde{u}, & x \in \mathbb{R}, t > 0,\\
\partial_t \hat{u} - D\partial_{xx}\hat{u} \leq \nu\hat{v}|_{y=0} - \mu\hat{u}, & x \in \mathbb{R}, t > 0,\\
\partial_t \tilde{v} - d\Delta\tilde{v} \geq \tilde{v}(f(\tilde{v}) - \alpha\phi * \hat{v}), & (x, y) \in \mathbb{R}\times\mathbb{R}^+, t > 0,\\
\partial_t \hat{v} - d\Delta\hat{v} \leq \hat{v}(f(\hat{v}) - \alpha\phi * \tilde{v}), & (x, y) \in \mathbb{R}\times\mathbb{R}^+, t > 0,\\
- d\partial_y\tilde{v}|_{y=0} \geq \mu\tilde{u} - \nu\tilde{v}|_{y=0}, & x \in \mathbb{R}, t > 0,\\
- d\partial_y\hat{v}|_{y=0} \leq \mu\hat{u} - \nu\hat{v}|_{y=0}, & x \in \mathbb{R}, t > 0,\\
\tilde{u}(x, 0) \geq u_0(x) \geq \hat{u}(x, 0), & x \in \mathbb{R},\\
\tilde{v}(x, y, 0) \geq v_0(x, y) \geq \hat{v}(x, y, 0), & (x, y) \in \mathbb{R}\times\mathbb{R}^+.
\end{cases}
\tag{2.1}
$$

Remark 2.1. *The coupling convolution term* $\tilde{v}\phi * \hat{v}$ $(\hat{v}\phi * \tilde{v})$ *of the third (fourth) differential inequality is a nonlinear term, and the negative sign in front of it determines that the lower (upper) solution should be applied within the convolution for the upper (lower) solution inequality.*

Theorem 2.1 (comparison principle). *If functions pairs* $\tilde{\mathbf{u}}$ *and* $\hat{\mathbf{u}}$ *are nonnegative and form the coupled upper and lower solutions of (1.1), then* $\tilde{u} \geq \hat{u}$ *and* $\tilde{v} \geq \hat{v}$ *for* $t > 0$.

Proof. Due to the boundedness of $\tilde{\mathbf{u}}$ and $\hat{\mathbf{u}}$, we select a nonnegative constant M as the uniform upper bound for them. To facilitate our analysis, we introduce the following time-dependent transformations for a yet-undetermined positive constant l:

$$\overline{\mathbf{u}} = (\overline{u}, \overline{v}) = \tilde{\mathbf{u}}e^{-lt} \text{ and } \underline{\mathbf{u}} = (\underline{u}, \underline{v}) = \hat{\mathbf{u}}e^{-lt}.$$

Next, for a given sufficiently small parameter $\varepsilon > 0$, we define an auxiliary functions pair

$$\check{\mathbf{u}} = (\check{u}, \check{v}) = \overline{\mathbf{u}} + \varepsilon(\Upsilon, \frac{\mu\Theta}{\nu}).$$

Here, the perturbation functions Υ and Θ are chosen as

$$\Upsilon = \vartheta(|x|) + t + \delta \text{ and } \Theta = \vartheta(|x|) + \vartheta(y) + t + \delta,$$

where $\delta > 0$ is a fixed parameter and smooth function $\vartheta : \mathbb{R} \to [0, +\infty)$ meets the following conditions:

(1) $\vartheta(0) = 0$ and $\vartheta(x) \to +\infty$ as $|x| \to +\infty$, (2.2)

(2) $\vartheta'(0) = 0$ and $\vartheta''(\cdot) \leq \min\{\frac{1}{2d}, \frac{1}{D}\}$. (2.3)

It is straightforward to verify $\check{u}(\cdot, 0) > \underline{u}(\cdot, 0)$ and $\check{v}(\cdot, 0) > \underline{v}(\cdot, 0)$.

To establish $\tilde{u} \geq \hat{u}$ and $\tilde{v} \geq \hat{v}$ for all $t > 0$, it is enough to show that perturbed functions $\check{u}$ and $\check{v}$ remain strictly above the transformed lower bound functions $\underline{\mathbf{u}}$ for all $t > 0$, respectively. Proceed by contradiction. Suppose, contrary to our assertion, that this inequality does not hold globally in time. Define the maximal time interval over which the strict inequality is preserved by

$$T = \sup\left\{\tau \geq 0 : \ \check{u} \geq \underline{u} \text{ and } \check{v} \geq \underline{v} \text{ for } t \in [0, \tau]\right\}.$$

According to the construction of assumption by contradiction, we can assert that $T \in [0, +\infty)$. By means of the continuity of functions pair $\check{\mathbf{u}}$, we know that $\check{\mathbf{u}} \to +\infty$ uniformly in time as $|x|$ or y tends to infinity. This implies that $T > 0$, and there exists at least one point in space where the difference $(\check{u} - \underline{u})$ or $(\check{v} - \underline{v})$ vanishes at time T. That is, the first time at which the perturbed upper function no longer strictly exceeds the lower function occurs at $t = T$. To proceed, we fix the constant $l = a + \alpha M$. We then examine two possible scenarios that could occur at time T:

(1) $\exists\, x_0 \in \mathbb{R}$ s.t. $\check{u}(x_0, T) = \underline{u}(x_0, T)$.
(2) $\exists\, (x_0, y_0) \in \mathbb{R} \times \mathbb{R}^+$ s.t. $\check{v}(x_0, y_0, T) = \underline{v}(x_0, y_0, T)$.

In the subsequent analysis, we will show that neither case can actually occur, thus contradicting the assumption that $T < +\infty$.

Assume that $\min_{\mathbb{R}}(\check{u} - \underline{u})(x, T) = 0$. Then we can find one point x_0 in $\mathbb{R}$ such that $\check{u}(x_0, T) = \underline{u}(x_0, T)$. Since $\check{u} - \underline{u} \geq 0$ in $\mathbb{R} \times [0, T]$ by the definition of T, $\check{u} - \underline{u}$ attains its minimum zero value at the interior point (x_0, T). Directly computing can yield that

$$\begin{aligned} & (\partial_t - D\partial_{xx} + (\mu + l))(\check{u} - \underline{u}) \\ = \ & e^{-lt}(\partial_t - D\partial_{xx} + \mu)(\tilde{u} - \hat{u}) + \varepsilon(1 - D\vartheta''(|x|)) + \varepsilon(\mu + l)\Upsilon \\ \geq \ & e^{-lt}\nu(\tilde{v} - \hat{v})|_{y=0} + \varepsilon(\mu + l)\Upsilon \\ = \ & \nu(\check{v} - \underline{v})|_{y=0} + l\varepsilon\Upsilon \geq 0 \end{aligned}$$

in $\mathbb{R} \times (0, T]$. By means of classic strong maximum principle, it follows that $(\check{u} - \underline{u})(x, t) \equiv 0$ in $\mathbb{R} \times [0, T]$. However, this contradicts the strict inequality $(\check{u} - \underline{u})(x, 0) > 0$. As a result, the minimum of the difference must be attained in the field domain and thus $\min\limits_{\mathbb{R}\times\mathbb{R}^+}(\check{v} - \underline{v})(\cdot, T) = 0$.

Similarly, we can obtain

$$\begin{aligned} (\partial_t - d\Delta)(\check{v} - \underline{v}) & = e^{-lt}(\partial_t - d\Delta - l)(\tilde{v} - \hat{v}) + \tfrac{\mu\varepsilon}{\nu}(1 - d(\vartheta''(|x|) + \vartheta''(y))) \\ & \geq e^{-lt}(\tilde{v}f(\tilde{v}) - \hat{v}f(\hat{v}) - \alpha\tilde{v}\phi * \hat{v} + \alpha\hat{v}\phi * \tilde{v} - l(\tilde{v} - \hat{v})) \\ & = e^{-lt}((\eta f'(\eta) + f(\eta) - l)(\tilde{v} - \hat{v}) - \alpha\tilde{v}\phi * \hat{v} + \alpha\hat{v}\phi * \tilde{v}) \\ & = e^{-lt}(a - 2b\eta - l - \alpha\phi * \tilde{v})(\tilde{v} - \hat{v}) + e^{-lt}\alpha\tilde{v}\phi * (\tilde{v} - \hat{v}) \\ & = (a - 2b\eta - l - \alpha\phi * \tilde{v})(\check{v} - \underline{v}) + \alpha\tilde{v}\phi * (\check{v} - \underline{v}) + \tfrac{\nu\varepsilon(l + 2b\eta - a + \alpha\phi*\tilde{v} - \alpha\tilde{v})\Theta}{\mu} \\ & \geq (a - 2b\eta - l - \alpha\phi * \tilde{v})(\check{v} - \underline{v}) \end{aligned}$$

in $\mathbb{R} \times \mathbb{R}^+ \times (0, T]$, where $\eta \in [0, M]$ is between $\hat{v}$ and $\tilde{v}$. Here, the final inequality is due to the nonnegativity and uniform boundedness of $\tilde{v}$, and thus

$$\begin{cases} (\partial_t - d\Delta + (l + 2b\eta - a + \alpha\phi * \tilde{v}))(\check{v} - \underline{v}) \geq 0, & \text{in } \mathbb{R} \times \mathbb{R}^+ \times (0, T], \\ (\check{v} - \underline{v})(x, y, 0) > 0, & \text{in } \mathbb{R} \times \mathbb{R}^+. \end{cases}$$

Depending on the parabolic strong maximum principle, we know the minimum of $\check{v} - \underline{v}$ cannot be attained in the interior of $\mathbb{R} \times \mathbb{R}^+$. Thus, there is some $x_0 \in \mathbb{R}$ satisfying $(\check{v} - \underline{v})(x_0, 0, T) = 0$. Applying Hopf lemma yields $-d\partial_y(\check{v} - \underline{v})(x_0, 0, T) < 0$. Substituting into the boundary condition gives

$$\nu(\check{v} - \underline{v})(x_0, 0, T) = d\partial_y(\check{v} - \underline{v})(x_0, 0, T) + \mu(\check{u} - \underline{u})(x_0, T) > 0,$$

which is a contradiction. Therefore, the original assumption is false. Taking $\varepsilon \to 0$ obtains the proof. □

Theorem 2.2. *If* $\tilde{\mathbf{u}}$ *and* $\hat{\mathbf{u}}$ *form the nonnegative coupled upper and lower solutions of (1.1), then the system (1.1) admits a unique global solution* (u, v).

