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Research article

The entire solution pairs of a class of Fermat-type difference equation systems on $\mathbb{C}^2$

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Abstract: For two-variable polynomials, we have identified periodicity characteristics that differ from those of the single-variable polynomials. In this paper, based on the research of [1], we have further investigated the finite-order entire solution pairs for the generalized quadratic trinomial Fermat-type functional equation systems on $\mathbb{C}^2$ and have derived results that are distinct from those in the single complex variable case.

Keywords: Nevanlinna theory; two-variable polynomials; finite-order entire solutions; Fermat-type functional equations on $\mathbb{C}^2$
Mathematics Subject Classification: 32A08, 32A15, 32A22

1. Introduction

For the Fermat-type functional equation $f(z)^n + g(z)^n = 1$ in the classical complex field, where $n \geq 2$ is a positive integer, Gross [2–4], Baker [5], and Montel [6] respectively studied its entire and meromorphic solutions. Montel [6] further demonstrated that, for the functional equation $f(z)^n + g(z)^m = 1$, no transcendental entire solutions exist when $n \geq 3$ and $m \geq 3$. Additionally, Yang [7] showed that the function equation $a(z)f(z)^n + b(z)g(z)^m = 1$ admits no non-constant entire solutions under the condition that $\frac{1}{m} + \frac{1}{n} < 1$, where $a(z)$ and $b(z)$ are small functions relative to $f(z)$ and $g(z)$, respectively.

Significant progress has recently been made in the study of solutions to various difference and functional equations, particularly concerning their precise forms, growth properties, and value distribution. For instance, Xu and Jiang [8] conducted a systematic investigation into the existence and explicit forms of entire and meromorphic solutions for systems of quadratic trinomial functional equations in two complex variables, providing important insights for handling multivariate polynomial-type equations. Concurrently, Chen and Han [9] developed innovative methods for dealing with nonlinear terms in their study of entire solutions to Eikonal-type equations. In the context of Fermat-type equations, Xu and Cao [10] extended classical formulations to partial difference and differential-difference equations, examining their solution structures and thereby enriching this research domain. Furthermore, the solution properties of more complex product-type nonlinear partial differential-difference equations have also attracted considerable scholarly attention (see [11]). Collectively, these works have substantially advanced the theory of solutions in several complex variables.

The present research, which systematically classifies solution pairs for generalized quadratic trinomial Fermat-type difference equation systems on $\mathbb{C}^2$, builds upon and extends this line of inquiry. While existing literature has primarily focused on single equations or specific system forms, a notable gap remains in understanding solution pairs for the more strongly coupled difference equation systems investigated here. Our work not only addresses solution existence but, more importantly, provides precise characterization of all finite-order transcendental entire solution pairs. This approach reveals unique periodic and quasi-periodic properties of solution pairs in multivariate settings, representing a fundamental advancement beyond both the single-variable case and previously examined equation types in the literature.

Herein, the definition of additive quasi-periodic functions is provided as follows.

Definition 1.1. Let $f(z)$ be a meromorphic function on $\mathbb{C}^m$, where $c(\neq 0) \in \mathbb{C}^m$ and $B \in \mathbb{C}^m$ are constants, and $m \geq 2$ is a positive integer. If

$$f(z+c) - f(z) = B,$$

then $f(z)$ is called an additive quasi-periodic function with period c and addend B. In particular, if $B \neq 0$, $f(z)$ is referred to as a strictly additive quasi-periodic function.

We note that for polynomial functions, the periodicity characteristic differs between single-variable and several-variable functions. The following example illustrates the difference.

Example 1.1. Let

$$p(z_1, z_2) = c_2^2 z_1^2 - 2c_1c_2z_1z_2 + c_1^2 z_2^2 + c_2z_1 - c_1z_2 + a,$$

where $c = (c_1, c_2)(\neq 0)$ and a are constants. Then we get

$$p(z+c) = p(z).$$

This shows that the quadratic polynomial $p(z_1, z_2)$ has a period of c. However, for the single-variable polynomials, periodicity necessitates that the polynomials reduce to constants.

We generalize Theorem 4.1 and its corollaries from reference [1] to $\mathbb{C}^2$, leading to new results.

Theorem 1.2. *Let $c = (c_1, c_2)$ be non-zero constants, $A_1 = a_{11}a_{12}$, $B_1 = a_{11}b_{12} + a_{12}b_{11}$, $C_1 = b_{11}b_{12}$, $A_2 = a_{21}a_{22}$, $B_2 = a_{21}b_{22} + a_{22}b_{21}$, $C_2 = b_{21}b_{22}$ and A_i, B_i, C_i are non-zero constants, $B_i^2 \neq 4A_iC_i$, $D_i(z)(\neq 0)$, $Q_i(z)$ are polynomials, $i = 1, 2$. If $M = a_{12}b_{11} - b_{12}a_{11}$, $N = a_{22}b_{21} - b_{22}a_{21}$, $D_1(z) = D_{11}(z)D_{12}(z)$, and $D_2(z) = D_{21}(z)D_{22}(z)$, then the finite order of transcendental entire solution pairs $(f_1(z), f_2(z))$ of Fermat-type difference equation systems*

$$\begin{cases} A_1 f_1(z+c)^2 + B_1 f_1(z+c) f_2(z) + C_1 f_2(z)^2 = D_1(z)e^{2Q_1(z)} \\ A_2 f_2(z+c)^2 + B_2 f_2(z+c) f_1(z) + C_2 f_1(z)^2 = D_2(z)e^{2Q_2(z)} \end{cases} \tag{1.1}$$

on $\mathbb{C}^2$ must be one of the following cases:

(i)

$$\begin{cases} f_1(z) = \frac{1}{N}\left(a_{22}D_{21}(z)e^{Q_1(z+c)+\theta_1+h_2} - a_{21}D_{22}(z)e^{Q_1(z+c)+\theta_1-h_2}\right), \\ f_2(z) = \frac{1}{M}\left(a_{12}D_{11}(z)e^{Q_1(z)+h_1} - a_{11}D_{12}(z)e^{Q_1(z)-h_1}\right), \end{cases}$$

where $Q_1(z)$ is a non-constant polynomial, θ_1, h_1 *and* h_2 *are constants satisfy*

$$\begin{cases} a_{22}D_{21}(z+c)e^{h_2} - a_{21}D_{22}(z+c)e^{-h_2} = 0, b_{11}D_{12}(z)e^{-h_1} - b_{12}D_{11}(z)e^{h_1} = 0, \\ a_{12}D_{11}(z+c)e^{h_1} - a_{11}D_{12}(z+c)e^{-h_1} \neq 0, b_{21}D_{22}(z)e^{-h_2} - b_{22}D_{21}(z)e^{h_2} \neq 0; \end{cases}$$

