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Research article

The bivariate Weibull distribution based on the GFGM copula

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Abstract: In statistical modeling, bivariate models are essential, especially when examining data with two associated variables. Bivariate distributions capture dependencies between variables, offering a more realistic depiction of real-world phenomena compared to univariate models that treat variables independently. This is particularly important in domains where variables frequently show non-trivial correlations, such as environmental science, reliability engineering, medicine, and finance. This motivates the proposal of a bivariate distribution that uses the generalized Farlie-Gumbel-Morgenstern (FGM) copula and Weibull marginal distribution, referred to as the GFGM-WD. The GFGM-WD describes bivariate lifetime data with weak to moderate correlation between variables. The suggested model was employed to investigate the reliability of dependent stress-strength models. Several properties of the GFGM-WD were derived, including the product moment, the coefficient of correlation between the inner variables, and the conditional expectation. Additionally, the statistical characteristics of the concomitants' k-record values from the GFGM-WD were discussed. We ran comprehensive Monte Carlo simulations to assess the suggested distribution's performance and used the maximum likelihood estimation and Bayesian methods to estimate its parameters. Finally, the distribution was tested on two actual medical datasets, showing that it outperformed other pre-existing bivariate models in terms of fitting accuracy.

Keywords: generalized FGM copula; stress-strength model; Bayesian estimation; entropy; extropy; bivariate data analysis; Weibull distribution; concomitants; k-record values
Mathematics Subject Classification: 60B12, 62G30

1. Introduction

Studying bivariate data plays a critical role in reliability theory. Many aspects of life, including physics, medicine, economics, biology, and environmental sciences, require statistical methods on bivariate distributions. Thus, we present a new bivariate Weibull distribution. The Weibull distribution is incredibly adaptable and can fit a variety of data set shapes. Like the normal distribution, the Weibull distribution is unimodal and describes probabilities for continuous data. However, unlike the normal distribution, it can also model skewed data. It can even model data that is skewed both left and right because of its tremendous adaptability. It can be used to approximately represent other distributions as well as the normal distribution. Because of its versatility, analysts apply it in several contexts, such as capability analysis, quality control, engineering, and medical investigations. For recent studies on the Weibull distribution and its extensions with applications, see Sarhan and Sobh [1] and Sarhan and Apaloo [2]. It is frequently used in reliability studies, warranty analysis, and life data to estimate the time to failure for systems and components. The univariate Weibull distribution (denoted by $WD(\alpha,\beta)$) has the following distribution function (DF) and probability density function (PDF), respectively:

$$H_Z(z;\alpha,\beta) = 1 - e^{-\left(\frac{z}{\beta}\right)^{\alpha}};\ z > 0,\ \alpha,\beta > 0$$

and

$$h_Z(z;\alpha,\beta) = \frac{\alpha}{\beta}\left(\frac{z}{\beta}\right)^{\alpha-1} e^{-\left(\frac{z}{\beta}\right)^{\alpha}};\ z > 0,\ \alpha,\beta > 0,$$

where α and β are the shape and scale parameters, respectively.

One technique for creating bivariate distributions that can be found in the statistical literature is the knowledge of a copula (see Nelsen [3]). Copulas can help characterize bivariate distributions when they contain an explicit dependency structure. It is a function that joins bivariate DFs with uniform [0,1] margins. Copulas can be applied in this way to look into bivariate distribution studies. The copula function will be selected based on the dependence structure between the two random variables (RVs). Copulas are useful in high-dimensional statistical applications because they make it simple to model and estimate the distribution of random vectors by estimating marginals and copula separately. For two marginal univariate distributions $H_Z(z) = P(Z \le z)$ and $H_T(t) = P(T \le t)$, a copula $C(u, v)$, and its PDF, i.e., $c(u, v) = \frac{\partial^2 C(u,v)}{\partial u \partial v}$, Sklar [4] presented the joint DF (JDF) and joint PDF (JPDF), respectively, as follows:

$$H_{Z,T}(z, t) = C(H_Z(z), H_T(t)) \tag{1.1}$$

and

$$h_{Z,T}(z, t) = h_Z(z)h_T(t)c(H_Z(z), H_T(t)). \tag{1.2}$$

Among the most well-known copulas are the Farlie-Gumbel-Morgenstern (FGM) copula and its extensions. The FGM family of bivariate distributions has been the subject of in-depth research. Fischer and Klein [5] focused on specific generalized FGM copulas generated by a single function defined on the unit interval. Gupta and Wong [6], among others, deduced a three- and five-parameter bivariate beta distribution from the FGM family. El-Sherpieny et al. [7] introduced a novel bivariate FGM-Weibull-G family of distributions based on the FGM copula. The bivariate generalized half-logistic distribution utilizing the FGM copula was proposed by Hassan and Chesneau [8]. Recently, Mansour et al. [9] introduced a novel bivariate distribution, which combines the Sarmanov copula with the Epanechnikov-Weibull marginal distribution. Related bivariate generalizations have also been proposed by Alsalafi et al. [10], Pathak et al. [11], and Shahbaz and Shahbaz [12], who respectively introduced a novel bivariate transmuted family capable of modeling both positive and negative dependencies, a flexible bivariate generalized lifetime distribution extending classical lifetime models, and a new bivariate exponentiated family that unifies and generalizes several existing lifetime frameworks.

To further enhance the dependence range attainable by the FGM copula, Huang and Kotz [13] proposed the generalized FGM (GFGM) copula. Through successive iterations, they showed that even a single additional term can substantially increase the attainable correlation between marginals. The GFGM copula is defined as

$$C(u, v; \gamma, \omega) = uv\left[1 + \gamma(1-u)(1-v) + \omega uv(1-u)(1-v)\right]. \tag{1.3}$$

The corresponding PDF of the copula (1.3) is given by

$$c(u, v; \gamma, \omega) = 1 + \gamma(1-2u)(1-2v) + \omega uv(2-3u)(2-3v). \tag{1.4}$$

By setting $\omega = 0$, the FGM copula can be obtained as a special instance of the GFGM copula (1.1)-(1.2). Huang and Kotz [13] proved that the natural parameter space Ω (which is the admissible set of the parameters γ and ω) is convex, where

$$\Omega = \left\{(\gamma, \omega) : -1 \le \gamma \le 1; -1 \le \gamma + \omega; \omega \le \frac{3 - \gamma + \sqrt{9 - 6\gamma - 3\gamma^2}}{2}\right\}.$$

Additionally, this copula's correlation coefficient is $\rho = \frac{\gamma}{3} + \frac{\omega}{12}$. This copula was the subject of extensive research by many different researchers. Abd Elgawad et al. [14,15], Alawady et al. [16], Barakat et al. [17–19], and Husseiny et al. [20] are a few among them.

In this study, a bivariate distribution known as the GFGM bivariate Weibull distribution (GFGM-WD) is created using the GFGM copula and Weibull marginal distributions. Almetwally et al. [21] used the FGM copula to tackle the same problem and obtained what is called the FGM bivariate Weibull (FGMBW). For characterizing bivariate data with a weak correlation between variables in lifetime data, the models GFGM-WD and FGMBW are both utilized. The GFGM-WD, however, can accommodate a wider variety of data set shapes because the GFGM family enhances the correlation between variables more effectively than the FGM family. In addition to this significant motive, our study also covers a study of reliability in a dependent stress-strength model based on the GFGM-WD and some additional aspects that are shown below.

Let $H_Z(z)$ and $h_Z(z)$ be a common continuous DF and PDF, respectively, for a sequence of independent and identically distributed RVs $\{Z_i,\ i \ge 1\}$. If $Z_j > Z_i$ for every $i < j$, the observation Z_j is referred to as an upper record value. Lower record values can be defined similarly. Due to the extremely long projected waiting times between two record values, the model of record values is inadequate in several circumstances. For instance, because record data is highly uncommon in real-world settings and the expected waiting time for each record after the first is infinite, statistical inference based on records faces significant challenges. The second or third greatest values are very significant in those circumstances. However, if we take into account the $k-$record (KR) value model (see Aly et al. [22], Berred [23], and Fashandi and Ahmadi [24]), we can avoid these problems. Dziubdziela and Kopocinski [25] provided the PDF of the nth upper KR values as

$$h_{n,k}(z) = \frac{k^n}{\Gamma(n)}(-\log \overline{H}_Z(z))^{n-1}\, \overline{H}_Z^{k-1}(z)h_Z(z),$$

where $\Gamma(.)$ is the gamma function and $\overline{H}_Z(z) = 1 - H_Z(z)$ is the survival function (or the reliability function $R(z_i) = \overline{H}_{Z_i}(z_i) = P(Z_i > z_i)$).

Let us assume that $(Z_i, T_i),\ i = 1, 2, ...,$ is a random bivariate sample with a shared continuous DF $H_{Z,T}(z, t) = P(Z \leq z, T \leq t)$. The second component associated with the KR value of the first one is referred to as the concomitant of that KR value when the investigator is only interested in examining the sequence of KR of the first component, Z. KR values and their concomitants have been the subject of numerous real-world studies, such as those conducted by Bdair and Raqab [26], Chacko and Mary [27], and Alawady et al. [28]. Given below is the PDF of $T_{[n,k]}$ (the nth upper concomitant of $Z_{n,k}$).

$$h_{[n,k]}(t) = \int_0^\infty h_{T|Z}(t|z)h_{n,k}(z)dz, \tag{1.5}$$

where $h_{T|Z}(t|z)$ is the conditional PDF of T given Z.

An RVs average reduction in uncertainty or variability is reflected by the mathematical information measure known as entropy. It was first established by Shannon [29] and is widely used in information theory, statistics, probability, and other fields. If Z is a continuous non-negative RV and has a PDF $h_Z(.)$, then its entropy is defined by

$$\mathcal{H}(Z) = -\int_0^\infty h_Z(z)\log h_Z(z)dz. \tag{1.6}$$

A recent generalization of classical entropy called weighted entropy (WE), which is a measure of the amount of information produced by Z, has been proposed in the literature.

$$\mathcal{H}^{(\omega)}(Z) = -\int_0^\infty z h_Z(z)\log h_Z(z)dz. \tag{1.7}$$

For more details about this measure, see Guiasu [30].

As a complementary measure of entropy, Lad et al. [31] presented extropy as a measure associated with order, structure, and predictability. The duality between entropy and extropy can be especially helpful in domains like artificial intelligence and pattern recognition, for example, where understanding both the “disorder” and “order” aspects of systems gives a more complete view of system behavior. This measure has drawn a lot of attention in the last five years, and is described by (Husseiny and Syam [32])

$$\zeta(Z) = -\frac{1}{2}\int_0^\infty h_Z^2(z)dz = -\frac{1}{2}\int_0^1 h_Z(H_Z^{-1}(v))dv \leq 0. \tag{1.8}$$

Kelbert et al. [33] defined the weighted extropy (WEX) as

$$\zeta^{(\omega)}(Z) = -\frac{1}{2}\int_0^\infty z h_Z^2(z)dz. \tag{1.9}$$

Motivation. Dependence between measured quantities is the rule rather than the exception in applied statistics. Mechanical components subject to common loads, environmental indicators driven by shared climatic forces, and correlated medical biomarkers all illustrate systems in which marginal (univariate) modeling is inadequate. Bivariate distributions provide a principled framework for quantifying such joint behavior. Yet much of the existing methodology either lacks the flexibility to realistically represent weak-to-moderate associations or sacrifices the analytical tractability needed for interpretation and inference. Thus, there remains a clear need for models that reconcile flexibility, interpretability, and computationally feasible parameter estimation.

Motivated by these considerations, we develop a bivariate lifetime distribution that incorporates Weibull marginals with a GFGM copula–henceforth referred to as the GFGM-WD. This construction retains the analytical convenience of the GFGM class while substantially broadening the range of attainable dependence structures. Given the ubiquity of Weibull models in reliability, fatigue, and biomedical survival studies, the GFGM-WD is particularly well suited for stress-strength investigations and analyses involving concomitant KR values. Its parameterization also supports practical estimation procedures, making the model attractive for applied work.

Prior studies have demonstrated the usefulness of Weibull and Weibull-FGM-based models in physics and dependability, including works by Barraza-Contreras et al. [34], El-Sherpieny et al. [7], Nagy et al. [35], Tovar-Falón et al. [36], and Teimour and Gupta [37]. The GFGM-WD framework provides a unified extension capable of strengthening and generalizing the results reported in these studies.

Novelty and contribution of the proposed model. The principal novelty of this study lies in developing a new GFGM-WD that addresses a well-recognized limitation in copula-based Weibull modeling. Traditional FGM copula–Weibull models are constrained by their limited dependence range (typically $|\rho| \leq 1/3$), restricting their ability to capture moderate-to-strong positive dependence structures commonly encountered in reliability, biomedical, and engineering applications. These conventional formulations typically exhibit restricted dependence ranges and limited flexibility for modeling nonlinear or asymmetric associations.

The proposed GFGM–WD model overcomes these limitations by introducing an additional dependence parameter ω within the GFGM copula framework. This structural enhancement significantly expands the copula's capacity to represent diverse dependence patterns while preserving the analytical tractability of the original FGM family. Compared to existing bivariate Weibull models based on FGM, Clayton, and Gumbel copulas, the GFGM copula maintains closed-form expressions while substantially broadening the attainable correlation range and enabling more flexible tail behavior, as demonstrated by our empirical results showing correlation coefficients up to approximately 0.44 (see Table 1), substantially exceeding the FGM copula's limitations.

Table 1. The coefficient of correlation, $\rho_{Z,T}$, in the GFGM-WD($\alpha_1,\beta_1;\ \alpha_2,\beta_2$).

$\rho_{Z,T}$	γ	ω	α_1	α_2	$\rho_{Z,T}$	γ	ω	α_1	α_2
-0.1694	-0.5	-0.4	1	1	0.0222	0.8	-1.6	1	1
-0.0316	-0.5	1.4	2	3	0.0340	0.8	-1.1	0.5	1
-0.0107	-0.5	2.1	5	6	0.3763	0.8	1.3	1.5	2
0.0086	-0.5	2.5	7	7	0.4038	0.8	1.6	2	2.5
0.0470	0.3	-0.7	8	5	-0.0024	0.9	-1.8	0.8	0.8
0.0616	0.3	-0.5	8	8	0.0624	0.9	-1.4	0.9	1
0.1064	0.3	0.2	8	9	0.4083	0.9	1.4	1.2	1.5
0.1375	0.3	0.7	9	9	**0.4390**	0.9	1.6	1.7	1.7

Beyond enhanced dependence modeling, the GFGM–WD offers superior interpretability and computational efficiency. The closed-form expressions for joint and marginal distributions facilitate straightforward parameter estimation and simulation, while all parameters retain clear physical interpretations related to scale, shape, and dependence strength. Consequently, the GFGM–WD provides a unified framework that bridges theoretical generality with practical applicability, offering a robust and computationally stable tool for modeling bivariate lifetime data with weak-to-moderate correlations. This represents a significant advancement beyond existing bivariate Weibull and FGM-based frameworks in terms of both modeling flexibility and interpretive clarity.

Organization of the paper: Section 2 develops the DF and PDF of the GFGM-WD, derives key structural properties (moments, correlation coefficient, conditional distributions, concomitants of KR, and the moment-generating function (MGF)), and illustrates its application to a dependent stress-strength model. Section 3 examines information and uncertainty measures (entropy, WE, extropy, and WEX) for $T_{[n,k]}$, supported by a numerical study. Section 4 presents maximum likelihood (ML) and Bayesian estimations of the model parameters, and compares their performance. Section 5 provides two real-data applications. Section 6 concludes the paper.