Proof. Similar to the above proof, we continue to use notation M as the uniform upper bound of $\tilde{\mathbf{u}}$ and $\hat{\mathbf{u}}$. Because $f(v)$ is continuously differentiable, one can select a positive constant L, satisfying

$$L + \xi f'(\xi) + f(\xi) \geq 0, \qquad \forall \xi \in [0, M].$$

In fact, it is enough that $L > \max\{\mu, \nu, 2bM\}$. Set $\overline{\mathbf{u}}^{(0)} = (\overline{u}^{(0)}, \overline{v}^{(0)}) = \tilde{\mathbf{u}}$ and $\underline{\mathbf{u}}^{(0)} = (\underline{u}^{(0)}, \underline{v}^{(0)}) = \hat{\mathbf{u}}$. Then define sequences $\{\overline{\mathbf{u}}^{(m)}\}_{m=0}^{\infty}$ and $\{\overline{\mathbf{u}}^{(m)}\}_{m=0}^{\infty}$ by iteratively solving the following scalar equations:

$$\begin{cases} \partial_t \overline{u}^{(m)} - D\partial_{xx}\overline{u}^{(m)} + L\overline{u}^{(m)} = L\overline{u}^{(m-1)} + \nu\overline{v}^{(m-1)}|_{y=0} - \mu\overline{u}^{(m-1)}, & \text{in } \mathbb{R}\times(0,+\infty), \\ \partial_t \underline{u}^{(m)} - D\partial_{xx}\underline{u}^{(m)} + L\underline{u}^{(m)} = L\underline{u}^{(m-1)} + \nu\underline{v}^{(m-1)}|_{y=0} - \mu\underline{u}^{(m-1)}, & \text{in } \mathbb{R}\times(0,+\infty), \\ \partial_t \overline{v}^{(m)} - d\Delta\overline{v}^{(m)} + L\overline{v}^{(m)} = L\overline{v}^{(m-1)} + \overline{v}^{(m-1)}f(\overline{v}^{(m-1)}) - \alpha\overline{v}^{(m)}\phi * \underline{v}^{(m-1)}, & \text{in } \mathbb{R}\times\mathbb{R}^+\times(0,+\infty), \\ \partial_t \underline{v}^{(m)} - d\Delta\underline{v}^{(m)} + L\underline{v}^{(m)} = L\underline{v}^{(m-1)} + \underline{v}^{(m-1)}f(\underline{v}^{(m-1)}) - \alpha\underline{v}^{(m)}\phi * \overline{v}^{(m-1)}, & \text{in } \mathbb{R}\times\mathbb{R}^+\times(0,+\infty), \\ -d\partial_y\overline{v}^{(m)}|_{y=0} + L\overline{v}^{(m)}|_{y=0} = L\overline{v}^{(m-1)}|_{y=0} + \mu\overline{u}^{(m-1)} - \nu\overline{v}^{(m-1)}|_{y=0}, & \text{in } \mathbb{R}\times(0,\infty), \\ -d\partial_y\underline{v}^{(m)}|_{y=0} + L\underline{v}^{(m)}|_{y=0} = L\underline{v}^{(m-1)}|_{y=0} + \mu\underline{u}^{(m-1)} - \nu\underline{v}^{(m-1)}|_{y=0}, & \text{in } \mathbb{R}\times(0,\infty), \\ \overline{u}^{(m)}(x,0) = \underline{u}^{(m)}(x,0) = u_0(x), & \text{in } \mathbb{R}, \\ \overline{v}^{(m)}(x,y,0) = \underline{v}^{(m)}(x,y,0) = v_0(x,y), & \text{in } \mathbb{R}\times\mathbb{R}^+. \end{cases} \tag{2.4}$$

Indeed, the unique solvability of the linear parabolic problem yields that the sequences $\{\overline{\mathbf{u}}^{(m)}\}_{m=0}^{\infty}$ and $\{\underline{\mathbf{u}}^{(m)}\}_{m=0}^{\infty}$ are well-defined. We also claim that they exhibit a monotonic behavior:

$$\hat{u} = \underline{u}^{(0)} \le \underline{u}^{(m)} \le \underline{u}^{(m+1)} \le \overline{u}^{(m+1)} \le \overline{u}^{(m)} \le \overline{u}^{(0)} = \tilde{u} \tag{2.5}$$

and

$$\hat{v} = \underline{v}^{(0)} \le \underline{v}^{(m)} \le \underline{v}^{(m+1)} \le \overline{v}^{(m+1)} \le \overline{v}^{(m)} \le \overline{v}^{(0)} = \tilde{v}, \tag{2.6}$$

where $\overline{\mathbf{u}}^{(m)}$ and $\underline{\mathbf{u}}^{(m)}$ serve as the coupled upper and lower solutions of (1.1) for each integer $m = 0, 1, 2, \cdots$.

Similar to [1, Proof of Theorem 3.1], we first show that $\underline{u}^{(0)} \le \underline{u}^{(1)} \le \overline{u}^{(1)} \le \overline{u}^{(0)}$ and $\underline{v}^{(0)} \le \underline{v}^{(1)} \le \overline{v}^{(1)} \le \overline{v}^{(0)}$. Let $z_u = \overline{u}^{(0)} - \overline{u}^{(1)}$ and $z_v = \overline{v}^{(0)} - \overline{v}^{(1)}$. We then derive that

$$\begin{cases} \partial_t z_u - D\partial_{xx}z_u + Lz_u \ge 0, & \text{in } \mathbb{R}\times(0,+\infty), \\ \partial_t z_v - d\Delta z_v + (L + \alpha\phi * \underline{v}^{(0)})z_v \ge 0, & \text{in } \mathbb{R}\times\mathbb{R}^+\times(0,+\infty), \\ -d\partial_y z_v|_{y=0} + Lz_v|_{y=0} \ge 0, & \text{in } \mathbb{R}\times(0,\infty), \\ z_u(x,0) \ge 0, & \text{in } \mathbb{R}, \\ z_v(x,y,0) \ge 0, & \text{in } \mathbb{R}\times\mathbb{R}^+. \end{cases}$$

Theorem 2.1 yields $z_u \ge 0$ in $\mathbb{R}\times[0,+\infty)$ and $z_v \ge 0$ in $\Omega\times[0,+\infty)$, i.e., $\overline{u}^{(1)} \le \overline{u}^{(0)}$ and $\overline{v}^{(1)} \le \overline{v}^{(0)}$. Similarly, we can also derive $\underline{u}^{(0)} \le \underline{u}^{(1)}$ and $\underline{v}^{(0)} \le \underline{v}^{(1)}$.

Next, we denote $z_1 = \overline{u}^{(1)} - \underline{u}^{(1)}$ and $z_2 = \overline{v}^{(1)} - \underline{v}^{(1)}$, thereby obtaining

$$\begin{cases} \partial_t z_1 - D\partial_{xx}z_1 + Lz_1 \ge 0, & \text{in } \mathbb{R}\times(0,+\infty), \\ \partial_t z_2 - d\Delta z_2 + (L + \alpha\phi * \underline{v}^{(0)})z_2 \ge 0, & \text{in } \mathbb{R}\times\mathbb{R}^+\times(0,+\infty), \\ -d\partial_y z_2|_{y=0} + Lz_2|_{y=0} \ge 0, & \text{in } \mathbb{R}\times(0,\infty), \\ z_1(x,0) = 0, & \text{in } \mathbb{R}, \\ z_2(x,y,0) = 0, & \text{in } \mathbb{R}\times\mathbb{R}^+. \end{cases}$$

Applying Theorem 2.1 deduces $\underline{u}^{(1)} \le \overline{u}^{(1)}$ and $\underline{v}^{(1)} \le \overline{v}^{(1)}$. Then we can further derive that

$$\partial_t\overline{u}^{(1)} - D\partial_{xx}\overline{u}^{(1)} + L\overline{u}^{(1)} = (L-\mu)\overline{u}^{(0)} + \nu\overline{v}^{(0)}|_{y=0} \ge (L-\mu)\overline{u}^{(1)} + \nu\overline{v}^{(1)}|_{y=0}$$

and

$$\partial_t\underline{u}^{(1)} - D\partial_{xx}\underline{u}^{(1)} + L\underline{u}^{(1)} = (L-\mu)\underline{u}^{(0)} + \nu\underline{v}^{(0)}|_{y=0} \le (L-\mu)\underline{u}^{(1)} + \nu\underline{v}^{(1)}|_{y=0}.$$