(ii)

$$\begin{cases} f_1(z) = \frac{1}{N}\left(a_{22}D_{21}(z)e^{Q_2(z)+h_2} - a_{21}D_{22}(z)e^{Q_2(z)-h_2}\right), \\ f_2(z) = \frac{1}{M}\left(a_{12}D_{11}(z)e^{Q_2(z+c)+\theta_2+h_1} - a_{11}D_{12}(z)e^{Q_2(z+c)+\theta_2-h_1}\right), \end{cases}$$

where $Q_2(z)$ is a non-constant polynomial, θ_2, h_1 *and* h_2 *are constants satisfy*

$$\begin{cases} a_{22}D_{21}(z+c)e^{h_2} - a_{21}D_{22}(z+c)e^{-h_2} \neq 0, b_{11}D_{12}(z)e^{-h_1} - b_{12}D_{11}(z)e^{h_1} \neq 0, \\ a_{12}D_{11}(z+c)e^{h_1} - a_{11}D_{12}(z+c)e^{-h_1} = 0, b_{21}D_{22}(z)e^{-h_2} - b_{22}D_{21}(z)e^{h_2} = 0; \end{cases}$$

(iii)

$$\begin{cases} f_1(z) = \frac{1}{N}\left(a_{22}D_{21}(z)e^{Q_2(z)+h_2} - a_{21}D_{22}(z)e^{Q_2(z)-h_2}\right), \\ f_2(z) = \frac{1}{M}\left(a_{12}D_{11}(z)e^{Q_1(z)+h_1} - a_{11}D_{12}(z)e^{Q_1(z)-h_1}\right), \end{cases}$$

where $Q_1(z)$ and $Q_2(z)$ are non-constant additive quasi-periodic polynomials with $2c$ as the period and C_{12} as the addend, *and h_1 and h_2 are constants satisfy*

$$\begin{cases} a_{22}D_{21}(z+c)e^{h_2} - a_{21}D_{22}(z+c)e^{-h_2} \neq 0, b_{11}D_{12}(z)e^{-h_1} - b_{12}D_{11}(z)e^{h_1} \neq 0, \\ a_{12}D_{11}(z+c)e^{h_1} - a_{11}D_{12}(z+c)e^{-h_1} \neq 0, b_{21}D_{22}(z)e^{-h_2} - b_{22}D_{21}(z)e^{h_2} \neq 0; \end{cases}$$

(iv)

$$\begin{cases} f_1(z) = \frac{1}{N}\left(a_{22}D_{21}(z)e^{Q_2(z)+h_2(z)} - a_{21}D_{22}(z)e^{Q_2(z)-h_2(z)}\right), \\ f_2(z) = \frac{1}{M}\left(a_{12}D_{11}(z)e^{Q_1(z)+h_1(z)} - a_{11}D_{12}(z)e^{Q_1(z)-h_1(z)}\right), \end{cases}$$

where $Q_1(z)$ and $Q_2(z)$ are arbitrary polynomials, *and $h_1(z)$ and $h_2(z)$ are non-constant additive quasi-periodic polynomials with $2c$ as period and C_{11} and $\pm C_{11}$ as addends*, *respectively*;

(v)

$$\begin{cases} f_1(z) = \frac{1}{N}\left(a_{22}D_{21}(z)e^{Q_2(z)+h_2(z)} - a_{21}D_{22}(z)e^{Q_2(z)-h_2(z)}\right), \\ f_2(z) = \frac{1}{M}\left(a_{12}D_{11}(z)e^{Q_1(z)+h_1(z)} - a_{11}D_{12}(z)e^{Q_1(z)-h_1(z)}\right), \end{cases}$$

where $Q_1(z)$ and $Q_2(z)$ are arbitrary polynomials, *and $h_1(z)$ and $h_2(z)$ are periodic polynomials with $4c$ as the period.*

From the proof of Theorem 1.2, we can immediately derive the following two corollaries.

Corollary 1.3. *Let $c = (c_1, c_2)$ be non-zero constants, $A_1 = a_{11}a_{12}$, $B_1 = a_{11}b_{12} + a_{12}b_{11}$, $C_1 = b_{11}b_{12}$, $A_2 = a_{21}a_{22}$, $B_2 = a_{21}b_{22} + a_{22}b_{21}$, $C_2 = b_{21}b_{22}$ and A_i, B_i, C_i are non-zero constants and $B_i^2 \neq 4A_iC_i$, $D_i(z)(\neq 0)$ are polynomials, $i = 1, 2$. If $M = a_{12}b_{11} - b_{12}a_{11}$, $N = a_{22}b_{21} - b_{22}a_{21}$, $D_1(z) = D_{11}(z)D_{12}(z)$, and $D_2(z) = D_{21}(z)D_{22}(z)$, then the finite order of transcendental entire solution pairs $(f_1(z), f_2(z))$ of Fermat-type difference equation systems*

$$\begin{cases} A_1 f_1(z+c)^2 + B_1 f_1(z+c) f_2(z) + C_1 f_2(z)^2 = D_1(z) \\ A_2 f_2(z+c)^2 + B_2 f_2(z+c) f_1(z) + C_2 f_1(z)^2 = D_2(z) \end{cases}$$

on $\mathbb{C}^2$ must be one of the following cases:

(i)

$$\begin{cases} f_1(z) = \frac{1}{N}\left(a_{22}D_{21}(z)e^{h_2(z)} - a_{21}D_{22}(z)e^{-h_2(z)}\right), \\ f_2(z) = \frac{1}{M}\left(a_{12}D_{11}(z)e^{h_1(z)} - a_{11}D_{12}(z)e^{-h_1(z)}\right), \end{cases}$$

where $h_1(z)$ and $h_2(z)$ are non-constant additive quasi-periodic polynomials with $2c$ as period and C_{11} and C_{12} as the addend, respectively;

(ii)

$$\begin{cases} f_1(z) = \frac{1}{N}\left(a_{22}D_{21}(z)e^{h_2(z)} - a_{21}D_{22}(z)e^{-h_2(z)}\right), \\ f_2(z) = \frac{1}{M}\left(a_{12}D_{11}(z)e^{h_1(z)} - a_{11}D_{12}(z)e^{-h_1(z)}\right), \end{cases}$$

where $h_1(z)$ and $h_2(z)$ are non-constant periodic polynomials with period $4c$.

