Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(11): 27837–27861.
DOI: 10.3934/math.20251223
Received: 15 October 2025
Revised: 13 November 2025
Accepted: 25 November 2025
Published: 28 November 2025

Research article

Continuous dependence and stability for a class of fractional partial differential equations with multiple parameters

Jia Zheng[1], **Xiuling Li**[1,*], **Yanni Pang**[2], **Hongying Wang**[1], **Tongchao Wang**[1] **and Jiaxuan Sun**[1,*]

[1] School of Statistics and Data Science, Jilin University of Finance and Economics, Changchun 130117, China

[2] School of Mathematics, Jilin University, Changchun 130021, China

* **Correspondence:** Email: lixiuling@jlufe.edu.cn, 5231091003@s.jlufe.edu.cn.

Abstract: In this paper, we studied the continuous dependence and stability of solutions for a class of fractional partial differential equations with multiple spatially varying coefficient parameters. The nonlocal operator was defined by a symmetric kernel, yielding a self-adjoint structure essential to the analysis. Using variational methods and the Minty–Browder theorem, we established the existence and uniqueness of weak solutions in the energy space X_0 for each admissible parameter vector w. We extended single-parameter stability to a multi-parameter framework by proving that the solution operator S_f is continuous with respect to w in the product space $\prod_i L^{q_i}(\Omega)$. Moreover, we derived an explicit global Lipschitz estimate for S_f and showed its Gâteaux differentiability under mild regularity assumptions on $f(x, u, w)$. Numerical simulations confirmed continuity, Lipschitz stability, and differentiability of S_f with respect to all parameters. These results provided rigorous guarantees for inverse problems and uncertainty quantification in multi-parameter fractional PDE models.

Keywords: fractional partial differential equations; variational methods; Lipschitz continuity; Gâteaux differentiability; dependence on parameters
Mathematics Subject Classification: 35R11, 35B30

1. Introduction

In recent years, as partial differential equations (PDEs) have been widely applied to describe nonlocal phenomena and model complex systems, research into the theoretical properties of their solutions has grown increasingly extensive. In particular, fractional nonlocal PDEs with spatially variable coefficients have emerged as a significant research focus in mathematical physics, as they are capable of more accurately capturing the non-uniform characteristics of physical parameters. Nowadays, fractional models have been widely applied in fields such as medical imaging, porous media, phase transition, and quantitative finance. They can describe abnormal transmission, long-range interactions, and memory effects. For instance, the use of variable-order fractional operators has significantly advanced adaptive denoising techniques in medical imaging [1], while mixed local–nonlocal diffusion equations have deepened our understanding of flow in heterogeneous porous structures. Likewise, phase transition problems with fractional Laplacians have clarified the effects of boundary reactions and symmetry properties in materials science [2], and jump-process-driven PDEs now play a central role in stochastic finance [3]. Beyond classical analysis, researchers have explored neural-network and neurosymbolic paradigms to construct exact analytical solutions for nonlinear PDEs, further enriching the toolbox for PDE modeling and computation [4–6].

Existence and uniqueness results for variable-order fractional p-Laplace evolution equations were established in [7, 8], while regularity and maximum principles for local–nonlocal mixed elliptic equations were derived in [9]. The connection between microscopic particle systems and macroscopic nonlocal dynamics was further clarified in [10]. Advances have extended the stability analysis of fractional PDEs to more complex scenarios. For instance, Colasuonno et al. [11] studied continuous dependence results for p-Laplace equations with operator variations, while Hamdani et al. [8] investigated nonlocal problems involving variable-order fractional $p(\cdot)$-Laplacian with two parameters. On the numerical side, Batiha et al. [12] proposed high-performance adaptive step size schemes for fractional differential equations, enhancing computational accuracy in multi-parameter settings. In the context of inverse problems and uncertainty quantification, Gao and Ng [13] introduced Wasserstein GAN-based uncertainty quantification in physics-informed neural networks, and Fu et al. [14] developed physics-informed kernel function neural networks for parametric PDEs. Furthermore, Rigas et al. [15] presented adaptive training techniques for physics-informed Kolmogorov-Arnold networks in multi-fidelity modeling, while Tripura and Chakraborty [16] utilized wavelet neural operators for solving parametric PDEs in computational mechanics.

Despite these advances, a critical gap persists in the literature. A thorough examination reveals that the overwhelming majority of stability results are confined to single-parameter perturbations or constant coefficients [11, 17]. While this provides a foundational understanding, it fails to address the far more complex and realistic scenario where a system is simultaneously subjected to multiple independent spatially varying parameters. This multi-parameter framework is the rule rather than the exception in practical applications such as inverse problems and uncertainty quantification, where one must identify or account for the uncertainty in several physical properties concurrently. Yet, a systematic theory guaranteeing solution stability and continuous dependence under such joint perturbations remains largely undeveloped.

Many inverse-problem and uncertainty-quantification workflows for fractional nonlocal models require quantitative sensitivity of solutions with respect to multiple spatially varying coefficients, rather than a single scalar parameter. While most stability results address single-parameter or constant-coefficient perturbations, practical models involve several independent distributed coefficients, and rigorous, computable guarantees under joint perturbations have been lacking. To address this need, we establish well-posedness in X_0 and qualitative continuity of the parameter-to-solution map S_f, derive a global Lipschitz estimate with an explicit constant for the full parameter vector w, and prove the Gâteaux differentiability of S_f together with its associated linearized nonlocal variational problem; numerical experiments corroborate these properties and report discrete counterparts of the constants. Taken together, these results provide a unified multiparameter stability/sensitivity theory that directly supports gradient-based identification, uncertainty propagation, and step-size control in data-driven settings.

The central requirement of any mathematical model is adaptability; that is, the existence, uniqueness, and stability of the solutions [18]. Among these, stability is particularly crucial for parameter identification, uncertainty quantification, and inverse problem solving [19]: Only by ensuring that small disturbances in the input data or coefficients cause controllable changes in the solution can numerical simulation and model inference be reliable. Although researchers have conducted relatively thorough discussions on the existence, uniqueness, and stability of solutions in the single-parameter case [17], practical problems typically rely on multiple spatially varying parameters changing simultaneously; thus, there is a need for a systematic stability theory to support identification, uncertainty quantification, and inverse problem solving in such multiparameter settings.

Here, we aim to address this gap by studying the fractional boundary value problem

$$\begin{cases} -L_K u = f(x, u(x), w_1(x), \ldots, w_n(x)), & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain, L_K denotes a non-local operator with a kernel K satisfying the general symmetry and integrability conditions, the bilinear energy form is symmetric and the associated nonlocal operator is self-adjoint on X_0. If, in addition, $K(y) = \kappa(|y|)$, then the energy is invariant under orthogonal transformations. Hence, whenever Ω and the data are invariant under a subgroup $G \subset O(N)$, uniqueness implies that the weak solution satisfies $u \circ g = u$ for all $g \in G$ [2]. The parameter functions $w_i \in L^{q_i}(\Omega)$ represent spatially distributed coefficients. The nonlinearity f is of Carathéodory type, subject to specific growth and one-sided Lipschitz assumptions, which will be detailed in the following sections.