Moreover, we find that

$$\begin{aligned}\partial_t\overline{v}^{(1)}-d\Delta\overline{v}^{(1)}&=L(\overline{v}^{(0)}-\overline{v}^{(1)})+\overline{v}^{(0)}f(\overline{v}^{(0)})-\alpha\overline{v}^{(1)}\phi*\underline{v}^{(0)}\\&\geq(L+\overline{\xi}f'(\overline{\xi})+f(\overline{\xi}))(\overline{v}^{(0)}-\overline{v}^{(1)})+\overline{v}^{(1)}f(\overline{v}^{(1)})-\alpha\overline{v}^{(1)}\phi*\underline{v}^{(1)}\\&\geq\overline{v}^{(1)}f(\overline{v}^{(1)})-\alpha\overline{v}^{(1)}\phi*\underline{v}^{(1)}\end{aligned}$$

and

$$\begin{aligned}\partial_t\underline{v}^{(1)}-d\Delta\underline{v}^{(1)}&=L(\underline{v}^{(0)}-\underline{v}^{(1)})+\underline{v}^{(0)}f(\underline{v}^{(0)})-\alpha\underline{v}^{(1)}\phi*\overline{v}^{(0)}\\&\leq(L+\underline{\xi}f'(\underline{\xi})+f(\underline{\xi}))(\underline{v}^{(0)}-\underline{v}^{(1)})+\underline{v}^{(1)}f(\underline{v}^{(1)})-\alpha\underline{v}^{(1)}\phi*\overline{v}^{(1)}\\&\leq\underline{v}^{(1)}f(\underline{v}^{(1)})-\alpha\underline{v}^{(1)}\phi*\overline{v}^{(1)},\end{aligned}$$

where $\overline{\xi}$ is between $\overline{v}^{(1)}$ and $\overline{v}^{(0)}$ while $\underline{\xi}$ is between $\underline{v}^{(0)}$ and $\underline{v}^{(1)}$. Hence, we deduce that

$$\begin{cases}\partial_t\overline{u}^{(1)}-D\partial_{xx}\overline{u}^{(1)}\geq\nu\overline{v}^{(1)}|_{y=0}-\mu\overline{u}^{(1)}, & \text{in } \mathbb{R}\times(0,+\infty),\\ \partial_t\underline{u}^{(1)}-D\partial_{xx}\underline{u}^{(1)}\leq\nu\underline{v}^{(1)}|_{y=0}-\mu\underline{u}^{(1)}, & \text{in } \mathbb{R}\times(0,+\infty),\\ \partial_t\overline{v}^{(1)}-d\Delta\overline{v}^{(1)}\geq\overline{v}^{(1)}f(\overline{v}^{(1)})-\alpha\overline{v}^{(1)}\phi*\underline{v}^{(1)}, & \text{in } \mathbb{R}\times\mathbb{R}^+\times(0,+\infty),\\ \partial_t\underline{v}^{(1)}-d\Delta\underline{v}^{(1)}\leq\underline{v}^{(1)}f(\underline{v}^{(1)})-\alpha\underline{v}^{(1)}\phi*\overline{v}^{(1)}, & \text{in } \mathbb{R}\times\mathbb{R}^+\times(0,+\infty),\\ -d\partial_y\overline{v}^{(1)}|_{y=0}\geq\mu\overline{u}^{(1)}-\nu\overline{v}^{(1)}|_{y=0}, & \text{in } \mathbb{R}\times(0,+\infty),\\ -d\partial_y\underline{v}^{(1)}|_{y=0}\leq\mu\underline{u}^{(1)}-\nu\underline{v}^{(1)}|_{y=0}, & \text{in } \mathbb{R}\times(0,+\infty),\\ \overline{u}^{(1)}(x,0)=\underline{u}^{(1)}(x,0)=u_0(x), & \text{in } \mathbb{R},\\ \overline{v}^{(1)}(x,y,0)=\underline{v}^{(1)}(x,y,0)=v_0(x,y), & \text{in } \mathbb{R}\times\mathbb{R}^+.\end{cases}$$

Then $\overline{\mathbf{u}}^{(1)}$ and $\underline{\mathbf{u}}^{(1)}$ form the coupled upper and lower solutions of system (1.1).

Let $\overline{\mathbf{u}}^{(m-1)}$ and $\underline{\mathbf{u}}^{(m-1)}$ be coupled upper and lower solutions for some $m>1$. Iteration yields

$$\underline{u}^{(m-1)}\leq\underline{u}^{(m)}\leq\overline{u}^{(m)}\leq\overline{u}^{(m-1)} \text{ and } \underline{v}^{(m-1)}\leq\underline{v}^{(m)}\leq\overline{v}^{(m)}\leq\overline{v}^{(m-1)},$$

implying monotonicity. Furthermore, $\overline{\mathbf{u}}^{(m)}$ and $\underline{\mathbf{u}}^{(m)}$ also form upper and lower solutions of system (1.1). The monotone convergence theorem yields that $\overline{\mathbf{u}}^{(m)}$ and $\underline{\mathbf{u}}^{(m)}$ converge pointwise to the functions pair denoted by $\overline{\mathbf{u}}$ and $\underline{\mathbf{u}}$, respectively. We next show that these limits coincide using integral representation for the associated linear problems under Robin and Neumann boundary conditions.

As in [1], we introduce the Green's functions associated with system (2.4). These functions represent the fundamental solutions of the corresponding parabolic problems for the road and the field components, respectively. Specifically, we define

$$\begin{aligned}G_u(x,t;\xi,\tau)&=\frac{e^{-Lt}}{2\sqrt{\pi D(t-\tau)}}e^{-\frac{(x-\xi)^2}{4D(t-\tau)}},\\ G_v(x,y,t;\xi_1,\xi_2,\tau)&=\frac{e^{-Lt}}{4d\pi(t-\tau)}(e^{-\frac{(x-\xi_1)^2+(y-\xi_2)^2}{4d(t-\tau)}}+e^{-\frac{(x-\xi_1)^2+(-y-\xi_2)^2}{4d(t-\tau)}}).\end{aligned}$$

For notational convenience, we define

$$\begin{aligned}I_u&=\int_{\mathbb{R}}G_u(x,t;\xi,0)u_0(\xi)\mathrm{d}\xi,\\ I_v&=\int_{\mathbb{R}\times\mathbb{R}^+}G_v(x,y,t;\xi_1,\xi_2,0)v_0(\xi_1,\xi_2)\mathrm{d}\boldsymbol{\xi},\end{aligned}$$

where $\boldsymbol{\xi} = (\xi_1, \xi_2)$. By the classic integral representation theory (see, e.g., [21]), the solution to system (2.4) can be represented using the Green's functions defined above:

$$
\begin{aligned}
\overline{u}^{(m)} &= I_u + \int_0^t \mathrm{d}\tau \int_{\mathbb{R}} G_u((L-\mu)\overline{u}^{(m-1)}(\xi,\tau) + \nu\overline{v}^{(m-1)}(\xi,0,\tau))\mathrm{d}\xi,\\
\underline{u}^{(m)} &= I_u + \int_0^t \mathrm{d}\tau \int_{\mathbb{R}} G_u((L-\mu)\underline{u}^{(m-1)}(\xi,\tau) + \nu\underline{v}^{(m-1)}(\xi,0,\tau))\mathrm{d}\xi,\\
\overline{v}^{(m)} &= I_v + \int_0^t \mathrm{d}\tau \int_{\mathbb{R}\times\mathbb{R}^+} G_v(\overline{v}^{(m-1)} f(\overline{v}^{(m-1)}) - \alpha\overline{v}^{(m)}\phi * \underline{v}^{(m-1)} + L\overline{v}^{(m-1)})\mathrm{d}\boldsymbol{\xi},\\
&\quad - \frac{1}{d}\int_0^t \mathrm{d}\tau \int_{\mathbb{R}} G_v|_{\xi_2=0}(\mu\overline{u}^{(m-1)}(\xi_1,\tau) + (L-\nu)\overline{v}^{(m-1)}(\xi_1,0,\tau))\mathrm{d}\xi_1,\\
\underline{v}^{(m)} &= I_v + \int_0^t \mathrm{d}\tau \int_{\mathbb{R}\times\mathbb{R}^+} G_v(\underline{v}^{(m-1)} f(\underline{v}^{(m-1)}) - \alpha\underline{v}^{(m)}\phi * \overline{v}^{(m-1)} + L\underline{v}^{(m-1)})\mathrm{d}\boldsymbol{\xi},\\
&\quad - \frac{1}{d}\int_0^t \mathrm{d}\tau \int_{\mathbb{R}} G_v|_{\xi_2=0}(\mu\underline{u}^{(m-1)}(\xi_1,\tau) + (L-\nu)\underline{v}^{(m-1)}(\xi_1,0,\tau))\mathrm{d}\xi_1.
\end{aligned}
$$

The dominated convergence theorem yields

$$
\begin{aligned}
\overline{u} &= I_u + \int_0^t \mathrm{d}\tau \int_{\mathbb{R}} G_u((L-\mu)\overline{u}(\xi,\tau) + \nu\overline{v}(\xi,0,\tau))\mathrm{d}\xi,\\
\underline{u} &= I_u + \int_0^t \mathrm{d}\tau \int_{\mathbb{R}} G_u((L-\mu)\underline{u}(\xi,\tau) + \nu\underline{v}(\xi,0,\tau))\mathrm{d}\xi,\\
\overline{v} &= I_v + \int_0^t \mathrm{d}\tau \int_{\mathbb{R}\times\mathbb{R}^+} G_v(\overline{v} f(\overline{v}) - \alpha\overline{v}\phi * \underline{v} + L\overline{v})\mathrm{d}\boldsymbol{\xi}\\
&\quad - \frac{1}{d}\int_0^t \mathrm{d}\tau \int_{\mathbb{R}} G_v|_{\xi_2=0}(\mu\overline{u}(\xi_1,\tau) + (L-\nu)\overline{v}(\xi_1,0,\tau))\mathrm{d}\xi_1,\\
\underline{v} &= I_v + \int_0^t \mathrm{d}\tau \int_{\mathbb{R}\times\mathbb{R}^+} G_v(\underline{v} f(\underline{v}) - \alpha\underline{v}\phi * \overline{v} + L\underline{v})\mathrm{d}\boldsymbol{\xi}\\
&\quad - \frac{1}{d}\int_0^t \mathrm{d}\tau \int_{\mathbb{R}} G_v|_{\xi_2=0}(\mu\underline{u}(\xi_1,\tau) + (L-\nu)\underline{v}(\xi_1,0,\tau))\mathrm{d}\xi_1.
\end{aligned}
\tag{2.7}
$$