Corollary 1.4. *Let $c = (c_1, c_2)$ be non-zero constants, $A_1 = a_{11}a_{12}$, $B_1 = a_{11}b_{12} + a_{12}b_{11}$, $C_1 = b_{11}b_{12}$, $A_2 = a_{21}a_{22}$, $B_2 = a_{21}b_{22} + a_{22}b_{21}$, $C_2 = b_{21}b_{22}$ and A_i, B_i, C_i are non-zero constants and $B_i^2 \neq 4A_iC_i$, $Q_i(z)$ are polynomials, $i = 1, 2$. If $M = a_{12}b_{11} - b_{12}a_{11}$, $N = a_{22}b_{21} - b_{22}a_{21}$, then the finite order of transcendental entire solution pairs $(f_1(z), f_2(z))$ of Fermat-type difference equation systems*

$$\begin{cases} A_1 f_1(z+c)^2 + B_1 f_1(z+c) f_2(z) + C_1 f_2(z)^2 = e^{2Q_1(z)} \\ A_2 f_2(z+c)^2 + B_2 f_2(z+c) f_1(z) + C_2 f_1(z)^2 = e^{2Q_2(z)} \end{cases}$$

on $\mathbb{C}^2$ must be one of the following cases:

(i)

$$\begin{cases} f_1(z) = \frac{1}{N}\left(a_{22}e^{Q_1(z+c)+\theta_1+h_2} - a_{21}e^{Q_1(z+c)+\theta_1-h_2}\right), \\ f_2(z) = \frac{1}{M}\left(a_{12}e^{Q_1(z)+h_1} - a_{11}e^{Q_1(z)-h_1}\right), \end{cases}$$

where $Q_1(z)$ is a non-constant polynomial, θ_1, h_1, and h_2 are constants that satisfy

$$\begin{cases} a_{22}e^{h_2} - a_{21}e^{-h_2} = 0, b_{11}e^{-h_1} - b_{12}e^{h_1} = 0, \\ a_{12}e^{h_1} - a_{11}e^{-h_1} \neq 0, b_{21}e^{-h_2} - b_{22}e^{h_2} \neq 0; \end{cases}$$

(ii)

$$\begin{cases} f_1(z) = \frac{1}{N}\left(a_{22}e^{Q_2(z)+h_2} - a_{21}e^{Q_2(z)-h_2}\right), \\ f_2(z) = \frac{1}{M}\left(a_{12}e^{Q_2(z+c)+\theta_2+h_1} - a_{11}e^{Q_2(z+c)+\theta_2-h_1}\right), \end{cases}$$

where $Q_2(z)$ is a non-constant polynomial, θ_2, h_1 and h_2 are constants satisfy

$$\begin{cases} a_{22}e^{h_2} - a_{21}e^{-h_2} \neq 0, b_{11}e^{-h_1} - b_{12}e^{h_1} \neq 0, \\ a_{12}e^{h_1} - a_{11}e^{-h_1} = 0, b_{21}e^{-h_2} - b_{22}e^{h_2} = 0; \end{cases}$$

(iii)
$$\begin{cases} f_1(z) = \frac{1}{N}\left(a_{22}e^{Q_2(z)+h_2} - a_{21}e^{Q_2(z)-h_2}\right), \\ f_2(z) = \frac{1}{M}\left(a_{12}e^{Q_1(z)+h_1} - a_{11}e^{Q_1(z)-h_1}\right), \end{cases}$$

where $Q_1(z)$ and $Q_2(z)$ are non-constant additive quasi-periodic polynomials with $2c$ as the period and C_{12} as the addend and satisfy

$$\begin{cases} a_{22}e^{h_2} - a_{21}e^{-h_2} \neq 0, b_{11}e^{-h_1} - b_{12}e^{h_1} \neq 0, \\ a_{12}e^{h_1} - a_{11}e^{-h_1} \neq 0, b_{21}e^{-h_2} - b_{22}e^{h_2} \neq 0; \end{cases}$$

(iv)
$$\begin{cases} f_1(z) = \frac{1}{N}\left(a_{22}e^{Q_2(z)+h_2(z)} - a_{21}e^{Q_2(z)-h_2(z)}\right), \\ f_2(z) = \frac{1}{M}\left(a_{12}e^{Q_1(z)+h_1(z)} - a_{11}e^{Q_1(z)-h_1(z)}\right), \end{cases}$$

where $Q_1(z)$ and $Q_2(z)$ are arbitrary polynomials, and $h_1(z)$ and $h_2(z)$ are non-constant additive quasi-periodic polynomials with $2c$ as period and C_{11} and $\pm C_{11}$ as addends, respectively;

(v)
$$\begin{cases} f_1(z) = \frac{1}{N}\left(a_{22}e^{Q_2(z)+h_2(z)} - a_{21}e^{Q_2(z)-h_2(z)}\right), \\ f_2(z) = \frac{1}{M}\left(a_{12}e^{Q_1(z)+h_1(z)} - a_{11}e^{Q_1(z)-h_1(z)}\right), \end{cases}$$

where $Q_1(z)$ and $Q_2(z)$ are arbitrary polynomials, $h_1(z)$ and $h_2(z)$ are periodic polynomials with $4c$ as period.

By comparing Theorem 1.2, Corollary 1.3, and Corollary 1.4 with Theorem 4.1, Corollaries 4.1 and 4.2 from [1], we can identify significant differences between them. This is because if a single-variable polynomial exhibits periodicity, it must degenerate into a constant. If it exhibits strict additive quasi-periodicity, it degenerates to a first-degree polynomial. However, several-variable polynomials do not possess such properties.

2. Preliminary result

The following lemma will be used in the proof of the theorem.

Lemma 2.1. (see [12], Lemma 3.1) *Let $f_j(z)(\not\equiv 0)$, $j = 1, 2, 3$ be meromorphic functions on $\mathbb{C}^m$ such that f_1 is not constant, and $f_1 + f_2 + f_3 = 1$, and such that*

$$\sum_{j=1}^{3}\left\{N_2\left(r, \frac{1}{f_k}\right) + 2\overline{N}(r, f_k)\right\} < \lambda T(r, f_1) + O\left\{\log^+ T(r, f_1)\right\},$$

where $\lambda < 1$ is a positive number. Then either $f_2 \equiv 1$ or $f_3 \equiv 1$.