Example 1 (fractional diffusion in heterogeneous media). A canonical physical instance of (1.1) is anomalous diffusion in a heterogeneous medium, where $u(x)$ denotes the concentration of a solute and several spatially varying fields jointly perturb the dynamics. A convenient multiparameter model that fits our analysis is

$$-L_K u = a(x)\, u(x) + \eta(x) + \sum_{i=1}^{n} \beta_i(x)\, w_i(x) \quad \text{in } \Omega \qquad u = 0 \text{ in } \Omega^c.$$

This affine structure satisfies the standard growth condition, the one-sided Lipschitz condition in t, and the multiparameter Lipschitz and differentiability conditions in w_i, namely $|\partial_t f| \leq a_+(x)$ and $\partial_{w_i} f = \beta_i(x) \in L^{s_i}(\Omega)$, with exponents (s_i, q_i, r) fulfilling $\frac{1}{s_i} + \frac{1}{q_i} + \frac{1}{r} = 1$. Consequently, the multiparameter stability and Gâteaux differentiability of the parameter-to-solution map S_f quantify how simultaneous perturbations in $(w_1, \ldots, w_n)$ impact u, which is critical for inverse identification and UQ in heterogeneous media.

Despite considerable progress for existence and stability theory for fractional PDEs with single-parameter or constant coefficients [11, 17], a unified quantitative framework for models with multiple spatially varying parameters remains lacking, yet is crucial in practical inverse problems and uncertainty quantification [19]. We systematically address this gap by establishing, under minimal Carathéodory and monotonicity assumptions, a multi-parameter stability and sensitivity theory for fractional nonlocal boundary value problems. Based on variational methods and the

Minty–Browder theorem [20–22], we prove the existence and uniqueness of weak solutions, derive explicit global Lipschitz estimates, and demonstrate Gâteaux differentiability of the solution operator with respect to all parameters. Compared with classical single-parameter results, our theory provides robust mathematical guarantees for well-posedness and sensitivity in genuinely multi-parameter, spatially heterogeneous settings. These advances not only generalize the classical stability theory, but also directly support reliable parameter identification and numerical modeling of nonlocal and fractional systems.

In this paper, the term "stability" is always understood as stability with respect to perturbations of the distributed parameter vector $w = (w_1, \dots, w_n)$. More precisely, we study the parameter-to-solution operator

$$S_f : \prod_{i=1}^{n} L^{q_i}_{\Sigma_i} \longrightarrow X_0$$

and prove that it is (i) qualitatively continuous and (ii) globally Lipschitz continuous with respect to w in the energy norm $\|\cdot\|$ of X_0. Thus, the stability results obtained here do not concern dynamical or asymptotic stability of time-dependent solutions, but rather the robustness of the stationary nonlocal boundary value problem (1.1) under small perturbations of the coefficient functions w_i.

The main structure of this article is as follows. In Section 2, we introduce the functional setting, notation, and fractional Sobolev embeddings. In Section 3, we state the standing assumptions and auxiliary lemmas, and prove existence–uniqueness and continuity of the solution operator. In Section 4, we present the major results: A global Lipschitz estimate with respect to all parameters and the Gâteaux differentiability together with the associated linearized nonlocal problem. Section 5 reports numerical experiments that verify the theory. In Section 6, we conclude the paper.

2. Preliminaries

Throughout the paper, we assume that $N \geq 1$, $0 < s \in (0,1)$ and that $\Omega \subset \mathbb{R}^N$ is a bounded Lipschitz domain. We denote by $\Omega^c = \mathbb{R}^N \setminus \Omega$ the complement of Ω, and introduce the interaction set

$$Q = \mathbb{R}^{2N} \setminus (\Omega^c \times \Omega^c) = \{(x,y) \in \mathbb{R}^{2N} : \text{ at least one of } x, y \text{ lies in } \Omega\}.$$

The nonlocal operator $\mathcal{L}_K$ in (1.1) is defined in terms of a measurable kernel $K : \mathbb{R}^N \setminus \{0\} \to (0,\infty)$ by

$$\mathcal{L}_K u(x) := \text{P.V.} \int_{\mathbb{R}^N} (u(x) - u(y))\, K(x-y)\, dy, \qquad x \in \Omega,$$

where P.V. denotes the Cauchy principal value. We assume that K is symmetric, $K(y) = K(-y)$, and satisfies the standard integrability and ellipticity conditions ensuring that the associated bilinear form is well defined and coercive; in particular,

$$\iint_Q |u(x) - u(y)|^2 K(x-y)\, dx\, dy < \infty$$

for all admissible functions u considered below.

The fractional Sobolev space associated with problem (1.1) is given by

$$X = \Big\{ u : \mathbb{R}^N \to \mathbb{R} \;\Big|\; u \text{ is measurable, } u \equiv 0 \text{ a.e. in } \Omega^c, \int_Q |u(x) - u(y)|^2 K(x-y)\, dx\, dy < \infty \Big\}.$$

We then introduce the closed subspace

$$X_0 = \{ u \in X : u(x) = 0 \text{ for a.e. } x \in \Omega^c \},$$

consisting of functions in X that vanish almost everywhere outside Ω. Endowed with the inner product

$$(u, v) := \iint_{\mathbb{R}^{2N}} (u(x) - u(y)) (v(x) - v(y)) K(x - y) \, dx \, dy,$$

and the induced norm

$$\|u\| := \left(\iint_{\mathbb{R}^{2N}} \left| u(x) - u(y) \right|^2 K(x - y) \, dx \, dy \right)^{1/2},$$

the space X_0 is a separable Hilbert space. We denote by X_0^* its topological dual.

It is standard that X_0 can be identified with the fractional Sobolev space $H_0^s(\Omega)$ associated with the operator $\mathcal{L}_K$ (up to equivalent norms). Thus the natural energy space for problem (1.1) is the fractional Sobolev space $H_0^s(\Omega)$, endowed with the norm induced by the bilinear form associated with $\mathcal{L}_K$. For notational convenience, we set

$$U := H_0^s(\Omega),$$

and refer to U as the state space of admissible solutions u. In the sequel, the parameter-to-solution map S_f will always be understood as a mapping

$$S_f : \mathcal{W} \to U,$$

where $\mathcal{W}$ denotes the set of admissible multiparameter fields $(w_1, \ldots, w_n)$ introduced below.

Next, we recall the fractional critical exponent

$$2_s^* = \frac{2N}{N - 2s},$$

which is finite whenever $N > 2s$. For every $p \in [1, 2_s^*]$ the Sobolev embedding

$$X_0 \to L^p(\Omega)$$

is continuous, and if $p < 2_s^*$, the embedding is also compact. In particular, for each $p \in [1, 2_s^*]$ we can define the optimal embedding constant

$$c_p := \inf\Big\{ \frac{\|u\|}{\|u\|_{L^p(\Omega)}} : u \in X_0 \setminus \{0\} \Big\}, \tag{2.1}$$

where $\|u\|_{L^p(\Omega)}$ denotes the usual L^p-norm of u on Ω.

Finally, we recall the notion of a weak solution for the boundary value problem (1.1). Let $(w_1, \ldots, w_n)$ be given parameter fields, and let $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a Carathéodory nonlinearity satisfying the growth condition

$$\left| f(x, t, w_1, \ldots, w_n) \right| \leq C(1 + |t|^{p-1}) \quad \text{for a.e. } x \in \Omega, \ \forall\, t \in \mathbb{R},$$

for some $p \in [2, 2_s^*]$ and constant $C > 0$ (here, $p' = \frac{p}{p-1}$ denotes the Hölder conjugate of p). We say that $u \in X_0$ is a weak solution of problem (1.1) if $f(\cdot, u(\cdot), w_1(\cdot), \dots, w_n(\cdot)) \in L^{p'}(\Omega)$ and

$$\iint_{\mathbb{R}^{2N}} (u(x) - u(y))\,(v(x) - v(y))\,K(x-y)\,dx\,dy \;=\; \int_\Omega f(x,\, u(x),\, w_1(x), \dots, w_n(x))\,v(x)\,dx$$

holds for all test functions $v \in X_0$. In this case, we also write

$$u = S_f(w_1, \dots, w_n) \in U$$

to emphasize the dependence of the weak solution on the multiparameter fields.