By computing the pairwise differences, we can derive the expressions as follows:

$$
\begin{aligned}
\overline{u} - \underline{u} &= \int_0^t \mathrm{d}\tau \int_{\mathbb{R}} G_u((L-\mu)(\overline{u}-\underline{u})(\xi,\tau) + \nu(\overline{v}-\underline{v})(\xi,0,\tau))\mathrm{d}\xi\\
&\le e^{-Lt}((L-\mu)\|\overline{u}-\underline{u}\|_{L^\infty(D_u)} + \nu\|\overline{v}-\underline{v}\|_{L^\infty(D_v)}) \int_0^t \mathrm{d}\tau \int_{\mathbb{R}} \frac{1}{2\sqrt{\pi D(t-\tau)}} e^{-\frac{(x-\xi)^2}{4D(t-\tau)}}\mathrm{d}\xi\\
&\le Lte^{-Lt}(\|\overline{u}-\underline{u}\|_{L^\infty(D_u)} + \|\overline{v}-\underline{v}\|_{L^\infty(D_v)}),
\end{aligned}
$$

and

$$\begin{aligned}
\overline{v}-\underline{v}=&\int_0^t \mathrm{d}\tau\int_{\mathbb{R}\times\mathbb{R}^+} G_v(\overline{v}f(\overline{v})-\underline{v}f(\underline{v})+L(\overline{v}-\underline{v})-\alpha\overline{v}\phi*\underline{v}+\alpha\underline{v}\phi*\overline{v})\mathrm{d}\boldsymbol{\xi}\\
&-\frac{1}{d}\int_0^t \mathrm{d}\tau\int_{\mathbb{R}} G_v|_{\xi_2=0}(\mu(\overline{u}-\underline{u})(\xi_1,\tau)+(L-\nu)(\overline{v}-\underline{v})(\xi_1,0,\tau))\mathrm{d}\xi_1\\
&\le e^{-Lt}(L+a)\|\overline{v}-\underline{v}\|_{L^\infty(D_v)}\int_0^t \mathrm{d}\tau\int_{\mathbb{R}\times\mathbb{R}^+}\frac{1}{4d\pi(t-\tau)}(e^{-\frac{(x-\xi_1)^2+(y-\xi_2)^2}{4d(t-\tau)}}+e^{-\frac{(x-\xi_1)^2+(-y-\xi_2)^2}{4d(t-\tau)}})\mathrm{d}\boldsymbol{\xi}\\
&+\frac{e^{-Lt}}{d}(\mu\|\overline{u}-\underline{u}\|_{L^\infty(D_u)}+(L-\nu)\|\overline{v}-\underline{v}\|_{L^\infty(D_v)})\int_0^t \mathrm{d}\tau\int_{\mathbb{R}}\frac{e^{-\frac{(x-\xi_1)^2+y^2}{4(t-\tau)}}}{2d\pi(t-\tau)}\mathrm{d}\xi_1\\
&\le (L+a)te^{-Lt}\|\overline{v}-\underline{v}\|_{L^\infty(D_v)}+\frac{2\sqrt{t}e^{-Lt}}{d\sqrt{d\pi}}(\mu\|\overline{u}-\underline{u}\|_{L^\infty(D_u)}+(L-\nu)\|\overline{v}-\underline{v}\|_{L^\infty(D_v)}),
\end{aligned}$$

where $D_u=\mathbb{R}\times[0,+\infty)$ and $D_v=\mathbb{R}\times\mathbb{R}^+\times[0,+\infty)$. By choosing large enough constant L, we can derive that

$$\|\overline{u}-\underline{u}\|_{L^\infty(D_u)}+\|\overline{v}-\underline{v}\|_{L^\infty(D_v)}\le CL(t+\sqrt{t})e^{-Lt}(\|\overline{u}-\underline{u}\|_{L^\infty(D_u)}+\|\overline{v}-\underline{v}\|_{L^\infty(D_v)})$$

for $t>0$, where positive constant $C=\max\{3,\frac{2}{d\sqrt{d\pi}}\}$. Notice that $t\le\sqrt{t}$ if $t\le 1$ otherwise $t>\sqrt{t}$. Thus,

$$\|\overline{u}-\underline{u}\|_{L^\infty(D_u)}+\|\overline{v}-\underline{v}\|_{L^\infty(D_v)}\le 2CL\sqrt{t}e^{-Lt}(\|\overline{u}-\underline{u}\|_{L^\infty(D_u)}+\|\overline{v}-\underline{v}\|_{L^\infty(D_v)})$$

for $t\in(0,1]$ and

$$\|\overline{u}-\underline{u}\|_{L^\infty(D_u)}+\|\overline{v}-\underline{v}\|_{L^\infty(D_v)}\le 2CLte^{-Lt}(\|\overline{u}-\underline{u}\|_{L^\infty(D_u)}+\|\overline{v}-\underline{v}\|_{L^\infty(D_v)})$$

for $t>1$.

Notice that $\sqrt{t}e^{-Lt}$ is monotonically increasing in $(0,\frac{1}{2L})$ and opposite in $(\frac{1}{2L},+\infty)$. Then we can choose $T\in(0,\frac{1}{4C^2L^2})$ such that $2CL\sqrt{t}e^{-Lt}<1$ for $t\in(0,T]$, thereby

$$\|\overline{u}-\underline{u}\|_{L^\infty(D_u)}=\|\overline{v}-\underline{v}\|_{L^\infty(D_v)}=0 \text{ in } (0,T].$$

Since the constants C and L do not depend on the initial data, the time horizon T could be prolonged up to 1. Conversely, since L is large enough, te^{-Lt} is monotonically decreasing in $[1,+\infty)$. Obviously, $2CLe^{-L}<1$ holds for large enough L, thereby $2CLte^{-Lt}<1$ for $t\in(1,+\infty)$. Therefore, we ensure the uniqueness for $t\in(0,\infty)$. The statement in Theorem 2.2 is proved. □

Remark 2.2. *Choose $M_u=\frac{\nu M_v}{\mu}$ and $M_v=\max\{\frac{\mu}{\nu}\|u_0\|_{L^\infty(\mathbb{R})},\|v_0\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)},\frac{a}{b}\}$. Then the pair (M_u,M_v) and $(0,0)$ form coupled upper and lower solutions of* (1.1)*. Depending on Theorem 2.2, the unique global solution (u,v) of* (1.1) *satisfies $0\le(u,v)\le(M_u,M_v)$.*

2.2. Asymptotic stability

Subsequently, to reveal the asymptotic stability of positive equilibrium to (1.1), we begin by analyzing the following auxiliary problem:

$$\begin{cases} w_1'=\nu w_2-\mu w_1, & t>0,\\ w_2'=w_2(a-bw_2-\alpha g^*), & t>0,\\ w_1(0)=w_{10}\ge 0,\\ w_2(0)=w_{20}\ge 0,\end{cases}\tag{2.8}$$

where $w_{i0}(i = 1, 2)$ are both constants.

Lemma 2.1. *Assume that $b > \alpha$ and r is any constant greater than $\frac{a}{b}$. For any fixed $g^* \in [0, r]$, the equilibrium (w_1^*, w_2^*) of (2.8) is globally asymptotically stable in $(0, +\infty)$, where $w_1^* = \frac{\nu w_2^*}{\mu}$ and $w_2^* \equiv \frac{a-\alpha g^*}{b}$.*

Proof. Define the Lyapunov function

$$V(w_1, w_2; t) = \frac{b\mu}{2\nu^2}(w_1 - w_1^*)^2 + w_2 - w_2^* - w_2^* \ln(\frac{w_2}{w_2^*}).$$

We check that $V \geq 0$ for all $t \geq 0$ and $V = 0$ if and only if $(w_1, w_2) = (w_1^*, w_2^*)$. Directly computing, we have

$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} &= \frac{\partial V}{\partial w_1} \cdot \frac{\partial w_1}{\partial t} + \frac{\partial V}{\partial w_2} \cdot \frac{\partial w_2}{\partial t} \\
&= \frac{b\mu}{\nu^2}(w_1 - w_1^*)w_1' + (1 - \frac{w_2^*}{w_2})w_2' \\
&= -\frac{b\mu^2}{\nu^2}(w_1 - w_1^*)^2 + \frac{b\mu}{\nu}(w_1 - w_1^*)(w_2 - w_2^*) - b(w_2 - w_2^*)^2.
\end{aligned}$$

Notice that $\mathrm{d}V/\mathrm{d}t$ is a negative definite quadratic form with respect to $w_1 - w_1^*$ and $w_2 - w_2^*$. It implies that $\mathrm{d}V/\mathrm{d}t < 0$ if $(w_1, w_2) \not\equiv (w_1^*, w_2^*)$. Hence, $\mathrm{d}V/\mathrm{d}t \leq 0$. By the Lyapunov-LaSalle invariance principle [22, Chapter X: Theorem 1.3], the equilibrium point (w_1^*, w_2^*) is globally asymptotically stable in $(0, +\infty)$. □

In what follows, we show the main long-time dynamics of the system.

Theorem 2.3. *Assume that $b > \alpha$. If the nonnegative initial datum $(u_0, v_0) \not\equiv (0, 0)$ exhibits local Hölder continuity, along with its second-order derivatives, and meets the compatibility condition*

$$-d\partial_y v_0(x, 0) = \mu u_0(x) - \nu v_0(x, 0), \tag{2.9}$$

then the solution (u, v) of (1.1) tends toward $(\frac{\nu a}{\mu(b+\alpha)}, \frac{a}{b+\alpha})$ uniformly in $\mathbb{R} \times \mathbb{R}^+$ as $t \to +\infty$.