3. Proof of Theorem 1.2

Let $(f_1(z), f_2(z))$ be finite-order transcendental entire solution pairs of the Fermat-type difference equation systems (1.1). First, these equation systems (1.1) can be rewritten as

$$\begin{cases} (a_{11}f_1(z+c) + b_{11}f_2(z))\,(a_{12}f_1(z+c) + b_{12}f_2(z)) = D_1(z)e^{2Q_1(z)}, \\ (a_{21}f_2(z+c) + b_{21}f_1(z))\,(a_{22}f_2(z+c) + b_{22}f_1(z)) = D_2(z)e^{2Q_2(z)}, \end{cases}$$

where $A_1 = a_{11}a_{12}$, $B_1 = a_{11}b_{12} + a_{12}b_{11}$, $C_1 = b_{11}b_{12}$, $A_2 = a_{21}a_{22}$, $B_2 = a_{21}b_{22} + a_{22}b_{21}$, $C_2 = b_{21}b_{22}$, and A_i, B_i, C_i are non-zero constants with $B_i^2 \neq 4A_iC_i$ for $i = 1, 2$. From the above definitions, there exist polynomials $h_1(z)$ and $h_2(z)$ such that

$$\begin{cases} a_{11}f_1(z+c) + b_{11}f_2(z) = D_{11}(z)e^{Q_1(z)+h_1(z)}, \\ a_{12}f_1(z+c) + b_{12}f_2(z) = D_{12}(z)e^{Q_1(z)-h_1(z)}, \\ a_{21}f_2(z+c) + b_{21}f_1(z) = D_{21}(z)e^{Q_2(z)+h_2(z)}, \\ a_{22}f_2(z+c) + b_{22}f_1(z) = D_{22}(z)e^{Q_2(z)-h_2(z)}, \end{cases} \tag{3.1}$$

where $D_{11}(z)$, $D_{12}(z)$, $D_{21}(z)$, and $D_{22}(z)$ are non-zero polynomials, with $D_1(z) = D_{11}(z)D_{12}(z)$ and $D_2(z) = D_{21}(z)D_{22}(z)$.

Let

$$r_1(z) = Q_1(z) + h_1(z), \qquad s_1(z) = Q_1(z) - h_1(z),$$

$$r_2(z) = Q_2(z) + h_2(z), \qquad s_2(z) = Q_2(z) - h_2(z),$$

noting that $B_i^2 \neq 4A_iC_i$ for $i = 1, 2$, we have $M = a_{12}b_{11} - b_{12}a_{11} \neq 0$ and $N = a_{22}b_{21} - b_{22}a_{21} \neq 0$. According to (3.1), we obtain

$$\begin{cases} f_1(z+c) = \frac{1}{M}\left(b_{11}D_{12}(z)e^{s_1(z)} - b_{12}D_{11}(z)e^{r_1(z)}\right), \\ f_2(z) = \frac{1}{M}\left(a_{12}D_{11}(z)e^{r_1(z)} - a_{11}D_{12}(z)e^{s_1(z)}\right), \\ f_1(z) = \frac{1}{N}\left(a_{22}D_{21}(z)e^{r_2(z)} - a_{21}D_{22}(z)e^{s_2(z)}\right), \\ f_2(z+c) = \frac{1}{N}\left(b_{21}D_{22}(z)e^{s_2(z)} - b_{22}D_{21}(z)e^{r_2(z)}\right). \end{cases} \tag{3.2}$$

Based on the first and third equations of Eq (3.2), as well as the second and fourth equations, we obtain

$$Ma_{22}D_{21}(z+c)e^{r_2(z+c)} - Ma_{21}D_{22}(z+c)e^{s_2(z+c)} - Nb_{11}D_{12}(z)e^{s_1(z)} + Nb_{12}D_{11}(z)e^{r_1(z)} = 0 \tag{3.3}$$

and

$$Na_{12}D_{11}(z+c)e^{r_1(z+c)} - Na_{11}D_{12}(z+c)e^{s_1(z+c)} - Mb_{21}D_{22}(z)e^{s_2(z)} + Mb_{22}D_{21}(z)e^{r_2(z)} = 0, \tag{3.4}$$

from (3.3) and (3.4), since the coefficients of each exponential term are not zero, it follows that

$$\frac{Ma_{22}D_{21}(z+c)}{Nb_{11}D_{12}(z)}e^{r_2(z+c)-s_1(z)} - \frac{Ma_{21}D_{22}(z+c)}{Nb_{11}D_{12}(z)}e^{s_2(z+c)-s_1(z)} + \frac{b_{12}D_{11}(z)}{b_{11}D_{12}(z)}e^{r_1(z)-s_1(z)} = 1 \tag{3.5}$$

and

$$\frac{Na_{12}D_{11}(z+c)}{Mb_{21}D_{22}(z)}e^{r_1(z+c)-s_2(z)} - \frac{Na_{11}D_{12}(z+c)}{Mb_{21}D_{22}(z)}e^{s_1(z+c)-s_2(z)} + \frac{b_{22}D_{21}(z)}{b_{21}D_{22}(z)}e^{r_2(z)-s_2(z)} = 1. \tag{3.6}$$

Firstly, we consider Eq (3.5). To apply Lemma 2.1, we classify and analyze the three items on the left side of the equation through the following four cases:

Case 1. We analyze the first term on the left side of Eq (3.5), specifically $r_2(z+c) - s_1(z)$ (or $\frac{Ma_{22}D_{21}(z+c)}{Nb_{11}D_{12}(z)}e^{r_2(z+c)-s_1(z)}$).

Subcase 1.1. If $\frac{Ma_{22}D_{21}(z+c)}{Nb_{11}D_{12}(z)}e^{r_2(z+c)-s_1(z)}$ is not a constant, then according to Lemma 2.1, we have

$$-\frac{Ma_{21}D_{22}(z+c)}{Nb_{11}D_{12}(z)}e^{s_2(z+c)-s_1(z)} \equiv 1 \quad \text{or} \quad \frac{b_{12}D_{11}(z)}{b_{11}D_{12}(z)}e^{r_1(z)-s_1(z)} \equiv 1,$$

if $-\frac{Ma_{21}D_{22}(z+c)}{Nb_{11}D_{12}(z)}e^{s_2(z+c)-s_1(z)} \equiv 1$, then by (3.5), we derive that

$$\frac{Ma_{22}D_{21}(z+c)}{Nb_{11}D_{12}(z)}e^{r_2(z+c)-s_1(z)} + \frac{b_{12}D_{11}(z)}{b_{11}D_{12}(z)}e^{r_1(z)-s_1(z)} = 0,$$

we obtain $Ma_{22}D_{21}(z+c)e^{r_2(z+c)-r_1(z)} = -Nb_{12}D_{11}(z)$. Consequently, we find that both $s_2(z+c)-s_1(z)$ and $r_2(z+c)-r_1(z)$ must be constants. Thus, we can further deduce that

$$s_2(z+c)-s_1(z)-r_2(z+c)+r_1(z) = -2(h_2(z+c)-h_1(z))$$

is a constant; if $\frac{b_{12}D_{11}(z)}{b_{11}D_{12}(z)}e^{r_1(z)-s_1(z)} \equiv 1$, then $r_1(z)-s_1(z) = 2h_1(z)$ must be a constant. From Eq (3.5), we also obtain

$$Ma_{22}D_{21}(z+c)e^{r_2(z+c)-s_2(z+c)} = Ma_{21}D_{22}(z+c),$$

which implies that $r_2(z+c)-s_2(z+c) = 2h_2(z+c)$ is also a constant. Consequently, we conclude that both $h_1(z)$ and $h_2(z)$ are constants.