3. Major lemmas

We next specify hypotheses guaranteeing that problem (1.1) admits exactly one weak solution and that this solution continuously depends on the input of the multi-parameter.

Let (Ω, Σ, μ) be a measure space, X a separable metric space, and Y a metric space. A mapping $f : \Omega \times X \to Y$ is called a Carathéodory map provided that

(a) for every fixed $\xi \in X$, the function $x \to f(x, \xi)$ is $(\Sigma, \mathcal{B}(Y))$-measurable (here $\mathcal{B}(Y)$ denotes the Borel σ-field on Y);

(b) for μ-almost every $x \in \Omega$, the map $\xi \to f(x, \xi)$ is continuous.

For $p \in [1, \infty]$, let $p' \in [1, \infty]$ be the conjugate exponent determined by $1/p + 1/p' = 1$. Observe that if $\frac{N}{2s} \le p \le \infty$, then

$$2 \;\le\; 2p' \;\le\; 2_s^*,$$

we will use the following standing assumptions on

$$f : \Omega \times \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ times}} \longrightarrow \mathbb{R} :$$

(F') There exist $r \in (1, 2_s^*)$, a function $\eta \in L^{r'}(\Omega)$, and constants $\alpha_1, \alpha_2, \dots, \alpha_{n+1} > 0$ such that, for $x \in \Omega$ and all $t \in \mathbb{R}$,

$$|f(x, t, w_1, \dots, w_n)| \;\le\; \eta(x) \;+\; \alpha_1\,|t|^{r-1} \;+\; \alpha_2\,|w_1|^{q_1/r'} \;+\; \cdots +\; \alpha_{n+1}\,|w_n|^{q_n/r'},$$

where $r' = \frac{r}{r-1}$ denotes the Hölder conjugate exponent of r.
(A') There exists a measurable function $a : \Omega \to \mathbb{R}$ such that, for $x \in \Omega$ and for all $t_1, t_2 \in \mathbb{R}$ with $w_i \in \Sigma_i$ $(i = 1, \dots, n)$,

$$\big(f(x, t_1, w_1, \dots, w_n) - f(x, t_2, w_1, \dots, w_n)\big)\,(t_1 - t_2) \;\le\; a(x)\,(t_1 - t_2)^2,$$

Remark 1. *By [23], assumption (F') ensures that the operator $N_f(u) = f(\cdot, u, w_1, \dots, w_n)$ defines a Nemytskii map.*

Lemma 3.1. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying the following scalar growth and one-sided Lipschitz conditions (which are the counterparts of (F') and (A') in the single-parameter case, see [17]):*

(F) *There exist $r \in (1, 2_s^*)$, a function $\eta \in L^{r'}(\Omega)$, and a constant $\alpha > 0$ such that*

$$|f(x,t)| \ \leq \ \eta(x) + \alpha\, |t|^{r-1} \quad \textit{for a.e. } x \in \Omega, \ \forall\, t \in \mathbb{R},$$

where $r' = \frac{r}{r-1}$ is the Hölder conjugate exponent of r.

(A) *There exists a function $a \in L^p(\Omega)$ with $\frac{N}{2s} \leq p \leq \infty$ such that the partial derivative $\partial_t f(x,t)$ exists for a.e. $x \in \Omega$, for all $t \in \mathbb{R}$, and satisfies*

$$|\partial_t f(x,t)| \ \leq \ a_+(x) \quad \textit{for a.e. } x \in \Omega, \ \forall\, t \in \mathbb{R},$$

where $a_+(x) := \max\{a(x), 0\}$ denotes the positive part of a.

Let $p' = \frac{p}{p-1}$ be the Hölder conjugate exponent of p. Suppose that

$$\|a_+\|_{L^p(\Omega)} \ < \ c_{2p'}^2, \tag{3.1}$$

where the constant $c_{2p'}$ is given in (2.1). Under these conditions, the boundary value problem

$$\begin{cases} -\mathcal{L}_K u \ = \ f(x,\, u(x)) & x \in \Omega, \\ \quad u = 0 & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{3.2}$$

admits a unique weak solution u_0 in X_0. Moreover, the norm of this solution is bounded from above by

$$\|u_0\| \ \leq \ \frac{c_{2_s^*}\, \|\eta\|_{L^{(2_s^*)'}(\Omega)}}{1 - c_{2p'}^{-2}\, \|a_+\|_{L^p(\Omega)}}\,. \tag{3.3}$$

In addition, when $\frac{N}{2s} < p \leq \infty$, the constant appearing in (3.1) is sharp: That is, there exists some $a \in L^p(\Omega)$ with $\|a_+\|_p = c_{2p'}^2$ such that if $f(x,u(x)) = a(x)\, u(x)$, then the problem (3.2) has at least two distinct weak solutions.

Proof. Let X_0 be the fractional Sobolev space defined in Section 2 with inner product and norm induced by K. Define $T : X_0 \to X_0^*$ by

$$\langle T(u), v\rangle := (u,v) - \int_\Omega f(x,u(x))\, v(x)\, dx, \qquad \forall\, u, v \in X_0.$$

By assumption (A), for all $u, v \in X_0$,

$$\begin{aligned} \langle T(u) - T(v), u - v\rangle &= \|u-v\|^2 - \int_\Omega (f(x,u) - f(x,v))(u-v)\, dx \\ &\geq \|u-v\|^2 - \int_\Omega a_+(x)\, |u-v|^2\, dx \\ &\geq \|u-v\|^2 - \|a_+\|_p\, \|u-v\|_{2p'}^2. \end{aligned}$$

Embedding constant and kernel symmetry. By (K1) and (K2), there exist $0 < \lambda \le \Lambda$, such that the energy is comparable with the classical fractional seminorm:

$$\lambda\, [w]^2_{H^s(\mathbb{R}^N)} \;\le\; \|w\|^2 \;\le\; \Lambda\, [w]^2_{H^s(\mathbb{R}^N)}, \qquad \forall\, w \in X_0,$$

hence, the (symmetric) bilinear form $(\cdot,\cdot)$ indeed defines the X_0-inner product. Combining this with the standard fractional Sobolev embedding $H^s_0(\Omega) \to L^{2p'}(\Omega)$ (constant $C_S = C_S(N, s, \Omega)$), we obtain

$$\|w\|_{2p'} \;\le\; C_S\, [w]_{H^s(\mathbb{R}^N)} \;\le\; C_S\, \lambda^{-1/2}\, \|w\|, \qquad \forall\, w \in X_0.$$

Since $c_{2p'}$ is defined by $\|w\|_{2p'} \le c^{-1}_{2p'}\|w\|$ (see (2.1)), we may take

$$c^{-1}_{2p'} = C_S\, \lambda^{-1/2} \qquad \Big(\text{equivalently } \; c_{2p'} = \lambda^{1/2}/C_S\Big),$$

which depends on only (N, s, Ω, λ); in particular, the symmetry (K1) guarantees that the energy form is symmetric so that this identification of $c_{2p'}$ is legitimate for the monotonicity argument.