2. The bivariate Weibull distribution based on the GFGM copula

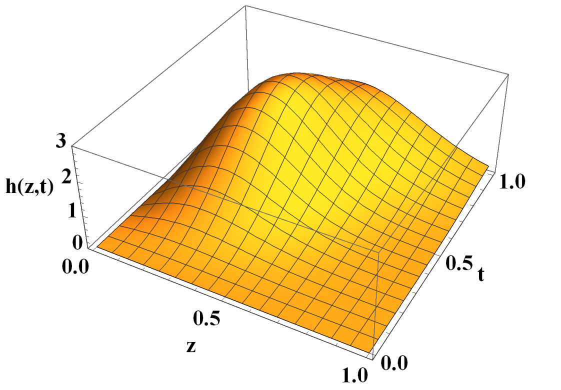
Let $Z \sim WD(\alpha_1,\beta_1)$ and $T \sim WD(\alpha_2,\beta_2)$. Thus, according to (1.1), the JDF of the bivariate Weibull distribution based on the GFGM copula, denoted by GFGM-WD($\alpha_1,\beta_1;\ \alpha_2,\beta_2$), is as follows:

$$H_{Z,T}(z,t) = (1-A_1)(1-A_2)\Big[1+\gamma A_1A_2+\omega A_1A_2(1-A_1)(1-A_2)\Big], \tag{2.1}$$

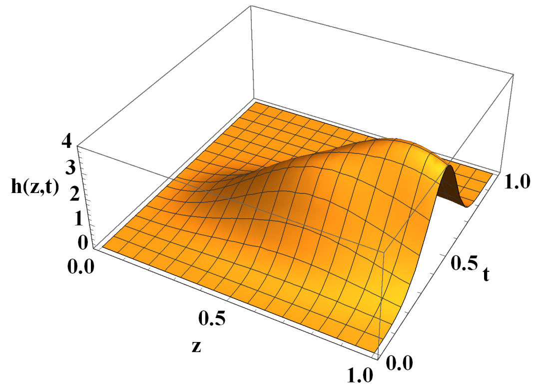
where $A_1 = e^{-\left(\frac{z}{\beta_1}\right)^{\alpha_1}}$ and $A_2 = e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}$. Moreover, according to (1.2), the corresponding JPDF of (2.1) is given by

$$\begin{aligned} h_{Z,T}(z,t) = {} & \frac{\alpha_1\alpha_2}{\beta_1\beta_2}\left(\frac{z}{\beta_1}\right)^{\alpha_1-1}\left(\frac{t}{\beta_2}\right)^{\alpha_2-1}A_1A_2\Big[\,1+\gamma(2A_1-1)(2A_2-1)+\omega(1-A_1)(1-A_2) \\ & \times(3A_1-1)(3A_2-1)\,\Big]. \end{aligned} \tag{2.2}$$

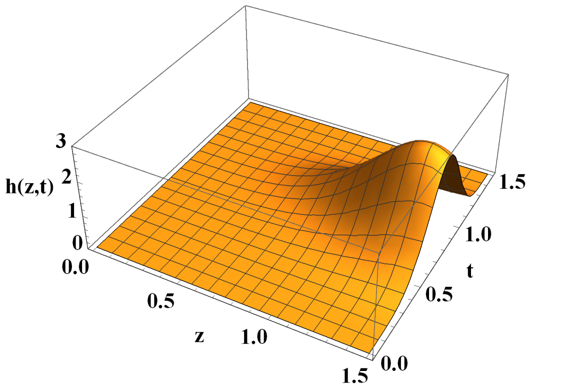
Figure 1 shows the three-dimensional surface plots for the JPDF of the GFGM-WD with different values of $\alpha_1,\beta_1,\alpha_2,\beta_2,\gamma$, and ω. The diversity in the surface shapes of this family for different values of its parameters reflects the various values of skewness and kurtosis that the family provides. This confirms the efficiency of this family in describing various categories of data.



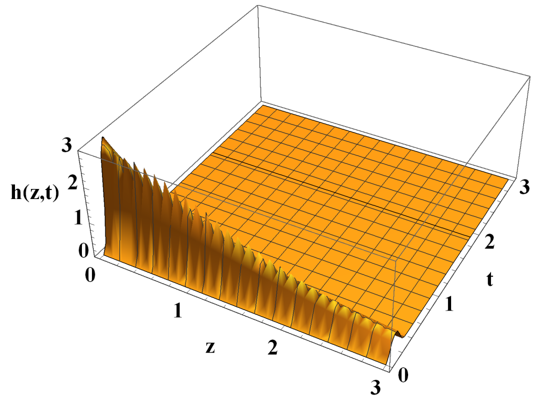
(a) $\alpha_1 = 2, \beta_1 = 0.5, \alpha_2 = 3, \beta_2 = 0.9, \gamma = 0.9, \omega = 1.2$



(b) $\alpha_1 = 3, \beta_1 = 0.9, \alpha_2 = 4, \beta_2 = 0.5, \gamma = 0.9, \omega = -1.4$



(c) $\alpha_1 = 5, \beta_1 = 1.5, \alpha_2 = 5, \beta_2 = 0.9, \gamma = -0.5, \omega = 1.4$



(d) $\alpha_1 = 1, \beta_1 = 2, \alpha_2 = 3, \beta_2 = 0.2, \gamma = -0.5, \omega = 2.5$

Figure 1. Some graphs of the JPDF of the GFGM-WD($\alpha_1, \beta_1; \alpha_2, \beta_2$).

2.1. Some statistical characteristics

2.1.1. Moments

Thus, it is easy to show that the product moments of the GFGM-WD($\alpha_1, \beta_1; \alpha_2, \beta_2$) are

$$\begin{aligned} E(Z^n T^m) = & \frac{\alpha_1 \alpha_2}{\beta_1 \beta_2} \int_0^\infty \int_0^\infty z^n t^m \left(\frac{z}{\beta_1}\right)^{\alpha_1 - 1} \left(\frac{t}{\beta_2}\right)^{\alpha_2 - 1} A_1 A_2 \Big[1 + \gamma (2A_1 - 1)(2A_2 - 1) + \omega (1 - A_1) \\ & \times (1 - A_2)(3A_1 - 1)(3A_2 - 1) \Big] dz dt \\ = & \frac{\alpha_1 \alpha_2}{\beta_1 \beta_2} \Bigg[\int_0^\infty \int_0^\infty z^n t^m \left(\frac{z}{\beta_1}\right)^{\alpha_1 - 1} \left(\frac{t}{\beta_2}\right)^{\alpha_2 - 1} A_1 A_2 dz dt + \gamma \int_0^\infty \int_0^\infty z^n t^m \left(\frac{z}{\beta_1}\right)^{\alpha_1 - 1} \left(\frac{t}{\beta_2}\right)^{\alpha_2 - 1} \\ & \times A_1 A_2 (2A_1 - 1)(2A_2 - 1) dz dt + \omega \int_0^\infty \int_0^\infty z^n t^m \left(\frac{z}{\beta_1}\right)^{\alpha_1 - 1} \left(\frac{t}{\beta_2}\right)^{\alpha_2 - 1} A_1 A_2 (1 - A_1) \\ & \times (1 - A_2)(3A_1 - 1)(3A_2 - 1) dz dt \Bigg] \end{aligned}$$

$$
\begin{aligned}
&= \beta_1^n\beta_2^m\Gamma\left(1+\frac{n}{\alpha_1}\right)\Gamma\left(1+\frac{m}{\alpha_2}\right)\left[1+\gamma\left(1-2^{\frac{-n}{\alpha_1}}\right)\left(1-2^{\frac{-m}{\alpha_2}}\right)+\omega\left(3^{\frac{-n}{\alpha_1}}-2^{1-\frac{n}{\alpha_1}}+1\right)\right.\\
&\times\left.\left(3^{\frac{-m}{\alpha_2}}-2^{1-\frac{m}{\alpha_2}}+1\right)\right],\ n,m=1,2,.... \qquad (2.3)
\end{aligned}
$$

Thus, by using (2.3) at $n = m = 1$, we get

$$
\begin{aligned}
E(ZT) &= \beta_1\beta_2\Gamma(1+\frac{1}{\alpha_1})\Gamma(1+\frac{1}{\alpha_2})\left[1+\gamma\left(1-2^{\frac{-1}{\alpha_1}}\right)\left(1-2^{\frac{-1}{\alpha_2}}\right)+\omega\left(3^{\frac{-1}{\alpha_1}}-2^{1-\frac{1}{\alpha_1}}+1\right)\right.\\
&\times\left.\left(3^{\frac{-1}{\alpha_2}}-2^{1-\frac{1}{\alpha_2}}+1\right)\right]. \qquad (2.4)
\end{aligned}
$$

The coefficient of correlation can be evaluated through the following expression:

$$
\rho_{Z,T} = \frac{E(ZT)-E(Z)E(T)}{\sqrt{Var(Z)Var(T)}}. \qquad (2.5)
$$

To determine the correlation between Z and T, we employed (2.4) and obtained the corresponding values of the expected values and variances as:

$$
E(Z) = \frac{\alpha_1}{\beta_1}\int_0^\infty z\left(\frac{z}{\beta_1}\right)^{\alpha_1-1}A_1dz = \beta_1\Gamma(1+\frac{1}{\alpha_1}), \qquad (2.6)
$$

and similarly, $E(T) = \beta_2\Gamma(1+\frac{1}{\alpha_2})$; and

$$
E(Z^2) = \frac{\alpha_1}{\beta_1}\int_0^\infty z^2\left(\frac{z}{\beta_1}\right)^{\alpha_1-1}A_1dz = \beta_1^2\Gamma(1+\frac{2}{\alpha_1}), \qquad (2.7)
$$

and similarly, $E(T^2) = \beta_2^2\Gamma(1+\frac{2}{\alpha_2})$. By substituting (2.4), (2.6), and (2.7) into (2.5), the coefficient of correlation between Z and T is given by

$$
\begin{aligned}
\rho_{Z,T} &= \frac{1}{\sqrt{(\Gamma(1+\frac{2}{\alpha_1})-(\Gamma(1+\frac{1}{\alpha_1}))^2)(\Gamma(1+\frac{2}{\alpha_2})-(\Gamma(1+\frac{1}{\alpha_2}))^2)}}\left[\Gamma(1+\frac{1}{\alpha_1})\Gamma(1+\frac{1}{\alpha_2})\right.\\
&\times\left.\left(\gamma\,(1-2^{\frac{-1}{\alpha_1}})(1-2^{\frac{-1}{\alpha_2}})+\omega\,(3^{\frac{-1}{\alpha_1}}-2^{1-\frac{1}{\alpha_1}}+1)(3^{\frac{-1}{\alpha_2}}-2^{1-\frac{1}{\alpha_2}}+1)\right)\right]. \qquad (2.8)
\end{aligned}
$$

Table 1 manifests the correlation, $\rho_{Z,T}$, for the GFGM-WD($\alpha_1,\beta_1;\alpha_2,\beta_2$), by using (2.8). The result of this table shows that the maximum and minimum values of $\rho_{Z,T}$ from the GFGM-WD($\alpha_1,\beta_1;\alpha_2,\beta_2$) are 0.438998, and −0.169444, respectively.

Mathematical justification, admissibility, and identifiability. We provide rigorous mathematical justification for coupling Weibull marginals with the GFGM copula, along with the admissibility and identifiability conditions required for the model to be well defined.

Marginal preservation. The Weibull marginals are preserved by construction since for each fixed $u \in [0, 1]$,

$$\int_0^1 c(u, v; \gamma, \omega)\, du = 1.$$

Indeed,

$$\int_0^1 (1 - 2u)\, du = 0 \;\text{ and }\; \int_0^1 u(2 - 3u)\, du = 0,$$

which imply that integrating $c(u, v; \gamma, \omega)$ with respect to either argument yields 1, ensuring correct marginal recovery.

Admissibility: nonnegativity of the copula density. The copula density must satisfy $c(u, v; \gamma, \omega) \geq 0$ for all $(u, v) \in [0, 1]^2$. Using the uniform norms

$$\sup_{u\in[0,1]} |1 - 2u| = 1 \;\text{ and }\; \sup_{u\in[0,1]} |u(2 - 3u)| = 1,$$

we obtain the sufficient condition

$$|\gamma| + |\omega| \leq 1,$$

which guarantees $c(u, v; \gamma, \omega) \geq 0$ and thus yields $\omega_{\max} = 1 - |\gamma|$. For sharper bounds, we may compute

$$\omega_{\max} = \sup\left\{\omega \geq 0 : \min_{(u,v)\in[0,1]^2} c(u, v; \gamma, \omega) \geq 0\right\},$$

which can be evaluated analytically or numerically to ensure the nonnegativity constraint holds exactly.

Identifiability. The full parameter vector $\underline{\theta} = (\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma, \omega)^\top$ is identifiable under:

(1) *Identifiable marginals:* distinct (α_1, β_1) and (α_2, β_2) yield distinct $H_Z(\cdot; \alpha_1, \beta_1)$ and $H_T(\cdot; \alpha_2, \beta_2)$;
(2) *Injective copula parameterization:* distinct (γ, ω) yield distinct copula densities $c(\cdot, \cdot; \gamma, \omega)$ on a set of positive measure.

Consequently, equal joint densities imply equal marginals (by integration) and hence equal copula densities (by division), which establishes $(\gamma, \omega) = (\gamma', \omega')$.

Practical implementation. During estimation, the constraints

$$\alpha_l > 0, \quad \beta_l > 0 \quad (l = 1, 2), \qquad |\gamma| < 1, \;\text{ and }\; |\omega| \leq \omega_{\max}$$

are enforced, together with numerical verification that $c(u, v; \gamma, \omega) \geq 0$ holds over a fine uniform grid in $[0, 1]^2$. The bound $|\gamma| + |\omega| \leq 1$ ensures a feasible starting region, and refined admissible ranges for ω are computed adaptively when exploring stronger dependence.

2.2. Conditional distribution and concomitants of KR values

After some straightforward algebraic manipulation, the conditional DF of T given $Z = z$ is given by

$$\begin{aligned} H_{T|Z}(t|z) &= \left(1 - e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\right)\left[1 + \gamma e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\left(2e^{-\left(\frac{z}{\beta_1}\right)^{\alpha_1}} - 1\right) + \omega e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\left(1 - e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\right)\right. \\ &\times \left.\left(1 - e^{-\left(\frac{z}{\beta_1}\right)^{\alpha_1}}\right)\left(3e^{-\left(\frac{z}{\beta_1}\right)^{\alpha_1}} - 1\right)\right]. \end{aligned}$$

Consequently, the GFGM-WD(α_1,β_1; α_2,β_2) regression curve for T given $Z = z$ is

$$\begin{aligned} E(T|Z = z) &= \int_0^\infty t\, h(T|Z = z)dt \\ &= \frac{\alpha_2}{\beta_2}\int_0^\infty t\left(\frac{t}{\beta_2}\right)^{\alpha_2 - 1} A_2\Big[\, 1 + \gamma\,(2A_1 - 1)\,(2A_2 - 1) + \omega\,(1 - A_1)\,(1 - A_2) \\ &\times (3A_1 - 1)\,(3A_2 - 1)\,\Big]dt \\ &= \beta_2\Gamma(1 + \frac{1}{\alpha_2})\left[1 - \gamma\left(2e^{-\left(\frac{z}{\beta_1}\right)^{\alpha_1}} - 1\right)\left(1 - 2^{\frac{-1}{\alpha_2}}\right) - \omega\left(1 - e^{-\left(\frac{z}{\beta_1}\right)^{\alpha_1}}\right)\left(3e^{-\left(\frac{z}{\beta_1}\right)^{\alpha_1}} - 1\right)\right. \\ &\times \left.\left(3^{\frac{-1}{\alpha_2}} - 2^{1-\frac{1}{\alpha_2}} + 1\right)\right], \end{aligned}$$

where the conditional expectation is non-linear with respect to z.