Subcase 1.2. If $\frac{Ma_{22}D_{21}(z+c)}{Nb_{11}D_{12}(z)}e^{r_2(z+c)-s_1(z)} = \Theta_1$ is a constant, we analyze the following two subcases based on Θ_1:

Subcase 1.2.1. If $\Theta_1 = 1$, then from (3.5) we have

$$\frac{Ma_{21}D_{22}(z+c)}{Nb_{12}D_{11}(z)}e^{s_2(z+c)-r_1(z)} \equiv 1,$$

this implies that both $r_2(z+c)-s_1(z)$ and $s_2(z+c)-r_1(z)$ are constants; thus, we get

$$r_2(z+c)-s_1(z)-s_2(z+c)+r_1(z) = 2(h_2(z+c)+h_1(z))$$

is also a constant.

Subcase 1.2.2. If $\Theta_1 \neq 1$, from Eq (3.5), if $-\frac{Ma_{21}D_{22}(z+c)}{Nb_{11}D_{12}(z)}e^{s_2(z+c)-s_1(z)}$ are non-constant functions, then by Lemma 2.1 we get $\Theta_1 = 1$ or $\frac{b_{12}D_{11}(z)}{b_{11}D_{12}(z)}e^{r_1(z)-s_1(z)} \equiv 1$. If $\Theta_1 = 1$, we obtain a contradiction; if $\frac{b_{12}D_{11}(z)}{b_{11}D_{12}(z)}e^{r_1(z)-s_1(z)} \equiv 1$, we have $-\frac{Ma_{21}D_{22}(z+c)}{Nb_{11}D_{12}(z)}e^{s_2(z+c)-s_1(z)}$ is a constant, we also get a contradiction; if $-\frac{Ma_{21}D_{22}(z+c)}{Nb_{11}D_{12}(z)}e^{s_2(z+c)-s_1(z)}$ is a constant, then $\frac{b_{12}D_{11}(z)}{b_{11}D_{12}(z)}e^{r_1(z)-s_1(z)}$ is also a constant. This implies that $r_2(z+c)-s_1(z)$, $s_2(z+c)-s_1(z)$, and $r_1(z)-s_1(z)$ are all constants. Similarly, for $\frac{b_{12}D_{11}(z)}{b_{11}D_{12}(z)}e^{r_1(z)-s_1(z)}$, we obtain the same conclusion. Thus, both $h_1(z)$ and $h_2(z)$ must be constants.

Case 2. We consider the second term on the left side of Eq (3.5), that is $s_2(z+c)-s_1(z)$ (or $-\frac{Ma_{21}D_{22}(z+c)}{Nb_{11}D_{12}(z)}e^{s_2(z+c)-s_1(z)}$).

Subcase 2.1. If $-\frac{Ma_{21}D_{22}(z+c)}{Nb_{11}D_{12}(z)}e^{s_2(z+c)-s_1(z)}$ is not a constant, then by applying Lemma 2.1, we have

$$\frac{Ma_{22}D_{21}(z+c)}{Nb_{11}D_{12}(z)}e^{r_2(z+c)-s_1(z)} \equiv 1 \quad \text{or} \quad \frac{b_{12}D_{11}(z)}{b_{11}D_{12}(z)}e^{r_1(z)-s_1(z)} \equiv 1,$$

if $\frac{Ma_{22}D_{21}(z+c)}{Nb_{11}D_{12}(z)}e^{r_2(z+c)-s_1(z)} \equiv 1$, then $r_2(z+c) - s_1(z)$ is a constant. From (3.5) we get

$$Ma_{21}D_{22}(z+c)e^{s_2(z+c)-r_1(z)} = Nb_{12}D_{11}(z),$$

then $s_2(z+c) - r_1(z)$ is also a constant, which implies that $r_2(z+c) - s_1(z) - s_2(z+c) + r_1(z) = 2(h_2(z+c) + h_1(z))$ is a constant; if $\frac{b_{12}D_{11}(z)}{b_{11}D_{12}(z)}e^{r_1(z)-s_1(z)} \equiv 1$, then $r_1(z) - s_1(z)$ is a constant, thus $h_1(z)$ is a constant. Again, from (3.5) we find $r_2(z+c) - s_2(z+c)$ is a constant; that is $h_2(z)$ is also a constant. Thus, both $h_1(z)$ and $h_2(z)$ are constants.

Subcase 2.2. If $-\frac{Ma_{21}D_{22}(z+c)}{Nb_{11}D_{12}(z)}e^{s_2(z+c)-s_1(z)} = \Theta_2$ is a constant, we also consider the following two subcases for Θ_2:

Subcase 2.2.1. If $\Theta_2 = 1$, then $s_2(z+c) - s_1(z)$ is a constant. According to Eq (3.5), we have

$$-\frac{Ma_{22}D_{21}(z+c)}{Nb_{12}D_{11}(z)}e^{r_2(z+c)-r_1(z)} \equiv 1,$$

it follows that $r_2(z+c) - r_1(z)$ is also a constant. Thus, we obtain

$$s_2(z+c) - s_1(z) - r_2(z+c) + r_1(z) = -2(h_2(z+c) - h_1(z))$$

which is a constant.

Subcase 2.2.2. If $\Theta_2 \neq 1$, according to Eq (3.5), as in the discussion of **Subcase** 1.2.2, we have $r_2(z+c) - s_1(z)$, $s_2(z+c) - s_1(z)$, and $r_1(z) - s_1(z)$ are constants. Then we also have $h_1(z)$ is a constant and $h_2(z)$, which is also a constant.

Case 3. We consider the third term on the left-hand side of Eq (3.5), which is $r_1(z) - s_1(z)$ (or $\frac{b_{12}D_{11}(z)}{b_{11}D_{12}(z)}e^{r_1(z)-s_1(z)}$).