Using the above estimate for $\|u - v\|_{2p'}$, we continue from the previous line to get

$$\langle T(u) - T(v), u - v\rangle \;\ge\; \Big(1 - c^{-2}_{2p'}\, \|a_+\|_p\Big)\, \|u - v\|^2.$$

Since $\|a_+\|_p < c^2_{2p'}$, T is strongly monotone with modulus $m := 1 - c^{-2}_{2p'}\|a_+\|_p > 0$. By (F) and standard properties of Carathéodory functions, T is hemicontinuous and coercive. Hence, by the Browder–Minty theorem, $T(u) = 0$ admits a unique solution $u_0 \in X_0$.

A priori bound. Testing the weak formulation with $v = u_0$ and arguing as usual yields

$$m\, \|u_0\|^2 \;\le\; \int_\Omega |f(x, 0)|\, |u_0|\, dx \;\le\; \|\eta\|_{(2^*_s)'}\, \|u_0\|_{2^*_s} \;\le\; c^{-1}_{2^*_s}\, \|\eta\|_{(2^*_s)'}\, \|u_0\|,$$

which gives

$$\|u_0\| \;\le\; \frac{c_{2^*_s}\, \|\eta\|_{(2^*_s)'}}{1 - c^{-2}_{2p'}\|a_+\|_p}.$$

There is sharpness when $\frac{N}{2s} < p \le \infty$. In this range, the embedding $X_0 \to L^{2p'}(\Omega)$ is compact, so the best constant $c_{2p'}$ is attained by some $\phi \in X_0 \setminus \{0\}$ with $\|\phi\|_{2p'} = c^{-1}_{2p'}\|\phi\|$. Set

$$a_*(x) := c^2_{2p'}\, \frac{|\phi(x)|^{2p'/p}}{\|\phi\|^{2p'/p}_{2p'}} \in L^p(\Omega), \qquad \|a_*\|_p = c^2_{2p'}.$$

A direct computation shows

$$\int_\Omega a_*(x)\, \phi(x)^2\, dx = c^2_{2p'}\, \|\phi\|^2_{2p'} = \|\phi\|^2,$$

so the linear problem $-\mathcal{L}_K u = a_*(x) u$ has at least two nontrivial weak solutions ($u = \pm\phi$), which proves the sharpness of the condition $\|a_+\|_p < c^2_{2p'}$. □

Assume that f satisfies (F') and (A'). For each $i = 1, \ldots, n$ we fix an exponent $q_i \in [1, \infty)$ and a (nonempty) linear subspace

$$L^{q_i}_{\Sigma_i} \subset L^{q_i}(\Omega),$$

which we interpret as the admissible parameter space for the i-th component; it is endowed with the subspace norm topology inherited from $L^{q_i}(\Omega)$. Then, for any fixed parameters $w_i \in L^{q_i}_{\Sigma_i}$, $i = 1, \ldots, n$, the function f automatically fulfills the scalar assumptions (F) and (A) used in Lemma 3.1. Consequently, according to Lemma 3.1, problem (1.1) admits a unique weak solution u in X_0. In this situation, we say that the solution u corresponds to the parameters w_i for $i = 1, \ldots, n$.

Based on this fact and within the framework of (F') and (A'), we define the single-valued solution operator

$$S_f : \mathcal{W} := L^{q_1}_{\Sigma_1} \times L^{q_2}_{\Sigma_2} \times \cdots \times L^{q_n}_{\Sigma_n} \longrightarrow U,$$

which assigns to any parameter tuple $(w_1, \ldots, w_n) \in \mathcal{W}$ the unique weak solution $u \in U$ to problem (1.1). If, for each $i = 1, \ldots, n$, the space $L^{q_i}_{\Sigma_i}$ is endowed with the subspace topology induced by the norm topology of $L^{q_i}(\Omega)$, then the Cartesian product $\mathcal{W} = L^{q_1}_{\Sigma_1} \times \cdots \times L^{q_n}_{\Sigma_n}$ carries the corresponding product topology. Within this topological setting, one can formulate a lemma on the continuous dependence of the solution on the parameters.

Lemma 3.2. *Assume (F') and (A') with $a \in L^p(\Omega)$ and $\frac{N}{2s} \le p \le \infty$ such that $\|a_+\|_p < c^2_{2p'}$. Then the solution operator*

$$S_f : L^{q_1}_{\Sigma_1} \times L^{q_2}_{\Sigma_2} \times \cdots \times L^{q_n}_{\Sigma_n} \to X_0,$$

is continuous.

Proof. Assume (F') and (A') hold with $a \in L^p(\Omega)$ and $\|a_+\|_p < c^2_{2p'}$. Consider sequences $\{w^k_i\}_{k\in\mathbb{N}} \subset L^{q_i}(\Omega)$, such that $w^k_i \to w^0_i$ in $L^{q_i}(\Omega)$ for each $i = 1, \ldots, n$. Set

$$u_k := S_f(w^k_1, \ldots, w^k_n), \qquad u_0 := S_f(w^0_1, \ldots, w^0_n).$$

We will show $u_k \to u_0$ strongly in X_0, which implies continuity of S_f.

For each k, define the operator $T_k : X_0 \to X_0^*$ by

$$\langle T_k(u), v\rangle = (u, v) - \int_\Omega f(x, u, w^k_1, \ldots, w^k_n)\, v\, dx, \qquad \forall\, u, v \in X_0.$$

By the definition of u_k as the weak solution, we have $T_k(u_k) = 0$. Using the coercivity assumption (A'), one can show that for all $u, v \in X_0$,

$$\begin{aligned}\langle T_k(u) - T_k(v), u - v\rangle &= \|u - v\|^2 - \int_\Omega (f(x, u, w^k) - f(x, v, w^k))(u - v)\, dx \\ &\ge \left(1 - c^{-2}_{2p'}\|a_+\|_p\right)\|u - v\|^2 .\end{aligned}$$

Hence, T_k is strongly monotone with modulus

$$m := 1 - c^{-2}_{2p'}\|a_+\|_p > 0.$$

By the continuity assumption (F'), T_k is also hemicontinuous, and standard monotone-operator theory ensures the uniqueness (and existence) of solution u_k.

Next, we derive a uniform bound for $\|u_k\|$. Testing the equation $T_k(u_k) = 0$ with $v = 0$ in the monotonicity inequality yields

$$m\,\|u_k\|^2 \;\le\; \langle T_k(u_k) - T_k(0),\, u_k\rangle \;=\; -\langle T_k(0),\, u_k\rangle \;=\; \int_\Omega f(x, 0, w_1^k, \ldots, w_n^k)\, u_k\, dx.$$

Using (F') and Hölder's inequality (together with the compact embedding $X_0 \hookrightarrow L^r(\Omega)$ for some r arising in (F')), one shows that

$$\int_\Omega f(x, 0, w_1^k, \ldots, w_n^k)\, u_k\, dx \;\le\; C\,\|u_k\|_{L^r(\Omega)} \;\le\; C\,\|u_k\|,$$

where the constant $C > 0$ depends on the norms $\|w_i^k\|_{q_i}$ but is independent of k (since $w_i^k \to w_i^0$). It follows that $\|u_k\| \le C/m$; hence, $\sup_k \|u_k\| < \infty$.

Because $\{u_k\}$ is bounded in the reflexive space X_0, by Banach–Alaoglu and compact embeddings, we can extract a subsequence (still denoted u_k) such that

$$u_k \rightharpoonup \tilde{u}_0 \quad \text{in } X_0, \qquad u_k \to \tilde{u}_0 \quad \text{in } L^r(\Omega), \qquad u_k \to \tilde{u}_0 \quad \text{a.e. in } \Omega,$$

for some $\tilde{u}_0 \in X_0$. Similarly, from $w_i^k \to w_i^0$ in $L^{q_i}(\Omega)$, we may assume (up to a further subsequence) that $w_i^k \to w_i^0$ a.e. in Ω, and there exist functions $h_1 \in L^r(\Omega)$ and $h_{i+1} \in L^{q_i}(\Omega)$ such that

$$|u_k(x)|,\ |\tilde{u}_0(x)| \;\le\; h_1(x), \qquad |w_i^k(x)|,\ |w_i^0(x)| \;\le\; h_{i+1}(x),$$

for a.e. $x \in \Omega$ and each $i = 1, \ldots, n$ (this is possible by standard diagonal arguments using the strong convergence in L^r and L^{q_i}).