2.2.1. Marginal DF of $T_{[n,k]}$

The following theorem, which is based on the GFGM-WD(α_1,β_1; α_2,β_2), provides a salutary form for the PDF $h_{[n,k]}(t)$.

Theorem 2.1. *Let $T_{[n,k]}$ be the concomitant of KR values based on the GFGM-WD(α_1,β_1; α_2,β_2). Then the PDF, DF, and survival function of $T_{[n,k]}$ are given, respectively, by*

$$\begin{aligned} h_{[n,k]}(t) &= \frac{\alpha_2}{\beta_2}\left(\frac{t}{\beta_2}\right)^{\alpha_2 - 1} e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\left(1 + \Xi^{(1)}_{n,k}\left(2e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}} - 1\right) + \Xi^{(2)}_{n,k}\left(1 - e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\right)\right. \\ &\times \left.\left(3e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}} - 1\right)\right), \end{aligned} \tag{2.9}$$

$$H_{[n,k]}(t) = \left(1 - e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\right)\left[1 + e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\left(\Xi^{(1)}_{n,k} + \Xi^{(2)}_{n,k}\left(1 - e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\right)\right)\right], \tag{2.10}$$

and

$$\overline{H}_{[n,k]}(t) = e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\left[1 - \left(1 - e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\right)\left(\Xi^{(1)}_{n,k} + \Xi^{(2)}_{n,k}\left(1 - e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\right)\right)\right], \tag{2.11}$$

where $\Xi^{(1)}_{n,k} = \gamma\left(2\left(\frac{k}{k+1}\right)^n - 1\right)$ *and* $\Xi^{(2)}_{n,k} = \omega\left(4\left(\frac{k}{k+1}\right)^n - 3\left(\frac{k}{k+2}\right)^n - 1\right)$.

Proof. By using (1.5) and (2.2), we get

$$
\begin{aligned}
h_{[n,k]}(t) = {} & \frac{\alpha_1\alpha_2 k^n}{\beta_1\beta_2\Gamma(n)} \int_0^\infty \left(\frac{t}{\beta_2}\right)^{\alpha_2-1} e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}} \left[1+\gamma\left(2e^{-\left(\frac{z}{\beta_1}\right)^{\alpha_1}}-1\right)\left(2e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}-1\right)\right. \\
& + \left.\omega\left(1-e^{-\left(\frac{z}{\beta_1}\right)^{\alpha_1}}\right)\left(1-e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\right)\left(3e^{-\left(\frac{z}{\beta_1}\right)^{\alpha_1}}-1\right)\left(3e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}-1\right)\right] \\
& \times \left(-\log\left(e^{-\left(\frac{z}{\beta_1}\right)^{\alpha_1}}\right)\right)^{n-1} \left(e^{-\left(\frac{z}{\beta_1}\right)^{\alpha_1}}\right)^{k-1} \left(\frac{z}{\beta_1}\right)^{\alpha_1-1} e^{-\left(\frac{z}{\beta_1}\right)^{\alpha_1}} dz \\
= {} & \frac{\alpha_2}{\beta_2}\left(\frac{t}{\beta_2}\right)^{\alpha_2-1} e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}} \left(1+\Xi^{(1)}_{n,k}\left(2e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}-1\right)+\Xi^{(2)}_{n,k}\left(1-e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\right)\right. \\
& \times \left.\left(3e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}-1\right)\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\Xi^{(1)}_{n,k} &= \frac{\alpha_1\gamma k^n}{\beta_1\Gamma(n)} \int_0^\infty \left(2e^{-\left(\frac{z}{\beta_1}\right)^{\alpha_1}}-1\right)\left(-\log\left(e^{-\left(\frac{z}{\beta_1}\right)^{\alpha_1}}\right)\right)^{n-1} \left(e^{-\left(\frac{z}{\beta_1}\right)^{\alpha_1}}\right)^{k-1} \left(\frac{z}{\beta_1}\right)^{\alpha_1-1} e^{-\left(\frac{z}{\beta_1}\right)^{\alpha_1}} dz \\
&= \gamma\left(2\left(\frac{k}{k+1}\right)^n-1\right)
\end{aligned}
$$

and

$$
\begin{aligned}
\Xi^{(2)}_{n,k} &= \frac{\alpha_1\omega k^n}{\beta_1\Gamma(n)} \int_0^\infty \left(4e^{-\left(\frac{z}{\beta_1}\right)^{\alpha_1}}-3e^{-\left(\frac{z}{\beta_1}\right)^{2\alpha_1}}-1\right)\left(-\log\left(e^{-\left(\frac{z}{\beta_1}\right)^{\alpha_1}}\right)\right)^{n-1} \left(e^{-\left(\frac{z}{\beta_1}\right)^{\alpha_1}}\right)^{k-1} \left(\frac{z}{\beta_1}\right)^{\alpha_1-1} \\
&\times\ e^{-\left(\frac{z}{\beta_1}\right)^{\alpha_1}} dz = \omega\left(4\left(\frac{k}{k+1}\right)^n-3\left(\frac{k}{k+2}\right)^n-1\right).
\end{aligned}
$$

This completes the proof. □

Relying on (2.9), the MGF of $T_{[n,k]}$ based on the GFGM-WD($\alpha_1,\beta_1;\ \alpha_2,\beta_2$) is given by

$$
\mathcal{M}_{[n,k]}(s) = \sum_{i=0}^{\infty}\frac{(s\beta_2)^i}{i!}\Gamma(1+\frac{i}{\alpha_2})\left[1-\Xi^{(1)}_{n,k}-\Xi^{(2)}_{n,k}+2^{-\frac{i}{\alpha_2}}\left(\Xi^{(1)}_{n,k}+2\ \Xi^{(2)}_{n,k}\right)-3^{-\frac{i}{\alpha_2}}\ \Xi^{(2)}_{n,k}\right].
$$

Using (2.9), the ℓth moment of $T_{[n,k]}$ based on the GFGM-WD($\alpha_1,\beta_1;\ \alpha_2,\beta_2$) is given by

$$
\mu^{(\ell)}_{[n,k]} = \beta_2^{(\ell)}\Gamma(1+\frac{\ell}{\alpha_2})\left[1-\Xi^{(1)}_{n,k}-\Xi^{(2)}_{n,k}+2^{-\frac{\ell}{\alpha_2}}\left(\Xi^{(1)}_{n,k}+2\ \Xi^{(2)}_{n,k}\right)-3^{-\frac{\ell}{\alpha_2}}\ \Xi^{(2)}_{n,k}\right]. \tag{2.12}
$$

Moreover, by putting $\ell=1$ and $\ell=2$ in (2.12), we get the mean and variance of $T_{[n,k]}$, respectively, by

$$
\mu_{[n,k]} = \beta_2\Gamma(1+\frac{1}{\alpha_2})\left[1-\Xi^{(1)}_{n,k}-\Xi^{(2)}_{n,k}+2^{-\frac{1}{\alpha_2}}\left(\Xi^{(1)}_{n,k}+2\ \Xi^{(2)}_{n,k}\right)-3^{-\frac{1}{\alpha_2}}\ \Xi^{(2)}_{n,k}\right]
$$

and

$$\begin{aligned}
\sigma^2_{[n,k]} &= (1+\Xi^{(1)}_{n,k})\beta_2^2\left[\Gamma(1+\frac{2}{\alpha_2})-(1+\Xi^{(1)}_{n,k})(\Gamma(1+\frac{1}{\alpha_2}))^2\right]+2\beta_2^2(\Xi^{(2)}_{n,k}-\Xi^{(1)}_{n,k})\left[\Gamma(1+\frac{2}{\alpha_2})\right.\\
&\times (1-2^{-(1+\frac{2}{\alpha_2})})-2(\Xi^{(2)}_{n,k}-\Xi^{(1)}_{n,k})(\Gamma(1+\frac{2}{\alpha_2})(1-2^{-(1+\frac{2}{\alpha_2})}))^2\Big]-3\beta_2^2\Xi^{(2)}_{n,k}\Big[\Gamma(1+\frac{2}{\alpha_2})\\
&\times \left(3^{-(1+\frac{2}{\alpha_2})}-2^{-\frac{2}{\alpha_2}}+1\right)+3\Xi^{(2)}_{n,k}\left(3^{-(1+\frac{1}{\alpha_2})}-2^{-\frac{1}{\alpha_2}}+1\right)(\Gamma(1+\frac{1}{\alpha_2}))^2\Big]-4\beta_2^2(1+\Xi^{(1)}_{n,k})\\
&\times (\Xi^{(2)}_{n,k}-\Xi^{(1)}_{n,k})(\Gamma(1+\frac{1}{\alpha_2}))^2(1-2^{-(1+\frac{1}{\alpha_2})})+6\beta_2^2\Xi^{(2)}_{n,k}(\Gamma(1+\frac{1}{\alpha_2}))^2\left(3^{-(1+\frac{1}{\alpha_2})}-2^{-\frac{1}{\alpha_2}}+1\right)\\
&\times \left[(1+\Xi^{(1)}_{n,k})+2(\Xi^{(2)}_{n,k}-\Xi^{(1)}_{n,k})(1-2^{-(1+\frac{1}{\alpha_2})})\right].
\end{aligned}$$

Practical interpretation and applications of concomitants of KR values. The theoretical results for the concomitants of KR values under the proposed GFGM-WD model have important implications for real-world problems in reliability, engineering, and biomedical data analysis. In reliability systems, the KR structure naturally models the joint behavior of sequential failure times for multiple dependent components monitored simultaneously. Specifically, the KR value from one component's failure times and its concomitant (the corresponding failure time of a dependent component) capture the dependence between successive failure events under shared environmental or load conditions. The derived expressions for the joint and marginal densities of record concomitants allow practitioners to quantify this dependence, evaluate conditional reliability measures, and estimate system-level performance indices such as mean residual life or conditional hazard functions. This is particularly relevant in designing redundant or load-sharing systems, where understanding the dependence between component lifetimes is essential for optimizing maintenance and replacement strategies.

In biomedical and survival analysis, concomitants of KR values provide a probabilistic framework for studying ordered survival times of patients in longitudinal clinical studies. Here, each record (upper or lower) represents a new extreme survival time, while the concomitant captures an associated outcome, such as tumor response or recovery time. The GFGM copula's dependence structure allows flexible modeling of correlated patient outcomes, capturing weak to moderate associations that often arise from shared treatment effects or genetic similarities. Consequently, the derived results facilitate the development of predictive tools for assessing patient prognosis based on prior record events, improving inference on survival dynamics in heterogeneous populations. Thus, the concomitant analysis under the proposed GFGM-WD framework bridges theoretical record-value properties and practical inference tools for reliability assessment and medical prognosis.

2.2.2. Bivariate reliability function

Sreelakshmi [38] introduced the relationship between the copula and reliability copula, which is defined as follows:

$$R(z,t) = 1 - H_Z(z) - H_T(t) + C\left(H_Z(z), H_T(t)\right).$$

The bivariate reliability function $R(z,t)$ for the GFGM-WD $(\alpha_1,\beta_1;\,\alpha_2,\beta_2)$ is

$$R(z,t)=e^{-\left(\frac{z}{\beta_1}\right)^{\alpha_1}}e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\left[1+\left(1-e^{-\left(\frac{z}{\beta_1}\right)^{\alpha_1}}\right)\left(1-e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\right)\left[\gamma+\omega\left(1-e^{-\left(\frac{z}{\beta_1}\right)^{\alpha_1}}\right)\left(1-e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\right)\right]\right].$$

2.2.3. Reliability in the dependent stress-strength model GFGM-WD

The life of a component with random strength Z and random stress T is described by the stress-strength model. When $Z > T$, the component performs satisfactorily, making $R = P(T < Z)$ a measure of component reliability. There are numerous uses for it, particularly in quality assurance, engineering, economics, physics, etc. When Z and T are independent RVs that belong to the same univariate distribution, a lot of work has been done to calculate the value of R. However, many applications call for the knowledge of R in dependent situations (Z and T). Here, we assume that the strength Z and stress T have some sort of dependence structure according to the GFGM-WD. Given (1.2) and (1.4), one may write

$$R = P(T < Z) = \int_0^\infty \int_0^z h_{Z,T}(z,t)dtdz = \int_0^\infty \int_0^z c(H_Z(z), H_T(t))h_Z(z)h_T(t)dtdz.$$

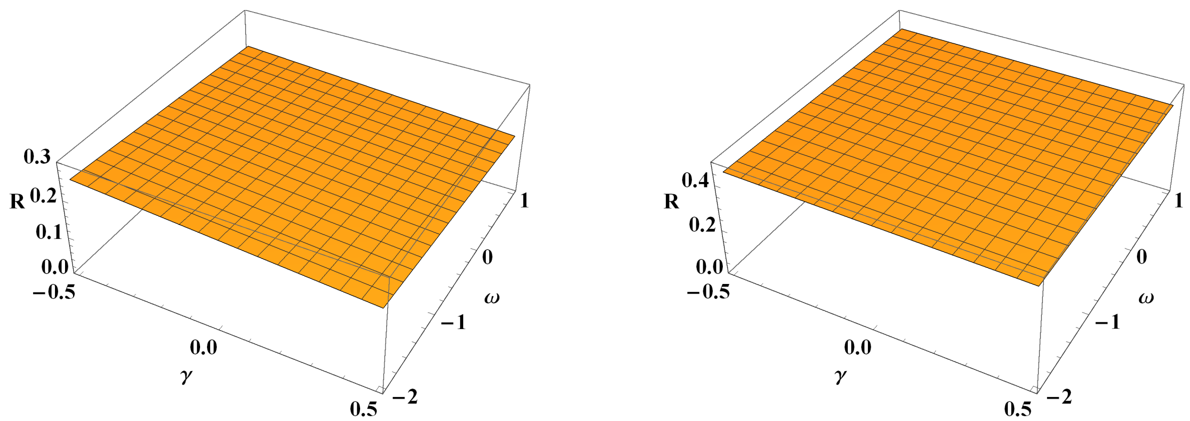
On putting $\alpha_1 = \alpha_2$ and $c = (\frac{\beta_1}{\beta_2})^{\alpha_2}$, and by using the probability integral transformations, we get

$$\begin{aligned} R = P(T < Z) &= \int_0^\infty \int_0^{H_T(H_Z^{-1}(v))} c(u,v)dudv \\ &= \int_0^1 \int_0^{1-(1-v)^c} [1 + \gamma(1-2u)(1-2v) + \omega uv\,(2-3u)\,(2-3v)]\, dudv. \end{aligned} \tag{2.13}$$

As $\beta_1 = \beta_2$, i.e., $c = 1$, we get, as expected, $R = \frac{1}{2}$. The integration (2.13) can be easily evaluated using MATHEMATICA 12 as

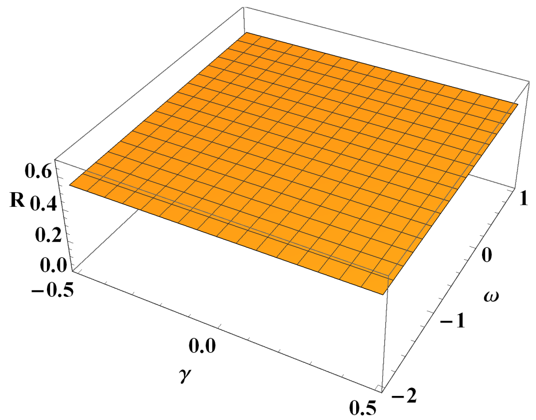
$$R = \frac{c[(c+3)(2c+3)(3c+1)(3c+2)(c(2c+\gamma+5)-\gamma+2) + 4\omega(c-1)c(c(11c+25)+11)]}{(c+1)(c+2)(c+3)(2c+1)(2c+3)(3c+1)(3c+2)}. \tag{2.14}$$

When $\omega = 0$, the relation (2.14) yields the reliability in the dependent stress-strength model FGMBW. Figures 2 and 3 show that $R > \frac{1}{2}$ (the component does not perform satisfactorily), while $R < \frac{1}{2}$ (the component performs satisfactorily) according to $\beta_2 < \beta_1$ and $\beta_2 > \beta_1$, respectively.