Subcase 3.1. If $\frac{b_{12}D_{11}(z)}{b_{11}D_{12}(z)}e^{r_1(z)-s_1(z)}$ is not a constant, then by Lemma 2.1, we have

$$\frac{Ma_{22}D_{21}(z+c)}{Nb_{11}D_{12}(z)}e^{r_2(z+c)-s_1(z)} \equiv 1 \quad \text{or} \quad -\frac{Ma_{21}D_{22}(z+c)}{Nb_{11}D_{12}(z)}e^{s_2(z+c)-s_1(z)} \equiv 1,$$

if $\frac{Ma_{22}D_{21}(z+c)}{Nb_{11}D_{12}(z)}e^{r_2(z+c)-s_1(z)} \equiv 1$, we find $r_2(z+c) - s_1(z)$ is a constant. Combining this with Eq (3.5), we also find $s_2(z+c) - r_1(z)$ is a constant; then $r_2(z+c) - s_1(z) - s_2(z+c) + r_1(z) = 2(h_2(z+c) + h_1(z))$ is a constant; if $-\frac{Ma_{21}D_{22}(z+c)}{Nb_{11}D_{12}(z)}e^{s_2(z+c)-s_1(z)} \equiv 1$, then $s_2(z+c) - s_1(z)$ is a constant. From (3.5) we get $r_2(z+c) - r_1(z)$ is a constant. Thus, $r_2(z+c) - r_1(z) - s_2(z+c) + s_1(z) = 2(h_2(z+c) - h_1(z))$ is a constant.

Subcase 3.2. If $\frac{b_{12}D_{11}(z)}{b_{11}D_{12}(z)}e^{r_1(z)-s_1(z)}$ is a constant, then $r_1(z) - s_1(z) = 2h_1(z)$ is a constant. For Θ_3, we also have the following two subcases:

Subcase 3.2.1. If $\Theta_3 = 1$, then from (3.5), we obtain

$$\frac{a_{22}D_{21}(z+c)}{a_{21}D_{22}(z+c)}e^{r_2(z+c)-s_2(z+c)} \equiv 1,$$

then we get $r_2(z+c) - s_2(z+c) = 2h_2(z+c)$ is a constant, which means $h_2(z)$ is a constant.

Subcase 3.2.2. If $\Theta_3 \neq 1$, then according to Eq (3.5), as in the discussion of **Subcase** 1.2.2, we find that $h_1(z)$ is a constant and $h_2(z)$ is also a constant. Therefore, in this case, both $h_1(z)$ and $h_2(z)$ are constants.

Case 4. $r_2(z+c) - s_1(z)$, $s_2(z+c) - s_1(z)$, and $r_1(z) - s_1(z)$ are constants. Therefore, in this case, we conclude that both $h_1(z)$ and $h_2(z)$ are constants.

In summary, for Eq (3.5), the following three cases hold: either $h_1(z)$ and $h_2(z)$ are both constants, or $h_2(z+c)+h_1(z)$ is a constant, or $h_2(z+c)-h_1(z)$ is a constant.

Next, applying the same method to Eq (3.6) as used for Eq (3.5), we find that for (3.6), the following three cases hold: either $h_1(z)$ and $h_2(z)$ are both constants, or $h_1(z+c)+h_2(z)$ is a constant, or $h_1(z+c)-h_2(z)$ is a constant.

We now proceed to a further classification of the cases by combining the results from Eqs (3.5) and (3.6). First, if either $h_1(z)$ or $h_2(z)$ is constant, this, combined with other results, also implies that both $h_1(z)$ and $h_2(z)$ must be constants.

Next, let

$$h_1(z)+h_2(z+c)=a_1, \qquad h_1(z)-h_2(z+c)=a_2,$$
$$h_2(z)+h_1(z+c)=a_3, \qquad h_2(z)-h_1(z+c)=a_4,$$

where a_i $(i=1,2,3,4)$ are constants.

If $h_1(z)+h_2(z+c)=a_1$ and $h_2(z)+h_1(z+c)=a_3$, by shifting these two equations separately and combining them with another one, we obtain

$$h_1(z+2c)-h_1(z)=a_3-a_1 \qquad \text{and} \qquad h_2(z+2c)-h_2(z)=a_1-a_3,$$

which indicates that both $h_1(z)$ and $h_2(z)$ are non-constant quasi-periodic polynomial functions with a period of $2c$ and addends of a_3-a_1 and a_1-a_3, respectively.

If $h_1(z)+h_2(z+c)=a_1$ and $h_2(z)-h_1(z+c)=a_4$, shifting and combining these equations yields

$$h_1(z+4c)=h_1(z) \qquad \text{and} \qquad h_2(z+4c)=h_2(z),$$

which indicates that both $h_1(z)$ and $h_2(z)$ are non-constant periodic polynomial functions with a period of $4c$.

If $h_1(z)-h_2(z+c)=a_2$ and $h_2(z)+h_1(z+c)=a_3$, then by shifting these two equations separately and combining them with another one, we obtain

$$h_1(z+4c)=h_1(z) \qquad \text{and} \qquad h_2(z+4c)=h_2(z),$$

showing that both $h_1(z)$ and $h_2(z)$ are non-constant periodic polynomial functions with a period of $4c$.

If $h_1(z)-h_2(z+c)=a_2$ and $h_2(z)-h_1(z+c)=a_4$, then by shifting these two equations separately and combining them with another one, we get

$$h_1(z+2c)-h_1(z)=-a_2-a_4 \qquad \text{and} \qquad h_2(z+2c)-h_2(z)=-a_2-a_4,$$

which indicates that both $h_1(z)$ and $h_2(z)$ are non-constant quasi-periodic polynomial functions with a period of $2c$ and an addend of $-a_2-a_4$.

In summary, the following three cases hold: Case A: Both $h_1(z)$ and $h_2(z)$ are constants; Case B: Both $h_1(z)$ and $h_2(z)$ are non-constant quasi-periodic polynomial functions with a period of $2c$ and addends of C_{11} and $\pm C_{11}$, respectively; Case C: Both $h_1(z)$ and $h_2(z)$ are non-constant periodic polynomial functions with a period of $4c$. We will now continue to discuss these three cases:

Case A. Both $h_1(z)$ and $h_2(z)$ are constants. Let $h_1(z)=h_1$ and $h_2(z)=h_2$, where h_1 and h_2 are constants. Then, from (3.3) and (3.4), we have

$$\left(Ma_{22}D_{21}(z+c)e^{h_2}-Ma_{21}D_{22}(z+c)e^{-h_2}\right)e^{Q_2(z+c)}$$

$$= \left(Nb_{11}D_{12}(z)e^{-h_1} - Nb_{12}D_{11}(z)e^{h_1}\right) e^{Q_1(z)} \tag{3.7}$$

and

$$\left(Na_{12}D_{11}(z+c)e^{h_1} - Na_{11}D_{12}(z+c)e^{-h_1}\right) e^{Q_1(z+c)}$$
$$= \left(Mb_{21}D_{22}(z)e^{-h_2} - Mb_{22}D_{21}(z)e^{h_2}\right) e^{Q_2(z)}, \tag{3.8}$$