Proof. First, we define

$$(\underline{u}(t), \underline{v}(t)) = (\inf_{\mathbb{R}} u, \inf_{\mathbb{R}\times\mathbb{R}^+} v) \text{ and } (\overline{u}(t), \overline{v}(t)) = (\sup_{\mathbb{R}} u, \sup_{\mathbb{R}\times\mathbb{R}^+} v).$$

Denote

$$I_u = [\liminf_{t\to+\infty} \underline{u}(t), \limsup_{t\to+\infty} \overline{u}(t)] \text{ and } I_v = [\liminf_{t\to+\infty} \underline{v}(t), \limsup_{t\to+\infty} \overline{v}(t)].$$

It is enough to show that $I_u = \{\frac{\nu a}{\mu(b+\alpha)}\}$ and $I_v = \{\frac{a}{b+\alpha}\}$. It follows from Remark 2.2 that $(0, 0) \leq (u, v) \leq (M_u, M_v)$. Thus, we have $I_u \subseteq [0, M_u]$ and $I_v \subseteq [0, M_v]$.

Let $(\underline{u}^1, \underline{v}^1) = (0, 0)$ and $(\overline{u}^1, \overline{v}^1)$ satisfy $\overline{u}^1 = \frac{\nu\overline{v}^1}{\mu}$ and

$$\begin{cases}
\overline{u}_t^1 = \nu\overline{v}^1 - \mu\overline{u}^1, & t > 0, \\
\overline{v}_t^1 = \overline{v}^1(a - b\overline{v}^1), & t > 0, \\
\overline{u}^1(0) = \|u_0\|_{L^\infty(\mathbb{R})}, \\
\overline{v}^1(0) = \|v_0\|_{L^\infty(\mathbb{R}\times\mathbb{R}^+)}.
\end{cases}$$

We can easily check that $(\overline{u}^1,\overline{v}^1)$ and $(\underline{u}^1,\underline{v}^1)$ are the upper and lower solutions of (1.1), respectively. Employing Theorem 2.1 yields $(\underline{u}^1,\underline{v}^1) \le (u,v) \le (\overline{u}^1,\overline{v}^1)$, and then $I_u \subseteq [\underline{u}^1,\overline{u}^1]$ and $I_v \subseteq [\underline{v}^1,\overline{v}^1]$. Combining Lemma 2.1, we have $\lim\limits_{t\to+\infty}(\overline{u}^1,\overline{v}^1) = (\frac{\nu a}{\mu b},\frac{a}{b})$. Hence, $I_u \subseteq [\underline{u}^1,\frac{\nu a}{\mu b}]$ and $I_v \subseteq [\underline{v}^1,\frac{a}{b}]$. Then for any given $\varepsilon > 0$, $\exists t_2 > 0$ s.t.

$$v \le \overline{v}(t) \le \frac{a}{b} + \varepsilon \text{ for all } (x,y) \in \mathbb{R}\times\mathbb{R}^+,\ t \ge t_2.$$

Let $(\underline{u}^2,\underline{v}^2)$ satisfy $\underline{u}^2 = \frac{\nu\underline{v}^2}{\mu}$ and

$$\begin{cases} \underline{u}^2_t = \nu\underline{v}^2 - \mu\underline{u}^2, & t > t_2,\\ \underline{v}^2_t = \underline{v}^2(a - b\underline{v}^2 - \alpha(\frac{a}{b}+\varepsilon)), & t > t_2,\\ \underline{u}^2(t_2) = \underline{u}(t_2), \\ \underline{v}^2(t_2) = \underline{v}(t_2). \end{cases}$$

Choose $(\overline{u}^2,\overline{v}^2) = (\frac{\nu}{\mu}(\frac{a}{b}+\varepsilon),\frac{a}{b}+\varepsilon)$. Then $(\overline{u}^2,\overline{v}^2)$ and $(\underline{u}^2,\underline{v}^2)$ serve as the upper and lower solutions of (1.1) for $t \ge t_2$. Employing Theorem 2.1 derives that $(\underline{u}^2,\underline{v}^2) \le (u,v) \le (\overline{u}^2,\overline{v}^2)$ in $\mathbb{R}\times\mathbb{R}^+\times[t_2,+\infty)$. Moreover, Lemma 2.1 yields that $(\underline{u}^2,\underline{v}^2) \to (\frac{\nu}{\mu}g(\frac{a}{b}+\varepsilon), g(\frac{a}{b}+\varepsilon))$ as $t \to +\infty$, where $g(z) = \frac{a-\alpha z}{b}$. It implies that $I_u \subseteq [\frac{\nu}{\mu}g(\frac{a}{b}+\varepsilon),\overline{u}^2]$ and $I_v \subseteq [g(\frac{a}{b}+\varepsilon),\overline{v}^2]$. By the continuity of g and arbitrariness of ε, we have $I_u \subseteq [\frac{\nu}{\mu}g(\frac{a}{b}),\frac{\nu a}{\mu b}]$ and $I_v \subseteq [g(\frac{a}{b}),\frac{a}{b}]$.

Let $\lambda_3 = \frac{g(a/b)}{2}$ and $\sigma_3 = \frac{a}{b}$. Define the sequences $\{\lambda_i\}_{i=3}^{\infty}$ and $\{\sigma_i\}_{i=3}^{\infty}$ by

$$\lambda_i = g(\sigma_{i-1}) \text{ and } \sigma_i = g(\lambda_{i-1}) \text{ for } i \ge 4.$$

Obviously, $0 < \lambda_3 < \lambda_4 < \sigma_3$. Notice that $g(\frac{a}{b+\alpha}) = \frac{a}{b+\alpha}$ and g is strictly decreasing. Hence, we obtain $0 < \lambda_3 < \lambda_4 < \frac{a}{b+\alpha} < \sigma_3$. On the other hand, since $0 < \lambda_3 < \frac{a}{b+\alpha}$, we obtain that $\frac{a}{b+\alpha} < \sigma_4 < \sigma_3$. Therefore, we deduce that $0 < \lambda_3 < \lambda_4 < \frac{a}{b+\alpha} < \sigma_4 < \sigma_3$. Repeating the above process, we can derive two monotone sequences, satisfying

$$0 < \lambda_3 < \lambda_4 < \cdots < \lambda_i < \frac{a}{b+\alpha} < \sigma_i < \cdots < \sigma_4 < \sigma_3 = \frac{a}{b}.$$

Next, we claim that $I_u \subseteq [\frac{\nu\lambda_i}{\mu},\frac{\nu\sigma_i}{\mu}]$ and $I_v \subseteq [\lambda_i,\sigma_i]$ for all $i \ge 3$. Obviously, $I_u \subseteq [\frac{\nu\lambda_3}{\mu},\frac{\nu\sigma_3}{\mu}]$ and $I_v \subseteq [\lambda_3,\sigma_3]$. Supposing that $I_u \subseteq [\frac{\nu\lambda_{i-1}}{\mu},\frac{\nu\sigma_{i-1}}{\mu}]$ and $I_v \subseteq [\lambda_{i-1},\sigma_{i-1}]$, it is sufficient to prove that $I_u \subseteq [\frac{\nu\lambda_i}{\mu},\frac{\nu\sigma_i}{\mu}]$ and $I_v \subseteq [\lambda_i,\sigma_i]$.

In fact, combining $\lambda_{i-1} < \lambda_i < \sigma_i < \sigma_{i-1}$ with the continuity of g, we can choose any sufficiently small positive constant ε such that

$$\lambda_{i-1} \le g(\sigma_{i-1}+\varepsilon) \le g(\lambda_{i-1}-\varepsilon) \le \sigma_{i-1}. \tag{2.10}$$

It follows from $I_u \subseteq [\frac{\nu\lambda_{i-1}}{\mu},\frac{\nu\sigma_{i-1}}{\mu}]$ and $I_v \subseteq [\lambda_{i-1},\sigma_{i-1}]$ that $\exists\, T_i > 0$ s.t.

$$\frac{\nu(\lambda_{i-1}-\varepsilon)}{\mu} < \underline{u}(t) \le \overline{u}(t) < \frac{\nu(\sigma_{i-1}+\varepsilon)}{\mu} \text{ and } \lambda_{i-1}-\varepsilon < \underline{v}(t) \le \overline{v}(t) < \sigma_{i-1}+\varepsilon \text{ for } t \ge T_i.$$

Let $(\overline{u}^i, \overline{v}^i)$ and $(\underline{u}^i, \underline{v}^i)$ satisfy

$$\begin{cases} \overline{u}^i_t = \nu\overline{v}^i - \mu\overline{u}^i, & t > T_i, \\ \overline{v}^i_t = \overline{v}^i(a - b\overline{v}^i - \alpha(\lambda_{i-1} - \varepsilon)), & t > T_i, \\ \overline{u}^i(T_i) = \frac{\nu(\sigma_{i-1}+\varepsilon)}{\mu}, \\ \overline{v}^i(T_i) = \sigma_{i-1} + \varepsilon, \end{cases}$$

and

$$\begin{cases} \underline{u}^i_t = \nu\underline{v}^i - \mu\underline{u}^i, & t > T_i, \\ \underline{v}^i_t = \underline{v}^i(a - b\underline{v}^i - \alpha(\sigma_{i-1} + \varepsilon)), & t > T_i, \\ \underline{u}^i(T_i) = \frac{\nu(\lambda_{i-1}-\varepsilon)}{\mu}, \\ \underline{v}^i(T_i) = \lambda_{i-1} - \varepsilon, \end{cases}$$

respectively. Combining Lemma 2.1 with (2.10), we derive that there exists $t_i \geq T_i$ such that $\overline{v}^i \leq \sigma_{i-1} + \varepsilon$ and $\underline{v}^i \geq \lambda_{i-1} - \varepsilon$ for $t_i \geq T_i$. Redefine $(\overline{u}^i, \overline{v}^i)$ and $(\underline{u}^i, \underline{v}^i)$ by $\overline{u}^i = \frac{\nu\overline{v}^i}{\mu}$, $\underline{u}^i = \frac{\nu\underline{v}^i}{\mu}$ and

$$\begin{cases} \overline{u}^i_t = \nu\overline{v}^i - \mu\overline{u}^i, \ \underline{u}^i_t = \nu\underline{v}^i - \mu\underline{u}^i, & t > t_i, \\ \overline{v}^i_t = \overline{v}^i(a - b\overline{v}^i - \alpha(\lambda_{i-1} - \varepsilon)), \ \underline{v}^i_t = \underline{v}^i(a - b\underline{v}^i - \alpha(\sigma_{i-1} + \varepsilon)), & t > t_i, \\ \overline{u}^i(t_i) = \frac{\nu(\sigma_{i-1}+\varepsilon)}{\mu}, \ \underline{u}^i(t_i) = \frac{\nu(\lambda_{i-1}-\varepsilon)}{\mu} \\ \overline{v}^i(t_i) = \sigma_{i-1} + \varepsilon, \ \underline{v}^i(t_i) = \lambda_{i-1} - \varepsilon. \end{cases}$$