Using the continuity and growth condition (F'), it follows by the dominated convergence theorem that for any fixed $v \in X_0$,

$$f(x, u_k(x), w_1^k(x), \ldots, w_n^k(x))\, v(x) \;\to\; f(x, \tilde{u}_0(x), w_1^0(x), \ldots, w_n^0(x))\, v(x) \quad \text{in } L^1(\Omega).$$

We can now pass to the limit in the weak formulation of the equation for u_k. Since u_k satisfies

$$(u_k, v) \;=\; \int_\Omega f(x, u_k, w_1^k, \ldots, w_n^k)\, v\, dx \qquad \forall v \in X_0,$$

letting $k \to \infty$ yields

$$(\tilde{u}_0, v) \;=\; \int_\Omega f(x, \tilde{u}_0, w_1^0, \ldots, w_n^0)\, v\, dx \qquad \forall v \in X_0.$$

That is, $\tilde{u}_0$ satisfies the weak equation with parameters w_i^0. By the uniqueness of solutions established in Lemma 3.1, it must be that $\tilde{u}_0 = u_0$. Hence, the whole sequence u_k converges weakly to u_0 and almost everywhere.

Finally, to show strong convergence, we compare the weak formulations for u_k and u_0. For each k, subtracting the equations

$$(u_k, v) - \int f(x, u_k, w^k) v = 0, \qquad \text{and} \qquad (u_0, v) - \int f(x, u_0, w^0) v = 0,$$

and choosing $v = u_k - u_0$ gives

$$\langle T_k(u_k) - T_k(u_0),\, u_k - u_0\rangle = \int_\Omega (f(x,u_0,w^0) - f(x,u_k,w^k))(u_k - u_0)\,dx.$$

By the strong monotonicity of T_k, the left-hand side is at least $m\|u_k - u_0\|^2$. The right-hand side tends to 0 as $k \to \infty$ because $f(x,u_0,w_i^k) \to f(x,u_0,w_i^0)$ in L^1 and $u_k \to u_0$ a.e. Thus,

$$m\,\|u_k - u_0\|^2 \;\le\; \int_\Omega (f(x,u_0,w^0) - f(x,u_k,w^k))(u_k - u_0)\,dx \;\longrightarrow\; 0,$$

as $k \to \infty$. We conclude $\|u_k - u_0\| \to 0$, i.e., $u_k \to u_0$ strongly in X_0. Therefore $S_f(w_1^k,\dots,w_n^k) \to S_f(w_1^0,\dots,w_n^0)$ in X_0, proving that S_f is continuous. □

We assume that the measurable kernel $K : \mathbb{R}^N\backslash\{0\} \to (0,\infty)$ satisfies

(K1) Symmetry: $K(y) = K(-y)$ for all $y \in \mathbb{R}^N\backslash\{0\}$.

(K2) Two-sided kernel bounds: there exist constants $0 < \lambda \le \Lambda$ such that

$$\lambda\,|y|^{-N-2s} \;\le\; K(y) \;\le\; \Lambda\,|y|^{-N-2s}, \qquad \forall\, y \in \mathbb{R}^N\backslash\{0\}.$$

Hence, the operator $-\mathcal{L}_K : X_0 \to X_0^*$ defined by $\langle -\mathcal{L}_K u, v\rangle = \mathcal{E}(u,v)$ is self-adjoint. If $K(y) = \kappa(|y|)$, then for any $R \in O(N)$, one has $\mathcal{E}(u \circ R, v \circ R) = \mathcal{E}(u,v)$ by a change of variables. As a consequence, if Ω and the data are G-invariant for some subgroup $G \subset O(N)$ and the weak solution is unique, then $u \circ g = u$ for all $g \in G$.

4. Major theorems

In this section, we strengthen the qualitative continuity of S_f by deriving a global quantitative Lipschitz estimate with respect to the multiparameter vector $(w_1,\dots,w_n)$ and, under mild additional smoothness in the parameters, we prove that S_f is Gâteaux differentiable and identify the linearized problem.

Throughout, keep the standing assumptions (K_1)–(K_2), (F') and (A') with $a \in L^p(\Omega)$, $\frac{N}{2s} \le p \le \infty$, and $\|a_+\|_p < c_{2p'}^2$. Let $r \in (1, 2_s^*)$ be the exponent in (F') and let $c_r > 0$ be the embedding constant of $X_0 \to L^r(\Omega)$.

4.1. A global Lipschitz estimate with respect to parameters

We impose the following quantitative Lipschitz condition in the parameter variables.

(W) Local Lipschitz in the parameters. For every $R > 0$, there exists a nondecreasing function $\phi(R) \ge 0$ and measurable weights $\beta_i \in L^{s_i}(\Omega)$ such that, for a.e. $x \in \Omega$, all $t \in \mathbb{R}$ and all parameter vectors $w, \tilde{w} \in \prod_i L^{q_i}(\Omega)$ with $\|w\|_{\prod_i L^{q_i}} \vee \|\tilde{w}\|_{\prod_i L^{q_i}} \le R$, one has

$$\left|f(x,t,w) - f(x,t,\tilde{w})\right| \;\le\; \phi(R)\sum_{i=1}^n \beta_i(x)\,|w_i(x) - \tilde{w}_i(x)|.$$

In particular, the original global Lipschitz condition (W) corresponds to the special case $\phi(R) \equiv 1$.

Theorem 4.1. *Assume the standing hypotheses on K and f in Sections 2 and 3, including the one-sided Lipschitz condition in t, the Carathéodory regularity in* (x, t, w)*, and* (W *above with exponents* s_i, q_i, r *satisfying* $\frac{1}{s_i} + \frac{1}{q_i} + \frac{1}{r} = 1$. *Let* $S_f : \prod_i L^{q_i}(\Omega) \to X_0$ *denote the parameter-to-solution map. Fix* $R > 0$ *and take any* $w, \tilde{w}$ *with* $\|w\|_{\prod_i L^{q_i}} \vee \|\tilde{w}\|_{\prod_i L^{q_i}} \leq R$. *Then the weak solutions* $u = S_f(w)$ *and* $\tilde{u} = S_f(\tilde{w})$ *satisfy the* local *Lipschitz stability estimate*

$$\|u - \tilde{u}\| \leq \frac{\phi(R)\, c_r}{1 - c_{2p'}^{-2}\, \|a_+\|_{L^p(\Omega)}} \sum_{i=1}^{n} \|\beta_i\|_{L^{s_i}(\Omega)}\, \|w_i - \tilde{w}_i\|_{L^{q_i}(\Omega)}. \tag{4.1}$$

In particular, S_f *is Lipschitz on every bounded subset of* $\prod_i L^{q_i}(\Omega)$.