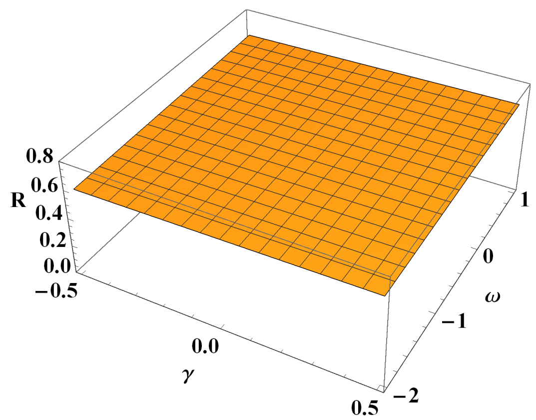


(a) $\alpha_1 = \alpha_2$ and $c = 0.25 < 1$ (b) $\alpha_1 = \alpha_2$ and $c = 0.75 < 1$

Figure 2. $R < \frac{1}{2}$ at $\beta_1 < \beta_2$.



(c) $\alpha_1 = \alpha_2$ and $c = 1.25 > 1$



(d) $\alpha_1 = \alpha_2$ and $c = 1.75 > 1$

Figure 3. $R > \frac{1}{2}$ at $\beta_1 > \beta_2$.

3. Information measures in $T_{[n,k]}$ based on the GFGM-WD($\alpha_1, \beta_1; \alpha_2, \beta_2$)

In terms of extropy, entropy, and information extraction, the GFGM-WD is superior to many bivariate distributions, including the exponential family, particularly for data sets with weak dependencies or changing hazard rates. In general, it can capture more structured information because of its adaptability to various lifetime patterns and moderate dependencies. Furthermore, the availability of closed-form equations for joint entropy, extropy, and associated information metrics aids in practical computation and interpretation, as well as informing dependability and maintenance planning. The measures of entropy, WE, extropy, and WEX for $T_{[n,k]}$ of the KR values from the GFGM-WD($\alpha_1, \beta_1; \alpha_2, \beta_2$) are obtained in this section.

3.1. Entropy in $T_{[n,k]}$

Let $T_{[n,k]}$ be the concomitant of KR values. Then from (1.6) and (2.5), and after some algebra using MATHEMATICA 12, the entropy in the GFGM-WD($\alpha_1, \beta_1; \alpha_2, \beta_2$) is given by

$$\begin{aligned}
\mathcal{H}(T_{[n,k]}) &= -\int_0^\infty h_{[n,k]}(t)\log h_{[n,k]}(t)dt \\
&= -\frac{\alpha_2}{\beta_2}\int_0^\infty \left(\frac{t}{\beta_2}\right)^{\alpha_2-1} e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\left(1+\Xi^{(1)}_{n,k}(2e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}-1)+\Xi^{(2)}_{n,k}\left(1-e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\right)\right. \\
&\times \left.\left(3e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}-1\right)\right)\log\left[\frac{\alpha_2}{\beta_2}\left(\frac{t}{\beta_2}\right)^{\alpha_2-1} e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\left(1+\Xi^{(1)}_{n,k}(2e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}-1)\right.\right. \\
&+ \left.\left.\Xi^{(2)}_{n,k}\left(1-e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\right)\left(3e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}-1\right)\right)\right]dt \\
&= (1+\Xi^{(1)}_{n,k})\mathcal{H}(T) - 2(\Xi^{(2)}_{n,k}-\Xi^{(1)}_{n,k})\psi(1) + 3\Xi^{(2)}_{n,k}\psi(2) + \delta_{[n,k]},
\end{aligned}$$

where

$$\mathcal{H}(T) = -E(\log h_T(T)) = -\frac{\alpha_2}{\beta_2}\int_0^\infty \left(\frac{t}{\beta_2}\right)^{\alpha_2-1} e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}} \log\left(\frac{\alpha_2}{\beta_2}\left(\frac{t}{\beta_2}\right)^{\alpha_2-1} e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\right)dt$$
$$= \left(1 - \log(\frac{\alpha_2}{\beta_2}) + EulerGamma(1 - \frac{1}{\alpha_2})\right)$$

is the Shannon entropy of T, and

$$\begin{aligned}\psi(1) &= \frac{\alpha_2}{\beta_2}\int_0^\infty \left(1 - e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\right)\left(\frac{t}{\beta_2}\right)^{\alpha_2-1} e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}} \log\left(\frac{\alpha_2}{\beta_2}\left(\frac{t}{\beta_2}\right)^{\alpha_2-1} e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\right)dt \\ &= \left[\frac{1}{2}\log\left(\frac{\alpha_2}{\beta_2}\right) + \frac{1}{2\alpha_2}\left(EulerGamma(1-\alpha_2) + \alpha_2\left(\log\left(2\left(\frac{1}{\beta_2}\right)^{\alpha_2-1}\right) - \log\left(\left(\frac{1}{\beta_2}\right)^{\alpha_2}\right)\right)\right.\right. \\ &+ \left.\left.\log\left(\frac{1}{2}\left(\frac{1}{\beta_2}\right)^{\alpha_2}\right)\right)\right] - \frac{3}{4},\end{aligned}$$

$$\begin{aligned}\psi(2) &= \frac{\alpha_2}{\beta_2}\int_0^\infty \left(1 - e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\right)^2\left(\frac{t}{\beta_2}\right)^{\alpha_2-1} e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}} \log\frac{\alpha_2}{\beta_2}\left(\frac{t}{\beta_2}\right)^{\alpha_2-1} e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}} dt \\ &= \left[\frac{1}{3}\log\left(\frac{\alpha_2}{\beta_2}\right) + \frac{1}{3\alpha_2}\left(EulerGamma(1-\alpha_2) + \alpha_2\left(\log\left(8\left(\frac{1}{\beta_2}\right)^{\alpha_2-1}\right) - \log\left(3\left(\frac{1}{\beta_2}\right)^{\alpha_2}\right)\right)\right.\right. \\ &+ \left.\left.\log\left(\frac{3}{8}\left(\frac{1}{\beta_2}\right)^{\alpha_2}\right)\right) - \frac{11}{18}\right],\end{aligned}$$

where the Euler gamma function $\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt$, and

$$\delta_{[n,k]} = -E\left(\log\left(1 + \Xi_{n,k}^{(1)}(2e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}} - 1) + \Xi_{n,k}^{(2)}\left(1 - e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\right)\left(3e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}} - 1\right)\right)\right).$$

3.2. Weighted entropy in $T_{[n,k]}$

From (1.7) and (2.5), and after some algebra with the aid of MATHEMATICA 12, the WE in the GFGM-WD($\alpha_1,\beta_1;\alpha_2,\beta_2$) is given by

$$\begin{aligned}\mathcal{H}^{(\omega)}(T_{[n,k]}) &= -\int_0^\infty t\, h_{[n,k]}(t)\log h_{[n,k]}(t)dt \\ &= -\frac{\alpha_2}{\beta_2}\int_0^\infty t\left(\frac{t}{\beta_2}\right)^{\alpha_2-1} e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\left(1 + \Xi_{n,k}^{(1)}\left(2e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}} - 1\right) + \Xi_{n,k}^{(2)}\left(1 - e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\right)\right. \\ &\times \left.\left(3e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}} - 1\right)\right)\log\left[\frac{\alpha_2}{\beta_2}\left(\frac{t}{\beta_2}\right)^{\alpha_2-1} e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\left(1 + \Xi_{n,k}^{(1)}\left(2e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}} - 1\right)\right.\right. \\ &+ \left.\left.\Xi_{n,k}^{(2)}\left(1 - e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\right)\left(3e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}} - 1\right)\right)\right]dt \\ &= (1 + \Xi_{n,k}^{(1)})\mathcal{H}^{(\omega)}(T) - 2(\Xi_{n,k}^{(2)} - \Xi_{n,k}^{(1)})\psi^{(\omega)}(1) + 3\Xi_{n,k}^{(2)}\psi^{(\omega)}(2) + \delta_{[n,k]}^{(\omega)},\end{aligned}$$

where

$$\begin{aligned}\mathcal{H}^{(\omega)}(T) &= -\frac{\alpha_2}{\beta_2}\int_0^\infty t\left(\frac{t}{\beta_2}\right)^{\alpha_2-1} e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}} \log\left(\frac{\alpha_2}{\beta_2}\left(\frac{t}{\beta_2}\right)^{\alpha_2-1} e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\right)dt\\ &= \left[\frac{\alpha_2}{\beta_2}\Gamma\left(\frac{1}{\alpha_2}\right)\log\left(\frac{\alpha_2}{\beta_2}\right)+\frac{\alpha_2^2}{\beta_2}\Gamma\left(\frac{1}{\alpha_2}\right)\left(\alpha_2\log\left(\left(\frac{1}{\beta_2}\right)^{\alpha_2-1}\right)-(\alpha_2-1)\log\left(\left(\frac{1}{\beta_2}\right)^{\alpha_2}\right)\right.\right.\\ &+ \left.\left.(\alpha_2-1)\text{PolyGamma}\left(0,1+\frac{1}{\alpha_2}\right)\right)-\frac{\alpha_2^2}{\beta_2(1+\alpha_2)}\Gamma\left(\frac{1}{\alpha_2}\right)\right],\end{aligned}$$

$$\begin{aligned}\psi^{(\omega)}(1) &= \frac{\alpha_2}{\beta_2}\int_0^\infty t\left(1-e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\right)\left(\frac{t}{\beta_2}\right)^{\alpha_2-1} e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}} \log\left(\frac{\alpha_2}{\beta_2}\left(\frac{t}{\beta_2}\right)^{\alpha_2-1} e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\right)dt\\ &= \left[\left(1-2^{-\frac{\alpha_2+1}{\alpha_2}}\right)\beta_2\Gamma\left(1+\frac{1}{\alpha_2}\right)\log\left(\frac{\alpha_2}{\beta_2}\right)-2^{-\frac{\alpha_2+1}{\alpha_2}}\frac{\alpha_2^2}{\beta_2}\Gamma\left(\frac{1}{\alpha_2}\right)\left(\left(2^{\frac{1}{\alpha_2}+1}-1\right)\alpha_2\right.\right.\\ &\times \log\left(\left(\frac{1}{\beta_2}\right)^{\alpha_2-1}\right)+2^{\frac{1}{\alpha_2}+1}\log\left(\left(\frac{1}{\beta_2}\right)^{\alpha_2}\right)-2^{\frac{1}{\alpha_2}+1}\alpha_2\log\left(\left(\frac{1}{\beta_2}\right)^{\alpha_2}\right)+\alpha_2\log\left(\left(\frac{1}{\beta_2}\right)^{\alpha_2}\right)\\ &- \log\left(2\left(\frac{1}{\beta_2}\right)^{\alpha_2}\right)+\alpha_2\log(2)+\left(2^{\frac{1}{\alpha_2}+1}-1\right)(\alpha_2-1)\text{PolyGamma}\left(0,1+\frac{1}{\alpha_2}\right)\Bigg)+\beta_2\\ &\times \left.\left(2^{-(2+\frac{1}{\alpha_2})}-1\right)\Gamma\left(2+\frac{1}{\alpha_2}\right)\right],\end{aligned}$$

$$\psi^{(\omega)}(2)=\frac{\alpha_2}{\beta_2}\int_0^\infty t\left(1-e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\right)^2\left(\frac{t}{\beta_2}\right)^{\alpha_2-1} e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}} \log\left(\frac{\alpha_2}{\beta_2}\left(\frac{t}{\beta_2}\right)^{\alpha_2-1} e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\right)dt,$$

where PolyGamma(n,z) gives the nth derivative of the digamma function $\psi^n(z)$, and

$$\delta^{(\omega)}_{[n,k]}=-E\left(T\log\left(1+\Xi^{(1)}_{n,k}(2e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}-1)+\Xi^{(2)}_{n,k}\left(1-e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\right)\left(3e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}-1\right)\right)\right).$$

3.3. Extropy in $T_{[n,k]}$

From (1.8), the extropy of $T_{[n,k]}$ is given by

$$\begin{aligned}\zeta(T_{[n,k]}) &= -\frac{1}{2}\int_0^\infty (h_{[n,k]}(t))^2\,dt\\ &= -\frac{1}{2}\int_0^\infty\left[\frac{\alpha_2}{\beta_2}\left(\frac{t}{\beta_2}\right)^{\alpha_2-1} e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\left(1+\Xi^{(1)}_{n,k}(2e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}-1)+\Xi^{(2)}_{n,k}\left(1-e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\right)\right.\right.\\ &\times \left.\left.\left(3e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}-1\right)\right)\right]^2 dt\\ &= \left(1+\Xi^{(1)}_{n,k}\right)^2\zeta(T)+2\left(\left(\Xi^{(1)}_{n,k}\right)^2+\Xi^{(1)}_{n,k}-\Xi^{(2)}_{n,k}-\Xi^{(1)}_{n,k}\Xi^{(2)}_{n,k}\right)\chi(1)-\left(2\left(\Xi^{(1)}_{n,k}\right)^2+2\left(\Xi^{(2)}_{n,k}\right)^2\right.\\ &- \left.3\Xi^{(2)}_{n,k}-7\Xi^{(1)}_{n,k}\Xi^{(2)}_{n,k}\right)\chi(2)+6\left(\left(\Xi^{(2)}_{n,k}\right)^2-\Xi^{(1)}_{n,k}\Xi^{(2)}_{n,k}\right)\chi(3)-\frac{9}{2}\left(\Xi^{(2)}_{n,k}\right)^2\chi(4),\end{aligned}$$

where $\zeta(T) = -\frac{1}{2}\int_0^\infty \left(\frac{\alpha_2}{\beta_2}\left(\frac{t}{\beta_2}\right)^{\alpha_2-1} e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\right)^2 dt = \left(-\frac{\alpha_2}{\beta_2} 2^{\frac{1}{\alpha_2}-3}\Gamma\left(2-\frac{1}{\alpha_2}\right)\right)$ is the extropy of T,

$$\chi(p) = \int_0^\infty \left(1-e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\right)^p \left(\frac{\alpha_2}{\beta_2}\left(\frac{t}{\beta_2}\right)^{\alpha_2-1} e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\right)^2 dt,\ p = 1,2,3,4,$$

and

$$\chi(1) = \int_0^\infty \left(1-e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\right)\left(\frac{\alpha_2}{\beta_2}\left(\frac{t}{\beta_2}\right)^{\alpha_2-1} e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\right)^2 dt = \frac{\alpha_2}{36\beta_2}\Gamma\left(2-\frac{1}{\alpha_2}\right)\left(9\times 2^{\frac{1}{\alpha_2}} - 4\times 3^{\frac{1}{\alpha_2}}\right).$$