Notice that $a - b\overline{v}^i - \alpha(\lambda_{i-1} - \varepsilon) \geq a - b\overline{v}^i - \alpha\varphi * \underline{v}^i$ and $a - b\underline{v}^i - \alpha(\sigma_{i-1} + \varepsilon) \leq a - b\underline{v}^i - \alpha\varphi * \overline{v}^i$. It is easy to verify that $(\overline{u}^i, \overline{v}^i)$ and $(\underline{u}^i, \underline{v}^i)$ are the upper and lower solutions of (1.1) for $t \geq t_i$. According to Theorem 2.1, we have $(\underline{u}^i, \underline{v}^i) \leq (u, v) \leq (\overline{u}^i, \overline{v}^i)$ in $\mathbb{R} \times \mathbb{R}^+ \times [t_i, +\infty)$. Moreover, it follows from Lemma 2.1 that

$$\lim_{t\to+\infty} (\overline{u}^i, \overline{v}^i) = (\frac{\nu g(\lambda_{i-1} - \varepsilon)}{\mu}, g(\lambda_{i-1} - \varepsilon)) \text{ and } \lim_{t\to+\infty} (\underline{u}^i, \underline{v}^i) = (\frac{\nu g(\sigma_{i-1} + \varepsilon)}{\mu}, g(\sigma_{i-1} + \varepsilon)).$$

Due to the continuity of g, let $\varepsilon \to 0$. We obtain $I_u \subseteq [\frac{\nu\lambda_i}{\mu}, \frac{\nu\sigma_i}{\mu}]$ and $I_v \subseteq [\lambda_i, \sigma_i]$. Therefore, the claim follows from the induction.

Moreover, the monotone convergence theorem yields that there exist $\lambda, \sigma \in (0, \frac{a}{b})$ such that $\lambda_i \to \lambda$ and $\sigma_i \to \sigma$ as $i \to \infty$. Then $I_u \subseteq [\frac{\nu\lambda}{\mu}, \frac{\nu\sigma}{\mu}]$ and $I_v \subseteq [\lambda, \sigma]$. Combining the definition of λ_i and σ_i, we further obtain $\lambda = g(\sigma)$ and $\sigma = g(\lambda)$, which implies $\lambda = \sigma = \frac{a}{b+\alpha}$. Hence, $I_u = \{\frac{\nu a}{\mu(b+\alpha)}\}$ and $I_v = \{\frac{a}{b+\alpha}\}$. The proof is completed. □

2.3. Asymptotic spreading speed

In this section, we will give the asymptotic spreading speed along the road for (1.1). First, we analyze the following auxiliary problem:

$$\begin{cases} \partial_t U - D\partial_{xx}U = \nu V|_{y=0} - \mu U, & \text{in } \mathbb{R} \times (0, +\infty), \\ \partial_t V - d\Delta V = V(a - bV - \alpha g^*), & \text{in } \mathbb{R} \times \mathbb{R}^+ \times (0, +\infty), \\ -d\partial_y V|_{y=0} = \mu U - \nu V|_{y=0}, & \text{in } \mathbb{R}, \\ U(x, 0) = U_0(x), & \text{in } \mathbb{R}, \\ V(x, y, 0) = V_0(x, y), & \text{in } \mathbb{R} \times \mathbb{R}^+. \end{cases} \tag{2.11}$$

Similar upper and lower bounds of the asymptotic spreading speed for (2.11) are well known [2]. The main results can be concluded into the lemma as follows:

Lemma 2.2 (Theorem 1.1 in [2]). *Assume that $b > \alpha$ and g^* satisfies the conditions in Lemma 2.1.*
(i) Spreading. If initial condition (U_0, V_0) with compact support is nonnegative and meets (2.9), then the asymptotic spreading speed $c_* = c_*(\mu, \nu, d, D) > 0$ exists, and the solution (U, V) satisfies:

- *For each $c > c_*$,*

$$\lim_{t\to+\infty} \sup_{|x|\geq ct}(U, V) = (0, 0).$$

- *For each $c < c_*$,*

$$\lim_{t\to+\infty} \inf_{|x|\leq ct}(U, V) = (\frac{\nu(a - \alpha g^*)}{\mu b}, \frac{a - \alpha g^*}{b}).$$

(ii) Velocity. If $D \in (0, +\infty)$ is variable and other parameters are constant, then we have:

- *If $D \leq 2d$, then $c_*(\mu, \nu, d, D) = c_{KPP} := 2\sqrt{d(a - \alpha g^*)}$.*
- *If $D > 2d$, then $c_*(\mu, \nu, d, D) > c_{KPP}$ and the limit $\frac{c_*(\mu,\nu,d,D)}{\sqrt{D}}$ exists as $D \to +\infty$ and is positive and real.*

The proof of the spreading speed for above auxiliary problem (2.11) lies beyond the scope of this paper. For detailed arguments, one can refer to Sections 5 and 6 of [2] and omit the proof here.

Theorem 2.4. *If the assumptions are the same as Lemma 2.2, then we have the following two statements:*
(i) Spreading. Assume that the nonnegative initial data (u_0, v_0) of system (1.1) possesses the compact support and satisfies the compatibility condition (2.9). Then the asymptotic spreading speed $c_* = c_*(\mu, \nu, d, D, \alpha) > 0$ exists, and therefore:

- *For each $c > c_*$,*

$$\lim_{t\to+\infty} \sup_{|x|\geq ct}(u, v) = (0, 0),$$

- *For each $c < c_*$,*

$$\lim_{t\to+\infty} \inf_{|x|\leq ct}(u, v) = (\frac{\nu a}{\mu(b + \alpha)}, \frac{a}{b + \alpha}).$$

(ii) Velocity. If $D \in (0, +\infty)$ is variable and other parameters are constant, then the following results are observed:

- *If $D \leq 2d$, then $c_*(\mu, \nu, d, D, \alpha) = c_{KPP} := \frac{2\sqrt{abd}}{\sqrt{b+\alpha}}$.*
- *If $D > 2d$, then $c_*(\mu, \nu, d, D, \alpha) > c_{KPP}$ and the limit $\frac{c_*(\mu,\nu,d,D,\alpha)}{\sqrt{D}}$ exists as $D \to +\infty$ and is positive and real.*

Proof. Let (u, v) be the solution to system (1.1) and satisfy conditions mentioned in the theorem. By Theorem 2.3, we know $\exists T > 0$ such that

$$\frac{a}{b + \alpha} - \varepsilon < v < \frac{a}{b + \alpha} + \varepsilon \text{ in } \mathbb{R} \times \mathbb{R}^+ \times [T, +\infty)$$

for any $\varepsilon > 0$. Then we can deduce that

$$v(a - bv - \alpha(\frac{a}{b + \alpha} + \varepsilon)) \leq \partial_t v - d\Delta v \leq v(a - bv - \alpha(\frac{a}{b + \alpha} - \varepsilon)) \text{ in } \mathbb{R} \times \mathbb{R}^+ \times (T, +\infty).$$

Next, introduce the auxiliary system:

$$\begin{cases}\partial_t U - D\partial_{xx} U = \nu V|_{y=0} - \mu U, & \text{in } \mathbb{R}\times(T,+\infty),\\ \partial_t V - \Delta V = V(a - bV - \alpha(\frac{a}{b+\alpha} + \varepsilon)), & \text{in } \mathbb{R}\times\mathbb{R}^+\times(T,+\infty),\\ -d\partial_y V|_{y=0} = \mu U - \nu V|_{y=0}, & \text{in } \mathbb{R}\times(T,+\infty),\\ U(x,T) = \inf_{\mathbb{R}} u_0, & \text{in } \mathbb{R},\\ V(x,y,T) = \inf_{\mathbb{R}\times\mathbb{R}^+} v_0, & \text{in } \mathbb{R}\times\mathbb{R}^+.\end{cases} \tag{2.12}$$

Employing Lemma 2.2, we deduce that

$$\lim_{t\to+\infty}\inf_{|x|\le ct}(U,V) = (\frac{\nu V_*}{\mu}, V_*) \text{ for all } c < c_*, \tag{2.13}$$

where c_* is the asymptotic spreading speed for (2.12) and $V_* = \frac{a-\alpha(\frac{a}{b+\alpha}+\varepsilon)}{b}$. Notice that $\underline{c}_{KPP} = 2\sqrt{d(a - \frac{a\alpha}{b+\alpha} - \alpha\varepsilon)}$ corresponds to the speed without road diffusion. Moreover, it is evident that (u,v) serves as an upper solution of system (2.12). Then the comparison principle for the road-field model [2, Proposition 3.2] yields that $(u,v) \ge (U,V)$ for $t \ge T$. It follows that

$$(\frac{\nu V_*}{\mu}, V_*) = \lim_{t\to+\infty}\inf_{|x|\le ct}(U,V) \le \lim_{t\to+\infty}\inf_{|x|\le ct}(u,v)$$

for $c < c_*$.