for (3.7), if $Ma_{22}D_{21}(z+c)e^{h_2} - Ma_{21}D_{22}(z+c)e^{-h_2} = 0$, then $Nb_{11}D_{12}(z)e^{-h_1} - Nb_{12}D_{11}(z)e^{h_1} = 0$; if $Ma_{22}D_{21}(z+c)e^{h_2} - Ma_{21}D_{22}(z+c)e^{-h_2} \neq 0$, then $Nb_{11}D_{12}(z)e^{-h_1} - Nb_{12}D_{11}(z)e^{h_1} \neq 0$, and $Q_2(z+c) - Q_1(z)$ is a constant; for (3.8), if $Na_{12}D_{11}(z+c)e^{h_1} - Na_{11}D_{12}(z+c)e^{-h_1} = 0$, then $Mb_{21}D_{22}(z)e^{-h_2} - Mb_{22}D_{21}(z)e^{h_2} = 0$; if $Na_{12}D_{11}(z+c)e^{h_1} - Na_{11}D_{12}(z+c)e^{-h_1} \neq 0$, then $Mb_{21}D_{22}(z)e^{-h_2} - Mb_{22}D_{21}(z)e^{h_2} \neq 0$, and $Q_1(z+c) - Q_2(z)$ is a constant. Thus, for **Case A**, we have the following four subcases:

Subcase A1.

$$\begin{cases} a_{22}D_{21}(z+c)e^{h_2} - a_{21}D_{22}(z+c)e^{-h_2} = 0, b_{11}D_{12}(z)e^{-h_1} - b_{12}D_{11}(z)e^{h_1} = 0, \\ a_{12}D_{11}(z+c)e^{h_1} - a_{11}D_{12}(z+c)e^{-h_1} = 0, b_{21}D_{22}(z)e^{-h_2} - b_{22}D_{21}(z)e^{h_2} = 0, \end{cases}$$

from the first and fourth equations, we have $a_{21}b_{22} - a_{22}b_{21} = 0$, which leads to a contradiction. Similarly, from the other two equations, we have $a_{12}b_{11} - a_{11}b_{12} = 0$, which also leads to a contradiction.

Subcase A2.

$$\begin{cases} a_{22}D_{21}(z+c)e^{h_2} - a_{21}D_{22}(z+c)e^{-h_2} = 0, b_{11}D_{12}(z)e^{-h_1} - b_{12}D_{11}(z)e^{h_1} = 0, \\ a_{12}D_{11}(z+c)e^{h_1} - a_{11}D_{12}(z+c)e^{-h_1} \neq 0, b_{21}D_{22}(z)e^{-h_2} - b_{22}D_{21}(z)e^{h_2} \neq 0, \end{cases}$$

then for (3.8), we have $Q_1(z+c) - Q_2(z)$ are constants. From (3.2), we conclude that if $h_1(z)$ and $h_2(z)$ are constants, then $Q_1(z)$ and $Q_2(z)$ must be non-constant polynomials. Otherwise, the solution pairs $(f_1(z), f_2(z))$ of the Fermat-type equations system (3.1) would not be transcendental, leading to a contradiction. Let $Q_2(z) = Q_1(z+c) + \theta_1$, where $Q_1(z)$ and $Q_2(z)$ are non-constant polynomials and θ_1 is a constant. Then, from (3.2), we conclude that the finite-order transcendental entire solutions pairs of the Fermat-type equations system (3.1) are:

$$\begin{cases} f_1(z) = \frac{1}{N}\left(a_{22}D_{21}(z)e^{Q_1(z+c)+\theta_1+h_2} - a_{21}D_{22}(z)e^{Q_1(z+c)+\theta_1-h_2}\right), \\ f_2(z) = \frac{1}{M}\left((a_{12}D_{11}(z)e^{Q_1(z)+h_1} - a_{11}D_{12}(z)e^{Q_1(z)-h_1}\right), \end{cases}$$

where θ_1, h_1, and h_2 are constants, which is the case (i) of Theorem 1.2.

Subcase A3.

$$\begin{cases} a_{22}D_{21}(z+c)e^{h_2} - a_{21}D_{22}(z+c)e^{-h_2} \neq 0, b_{11}D_{12}(z)e^{-h_1} - b_{12}D_{11}(z)e^{h_1} \neq 0, \\ a_{12}D_{11}(z+c)e^{h_1} - a_{11}D_{12}(z+c)e^{-h_1} = 0, b_{21}D_{22}(z)e^{-h_2} - b_{22}D_{21}(z)e^{h_2} = 0, \end{cases}$$

from (3.7), we have $Q_2(z+c) - Q_1(z)$ is a constant. Let $Q_1(z) = Q_2(z+c) + \theta_2$, where $Q_1(z)$ and $Q_2(z)$ are non-constant polynomials and θ_2 is a constant. Therefore, we conclude that the finite-order transcendental entire solutions pairs of the Fermat-type equations system (3.1) are:

$$\begin{cases} f_1(z) = \frac{1}{N}\left(a_{22}D_{21}(z)e^{Q_2(z)+h_2} - a_{21}D_{22}(z)e^{Q_2(z)-h_2}\right), \\ f_2(z) = \frac{1}{M}\left(a_{12}D_{11}(z)e^{Q_2(z+c)+\theta_2+h_1} - a_{11}D_{12}(z)e^{Q_2(z+c)+\theta_2-h_1}\right), \end{cases}$$

where θ_2, h_1, and h_2 are constants, which is the case (ii) of Theorem 1.2.

Subcase A4.

$$\begin{cases} a_{22}D_{21}(z+c)e^{h_2} - a_{21}D_{22}(z+c)e^{-h_2} \neq 0, b_{11}D_{12}(z)e^{-h_1} - b_{12}D_{11}(z)e^{h_1} \neq 0, \\ a_{12}D_{11}(z+c)e^{h_1} - a_{11}D_{12}(z+c)e^{-h_1} \neq 0, b_{21}D_{22}(z)e^{-h_2} - b_{22}D_{21}(z)e^{h_2} \neq 0, \end{cases}$$

then we have $Q_2(z+c) - Q_1(z)$ and $Q_1(z+c) - Q_2(z)$ are constants. Let

$$Q_2(z+c) - Q_1(z) = b_1 \quad \text{and} \quad Q_1(z+c) - Q_2(z) = b_2,$$

where b_1 and b_2 are constants. Then, it follows that

$$Q_1(z+2c) - Q_1(z) = b_1 + b_2 \quad \text{and} \quad Q_2(z+2c) - Q_2(z) = b_1 + b_2,$$

which indicates that both $Q_1(z)$ and $Q_2(z)$ are non-constant quasi-periodic polynomial functions with a period of $2c$ and an addend of $b_1 + b_2$. Similarly, from Eq (3.2), we conclude that the finite-order transcendental entire solutions pairs of the Fermat-type equations system (3.1) are:

$$\begin{cases} f_1(z) = \frac{1}{N}\left(a_{22}D_{21}(z)e^{Q_2(z)+h_2} - a_{21}D_{22}(z)e^{Q_2(z)-h_2}\right), \\ f_2(z) = \frac{1}{M}\left(a_{12}D_{11}(z)e^{Q_1(z)+h_1} - a_{11}D_{12}(z)e^{Q_1(z)-h_1}\right), \end{cases}$$

where $Q_1(z)$ and $Q_2(z)$ are non-constant quasi-periodic polynomial functions with a period of $2c$ and addend of C_{12}, which is the case (iii) of Theorem 1.2.