3.4. Weighted extropy in $T_{[n,k]}$

By using (1.9) and (2.5), and after some algebra with the aid of MATHEMATICA 12, the WEX in the GFGM-WD($\alpha_1,\beta_1;\ \alpha_2,\beta_2$) is given by

$$\begin{aligned}
\zeta^{(\omega)}(T_{[n,k]}) &= -\frac{1}{2}\int_0^\infty t\, h^2_{[n,k]}(t)dt \\
&= -\frac{1}{2}\int_0^\infty t\left[\frac{\alpha_2}{\beta_2}\left(\frac{t}{\beta_2}\right)^{\alpha_2-1} e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\left(1+\Xi^{(1)}_{n,k}\left(2e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}-1\right)+\Xi^{(2)}_{n,k}\left(1-e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\right)\right.\right. \\
&\times \left.\left.\left(3e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}-1\right)\right)\right]^2 dt \\
&= \left(1+\Xi^{(1)}_{n,k}\right)^2\zeta^{(\omega)}(T) + 2\left(\left(\Xi^{(1)}_{n,k}\right)^2+\Xi^{(1)}_{n,k}-\Xi^{(2)}_{n,k}-\Xi^{(1)}_{n,k}\Xi^{(2)}_{n,k}\right)\chi^{(\omega)}(1) - \left(2\left(\Xi^{(1)}_{n,k}\right)^2\right. \\
&\left. + 2\left(\Xi^{(2)}_{n,k}\right)^2 - 3\Xi^{(2)}_{n,k} - 7\Xi^{(1)}_{n,k}\Xi^{(2)}_{n,k}\right)\chi^{(\omega)}(2) + 6\left(\left(\Xi^{(2)}_{n,k}\right)^2-\Xi^{(1)}_{n,k}\Xi^{(2)}_{n,k}\right)\chi^{(\omega)}(3) \\
&- \frac{9}{2}\left(\Xi^{(2)}_{n,k}\right)^2\chi^{(\omega)}(4),
\end{aligned}$$

where $\zeta^{(\omega)}(T) = -\frac{1}{2}\int_0^\infty t\left(\frac{\alpha_2}{\beta_2}\left(\frac{t}{\beta_2}\right)^{\alpha_2-1} e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\right)^2 dt = \frac{-\alpha_2}{8}$ is the WEX of T,

$$\chi^{(\omega)}(p) = \int_0^\infty t\left(1-e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\right)^p\left(\frac{\alpha_2}{\beta_2}\left(\frac{t}{\beta_2}\right)^{\alpha_2-1} e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\right)^2 dt,\ p = 1,2,3,4,\ \text{and}$$
$$\chi^{(\omega)}(1) = \int_0^\infty t\left(1-e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\right)\left(\frac{\alpha_2}{\beta_2}\left(\frac{t}{\beta_2}\right)^{\alpha_2-1} e^{-\left(\frac{t}{\beta_2}\right)^{\alpha_2}}\right)^2 dt = \frac{5\alpha_2}{36}.$$

3.5. Numerical study with discussion

Entropy, WE, extropy, and WEX for $T_{[n,k]}$ in the GFGM-WD($\alpha_1,\beta_1;\ \alpha_2,\beta_2$) are shown in Tables 2–5. MATHEMATICA 12 is used to perform the computations. These tables can be used to derive the following qualities.

Table 2. Entropy for $T_{[n,k]}$ based on the GFGM-WD$(\alpha_1, \beta_1; 1, 2)$.

		$\gamma = 0.9$						$\gamma = -0.5$			
k	**n**	$\omega = -1.8$	$\omega = -1.4$	$\omega = 1.2$	$\omega = 1.4$	**k**	**n**	$\omega = -0.4$	$\omega = 1.4$	$\omega = 2.5$	$\omega = 3$
5	1	1.4753	1.4451	1.2196	1.1995	5	1	1.8621	1.76202	1.6936	1.6605
5	2	1.6591	1.62563	1.3714	1.3486	5	2	1.8133	1.67189	1.5720	1.5225
5	3	1.7508	1.7233	1.5185	1.5006	5	3	1.7594	1.6256	1.5325	1.4867
5	4	1.7919	1.7742	1.6485	1.6380	5	4	1.7039	1.6115	1.5499	1.5204
5	5	1.8030	1.7972	1.7578	1.7547	5	5	1.6494	1.6178	1.5979	1.5887
5	6	1.7959	1.8029	1.8468	1.8500	5	6	1.5975	1.6354	1.6577	1.6677
5	7	1.7778	1.7982	1.9175	1.9257	5	7	1.5491	1.6581	1.7185	1.7446
5	8	1.7534	1.7872	1.9726	1.9844	5	8	1.5049	1.6823	1.7748	1.8132
5	9	1.7258	1.7729	2.0148	2.0291	5	9	1.4649	1.7058	1.8241	1.8715
5	10	1.6972	1.7570	2.0468	2.0624	5	10	1.4293	1.7274	1.8660	1.9194
7	1	1.4089	1.3809	1.1741	1.1557	7	1	1.8749	1.7897	1.7322	1.7046
7	2	1.5877	1.55206	1.2788	1.2539	7	2	1.8424	1.7058	1.6092	1.5616
7	3	1.6948	1.6597	1.3902	1.3657	7	3	1.8054	1.6493	1.5375	1.4817
7	4	1.7582	1.7278	1.4981	1.4777	7	4	1.7659	1.6166	1.5101	1.4585
7	5	1.7933	1.7698	1.5975	1.5827	7	5	1.7254	1.6026	1.5182	1.4770
7	6	1.8094	1.7941	1.6862	1.6772	7	6	1.6849	1.6020	1.5473	1.5213
7	7	1.8124	1.8061	1.7636	1.7602	7	7	1.6452	1.6105	1.5886	1.5784
7	8	1.8063	1.8095	1.8210	1.8315	7	8	1.6069	1.6245	1.6351	1.6399
7	9	1.7938	1.8069	1.8862	1.8920	7	9	1.5705	1.6417	1.6825	1.70046
7	10	1.7772	1.8002	1.9334	1.9425	7	10	1.5362	1.6602	1.7281	1.7571
9	1	1.3619	1.3364	1.1504	1.1341	9	1	1.8816	1.8086	1.7599	1.7367
9	2	1.5298	1.4948	1.2271	1.2025	9	2	1.8578	1.7334	1.6463	1.6035
9	3	1.6406	1.6032	1.3128	1.2861	9	3	1.8305	1.6763	1.5656	1.5104
9	4	1.7143	1.6784	1.4006	1.3752	9	4	1.8009	1.6363	1.5175	1.4578
9	5	1.7626	1.7306	1.4858	1.4639	9	5	1.7697	1.6111	1.4978	1.4411
9	6	1.7929	1.7661	1.5661	1.5487	9	6	1.7378	1.5981	1.5003	1.4521
9	7	1.8101	1.7895	1.6400	1.6273	9	7	1.7057	1.5945	1.5189	1.4824
9	8	1.8175	1.8037	1.7070	1.6990	9	8	1.6739	1.5979	1.5480	1.5244
9	9	1.8177	1.8111	1.7668	1.7633	9	9	1.6427	1.6063	1.5832	1.5726
9	10	1.8124	1.8134	1.8199	1.8204	9	10	1.6124	1.6180	1.6213	1.6229

Table 3. The WE for $T_{[n,k]}$ based on the GFGM-WD$(\alpha_1, \beta_1; 1, 2)$.

		$\gamma = 0.9$						$\gamma = -0.5$			
k	**n**	$\omega = -1.8$	$\omega = -1.4$	$\omega = 1.2$	$\omega = 1.4$	**k**	**n**	$\omega = -0.4$	$\omega = 1.4$	$\omega = 2.5$	$\omega = 3$
5	1	4.4812	4.2899	2.8913	2.7674	5	1	6.3919	5.6776	5.2092	4.9870
5	2	5.4335	5.2118	3.5902	3.4479	5	2	6.1004	5.1282	4.4704	4.1507
5	3	5.8807	5.6920	4.3457	4.2316	5	3	5.7871	4.8876	4.2823	3.9892
5	4	6.0432	5.9195	5.0679	4.9986	5	4	5.4707	4.8574	4.4575	4.2682
5	5	6.0367	5.9953	5.7209	5.6995	5	5	5.1635	4.9552	4.8252	4.7653
5	6	5.9295	5.9790	6.2939	6.3176	5	6	4.8732	5.1213	5.2693	5.3357
5	7	5.7646	5.9083	6.7876	6.8517	5	7	4.6044	5.3165	5.7229	5.9013
5	8	5.5700	5.8075	7.2084	7.3073	5	8	4.3594	5.5169	6.1515	6.4251
5	9	5.3641	5.6925	7.5644	7.6928	5	9	4.1389	5.7089	6.5395	6.8921
5	10	5.1590	5.5736	7.8640	8.0172	5	10	3.9425	5.8857	6.8818	7.2990
7	1	4.1405	3.9657	2.6903	2.5771	7	1	6.4718	5.8559	5.4562	5.2679
7	2	5.0827	4.8512	3.1346	2.9801	7	2	6.2751	5.3201	4.6779	4.3664
7	3	5.6339	5.3987	3.6688	3.5160	7	3	6.0570	4.9910	4.2601	3.9024
7	4	5.9422	5.7332	4.2258	4.0964	7	4	5.8279	4.8240	4.1397	3.8047
7	5	6.0903	5.9258	4.7694	4.6732	7	5	5.5956	4.7785	4.2331	3.9705
7	6	6.1296	6.0213	5.2810	5.2213	7	6	5.3657	4.8183	4.4639	4.2970
7	7	6.0947	6.0498	5.7518	5.7285	7	7	5.1422	4.9141	4.7714	4.7056
7	8	6.0093	6.0321	6.1788	6.1810	7	8	4.9280	5.0434	5.1131	5.1446
7	9	5.8906	5.9833	6.5622	6.6052	7	9	4.7250	5.1903	5.4619	5.5825
7	10	5.7512	5.9141	6.9041	6.9757	7	10	4.5345	5.3436	5.8012	6.0013
9	1	3.8952	3.7375	2.5940	2.4931	9	1	6.5140	5.9815	5.6389	5.4784
9	2	4.7866	4.5623	2.8940	2.7424	9	2	6.3691	5.4897	4.9023	4.6110
9	3	5.3678	5.1201	3.2826	3.1165	9	3	6.2065	5.14021	4.4112	4.0540
9	4	5.7449	5.5023	3.7125	3.5539	9	4	6.0324	4.9126	4.1400	3.7579
9	5	5.9794	5.7584	4.1541	4.0153	9	5	5.8518	4.7852	4.0525	3.6912
9	6	6.1112	5.9236	4.5892	4.4766	9	6	5.6685	4.7370	4.1067	3.7999
9	7	6.1675	6.0217	5.0071	4.9236	9	7	5.4856	4.7489	4.2615	4.0284
9	8	6.1681	6.0698	5.40163	5.3480	9	8	5.3055	4.8046	4.4818	4.3303
9	9	6.1275	6.0807	5.7697	5.7453	9	9	5.1299	4.8906	4.7407	4.6716
9	10	6.0569	6.0640	6.1101	6.1137	9	10	4.9601	4.9964	5.0185	5.0285

Table 4. Extropy for $T_{[n,k]}$ based on the GFGM-WD(α_1, β_1; 1, 2).

		$\gamma = 0.9$						$\gamma = -0.5$			
k	**n**	$\omega = -1.8$	$\omega = -1.4$	$\omega = 1.2$	$\omega = 1.4$	**k**	**n**	$\omega = -0.4$	$\omega = 1.4$	$\omega = 2.5$	$\omega = 3$
5	1	-0.1716	-0.1755	-0.2034	-0.2057	5	1	-0.0994	-0.1113	-0.1196	-0.12361
5	2	-0.1385	-0.1428	-0.1750	-0.1777	5	2	-0.1068	-0.1241	-0.1363	-0.1422
5	3	-0.1209	-0.1244	-0.1502	-0.1525	5	3	-0.1152	-0.1318	-0.1432	-0.1488
5	4	-0.1121	-0.1143	-0.1299	-0.1312	5	4	-0.1240	-0.1356	-0.1432	-0.1468
5	5	-0.1083	-0.1090	-0.1138	-0.1142	5	5	-0.1327	-0.1367	-0.1392	-0.1403
5	6	-0.1076	-0.1068	-0.1015	-0.1012	5	6	-0.1401	-0.1362	-0.1333	-0.1321
5	7	-0.1088	-0.1064	-0.0924	-0.0914	5	7	-0.1487	-0.1348	-0.1271	-0.1237
5	8	-0.1111	-0.1071	-0.0857	-0.0844	5	8	-0.1556	-0.1330	-0.1211	-0.1163
5	9	-0.1140	-0.1084	-0.0801	-0.0795	5	9	-0.1619	-0.1311	-0.1159	-0.1099
5	10	-0.1171	-0.1101	-0.0777	-0.0762	5	10	-0.1674	-0.1293	-0.1115	-0.1049
7	1	-0.1834	-0.1870	-0.2123	-0.2143	7	1	-0.0975	-0.1076	-0.1145	-0.1178
7	2	-0.1521	-0.1568	-0.1910	-0.1940	7	2	-0.1024	-0.1190	-0.1307	-0.1364
7	3	-0.1325	-0.1371	-0.1711	-0.1741	7	3	-0.1081	-0.1273	-0.1409	-0.1475
7	4	-0.1203	-0.1242	-0.1532	-0.1557	7	4	-0.1143	-0.1328	-0.1457	-0.1520
7	5	-0.1130	-0.1159	-0.1375	-0.1393	7	5	-0.1207	-0.1360	-0.1464	-0.1514
7	6	-0.1089	-0.1107	-0.1241	-0.1252	7	6	-0.1271	-0.1375	-0.1443	-0.1475
7	7	-0.1069	-0.1077	-0.1129	-0.1133	7	7	-0.1335	-0.1378	-0.1406	-0.1419
7	8	-0.1066	-0.10618	-0.1037	-0.1035	7	8	-0.1396	-0.1373	-0.1360	-0.1354
7	9	-0.1073	-0.1057	-0.0963	-0.0956	7	9	-0.1453	-0.1363	-0.1311	-0.1288
7	10	-0.1087	-0.1059	-0.0904	-0.0894	7	10	-0.1508	-0.1350	-0.1263	-0.1226
9	1	-0.1915	-0.1948	-0.2173	-0.2191	9	1	-0.0965	-0.1051	-0.1101	-0.1137
9	2	-0.1627	-0.1673	-0.2006	-0.2034	9	2	-0.1001	-0.1151	-0.1256	-0.1308
9	3	-0.1429	-0.1478	-0.1843	-0.1875	9	3	-0.1043	-0.1230	-0.1365	-0.1431
9	4	-0.1293	-0.1339	-0.1690	-0.1721	9	4	-0.1089	-0.1291	-0.1435	-0.1506
9	5	-0.1110	-0.1240	-0.1549	-0.1576	9	5	-0.1138	-0.1334	-0.1472	-0.1539
9	6	-0.1137	-0.1171	-0.1421	-0.1443	9	6	-0.1188	-0.1362	-0.1482	-0.1539
9	7	-0.1097	-0.1122	-0.1308	-0.1324	9	7	-0.1239	-0.1378	-0.1471	-0.1516
9	8	-0.1073	-0.1090	-0.1209	-0.1219	9	8	-0.1289	-0.1385	-0.1447	-0.1476
9	9	-0.1062	-0.1070	-0.1124	-0.1128	9	9	-0.1339	-0.1385	-0.1414	-0.1427
9	10	-0.1059	-0.1058	-0.1050	-0.1050	9	10	-0.1387	-0.1380	-0.1376	-0.1374

Table 5. WEX for $T_{[n,k]}$ based on the GFGM-WD$(\alpha_1, \beta_1; 2, 2)$.