Moreover, for the other auxiliary problem

$$\begin{cases}\partial_t U - D\partial_{xx} U = \nu V|_{y=0} - \mu U, & \text{in } \mathbb{R}\times(T,+\infty),\\ \partial_t V - \Delta V = V(a - bV - \alpha(\frac{a}{b+\alpha} - \varepsilon)), & \text{in } \mathbb{R}\times\mathbb{R}^+\times(T,+\infty),\\ -d\partial_y V|_{y=0} = \mu U - \nu V|_{y=0}, & \text{in } \mathbb{R}\times(T,+\infty),\\ U(x,T) = \sup_{\mathbb{R}} u_0, & \text{in } \mathbb{R},\\ V(x,y,T) = \sup_{\mathbb{R}\times\mathbb{R}^+} v_0, & \text{in } \mathbb{R}\times\mathbb{R}^+,\end{cases} \tag{2.14}$$

we can similarly deduce that

$$\lim_{t\to+\infty}\inf_{|x|\le ct}(U,V) = (\frac{\nu V^*}{\mu}, V^*) \text{ for all } c < c^*, \tag{2.15}$$

where $V^* = \frac{a-\alpha(\frac{a}{b+\alpha}-\varepsilon)}{b}$ and c^* is the asymptotic speeding speed for (2.12). Here, $\underline{c}_{KPP} = 2\sqrt{d(a - \frac{a\alpha}{b+\alpha} + \alpha\varepsilon)}$ corresponds to the speed of (2.12) without road diffusion. Notice that (u,v) serves exactly as a lower solution of (2.14). By Proposition 3.2 in [2], $(U,V) \ge (u,v)$ for $t \ge T$. Thus,

$$\lim_{t\to+\infty}\inf_{|x|\le ct}(u,v) \le \lim_{t\to+\infty}\inf_{|x|\le ct}(U,V) = (\frac{\nu V^*}{\mu}, V^*) \text{ for all } c < c^*.$$

Letting $\varepsilon \to 0$ yields that $V_* = V^* = \frac{a}{b+\alpha}$, $\underline{c}_{KPP} = \overline{c}_{KPP} = \frac{2\sqrt{abd}}{\sqrt{b+\alpha}} := c_{KPP}$ and $c_* = c^* := C_*$. Then one can obtain

$$\lim_{t\to+\infty}\inf_{|x|\le ct}(u,v) = (\frac{\nu a}{\mu(b+\alpha)}, \frac{a}{b+\alpha}) \text{ for all } c < C_*,$$

where C_* is the asymptotic spreading speed for (2.11) with $g^* = \frac{a}{b+\alpha}$ and c_{KPP} corresponding to the speed without road diffusion. By means of Lemma 2.2, we derive that

$$\lim_{t\to+\infty} \sup_{|x|\geq ct}(U, V) = (0, 0) \text{ for each } c > C_*.$$

Combining $(U, V) \geq (u, v)$ for $t \geq T$ with the nonnegativity of (u, v), we obtain

$$\lim_{t\to+\infty} \sup_{|x|\geq ct}(u, v) = (0, 0) \text{ for each } c > C_*.$$

The rest proof can be immediately verified by the similar discussion in [2, Section 5]. The proof is completed. □

3. Numerical simulations

This part involves computational simulations aimed at reproducing the evolving patterns of various subgroups, considering their movement along transportation routes and their activity within the field. Simulating the long-term behavior is numerically challenging due to the infinite spatial domain. To overcome this, we restrict the spatial domain to a finite region $[-L_x, L_x] \times [0, L_y]$ for the subpopulation in the field and $[-L_x, L_x]$ for that on the road. We investigate long-term propagation dynamics under Neumann boundary conditions at the boundaries of the new finite domain, which effectively approximate the original domain's behavior, provided that u and v remain sufficiently close to zero near these boundaries.

We choose the following parameter values: $a = 1, b = 1, \alpha = 0.4, \nu = 1$, and $\mu = 1.4$, and the initial data:

$$u_0 = \begin{cases} \delta\nu/\mu, & x \in [-50, 50], \\ 0, & \text{otherwise}, \end{cases} \quad \text{and } v_0 = \begin{cases} \delta, & (x, y) \in [-50, 50] \times [0, 20], \\ 0, & \text{otherwise}, \end{cases}$$

where $\delta = 0.5$. It is easy to check that the initial data selected meets the compatibility condition (2.9). According to Theorem 2.3, we calculate $\lim_{t\to\infty} (u, v) = (\frac{\nu a}{\mu(b+\alpha)}, \frac{a}{b+\alpha}) = (0.5102, 0.7143)$. It is evident that mobility is more pronounced along the road compared to the field. Therefore, set $D = 100$ and $d = 10$. For the numerical simulations, we adopt $L_x = 2000$, $L_y = 200$, and $\sigma = 25$. The main differential equations are discretized through classical finite difference scheme, while Robin boundary condition is approximated with a second-order finite difference method. Moreover, we adopt spatial step sizes $dx = dy = 1$ and a temporal increment of $dt = 0.001$, which yield stable and accurate numerical results.

Our investigation emphasizes the behavior of the subpopulations over the initial 6000 time steps. The nonlocal effect is characterized by a kernel function

$$\phi(x, y) = \begin{cases} \frac{1}{2\pi\sigma} e^{-\frac{x^2+y^2}{2\sigma}}, & y > 0, \\ 0, & y \leq 0, \end{cases}$$

where σ is a scaling parameter that determines the spatial extent of interactions. As [1], the kernel is well-defined and satisfies the series of conditions.

Figures 1 and 2 depict the diffusion of both subpopulations, which gradually occupy initially empty regions. Central densities in the invaded zones converge toward the steady state $(0.5102, 0.7143)$, which align with the theoretical prediction provided in Theorem 2.3.

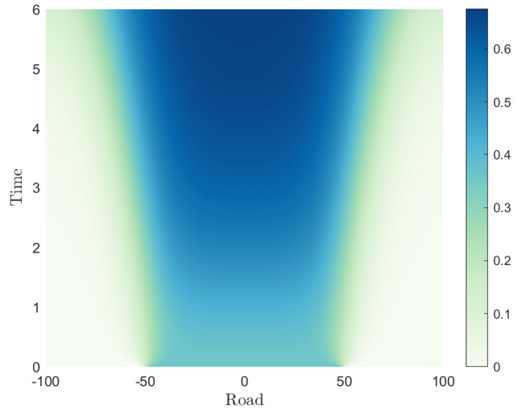

Figure 1. The road population density under default parameters.

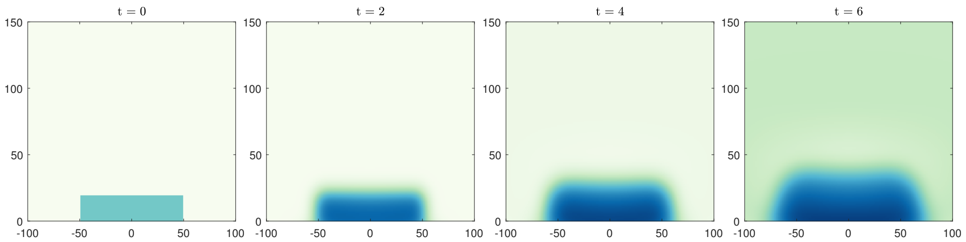

Figure 2. The field population density at times $t = 0, 2, 4, 6$ under default parameters.

We are interested in the influence of the nonlocal interaction on the spreading of populations. Figure 3 demonstrates that intensified nonlocal interaction lows down population density over the entire domain, but it has a negligible decrease on the speed of population spreading. Here, the speed is numerically approximated as $(L_s - 50)/6$, where $L_s(> 0)$ is the location at which $u(L_s, 6) \simeq 0.001$, i.e., $L_s = \sup_{x\in[-2000,2000]}\{|x| : u(x, 6) \simeq 0.001\}$. Figure 4 varies the width of the nonlocal kernel and shows that different kernel widths impose marginal effects on the population density distribution while slightly increasing the invasion speed, likely as a result of the nonlocal interactions in the field. As shown in Figure 5, the spreading speed along the road remains constant when $D \leq 2d$ and exceeds the velocity if $D > 2d$.

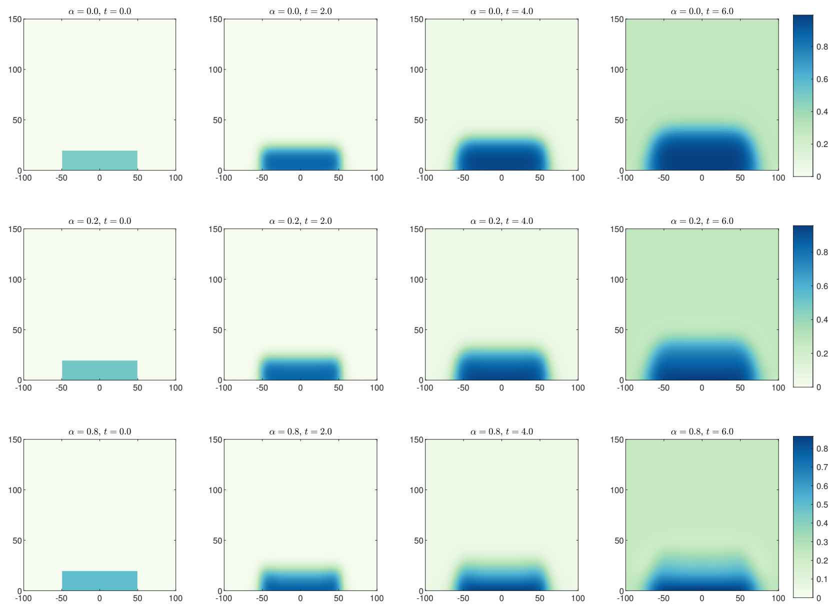

Figure 3. The filed population density at times $t = 0, 2, 4, 6$ for varying strengths of the intraspecific competition (α) with other default values.