Proof. Let $w = (w_1, \dots, w_n)$ and $\tilde{w} = (\tilde{w}_1, \dots, \tilde{w}_n)$ be two admissible parameter tuples, and let

$$u = S_f(w), \qquad \tilde{u} = S_f(\tilde{w})$$

denote the corresponding weak solutions in X_0 given by Lemma 3.1. By definition of weak solution, for every $v \in X_0$, we have

$$\iint_{\mathbb{R}^{2N}} (u(x) - u(y))\,(v(x) - v(y))\,K(x-y)\,dx\,dy = \int_\Omega f(x, u(x), w(x))\,v(x)\,dx,$$

and

$$\iint_{\mathbb{R}^{2N}} (\tilde{u}(x) - \tilde{u}(y))\,(v(x) - v(y))\,K(x-y)\,dx\,dy = \int_\Omega f(x, \tilde{u}(x), \tilde{w}(x))\,v(x)\,dx.$$

Subtracting these two identities and choosing the test function

$$v := u - \tilde{u}$$

gives

$$\iint_{\mathbb{R}^{2N}} ((u - \tilde{u})(x) - (u - \tilde{u})(y))^2\,K(x-y)\,dx\,dy = \int_\Omega (f(x, u, w) - f(x, \tilde{u}, \tilde{w}))\,(u - \tilde{u})\,dx.$$

By the definition of the energy norm on X_0, the left-hand side is exactly $\|u - \tilde{u}\|^2$. Hence,

$$\|u - \tilde{u}\|^2 = \int_\Omega (f(x, u, w) - f(x, \tilde{u}, \tilde{w}))\,(u - \tilde{u})\,dx. \tag{4.2}$$

We split the difference of the nonlinear terms as

$$f(x, u, w) - f(x, \tilde{u}, \tilde{w}) = \underbrace{f(x, u, w) - f(x, \tilde{u}, w)}_{=:I_1(x)} + \underbrace{f(x, \tilde{u}, w) - f(x, \tilde{u}, \tilde{w})}_{=:I_2(x)}.$$

Inserting this into (4.2) yields

$$\|u - \tilde{u}\|^2 = \int_\Omega I_1(x)\,(u - \tilde{u})\,dx + \int_\Omega I_2(x)\,(u - \tilde{u})\,dx.$$

By the one-sided Lipschitz condition in t encoded in assumption (A'), together with the bound

$$|\partial_t f(x, t, w)| \leq a_+(x) \qquad \text{for a.e. } x \in \Omega,\ \forall\, t \in \mathbb{R},$$

it follows (e.g., by the mean value theorem in t) that for a.e. $x \in \Omega$,

$$\big(f(x,u(x),w(x)) - f(x,\tilde{u}(x),w(x))\big)\,(u(x)-\tilde{u}(x)) \;\le\; a_+(x)\,|u(x)-\tilde{u}(x)|^2.$$

Therefore,

$$\int_\Omega I_1(x)\,(u-\tilde{u})\,dx \;\le\; \int_\Omega a_+(x)\,|u-\tilde{u}|^2\,dx.$$

For the second term, we simply take absolute values:

$$\left|\int_\Omega I_2(x)\,(u-\tilde{u})\,dx\right| \;\le\; \int_\Omega \left|f(x,\tilde{u},w) - f(x,\tilde{u},\tilde{w})\right|\,|u-\tilde{u}|\,dx.$$

Combining the last two estimates with (4.2), we arrive at

$$\|u-\tilde{u}\|^2 \;\le\; \int_\Omega a_+(x)\,|u-\tilde{u}|^2\,dx \;+\; \int_\Omega \left|f(x,\tilde{u},w) - f(x,\tilde{u},\tilde{w})\right|\,|u-\tilde{u}|\,dx. \tag{4.3}$$

We now estimate the first term on the right-hand side of (4.3). Using Hölder's inequality with exponents p and p' (where $p' = \frac{p}{p-1}$ is the Hölder conjugate of p), we obtain

$$\int_\Omega a_+(x)\,|u-\tilde{u}|^2\,dx \;\le\; \|a_+\|_{L^p(\Omega)}\,\||u-\tilde{u}|^2\|_{L^{p'}(\Omega)}.$$

Since

$$\||u-\tilde{u}|^2\|_{L^{p'}(\Omega)} = \|u-\tilde{u}\|^2_{L^{2p'}(\Omega)},$$

we may use the continuous embedding $X_0 \hookrightarrow L^{2p'}(\Omega)$ with embedding constant $c_{2p'}$ (see (2.1)) to conclude that

$$\|u-\tilde{u}\|_{L^{2p'}(\Omega)} \;\le\; c_{2p'}^{-1}\,\|u-\tilde{u}\|.$$

Hence,

$$\int_\Omega a_+(x)\,|u-\tilde{u}|^2\,dx \;\le\; \|a_+\|_{L^p(\Omega)}\,c_{2p'}^{-2}\,\|u-\tilde{u}\|^2.$$

Substituting this bound into (4.3) gives

$$(1 - c_{2p'}^{-2}\|a_+\|_{L^p(\Omega)})\,\|u-\tilde{u}\|^2 \;\le\; \int_\Omega \left|f(x,\tilde{u},w) - f(x,\tilde{u},\tilde{w})\right|\,|u-\tilde{u}|\,dx. \tag{4.4}$$

By assumption (3.1) we have $1 - c_{2p'}^{-2}\|a_+\|_{L^p(\Omega)} > 0$.

We now estimate the right-hand side of (4.4). Using Hölder's inequality with the pair of exponents (r,r') (where $r' = \frac{r}{r-1}$ is the Hölder conjugate of r) yields

$$\int_\Omega \left|f(x,\tilde{u},w) - f(x,\tilde{u},\tilde{w})\right|\,|u-\tilde{u}|\,dx \;\le\; \left\|f(x,\tilde{u},w) - f(x,\tilde{u},\tilde{w})\right\|_{L^r(\Omega)}\,\|u-\tilde{u}\|_{L^{r'}(\Omega)}.$$

The continuous embedding $X_0 \hookrightarrow L^{r'}(\Omega)$ with embedding constant $c_{r'}$ implies

$$\|u-\tilde{u}\|_{L^{r'}(\Omega)} \le c_{r'}^{-1}\|u-\tilde{u}\|.$$

Therefore,

$$\int_\Omega \left| f(x,\tilde{u},w) - f(x,\tilde{u},\tilde{w}) \right| |u-\tilde{u}|\, dx \;\le\; c_{r'}^{-1} \left\| f(x,\tilde{u},w) - f(x,\tilde{u},\tilde{w}) \right\|_{L^r(\Omega)} \|u-\tilde{u}\|.$$

Substituting this estimate into (4.4) and assuming $u \neq \tilde{u}$ (otherwise, the desired inequality is trivial), we can divide both sides by $\|u-\tilde{u}\|$ to obtain

$$(1 - c_{2p'}^{-2} \|a_+\|_{L^p(\Omega)})\, \|u-\tilde{u}\| \;\le\; c_{r'}^{-1} \left\| f(x,\tilde{u},w) - f(x,\tilde{u},\tilde{w}) \right\|_{L^r(\Omega)}. \tag{4.5}$$

Next we apply assumption (W). Let $R > 0$ be such that the parameters w and $\tilde{w}$ belong to the same bounded set, say

$$\|w_i\|_{L^{q_i}(\Omega)} \le R, \qquad \|\tilde{w}_i\|_{L^{q_i}(\Omega)} \le R, \qquad i = 1, \dots, n.$$

By (W), there exists a nondecreasing function $\phi : [0,\infty) \to [0,\infty)$, such that

$$\left\| f(x,\tilde{u},w) - f(x,\tilde{u},\tilde{w}) \right\|_{L^r(\Omega)} \;\le\; \phi(R) \sum_{i=1}^n \|\beta_i\|_{L^{s_i}(\Omega)} \|w_i - \tilde{w}_i\|_{L^{q_i}(\Omega)}.$$