Case B. Both $h_1(z)$ and $h_2(z)$ are non-constant quasi-periodic polynomial functions with a period of $2c$ and addends of C_{11} and $\pm C_{11}$, respectively. Therefore, the finite-order transcendental entire solutions pairs $(f_1(z), f_2(z))$ of the Fermat-type equations system (3.1) are:

$$\begin{cases} f_1(z) = \frac{1}{N}\left(a_{22}D_{21}(z)e^{Q_2(z)+h_2(z)} - a_{21}D_{22}(z)e^{Q_2(z)-h_2(z)}\right), \\ f_2(z) = \frac{1}{M}\left(a_{12}D_{11}(z)e^{Q_1(z)+h_1(z)} - a_{11}D_{12}(z)e^{Q_1(z)-h_1(z)}\right), \end{cases}$$

where $Q_1(z)$ and $Q_2(z)$ are arbitrary polynomials, which is the case (iv) of Theorem 1.2.

Case C. Both $h_1(z)$ and $h_2(z)$ are non-constant periodic polynomial functions with a period of $4c$. Therefore, the finite-order transcendental entire solutions pairs $(f_1(z), f_2(z))$ of the Fermat-type equations system (3.1) are:

$$\begin{cases} f_1(z) = \frac{1}{N}\left(a_{22}D_{21}(z)e^{Q_2(z)+h_2(z)} - a_{21}D_{22}(z)e^{Q_2(z)-h_2(z)}\right), \\ f_2(z) = \frac{1}{M}\left(a_{12}D_{11}(z)e^{Q_1(z)+h_1(z)} - a_{11}D_{12}(z)e^{Q_1(z)-h_1(z)}\right), \end{cases}$$

where $Q_1(z)$ and $Q_2(z)$ are arbitrary polynomials, which is the case (v) of Theorem 1.2.

Therefore, we have completed the proof of Theorem 1.2.

4. Conclusions

In this paper, we have systematically investigated the finite-order transcendental entire solution pairs for a class of generalized quadratic trinomial Fermat-type difference equation systems on $\mathbb{C}^2$. By employing Nevanlinna theory and difference algebra techniques, we have successfully characterized the precise forms of all admissible solutions, as detailed in Theorem 1.2 and its Corollaries 1.3 and 1.4.

Our results reveal a fundamental distinction between the single-variable and several-variable cases. In the context of one complex variable, the periodicity or strict additive quasi-periodicity of a polynomial forces it to be constant or linear, respectively. However, as demonstrated in Example 1.1 and embodied in the solution forms of our main results, several-variable polynomials can exhibit non-trivial periodic and additive quasi-periodic behaviors without degenerating. This key difference leads to a significantly richer and more diverse structure of solutions for the considered system on $\mathbb{C}^2$, encompassing not only solutions with constant $h_i(z)$ but also those involving non-constant periodic or quasi-periodic polynomials $h_i(z)$.

This work extends and deepens the findings in [1] for the higher-dimensional case. Future research could focus on extending these results to more general types of functional equations, investigating systems on $\mathbb{C}^m$ for $m > 2$, or exploring the properties of meromorphic solution pairs.

Author contributions

This study was conceptualized, designed, and analyzed by Wang Zhuo, who also prepared the original draft. Zhang Qingcai was responsible for validating the research findings and reviewing and editing the manuscript. Both authors have reviewed and approved the final version.

Use of Generative-AI tools declaration

The authors declare that no generative AI tools were used in the preparation or writing of this research.

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Conflict of interest

The authors declare that there is no conflicts of interests regarding the publication of this paper.

References

1. Z. Wang, Q. C. Zhang, The entire solutions of some binomial difference equations, *Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A Mat.*, **118** (2024), 147. https://doi.org/10.1007/s13398-024-01641-9
2. F. Gross, On the functional equation $f^n + g^n = h^n$, *Amer. Math. Monthly*, **73** (1966), 1093–1096. https://doi.org/10.2307/2314644

3. F. Gross, On the equation $f^n + g^n = 1$, *Bull. Amer. Math. Soc.*, **72** (1966), 86–88. https://doi.org/10.1090/S0002-9904-1966-11429-5

4. F. Gross, On the equation $f^n + g^n = 1$. II, *Bull. Amer. Math. Soc.*, **74** (1968), 647–648. https://doi.org/10.1090/S0002-9904-1968-11975-5

5. I. N. Baker, On a class of meromorphic functions, *Proc. Amer. Math. Soc.*, **17** (1966), 819–822. https://doi.org/10.1090/S0002-9939-1966-0197732-X

6. P. Montel, *Leçons sur les familles normales de fonctions analytiques et leurs applications*, Paris: Gauthier-Viuars, 1927.

7. C. C. Yang, A generalization of a theorem of P. Montel on entire functions, *Proc. Amer. Math. Soc.*, **26** (1970), 332–334. https://doi.org/10.1090/S0002-9939-1970-0264080-X

8. H. Y. Xu, Y. Y. Jiang, Results on entire and meromorphic solutions for several systems of quadratic trinomial functional equations with two complex variables, *Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A Mat.*, **116** (2022), 8. https://doi.org/10.1007/s13398-021-01154-9

9. W. Chen, Q. Han, On entire solutions to eikonal-type equations, *J. Math. Anal. Appl.*, **506** (2022), 124704. https://doi.org/10.1016/j.jmaa.2020.124704

10. L. Xu, T. B. Cao, Solutions of complex Fermat-type partial difference and differential-difference equations, *Mediterr. J. Math.*, **15** (2018), 227. https://doi.org/10.1007/s00009-018-1274-x

11. Y. H. Xu, X. L. Liu, H. Y. Xu, N. Li, The study of solutions of several systems of the product type nonlinear partial differential difference equations, *ScienceAsia*, **50** (2024), 2024032. https://doi.org/10.2306/scienceasia1513-1874.2024.032

12. P. C. Hu, P. Li, C. C. Yang, *Unicity of meromorphic mappings*, New York: Springer, 2003. https://doi.org/10.1007/978-1-4757-3775-2

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