		$\gamma = -0.9$						$\gamma = 0.25$			
k	**n**	$\omega = 0.4$	$\omega = 0.8$	$\omega = 1.2$	$\omega = 1.4$	**k**	**n**	$\omega = -0.7$	$\omega = -0.2$	$\omega = 0.3$	$\omega = 0.8$
5	1	-0.3017	-0.3004	-0.2994	-0.2990	5	1	-0.2395	-0.2414	-0.2438	-0.2466
5	2	-0.2757	-0.2758	-0.2763	-0.2767	5	2	-0.2425	-0.2442	-0.2467	-0.2501
5	3	-0.2597	-0.2607	-0.2622	-0.2631	5	3	-0.2460	-0.2472	-0.2491	-0.2516
5	4	-0.2493	-0.2505	-0.2519	-0.2527	5	4	-0.2494	-0.2501	-0.2511	-0.2524
5	5	-0.2424	-0.2430	-0.2435	-0.2438	5	5	-0.2525	-0.2528	-0.2530	-0.2533
5	6	-0.2377	-0.2369	-0.2363	-0.2359	5	6	-0.2554	-0.2551	-0.2548	-0.2546
5	7	-0.2342	-0.2320	-0.2300	-0.2291	5	7	-0.2580	-0.2571	-0.2565	-0.2563
5	8	-0.2317	-0.2279	-0.2247	-0.2233	5	8	-0.2605	-0.2588	-0.2581	-0.2584
5	9	-0.2297	-0.2244	-0.2202	-0.2185	5	9	-0.2627	-0.2603	-0.2596	-0.2608
5	10	-0.2282	-0.2214	-0.2164	-0.2145	5	10	-0.2648	-0.2615	-0.2610	-0.2632
7	1	-0.3111	-0.3096	-0.3083	-0.3078	7	1	-0.2389	-0.2407	-0.2429	-0.2454
7	2	-0.2880	-0.2872	-0.2869	-0.2870	7	2	-0.2406	-0.2426	-0.2454	-0.2490
7	3	-0.2720	-0.2723	-0.2733	-0.2740	7	3	-0.2430	-0.2447	-0.2474	-0.2501
7	4	-0.2606	-0.2618	-0.2635	-0.2646	7	4	-0.2456	-0.2470	-0.2491	-0.2520
7	5	-0.2525	-0.2539	-0.2558	-0.2568	7	5	-0.2482	-0.2491	-0.2506	-0.2526
7	6	-0.2465	-0.2477	-0.2491	-0.2498	7	6	-0.2506	-0.2512	-0.2520	-0.2530
7	7	-0.2419	-0.2425	-0.2431	-0.2435	7	7	-0.2528	-0.2530	-0.2533	-0.2537
7	8	-0.2384	-0.2381	-0.2378	-0.2376	7	8	-0.2549	-0.2547	-0.2546	-0.2545
7	9	-0.2357	-0.2343	-0.2329	-0.2323	7	9	-0.2568	-0.2563	-0.2559	-0.2556
7	10	-0.2335	-0.2309	-0.2286	-0.2275	7	10	-0.2587	-0.2577	-0.2571	-0.2570
9	1	-0.3171	-0.3156	-0.3143	-0.3137	9	1	-0.2387	-0.2404	-0.2423	-0.2445
9	2	-0.2966	-0.2954	-0.2946	-0.2944	9	2	-0.2396	-0.2417	-0.2445	-0.2480
9	3	-0.2814	-0.2811	-0.2814	-0.2818	9	3	-0.2413	-0.2433	-0.2463	-0.2502
9	4	-0.2699	-0.2705	-0.2718	-0.2727	9	4	-0.2433	-0.2450	-0.2478	-0.2515
9	5	-0.2612	-0.2625	-0.2643	-0.2655	9	5	-0.2454	-0.2470	-0.2491	-0.2522
9	6	-0.2545	-0.2561	-0.2581	-0.2592	9	6	-0.2474	-0.2485	-0.2503	-0.2527
9	7	-0.2492	-0.2507	-0.2526	-0.2536	9	7	-0.2494	-0.2502	-0.2514	-0.2530
9	8	-0.2450	-0.2462	-0.2476	-0.2483	9	8	-0.2512	-0.2517	-0.2525	-0.2534
9	9	-0.2416	-0.2423	-0.2429	-0.2433	9	9	-0.2529	-0.2532	-0.2535	-0.2538
9	10	-0.2389	-0.2388	-0.2387	-0.2386	9	10	-0.2546	-0.2545	-0.2545	-0.2545

- Generally, for $\gamma = 0.9$, $\alpha_2 = 1$, and $\beta_2 = 2$, the value of $\mathcal{H}(T_{[n,k]})$ increases at $k \geq n$, and decreases at $k < n$ with $\omega < 0$. Also, with fixed k and $\omega > 0$, the value of $\mathcal{H}(T_{[n,k]})$ increases as the value of n increases. On the other hand, for $\gamma = -0.5$, $\alpha_2 = 1$, and $\beta_2 = 2$, the value of $\mathcal{H}(T_{[n,k]})$ decreases as the value of $n \leq \frac{k+1}{2}$, and increases for $n > \frac{k+1}{2}$ with $\omega > 0$. With fixed k and $\omega < 0$, the value of $\mathcal{H}(T_{[n,k]})$ decreases as the value of n increases (see Table 2).
- For $\gamma = 0.9$, $\alpha_2 = 1$, and $\beta_2 = 2$, the value of $\mathcal{H}^{(\omega)}(T_{[n,k]})$ increases at $k > n$, and decreases at $k \leq n$ ($\omega < 0$). Also, with fixed k and $\omega > 0$, the value of $\mathcal{H}^{(\omega)}(T_{[n,k]})$ increases as the value of n increases. On the other hand, for $\gamma = -0.5$, $\alpha_2 = 1$, and $\beta_2 = 2$, the value of $\mathcal{H}^{(\omega)}(T_{[n,k]})$ decreases as the value of $k \geq n$, and increases for $k < n$ ($\omega > 0$). With fixed k and $\omega < 0$, the value of $\mathcal{H}^{(\omega)}(T_{[n,k]})$ decreases as the value of n increases (see Table 3).
- For $\gamma = 0.9$, $\alpha_2 = 1$, and $\beta_2 = 2$, the value of $\zeta(T_{[n,k]})$ increases as the value of n increases, and for $\gamma = -0.5$, $\alpha_2 = 1$, and $\beta_2 = 2$, the value of $\zeta(T_{[n,k]})$ decreases as the value of $k \geq n$, and increases for $k < n$ ($\omega > 0$). For $\omega < 0$, the value of $\zeta(T_{[n,k]})$ decreases as the value of n increases (see Table 4).
- For $\gamma = -0.9$ and $\alpha_2 = \beta_2 = 2$, the value of $\zeta^{(\omega)}(T_{[n,k]})$ increases as the value of n increases, and for $\gamma = 0.25$ and $\alpha_2 = \beta_2 = 2$, the value of $\zeta^{(\omega)}(T_{[n,k]})$ decreases as the value of n increases (see Table 5).

4. Methods of estimation

4.1. Maximum likelihood estimation and properties

Let $(Z_1, T_1), (Z_2, T_2), \dots, (Z_n, T_n)$ be a random sample from the GFGM–WD with the JPDF

$$h(z, t; \underline{\theta}) = c(H_Z(z; \alpha_1, \beta_1), H_T(t; \alpha_2, \beta_2); \gamma, \omega)\, h_Z(z; \alpha_1, \beta_1)\, h_T(t; \alpha_2, \beta_2),$$

where $\underline{\theta} = (\alpha_1, \beta_1, \alpha_2, \beta_2, \gamma, \omega)^\top$ is the parameter vector. The likelihood function for the observed sample is

$$L(\underline{\theta}) = \prod_{i=1}^{n} h_{Z,T}(z_i, t_i; \underline{\theta}) = \prod_{i=1}^{n} \left[c(H_Z(z_i), H_T(t_i); \gamma, \omega)\, h_Z(z_i; \alpha_1, \beta_1)\, h_T(t_i; \alpha_2, \beta_2) \right].$$

Taking logarithms gives

$$\ell(\underline{\theta}) = \sum_{i=1}^{n} \left[\log h_Z(z_i; \alpha_1, \beta_1) + \log h_T(t_i; \alpha_2, \beta_2) + \log c(H_Z(z_i), H_T(t_i); \gamma, \omega) \right].$$

Substituting the explicit forms of the marginals and copula yields

$$\begin{aligned}\ell(\underline{\theta}) = {} & n(\log \alpha_1 + \log \alpha_2 - \log \beta_1 - \log \beta_2) + (\alpha_1 - 1) \sum_{i=1}^{n} \log\left(\frac{z_i}{\beta_1}\right) + (\alpha_2 - 1) \sum_{i=1}^{n} \log\left(\frac{t_i}{\beta_2}\right) \\ & - \sum_{i=1}^{n} \left[\left(\frac{z_i}{\beta_1}\right)^{\alpha_1} + \left(\frac{t_i}{\beta_2}\right)^{\alpha_2} \right] + \sum_{i=1}^{n} \log\Big[1 + \gamma(1 - 2H_Z(z_i))(1 - 2H_T(t_i)) \\ & + \omega\, H_Z(z_i) H_T(t_i)(2 - 3H_Z(z_i))(2 - 3H_T(t_i)) \Big].\end{aligned}$$

The MLEs are obtained by solving $\partial\ell(\underline{\theta})/\partial(\underline{\theta}) = 0$ for all parameters. Let

$$\Phi_i = 1 + \gamma(1 - 2H_Z(z_i))(1 - 2H_T(t_i)) + \omega\, H_Z(z_i) H_T(t_i)(2 - 3H_Z(z_i))(2 - 3H_T(t_i)).$$

(i) For α_1:

$$\frac{\partial \ell}{\partial \alpha_1} = \frac{n}{\alpha_1} + \sum_{i=1}^{n} \log\left(\frac{z_i}{\beta_1}\right) - \sum_{i=1}^{n}\left(\frac{z_i}{\beta_1}\right)^{\alpha_1} \log\left(\frac{z_i}{\beta_1}\right) + \sum_{i=1}^{n} \frac{\Psi_{1i}}{\Phi_i},$$

where

$$\Psi_{1i} = \frac{\partial \Phi_i}{\partial H_Z(z_i)} \frac{\partial H_Z(z_i)}{\partial \alpha_1}, \qquad \frac{\partial H_Z(z_i)}{\partial \alpha_1} = (1 - H_Z(z_i))\left(\frac{z_i}{\beta_1}\right)^{\alpha_1} \log\left(\frac{z_i}{\beta_1}\right),$$

and

$$\frac{\partial \Phi_i}{\partial H_Z(z_i)} = -2\gamma(1 - 2H_T(t_i)) + \omega\, H_T(t_i)(2 - 3H_T(t_i))(2 - 6H_Z(z_i)).$$

(ii) For β_1:

$$\frac{\partial \ell}{\partial \beta_1} = -\frac{n}{\beta_1} - \frac{(\alpha_1 - 1)}{\beta_1} + \frac{\alpha_1}{\beta_1} \sum_{i=1}^{n}\left(\frac{z_i}{\beta_1}\right)^{\alpha_1} + \sum_{i=1}^{n} \frac{\frac{\partial \Phi_i}{\partial \beta_1}}{\Phi_i},$$

where

$$\frac{\partial \Phi_i}{\partial \beta_1} = \frac{\partial \Phi_i}{\partial H_Z(z_i)} \frac{\partial H_Z(z_i)}{\partial \beta_1}, \qquad \frac{\partial H_Z(z_i)}{\partial \beta_1} = -\,\alpha_1\, \frac{(1 - H_Z(z_i))}{\beta_1}\left(\frac{z_i}{\beta_1}\right)^{\alpha_1}.$$

(iii) For α_2 and β_2: The corresponding expressions follow by symmetry, interchanging $(z_i, \alpha_1, \beta_1, H_Z)$ with $(t_i, \alpha_2, \beta_2, H_T)$.

(iv) For γ:

$$\frac{\partial \ell}{\partial \gamma} = \sum_{i=1}^{n} \frac{(1 - 2H_Z(z_i))(1 - 2H_T(t_i))}{\Phi_i}.$$

(v) For ω:

$$\frac{\partial \ell}{\partial \omega} = \sum_{i=1}^{n} \frac{H_Z(z_i)H_T(t_i)(2 - 3H_Z(z_i))(2 - 3H_T(t_i))}{\Phi_i}.$$

The vector of MLEs $\widehat{\underline{\theta}}$ is obtained by numerically solving $\partial \ell / \partial \underline{\theta} = 0$.

Newton–Raphson iterative scheme

Because the score equations are nonlinear, the estimates are obtained iteratively using the **Newton–Raphson algorithm**. Let $\mathbf{s}(\underline{\theta}) = \nabla \ell(\underline{\theta})$ denote the score vector and $\mathbf{D}(\underline{\theta}) = -\nabla^2 \ell(\underline{\theta})$ the observed information matrix. The update rule is

$$\underline{\theta}^{(r+1)} = \underline{\theta}^{(r)} + \mathbf{D}^{-1}\left(\underline{\theta}^{(r)}\right) \mathbf{s}\left(\underline{\theta}^{(r)}\right), \qquad r = 0, 1, 2, \ldots$$

Implementation details:

- **Initialization:** $\underline{\theta}^{(0)} = (\hat{\alpha}_1^{\text{marg}}, \hat{\beta}_1^{\text{marg}}, \hat{\alpha}_2^{\text{marg}}, \hat{\beta}_2^{\text{marg}}, 0, 0)^\top$, where marginal MLEs are obtained from independent Weibull fits.
- **Convergence criteria:**

$$||\underline{\theta}^{(r+1)} - \underline{\theta}^{(r)}||_2 < 10^{-6} \quad \text{and} \quad |\ell(\underline{\theta}^{(r+1)}) - \ell(\underline{\theta}^{(r)})| < 10^{-8}.$$

- **Parameter constraints:** We enforce $\alpha_i, \beta_i > 0$ and $(\gamma, \omega) \in \Omega$.

Existence and uniqueness of MLEs

Under the regularity conditions of Cox and Hinkley [39], the MLE exists and is unique within $\Theta = (0, \infty)^4 \times \Omega$ when:

(1) $\ell(\underline{\theta})$ is strictly concave in each parameter,
(2) $\mathbf{s}(\underline{\theta})$ is bounded and continuous, and
(3) the Fisher information matrix $\mathcal{I}(\underline{\theta}) = \mathbb{E}[-\nabla^2 \ell(\underline{\theta})]$ is positive definite for all $\underline{\theta} \in \Theta$.

For the bivariate GFGM–WD model:

- **Log-concavity:** The Weibull marginal log-likelihoods are strictly concave for $\alpha_i > 0$. The copula term $\log[c(H_Z, H_T; \gamma, \omega)]$ is concave in (γ, ω) within Ω, producing a unique maximum.
- **Boundedness:** The score components are bounded since the Weibull density decays exponentially and the copula derivatives remain finite for all admissible (γ, ω).
- **Positive definiteness:** The marginal Fisher information matrices are positive definite (cf. Smith [40]), and the joint matrix retains this property for large n.

Hence, $\ell(\underline{\theta})$ attains a unique global maximum at $\widehat{\underline{\theta}}$, which can be efficiently located via the Newton–Raphson algorithm.

Remark 4.1. *In finite samples, the observed information matrix* $\widehat{\mathbf{D}} = -\nabla^2 \ell(\widehat{\underline{\theta}})$ *is checked to ensure all eigenvalues are positive, guaranteeing numerical stability and the validity of the MLE.*

This completes the MLE framework for the proposed GFGM-WD model.

4.2. Bayesian estimation

The Bayesian estimation approach provides a flexible inferential framework by combining sample information with prior beliefs about the parameters. Unlike the classical MLE method, which depends solely on observed data, Bayesian inference integrates prior distributions and updates them via Bayes' theorem to obtain posterior distributions that reflect both prior knowledge and empirical evidence.

Let $\underline{\theta} = (\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma, \omega)^\top$ denote the parameter vector of the proposed GFGM-WD.