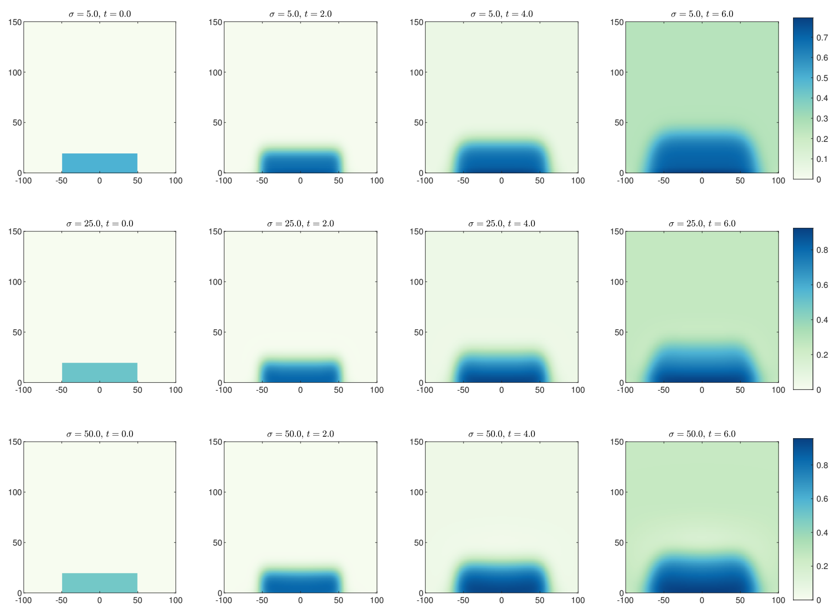

Figure 4. The field population density under default parameters with varying values of kernel width σ.

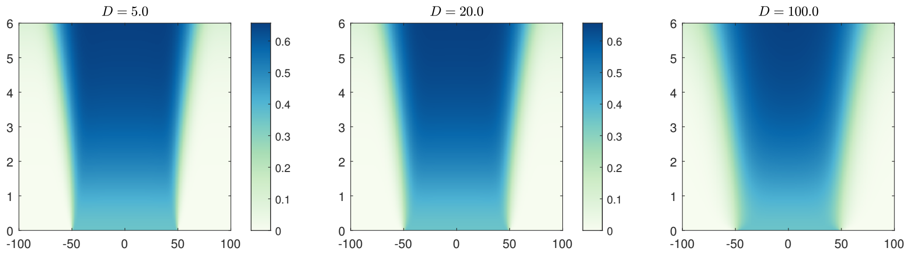

Figure 5. The road population density under default parameters with varying values of D.

4. Conclusions

In this study, we investigated a coupled road-field population model incorporating nonlocal interaction, based on the model newly developed by Berestycki et al. in 2013 [2]. This model facilitates the analysis of scenarios involving the rapid movement of diseases or ecological populations along specific transport routes. Compared with the model in [2], we take into account the nonlocal interactions among individuals in the field and present its effect on the dynamic behaviors. A stronger uniform convergence for $\lim_{t\to\infty}(u, v) = (\frac{\nu a}{\mu(b+\alpha)}, \frac{a}{b+\alpha})$ has been achieved under an additional condition $b > \alpha$, which is well justified. Numerical experiments with multiple group conditions (different nonlocal competition strengths α) further show a hint that the convergence is valid (Figure 3).

Moreover, we find a spreading speed threshold $c_*(\mu, \nu, d, D, \alpha)$ and state that the subpopulations both die out when the invasion velocity along the road c exceeds the threshold c_* and persist if $c < c_*$. Theorem 2.4 also analyzes how road diffusion accelerates subpopulations' spread and invasion speed. Specifically, when D is relatively low (i.e., below twice field diffusion rate $2d$), the spread along the road is mainly governed by the dynamics in the field. In this regime, the invasion speed coincides with the classical KPP-type speed, although it is slowed down due to the effect of nonlocal interactions. On the other hand, when $D > 2d$, the invasion speed exceeds the KPP speed and grows as the square root of the road diffusion, highlighting the accelerating influence of fast diffusion along the road.

Our numerical experiments (Figure 3) also present that the propagation speed seems to slightly decrease with the strength of nonlocal effect but not so obviously if D is large enough, whereas an increase in the kernel width σ leads to slight acceleration in the spreading process (Figure 4). A precise mathematical analysis of these numerical observations remains to be conducted and will be considered as a future extension.

Author contributions

You Zhou: Writing-original draft preparation, writing-reviewing and editing, methodology, visualization and funding acquisition; Zhi Ling: Writing-reviewing and editing, supervision and funding acquisition. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors would like to express their sincere thanks to the anonymous reviewers for their helpful comments. The work was partially supported by the National Natural Science Foundation of China (No. 12171418 & 12371505) and the Postgraduate Research & Practice Innovation Program of Jiangsu Province (No. KYCX24_3707).

Conflict of interest

The authors declare that there are no conflicts of interest.

References

1. Y. Zhou, Y. Xu, C. R. Tian, Z. Ling, Dynamic behaviors of a reaction–diffusion competition system with road diffusion and nonlocal interaction, *Chaos Soliton Fract.*, **198** (2025), 116458. https://doi.org/10.1016/j.chaos.2025.116458
2. H. Berestycki, J. M. Roquejoffre, L. Rossi, The influence of a line with fast diffusion on Fisher-KPP propagation, *J. Math. Biol.*, **66** (2013), 743–766. https://doi.org/10.1007/s00285-012-0604-z
3. H. W. McKenzie, E. H. Merrill, R. J. Spiteri, M. A. Lewis, How linear features alter predator movement and the functional response, *Interface Focus*, **2** (2012), 205–216. https://doi.org/10.1098/rsfs.2011.0086
4. A. Siegfried, *Itinéraires de contagions. Épidémies et idéologies*, Paris: Librairie Armand Colin, 1960.
5. H. Berestycki, J. M. Roquejoffre, L. Rossi, Fisher-KPP propagation in the presence of a line: Further effects, *Nonlinearity*, **26** (2013), 2623. https://doi.org/10.1088/0951-7715/26/9/2623
6. H. Berestycki, J. M. Roquejoffre, L. Rossi, The shape of expansion induced by a line with fast diffusion in Fisher-KPP equations, *Commun. Math. Phys.*, **343** (2016), 207–232. https://doi.org/10.1007/s00220-015-2517-3
7. H. Berestycki, R. Ducasse, L. Rossi, Generalized principal eigenvalues for heterogeneous road-field systems, *Commun. Contemp. Math.*, **22** (2020), 1950013. https://doi.org/10.1142/S0219199719500135
8. H. Berestycki, R. Ducasse, L. Rossi, Influence of a road on a population in an ecological niche facing climate change, *J. Math. Biol.*, **81** (2020), 1059–1097. https://doi.org/10.1007/s00285-020-01537-3
9. L. Dietrich, Existence of travelling waves for a reaction-diffusion system with a line of fast diffusion, *Appl. Math. Res. Express*, **2015** (2015), 204–252. https://doi.org/10.1093/amrx/abv002

10. L. Dietrich, Velocity enhancement of reaction-diffusion fronts by a line of fast diffusion, *Trans. Amer. Math. Soc.*, **369** (2017), 3221–3252. https://doi.org/10.1090/tran/6776
11. A. Tellini, Propagation speed in a strip bounded by a line with different diffusion, *J. Differ. Equ.*, **260** (2016), 5956–5986. https://doi.org/10.1016/j.jde.2015.12.028
12. T. Giletti, L. Monsaingeon, M. L. Zhou, A KPP road-field system with spatially periodic exchange terms, *Nonlinear Anal.*, **128** (2015), 273–302. https://doi.org/10.1016/j.na.2015.07.021
13. A. Pauthier, The influence of nonlocal exchange terms on Fisher-KPP propagation driven by a line of fast diffusion, *Commun. Math. Sci.*, **14** (2016), 535–570. https://doi.org/10.4310/CMS.2016.V14.N2.A10
14. M. Alfaro, R. Ducasse, S. Tréton, The field-road diffusion model: Fundamental solution and asymptotic behavior, *J. Differ. Equ.*, **367** (2023), 332–365. https://doi.org/10.1016/j.jde.2023.05.002
15. Z. Ling, L. Zhang, Y. Zhou, Long time behavior of a river-field population model with nonlocal reaction-diffusion, *Math. Methods Appl. Sci.*, **48** (2025), 15082–15091. https://doi.org/10.1002/mma.70001
16. M. Alfaro, M. Mourragui, S. Tréton, Bridging bulk and surface: an interacting particle system towards the field-road diffusion model, *Stoch. Partial Differ. Equ. Anal. Comput.*, **13** (2025), 1742–1793. https://doi.org/10.1007/s40072-025-00367-y
17. M. Alfaro, C. Chainais-Hillairet, Long time behavior of the field-road diffusion model: An entropy method and a finite volume scheme, *J. Numer. Math.*, **33** (2025), 241–265. https://doi.org/10.1515/jnma-2023-0137
18. K. Deng, Y. X. Wu, Global stability for a nonlocal reaction-diffusion population model, *Nonlinear Anal. Real World Appl.*, **25** (2015), 127–136. https://doi.org/10.1016/j.nonrwa.2015.03.006
19. K. Deng, Dynamical behavior of solutions of a semilinear heat equation with nonlocal singularity, *SIAM J. Math. Anal.*, **26** (1995), 98–111. https://doi.org/10.1137/S0036141091223881
20. K. Deng, M. K. Kwong, H. A. Levine, The influence of nonlocal nonlinearities on the long time behavior of solutions of Burgers' equation, *Quart. Appl. Math.*, **50** (1992), 173–200. https://doi.org/10.1090/QAM/1146631
21. C. V. Pao, *Nolinear parabolic and elliptic equations*, New York: Springer, 1992. https://doi.org/10.1007/978-1-4615-3034-3
22. J. K. Hale, *Ordinary differential equations*, 2 Eds., Florida: Krieger Publishing Company, 1980.

AIMS Press