Inserting this bound into (4.5) gives

$$(1 - c_{2p'}^{-2} \|a_+\|_{L^p(\Omega)})\, \|u-\tilde{u}\| \;\le\; c_{r'}^{-1}\, \phi(R) \sum_{i=1}^n \|\beta_i\|_{L^{s_i}(\Omega)} \|w_i - \tilde{w}_i\|_{L^{q_i}(\Omega)}.$$

Rearranging the last inequality, we obtain

$$\|u-\tilde{u}\| \;\le\; \frac{c_{r'}^{-1}\, \phi(R)}{1 - c_{2p'}^{-2} \|a_+\|_{L^p(\Omega)}} \sum_{i=1}^n \|\beta_i\|_{L^{s_i}(\Omega)} \|w_i - \tilde{w}_i\|_{L^{q_i}(\Omega)}.$$

This is precisely the Lipschitz estimate (4.1) for the solution operator S_f on the bounded parameter set of radius R. Hence, S_f is Lipschitz continuous on bounded subsets of $\mathcal{W}$, with Lipschitz constant depending on R only through $\phi(R)$. □

Remark 2. *(Applicability beyond globally Lipschitz f) Condition* (W *accommodates many practical, possibly* nonsmooth in *w nonlinearities used in modeling (e.g., piecewise-affine responses, saturation/clipping, thresholds), provided they are locally Lipschitz in w uniformly in (x,t) on bounded parameter ranges. Note that* (W *imposes no spatial regularity on the coefficient functions $w_i(\cdot)$ (they may be discontinuous in x), and the weights $\beta_i \in L^{s_i}(\Omega)$ play the same role as in the global case. When f is globally Lipschitz in w, one recovers the original Theorem 4.1 with $\phi(R) \equiv 1$.*

4.2. Directional differentiability of the parameter-to-solution map

We next show that, under mild smoothness in (t,w), the map S_f is Gâteaux differentiable and its derivative is the unique solution of a strongly monotone linear nonlocal problem.

(D) There exist Carathéodory functions $\partial_t f, \partial_{w_i} f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that for x and all (t,w):

- $|\partial_t f(x,t,w)| \le a_+(x)$;

- $|\partial_{w_i} f(x,t,w)| \le \beta_i(x)$ with the same $\beta_i \in L^{s_i}(\Omega)$ as in *(W)*.

Theorem 4.2. *Assume* (F'), (A') *with* $\|a_+\|_p < c_{2p'}^2$, (W), *and* (D). *Fix* $w = (w_1,\dots,w_n) \in \prod_i L^{q_i}_{\Sigma_i}$ *and let* $u = S_f(w)$. *For any direction* $h = (h_1,\dots,h_n) \in \prod_i L^{q_i}(\Omega)$, *the directional derivative*

$$S_f'(w)[h] := \lim_{\tau\to 0} \frac{S_f(w+\tau h) - S_f(w)}{\tau} \in X_0$$

exists. Denoting $z = S_f'(w)[h]$, *z is the unique weak solution of the linear variational problem*

$$(z,v) - \int_\Omega \partial_t f(x,u(x),w(x))\, z\, v\, dx = \sum_{i=1}^n \int_\Omega \partial_{w_i} f(x,u(x),w(x))\, h_i\, v\, dx \qquad \forall v \in X_0, \tag{4.6}$$

and satisfies the a priori estimate

$$\|z\| \;\le\; \frac{c_r}{1 - c_{2p'}^{-2}\|a_+\|_p} \sum_{i=1}^n \|\partial_{w_i} f(\cdot,u,w)\|_{s_i}\, \|h_i\|_{q_i} \;\le\; \frac{c_r}{1 - c_{2p'}^{-2}\|a_+\|_p} \sum_{i=1}^n \|\beta_i\|_{s_i}\, \|h_i\|_{q_i}. \tag{4.7}$$

In particular, if we introduce the weighted parameter seminorm

$$\|h\|_{\mathcal{W}_\beta} := \sum_{i=1}^n \|\beta_i\|_{s_i}\, \|h_i\|_{q_i}, \tag{4.8}$$

then for any nonzero direction h, the associated unit vector of the directional derivative in the parameter space is given explicitly by

$$\widehat{h} := \frac{h}{\|h\|_{\mathcal{W}_\beta}} = \left(\frac{h_1}{\|h\|_{\mathcal{W}_\beta}},\dots,\frac{h_n}{\|h\|_{\mathcal{W}_\beta}}\right), \tag{4.9}$$

which satisfies $\|\widehat{h}\|_{\mathcal{W}_\beta} = 1$. *Moreover, inserting* (4.9) *into* (4.7) *yields the uniform bound*

$$\left\|S_f'(w)[\widehat{h}]\right\| \;\le\; \frac{c_r}{1 - c_{2p'}^{-2}\|a_+\|_p} \qquad \text{for every unit vector } \widehat{h} \text{ in } (\mathcal{W}_\beta, \|\cdot\|_{\mathcal{W}_\beta}). \tag{4.10}$$

Proof. Let $w \in \prod_i L^{q_i}(\Omega)$ be fixed and $u := S_f(w) \in X_0$ the unique weak solution. For a direction $h = (h_1,\dots,h_n) \in \prod_i L^{q_i}(\Omega)$ and $\tau \in \mathbb{R}$ set

$$u_\tau := S_f(w+\tau h), \qquad z_\tau := \frac{u_\tau - u}{\tau}.$$

Subtracting the weak formulations for u_τ and u, dividing by τ, and testing with $v \in X_0$ yield

$$(z_\tau, v) = \int_\Omega \frac{f(x,u_\tau,w+\tau h) - f(x,u,w)}{\tau}\, v\, dx \qquad \forall v \in X_0. \tag{4.11}$$

By the Carathéodory regularity in *(D)*, for a.e. $x \in \Omega$, there exist points $(\xi_\tau(x), \eta_\tau(x))$ on the segment joining $(u(x), w(x))$ and $(u_\tau(x), w(x)+\tau h(x))$, such that

$$\frac{f(x,u_\tau,w+\tau h) - f(x,u,w)}{\tau} = \partial_t f(x,\xi_\tau,\eta_\tau)\, z_\tau + \sum_{i=1}^n \partial_{w_i} f(x,\xi_\tau,\eta_\tau)\, h_i.$$

Substituting this identity into (4.11), we obtain the variational identity

$$(z_\tau, v) - \int_\Omega \partial_t f(x, \xi_\tau, \eta_\tau)\, z_\tau v\, dx = \sum_{i=1}^{n} \int_\Omega \partial_{w_i} f(x, \xi_\tau, \eta_\tau)\, h_i v\, dx \qquad \forall v \in X_0. \tag{4.12}$$

Uniform bound for z_τ. Choosing $v = z_\tau$ in (4.12) gives

$$\|z_\tau\|^2 - \int_\Omega \partial_t f(x, \xi_\tau, \eta_\tau)\, |z_\tau|^2\, dx = \sum_{i=1}^{n} \int_\Omega \partial_{w_i} f(x, \xi_\tau, \eta_\tau)\, h_i z_\tau\, dx. \tag{4.13}$$

Using $|\partial_t f| \le a_+(x)$ and $a_+ \ge 0$, we estimate the left-hand side from below:

$$\begin{aligned} \|z_\tau\|^2 - \int_\Omega \partial_t f(x, \xi_\tau, \eta_\tau)\, |z_\tau|^2\, dx &\ge \|z_\tau\|^2 - \int_\Omega |\partial_t f(x, \xi_\tau, \eta_\tau)|\, |z_\tau|^2\, dx \\ &\ge \|z_\tau\|^2 - \int_\Omega a_+(x)\, |z_\tau|^2\, dx. \end{aligned}$$