We assume independent priors for all parameters. For the Weibull shape parameters $\alpha_l > 0$ ($l = 1, 2$), gamma priors are assigned:

$$\Pi(\alpha_l) \propto \frac{d_l^{c_l}}{\Gamma(c_l)} \alpha_l^{c_l - 1} e^{-d_l \alpha_l}, \qquad \alpha_l > 0,\ c_l, d_l > 0.$$

For the scale parameters $\beta_l > 0$ ($l = 1, 2$), we also assume gamma priors:

$$\Pi(\beta_l) \propto \frac{b_l^{a_l}}{\Gamma(a_l)} \beta_l^{a_l - 1} e^{-b_l \beta_l}, \qquad \beta_l > 0,\ a_l, b_l > 0.$$

For the dependence parameters, we adopt independent uniform priors

$$\gamma \sim \text{Uniform}(-1, 1) \ \text{ and } \ \omega \sim \text{Uniform}(-\omega_{\max}, \omega_{\max}),$$

where $\omega_{\max}$ denotes the maximum admissible value that ensures $c(u, v; \gamma, \omega) \geq 0$ for all $u, v \in [0, 1]$. Hence, the joint prior density is

$$\Pi(\underline{\theta}) \propto \prod_{l=1}^{2} \left[\frac{d_l^{c_l}}{\Gamma(c_l)} \alpha_l^{c_l - 1} e^{-d_l \alpha_l} \right] \prod_{l=1}^{2} \left[\frac{b_l^{a_l}}{\Gamma(a_l)} \beta_l^{a_l - 1} e^{-b_l \beta_l} \right] \times \frac{1}{2} \times \frac{1}{2\omega_{\max}} \mathbb{I}_{(-1,1)}(\gamma)\, \mathbb{I}_{(-\omega_{\max}, \omega_{\max})}(\omega),$$

where $\mathbb{I}_{\mathbf{A}}(x)$ denotes the usual indicator function of the set $\mathbf{A}$. The hyperparameters (a_l, b_l, c_l, d_l) are chosen to reflect prior information, typically aligning prior means and variances with marginal MLEs (see Gupta and Kundu [41], Dey et al. [42], and Hamdy and Almetwally [43]).

Using the copula density of the GFGM–WD model, which is given by (2.2), and substituting these expressions, we obtain

$$\begin{aligned} L(\underline{\theta}) = {} & \left(\frac{\alpha_1 \alpha_2}{\beta_1 \beta_2} \right)^n \prod_{i=1}^{n} \left(\frac{z_i}{\beta_1} \right)^{\alpha_1 - 1} \left(\frac{t_i}{\beta_2} \right)^{\alpha_2 - 1} \exp\left[-\left(\left(\frac{z_i}{\beta_1} \right)^{\alpha_1} + \left(\frac{t_i}{\beta_2} \right)^{\alpha_2} \right) \right] \\ & \times \prod_{i=1}^{n} \left[1 + \gamma (1 - 2H_Z(z_i))(1 - 2H_T(t_i)) + \omega\, H_Z(z_i) H_T(t_i)(2 - 3H_Z(z_i))(2 - 3H_T(t_i)) \right]. \end{aligned}$$

The posterior density function, up to a proportionality constant, is

$$\begin{aligned} \Pi(\underline{\theta} \mid z, t) & \propto \Pi(\underline{\theta})\, L(\underline{\theta}) \\ & \propto \left[\prod_{l=1}^{2} \frac{d_l^{c_l}}{\Gamma(c_l)} \alpha_l^{c_l - 1} e^{-d_l \alpha_l} \right] \left[\prod_{l=1}^{2} \frac{b_l^{a_l}}{\Gamma(a_l)} \beta_l^{a_l - 1} e^{-b_l \beta_l} \right] \frac{1}{4\omega_{\max}} \left(\frac{\alpha_1 \alpha_2}{\beta_1 \beta_2} \right)^n \\ & \quad \times \prod_{i=1}^{n} \left(\frac{z_i}{\beta_1} \right)^{\alpha_1 - 1} \left(\frac{t_i}{\beta_2} \right)^{\alpha_2 - 1} \exp\left[-\left(\left(\frac{z_i}{\beta_1} \right)^{\alpha_1} + \left(\frac{t_i}{\beta_2} \right)^{\alpha_2} \right) \right] \\ & \quad \times \prod_{i=1}^{n} \left[1 + \gamma (1 - 2H_Z(z_i))(1 - 2H_T(t_i)) + \omega\, H_Z(z_i) H_T(t_i)(2 - 3H_Z(z_i))(2 - 3H_T(t_i)) \right] \\ & \quad \times \mathbb{I}_{(-1,1)}(\gamma)\, \mathbb{I}_{(-\omega_{\max}, \omega_{\max})}(\omega). \end{aligned}$$

Numerical computation via MCMC

Because the posterior distribution lacks a closed-form normalizing constant, we employ a Markov chain Monte Carlo (MCMC) method based on the Metropolis–Hastings (MH) algorithm.

Algorithm (Metropolis–Hastings sampling):

(1) **Initialization:** Set $\underline{\theta}^{(0)} = (\hat{\alpha}_1^{\text{MLE}}, \hat{\alpha}_2^{\text{MLE}}, \hat{\beta}_1^{\text{MLE}}, \hat{\beta}_2^{\text{MLE}}, \hat{\gamma}^{\text{MLE}}, \hat{\omega}^{\text{MLE}})$ using the MLEs from Section 4.1. Multiple chains with dispersed starting points enhance robustness.

(2) **Proposal generation:** At iteration r, propose $\underline{\theta}^* \sim \mathcal{N}(\underline{\theta}^{(r-1)}, \Sigma)$, with covariance Σ tuned for an acceptance rate of 30%–40%. For constrained parameters ($\alpha_l, \beta_l > 0$, $\gamma \in (-1,1)$, $\omega \in (-\omega_{\max}, \omega_{\max})$), transformations (log or logit) or rejection of invalid proposals are employed.

(3) **Acceptance step:**

$$A = \min\left\{1, \frac{\Pi(\underline{\theta}^* \mid z, t)}{\Pi(\underline{\theta}^{(r-1)} \mid z, t)}\right\} \quad \text{and} \quad \underline{\theta}^{(r)} = \begin{cases} \underline{\theta}^*, & \text{with probability } A, \\ \underline{\theta}^{(r-1)}, & \text{otherwise.} \end{cases}$$

(4) **Convergence check:** Run several chains and assess convergence using the Gelman–Rubin statistic ($\widehat{R} < 1.1$), the effective sample size (ESS$>$ 400), and trace plots indicating stable mixing. Typically, 50,000 iterations are run with the first 10–20% discarded as burn-in.

(5) **Posterior summaries:** Bayes estimates (posterior means) under squared-error loss are

$$\hat{\alpha}_l = \frac{1}{M}\sum_{r=1}^{M} \alpha_l^{(r)}, \quad \hat{\beta}_l = \frac{1}{M}\sum_{r=1}^{M} \beta_l^{(r)}, \quad \hat{\gamma} = \frac{1}{M}\sum_{r=1}^{M} \gamma^{(r)}, \text{ and } \hat{\omega} = \frac{1}{M}\sum_{r=1}^{M} \omega^{(r)},$$

where M is the number of post-burn-in iterations. Credible intervals are obtained from the 2.5th and 97.5th percentiles of the posterior samples.

Remarks

- The Bayesian approach provides both point estimates and complete posterior uncertainty quantification.
- The MH tuning matrix Σ strongly influences convergence, and adaptive tuning can enhance mixing and efficiency.
- Tables 6–9 present the posterior means and 95% credible intervals obtained from this MCMC estimation.

This completes the Bayesian estimation framework for the parameters of the proposed GFGM-WD.

4.3. Monte Carlo simulation

This subsection presents a numerical comparison of the performance of the ML and Bayesian estimators. The analytical efficiency of both estimation methods is evaluated for the parameters of the GFGM-WD. To perform this assessment, 1000 samples were generated from the GFGM-WD model using the Mathcad software package. The study considers various scenarios by selecting different parameter values for $\gamma = 0.9, -0.5$, $\omega = 1.59, 3.423$, and varying sample sizes $n = 20, 50, 100, 150$. Specifically, the parameter configurations used in the simulation study are as follows:
In Table 6, $\beta_1 = 1$, $\beta_2 = 2$, with known $\alpha_1 = 1$, $\alpha_2 = 2$.
In Table 7, $\beta_1 = 2$, $\beta_2 = 0.3$, with known $\alpha_1 = 2$, $\alpha_2 = 0.3$.
In Table 8, $\beta_1 = 1.5$, $\beta_2 = 0.9$, with known $\alpha_1 = 1.5$, $\alpha_2 = 0.9$.
In Table 9, $\beta_1 = 0.2$, $\beta_2 = 2$, with known $\alpha_1 = 0.2$, $\alpha_2 = 2$.

The simulation results of bias and mean squared error (MSE), based on 5000 iterations of the Monte Carlo simulation, are shown in Tables 6–9. The following conclusions can be drawn from the tables:

- It can be shown that the ML and Bayesian estimates of unknown parameters are fairly good in terms of bias and MSE.
- With an increase in sample size, it is shown that MSEs decrease and the estimated values of the parameters approach the nominal values of the parameters.
- Bayesian estimates are smaller than MLE.
- For $\gamma = 0.9$ and $\omega = 1.59$, both ML and Bayesian estimate values are smaller than the case of $\gamma = -0.5$ and $\omega = 3.423$.

Table 6. ML and Bayesian estimation methods for the parameters of the GFGM-WD model.

		$\beta_1 = 1, \beta_2 = 2$ with known $\alpha_1 = 1, \alpha_2 = 2$							
		$\gamma = 0.9, \omega = 1.59$				$\gamma = -0.5, \omega = 3.423$			
		ML estimate		Bayes		ML estimate		Bayes	
n		*Bias*	*MSE*	*Bias*	*MSE*	*Bias*	*MSE*	*Bias*	*MSE*
20	β_1	0.055	0.255	0.045	0.155	0.088	0.257	0.068	0.155
	β_2	0.088	0.599	0.068	0.399	0.112	0.784	0.012	0.681
	γ	0.064	0.316	0.054	0.215	0.084	0.457	0.064	0.353
	ω	0.045	0.435	0.034	0.372	0.054	0.457	0.023	0.433
50	β_1	0.028	0.333	0.021	0.231	0.079	0.241	0.079	0.140
	β_2	0.072	0.487	0.062	0.382	0.097	0.698	0.087	0.593
	γ	0.051	0.287	0.038	0.183	0.069	0.421	0.059	0.320
	ω	0.034	0.421	0.022	0.354	0.041	0.443	0.014	0.421
100	β_1	0.019	0.298	0.005	0.193	0.063	0.239	0.053	0.133
	β_2	0.065	0.414	0.054	0.311	0.083	0.631	0.073	0.530
	γ	0.047	0.245	0.031	0.143	0.057	0.401	0.037	0.399
	ω	0.021	0.411	0.019	0.341	0.036	0.427	0.014	0.322
150	β_1	0.010	0.167	0.007	0.167	0.059	0.214	0.049	0.112
	β_2	0.054	0.389	0.034	0.389	0.076	0.543	0.056	0.432
	γ	0.038	0.212	0.028	0.212	0.053	0.289	0.043	0.121
	ω	0.036	0.234	0.015	0.331	0.034	0.299	0.009	0.299

Table 7. ML and Bayesian estimation methods for the parameters of the GFGM-WD model.

		$\beta_1 = 2, \beta_2 = 0.3$ with known $\alpha_1 = 2, \alpha_2 = 0.3$							
		$\gamma = 0.9, \omega = 1.59$				$\gamma = -0.5, \omega = 3.423$			
		ML estimate		Bayes		ML estimate		Bayes	
n		*Bias*	*MSE*	*Bias*	*MSE*	*Bias*	*MSE*	*Bias*	*MSE*
20	β_1	0.074	0.542	0.064	0.441	0.092	0.652	0.082	0.351
	β_2	0.106	0.321	0.099	0.220	0.122	0.382	0.102	0.281
	γ	0.040	0.242	0.032	0.141	0.053	0.384	0.023	0.182
	ω	0.065	0.237	0.061	0.233	0.076	0.257	0.072	0.245
50	β_1	0.066	0.532	0.054	0.431	0.086	0.583	0.066	0.481
	β_2	0.094	0.286	0.085	0.184	0.115	0.351	0.105	0.250
	γ	0.030	0.230	0.022	0.123	0.044	0.341	0.024	0.240
	ω	0.042	0.221	0.038	0.211	0.053	0.276	0.049	0.266
100	β_1	0.057	0.508	0.043	0.401	0.071	0.542	0.061	0.441
	β_2	0.081	0.240	0.076	0.434	0.094	0.298	0.084	0.193
	γ	0.022	0.199	0.011	0.091	0.038	0.308	0.018	0.204
	ω	0.054	0.211	0.028	0.201	0.049	0.266	0.041	0.257
150	β_1	0.040	0.460	0.034	0.354	0.060	0.483	0.050	0.381
	β_2	0.073	0.210	0.067	0.109	0.076	0.252	0.056	0.152
	γ	0.020	0.182	0.009	0.081	0.030	0.262	0.020	0.161
	ω	0.038	0.199	0.21	0.189	0.041	0.254	0.037	0.243

Table 8. ML and Bayesian estimation methods for the parameters of the GFGM-WD model.

		$\beta_1 = 1.5, \beta_2 = 0.9$ with known $\alpha_1 = 1.5, \alpha_2 = 0.9$							
		$\gamma = 0.9, \omega = 1.59$				$\gamma = -0.5, \omega = 3.423$			
		ML estimate		Bayes		ML estimate		Bayes	
n		*Bias*	*MSE*	*Bias*	*MSE*	*Bias*	*MSE*	*Bias*	*MSE*
20	β_1	0.047	0.248	0.027	0.144	0.068	0.279	0.048	0.178
	β_2	0.059	0.288	0.049	0.183	0.079	0.304	0.069	0.202
	γ	0.111	0.425	0.101	0.322	0.111	0.532	0.101	0.431
	ω	0.065	0.365	0.061	0.355	0.077	0.387	0.072	0.378
50	β_1	0.038	0.236	0.028	0.134	0.059	0.248	0.049	0.146
	β_2	0.045	0.268	0.035	0.166	0.055	0.285	0.045	0.184
	γ	0.086	0.411	0.076	0.310	0.102	0.477	0.099	0.376
	ω	0.061	0.345	0.056	0.332	0.071	0.375	0.065	0.367
100	β_1	0.025	0.222	0.015	0.121	0.042	0.231	0.032	0.130
	β_2	0.032	0.255	0.022	0.154	0.043	0.264	0.033	0.163
	γ	0.075	0.401	0.065	0.300	0.079	0.468	0.069	0.364
	ω	0.059	0.321	0.055	0.311	0.066	0.367	0.061	0.356
150	β_1	0.018	0.201	0.008	0.100	0.033	0.224	0.023	0.121
	β_2	0.021	0.235	0.011	0.133	0.025	0.241	0.015	0.140
	γ	0.069	0.388	0.059	0.287	0.071	0.405	0.061	0.301
	ω	0.049	0.299	0.044	0.287	0.054	0.344	0.041	0.276

Table 9. ML and Bayesian estimation methods for the parameters of the GFGM-WD model.

		$\beta_1 = 0.2, \beta_2 = 2$ with known $\alpha_1 = 0.2, \alpha_2 = 2$							
		$\gamma = 0.9, \omega = 1.59$				$\gamma = -0.5, \omega = 3.423$			
		ML estimate		Bayes		ML estimate		Bayes	
n		*Bias*	*MSE*	*Bias*	*MSE*	*Bias*	*MSE*	*Bias*	*MSE*
20	β_1	0.076	0.447	0.056	0.246	0.082	0.501	0.072	0.400
	β_2	0.088	0.568	0.038	0.363	0.089	0.644	0.039	0.443
	γ	0.068	0.264	0.048	0.162	0.079	0.337	0.069	0.134
	ω	0.055	0.324	0.019	0.278	0.076	0.389	0.045	0.333
50	β_1	0.065	0.413	0.055	0.312	0.075	0.472	0.045	0.271
	β_2	0.075	0.552	0.035	0.451	0.077	0.579	0.067	0.373
	γ	0.059	0.232	0.039	0.131	0.071	0.331	0.031	0.227
	ω	0.048	0.322	0.044	0.312	0.068	0.363	0.061	0.355
100	β_1	0.055	0.377	0.035	0.276	0.065	0.411	0.055	0.301
	β_2	0.074	0.513	0.054	0.411	0.062	0.502	0.032	0.366
	γ	0.052	0.204	0.042	0.102	0.068	0.277	0.058	0.170
	ω	0.039	0.311	0.031	0.302	0.062	0.346	0.054	0.341
150	β_1	0.048	0.335	0.028	0.232	0.058	0.378	0.048	0.272
	β_2	0.052	0.477	0.042	0.372	0.056	0.473	0.046	0.371
	γ	0.044	0.198	0.024	0.091	0.059	0.241	0.049	0.140
	ω	0.029	0.289	0.021	0.278	0.059	0.338	0.052	0.331

5. Real data application

Example 5.1 (Diabetic nephropathy data)**.** *This example aims to examine the extropy and WEX of a real-world data set based on the GFGM-WD. Using diabetic nephropathy data, a review of medical information is performed, revealing a poor correlation between the two RVs (bivariate data). This data was obtained from Dr. Path Lal's lab database between January 2012 and August 2013. Researchers measured glucose levels in 132 patients with type 2 diabetic nephropathy throughout their lives, from childhood to adulthood, according to Grover et al. [44]. The RV Z represents the mean duration of diabetes, whereas the RV T represents the mean serum creatinine level (SrCr), based on studies of 19 patients (cf. El-Sherpieny et al. [45]). The FGMBW, GFGM-Chen distribution (cf. Chen [46]), denoted by GFGM-CHD, GFGM-Epanechnikov-exponential distribution (cf. Alkhazaalh and Al-Zoubi [47]), denoted by GFGM-EED, and GFGM-generalized exponential* *distribution, denoted by GFGM-GED have all been compared to the GFGM-WD based on Akaike information criterion (AIC), corrected AIC (AICc), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), and consistent AIC (CAIC). Comparing the GFGM-WD against these distributions, it is found that its AIC and BIC are the lowest, and the findings are shown in Table 10. Moreover, Table 11 shows estimated values of the extropy and WEX for the model GFGM-WD*$(3.3486, 19.2634; 9.1617, 1.8278)$ *for the concomitant* $T_{[n,k]}$.

Table 10. Comparison of goodness-of-fit measures for competing bivariate distributions applied to diabetic nephropathy data.

	$-\log L$	AIC	AICc	BIC	HQIC	CAIC
GFGM-WD	-55.07	122.148	129.148	127.815	123.107	129.148
GFGM-CHD	-56.30	124.599	131.599	130.266	125.558	131.599
FGMBW	-57.72	125.433	130.048	130.155	126.232	130.048
GFGM-EED	-98.07	204.142	206.999	207.92	204.781	206.999
GFGM-GED	-148.58	309.154	316.154	314.821	310.113	316.154

Table 11. Extropy and WEX of the GFGM-WD at $\widehat{\alpha}_2 = 9.1617$ and $\widehat{\beta}_2 = 1.8278$.

k	2	2	2	2	2	4	4	4	4	4
n	1	3	5	8	10	1	3	5	8	10
$\widehat{\zeta}(T_{[n,k]})$	-0.7251	-0.5793	-0.5798	-0.6989	-0.7502	-0.7641	-0.7079	-0.6039	-0.5584	-0.5922
$\zeta^{(\omega)}(T_{[n,k]})$	-1.2804	-1.0231	-1.0729	-1.3374	-1.4450	-1.3650	-1.2263	-1.0525	-1.0145	-1.1031

Figure 4 presents quantile–quantile (Q–Q) and probability–probability (P–P) plots for the marginal Weibull distributions fitted to the real dataset. The left panel displays Q–Q plots, where empirical quantiles are plotted against theoretical quantiles of the fitted Weibull distributions. The close alignment of points along the 45° reference line indicates good agreement between empirical and theoretical quantiles, suggesting that the Weibull distribution provides an appropriate marginal representation for both variables. The right panel shows P–P plots comparing empirical DFs with the fitted Weibull DFs. The points closely following the diagonal line demonstrate strong concordance between empirical and theoretical probabilities. These graphical diagnostics, supported by high correlation coefficients (exceeding 0.95 in this dataset), support the suitability of the Weibull marginals and confirm the adequacy of the proposed GFGM-WD model in capturing the empirical distributional characteristics.

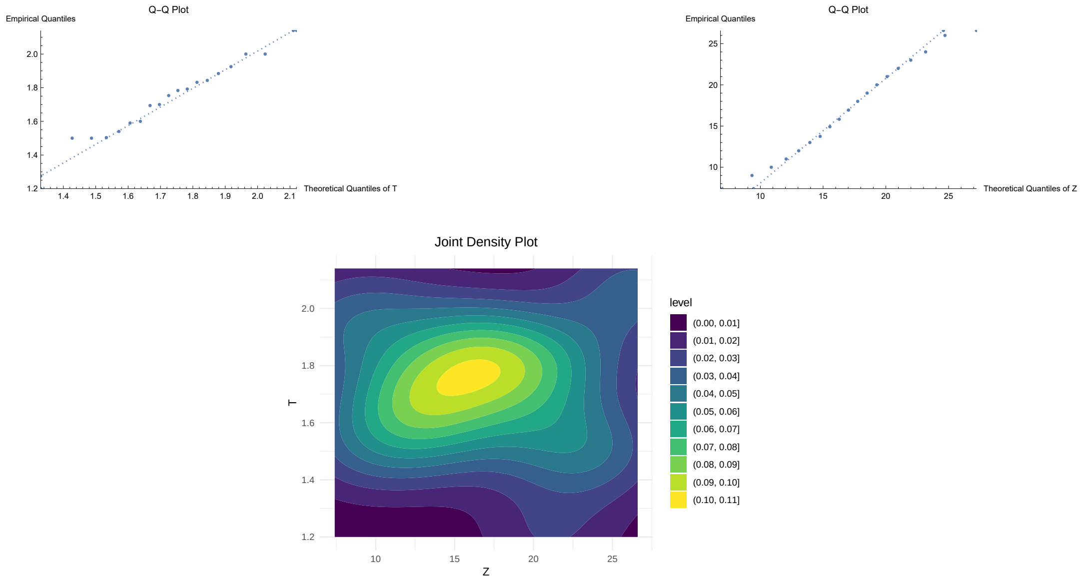


Figure 4. Q-Q plot and density plot for diabetic nephropathy data.

Example 5.2 (Cholesterol data set)**.** *This data set includes cholesterol levels measured at 5 and 25 weeks after treatment in 30 patients (see Shoaee [47]). We fit the data based on the GFGM-WD*$(\alpha_1, \beta_1; \alpha_2, \beta_2)$*. The ML estimates of the parameters are* $\widehat{\alpha}_1 = 2.897, \widehat{\beta}_1 = 1.203, \widehat{\alpha}_2 = 2.545, \widehat{\beta}_2 = 1.087, \widehat{\gamma} = 1$, *and* $\widehat{\omega} = 0.707$*. Based on AIC and BIC, the GFGM-WD has been compared to the GFGM-CHD, GFGM-EED, and GFGM-GED families. The results are displayed in Table 12 and indicate that the GFGM-WD has the lowest AIC and BIC when compared to those families. Table 13 shows estimated values of the entropy, extropy, and WEX for the model GFGM-WD*$(2.897, 1.203; 2.545, 1.087)$ *for the concomitant* $T_{[n,k]}$.

Table 12. Model comparison for cholesterol data using information criteria.

	$-\log L$	AIC	AICc	BIC	HQIC	CAIC
GFGM-WD	-26.82	65.6323	69.2845	74.0395	68.3218	69.2845
GFGM-GED	-27.88	67.7564	71.4085	76.1635	70.4459	71.4085
GFGM-CHD	-28.97	69.9335	73.5857	78.3407	72.6231	73.5857
GFGM-EED	-54.34	116.688	118.288	122.293	118.481	118.288

Table 13. Entropy, extropy, and WEX of the GFGM-WD at $\widehat{\alpha}_2 = 2.545$ and $\widehat{\beta}_2 = 1.087$.

k	2	2	2	2	2	4	4	4	4	4
n	1	3	5	8	10	1	3	5	8	10
$\mathcal{H}(T_{[n,k]})$	0.2298	0.2828	0.2187	0.1350	0.1066	0.1612	0.2679	0.2842	0.2401	0.2016
$\widehat{\zeta}(T_{[n,k]})$	-0.3735	-0.3402	-0.3639	-0.3942	-0.4035	-0.40601	-0.3527	-0.3405	-0.3558	-0.3703
$\zeta^{(\omega)}(T_{[n,k]})$	-0.3080	-0.3612	-0.4291	-0.4886	-0.5051	-0.3101	-0.3191	-0.3516	-0.4105	-0.4430

Figure 5 illustrates the dependence structure captured by the proposed GFGM-WD using contour and three-dimensional surface plots. The contour plots reveal a structured pattern characteristic of the GFGM copula's dependence form, with contour density indicating regions of higher joint probability concentration. The three-dimensional surface plot complements this view by displaying the overall shape of the JPDF, highlighting modal intensity and tail behavior. The smooth curvature and well-defined modal regions demonstrate the models capability to capture both the strength and direction of dependence effectively. These visualizations, corresponding to the estimated parameters $\hat{\gamma}$ and $\hat{\omega}$, confirm the robustness of the GFGM-WD model in representing real-world phenomena with distinct marginal behaviors and moderate dependence within a unified probabilistic framework.

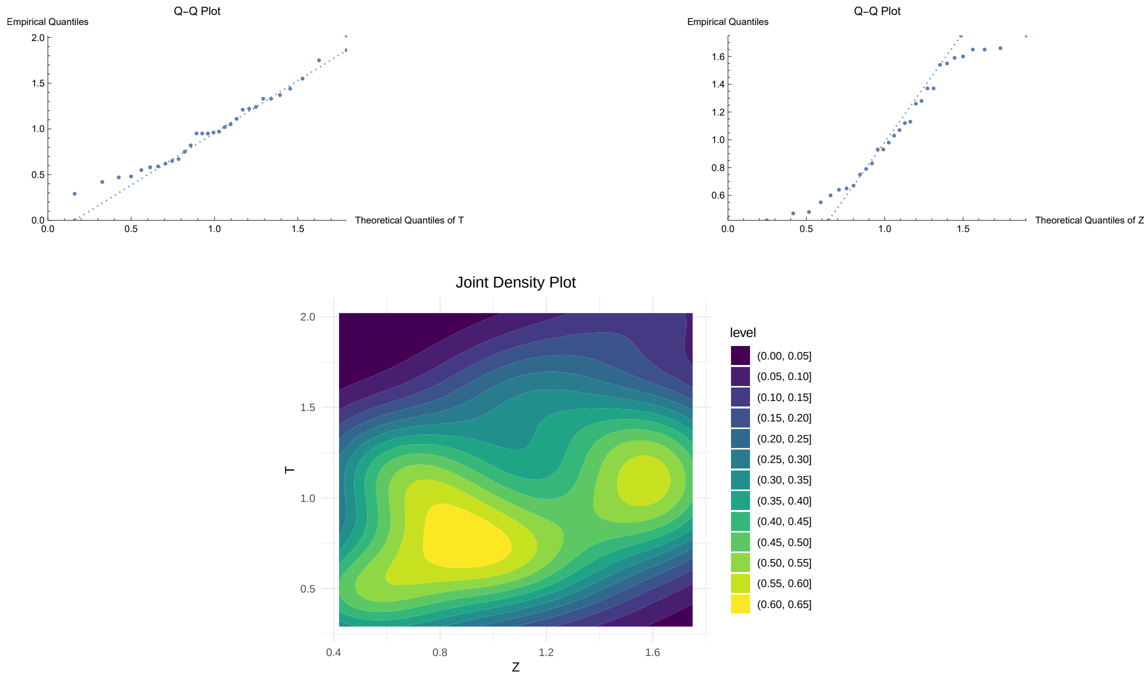


Figure 5. Some summary plots of the cholesterol data set.

6. Discussion and concluding remarks

In this study, we introduced a new bivariate Weibull distribution, termed the *GFGM bivariate Weibull distribution* (GFGM–WD), constructed by coupling Weibull marginals with the generalized Farlie–Gumbel–Morgenstern (GFGM) copula. This model offers a flexible yet analytically tractable framework for describing bivariate lifetime data exhibiting weak to moderate dependence, as commonly observed in reliability, medical, and engineering applications. The closed-form expressions of its joint, marginal, and conditional distributions allow for straightforward computation of several reliability measures and statistical properties.

Practical interpretation of the model parameters. The parameters of the proposed GFGM–WD model possess clear physical interpretations that enhance its applicability in practical contexts. The Weibull shape parameters α_l ($l = 1, 2$) describe the behavior of component failure rates: $\alpha_l < 1$ corresponds to decreasing failure rates (infant mortality), $\alpha_l = 1$ to constant failure rates (exponential-type behavior), and $\alpha_l > 1$ to increasing failure rates (ageing effects). The scale parameters β_l $(l = 1, 2)$ determine the characteristic lifetimes, where larger β_l values imply longer expected survival times. The dependence parameters γ and ω jointly govern the strength and type of association between the two components. Specifically, γ controls linear dependence in the classical FGM sense, while ω introduces nonlinear dependence effects that expand the attainable correlation range and allow for moderate positive or negative association. Positive (γ, ω) values indicate simultaneous degradation or shared environmental stress, whereas negative values suggest compensatory or competitive reliability behavior. Together, these parameters provide meaningful insight into correlated system performance and joint survival probabilities, which are essential for reliability design and medical prognosis.

From a methodological perspective, the GFGM–WD retains analytical simplicity while extending the dependence flexibility of the classical FGM-Weibull models. The derived reliability and uncertainty measures including entropy, weighted entropy (WE), extropy, and weighted extropy (WEX) were examined both theoretically and numerically. Parameter estimation was conducted using maximum likelihood and Bayesian approaches, with Bayesian inference showing improved stability and efficiency. A Monte Carlo simulation study confirmed the consistency and precision of the estimators, and two real medical datasets demonstrated the superior fitting capability of the proposed model compared to existing bivariate Weibull frameworks.

Future study. While the GFGM–WD is well suited for datasets exhibiting moderate dependence without significant tail behavior, it still inherits some limitations from the GFGM family, such as restricted dependence range and challenges in extending to higher dimensions. In future research, these limitations can be addressed by: (i) embedding stronger dependence structures using other copulas (e.g., Clayton, Joe, or Archimax families), (ii) extending the current formulation to trivariate or multivariate versions, and (iii) adapting the model to handle censored or heterogeneous lifetime data. Moreover, issues of parameter identifiability and copula-based estimation (as discussed by Genest and Rivest [48]) warrant further exploration to improve robustness and interpretability in more complex reliability systems.

Overall, the proposed GFGM–WD constitutes a significant advancement in modeling dependent lifetime phenomena, offering a balance between theoretical generality, computational tractability, and practical interpretability.

Author contributions

The authors contributed equally to the paper. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Data availability

The data used to support the findings of this study are available within the article.

Conflict of interest

The authors have declared that no competing interests exist.

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