By Hölder's inequality with exponents (p, p') and the embedding $X_0 \to L^{2p'}(\Omega)$, we obtain

$$\begin{aligned} \int_\Omega a_+(x)\, |z_\tau|^2\, dx \le \|a_+\|_{L^p(\Omega)}\, \||z_\tau|^2\|_{L^{p'}(\Omega)} &= \|a_+\|_p\, \|z_\tau\|^2_{L^{2p'}(\Omega)} \\ \le \|a_+\|_p\, c_{2p'}^{-2}\, \|z_\tau\|^2, \end{aligned}$$

where we used $\|v\|_{L^{2p'}(\Omega)} \le c_{2p'}^{-1} \|v\|$ for all $v \in X_0$. Therefore,

$$\|z_\tau\|^2 - \int_\Omega a_+(x)\, |z_\tau|^2\, dx \;\ge\; (1 - c_{2p'}^{-2} \|a_+\|_p)\, \|z_\tau\|^2.$$

For the right-hand side of (4.13), we use $|\partial_{w_i} f| \le \beta_i$ and Hölder's inequality with exponents (s_i, q_i, r) (so that $1/s_i + 1/q_i + 1/r = 1$):

$$\begin{aligned} \left| \int_\Omega \partial_{w_i} f(x, \xi_\tau, \eta_\tau)\, h_i z_\tau\, dx \right| &\le \int_\Omega \beta_i(x)\, |h_i|\, |z_\tau|\, dx \\ &\le \|\beta_i\|_{s_i}\, \|h_i\|_{q_i}\, \|z_\tau\|_{L^r(\Omega)}. \end{aligned}$$

Summing over i and combining with the previous lower bound, we obtain

$$(1 - c_{2p'}^{-2} \|a_+\|_p)\, \|z_\tau\|^2 \;\le\; \sum_{i=1}^{n} \|\beta_i\|_{s_i}\, \|h_i\|_{q_i}\, \|z_\tau\|_{L^r(\Omega)}.$$

Using the continuous embedding $X_0 \hookrightarrow L^r(\Omega)$, there exists $c_r > 0$, such that $\|v\|_{L^r(\Omega)} \le c_r \|v\|$ for all $v \in X_0$. Hence,

$$(1 - c_{2p'}^{-2} \|a_+\|_p)\, \|z_\tau\|^2 \;\le\; c_r \left(\sum_{i=1}^{n} \|\beta_i\|_{s_i}\, \|h_i\|_{q_i} \right) \|z_\tau\|.$$

If $z_\tau \neq 0$, we divide both sides by $(1 - c_{2p'}^{-2} \|a_+\|_p)\, \|z_\tau\|$ (which is positive by the assumption $\|a_+\|_p < c_{2p'}^2$) to obtain

$$\|z_\tau\| \;\le\; \frac{c_r}{1 - c_{2p'}^{-2} \|a_+\|_p} \sum_{i=1}^{n} \|\beta_i\|_{s_i}\, \|h_i\|_{q_i}, \tag{4.14}$$

and the same estimate is trivially true when $z_\tau = 0$. Thus, $\{z_\tau\}$ is bounded in X_0.
Passage to the limit. Since $\{z_\tau\}$ is bounded in the Hilbert space X_0, there exists a subsequence (still denoted z_τ) and $z \in X_0$ such that $z_\tau \rightharpoonup z$ weakly in X_0. By Lemma 3.2 we know that $u_\tau \to u$ in X_0 as $\tau \to 0$, hence (up to a subsequence) $\xi_\tau \to u$ and $\eta_\tau \to w$ a.e. in Ω. Using the Carathéodory property of the partial derivatives and dominated convergence in (4.12), we can pass to the limit and obtain, for every $v \in X_0$,

$$(z, v) - \int_\Omega \partial_t f(x, u, w)\, zv\, dx = \sum_{i=1}^{n} \int_\Omega \partial_{w_i} f(x, u, w)\, h_i v\, dx,$$

that is, z is a weak solution of the linearized variational problem (4.6).

Finally, define the bilinear form and linear functional

$$A(u; z, v) := (z, v) - \int_\Omega \partial_t f(x, u, w)\, zv\, dx, \qquad R(u, w; h, v) := \sum_{i=1}^{n} \int_\Omega \partial_{w_i} f(x, u, w)\, h_i v\, dx.$$

The previous estimates show that $A(u; \cdot, \cdot)$ is coercive on X_0 with coercivity constant $1 - c_{2p'}^{-2} \|a_+\|_p > 0$, whereas $R(u, w; h, \cdot)$ is bounded. By the Lax–Milgram lemma, the linear problem (4.6) admits a unique solution $z \in X_0$, which coincides with the weak limit of z_τ, and satisfies the a priori bound (4.7). This concludes the proof. □

Remark 3. *The differentiability result identifies the directional derivative $S'_f(w)[h]$ as the solution of a linearized fractional PDE. This characterization provides exactly the sensitivity information needed in gradient-based optimization, data assimilation, and optimal control. In inverse problems, it enables the computation of Fréchet derivatives for use in Newton-type or adjoint-based reconstruction schemes.*

5. Numerical experiments

In this section, we mainly conduct numerical simulations on the major conclusions of the article, verifying the three core results of this paper: (i) The continuity of the solution operator S_f (Lemma (3.2)), (ii) the global Lipschitz estimate with respect to multiparameters (Theorem (4.1)), and (iii) the Gâteaux differentiability of S_f together with its linearized variational problem (Theorem (4.2)).

We work on $\Omega = (0, 1)^2$. Let $m \in \mathbb{N}$ and set $h = 1/(m + 1)$; the discrete nodal set has $M = m^2$ interior points. The nonlocal operator is assembled in the symmetric interaction form associated with a kernel $K(y) \asymp |y|^{-(N+2s)}$ ($s \in (0, 1)$), together with the homogeneous exterior condition $u \equiv 0$ on Ω^c. The discrete energy norm is

$$\|u_h\|_{E,h}^2 = \tfrac{1}{2}\, u_h^\top (-A_h) u_h,$$

and the discrete L^p norms $\|\cdot\|_{L_h^p}$ are computed with nodal weights.

For forcing, we take the affine choice.

$$f(x, t, w) = a(x)\, t + \eta(x) + \sum_{i=1}^{n} \beta_i(x)\, w_i(x), \tag{5.1}$$

which satisfies (F'), (A'), (W), and (D); here, $|\partial_t f| \le a_+(x)$ and $\partial_{w_i} f = \beta_i(x) \in L^{s_i}(\Omega)$. Throughout the experiments, we use $r = 2$ and $q_i = s_i = 4$, so that $1/s_i + 1/q_i + 1/r = 1$. The coefficients are smooth and bounded;for example, $a(x) = a_0 + \alpha_a \sin(2\pi x_1)\sin(2\pi x_2)$, $\beta_i(x) = b_{0,i} + \alpha_{b,i}\cos(2\pi x_1)\cos(2\pi x_2)$, and $\eta \equiv 0$, with amplitudes chosen so that $1 - c_{2p'}^{-2}\|a_+\|_{L^p} > 0$.

5.1. Continuity of S_f

Fix a baseline parameter tuple $w^{(0)} = (w_1^{(0)}, \ldots, w_n^{(0)})$ and a direction $d = (d_1, \ldots, d_n)$. For a geometric sequence $\varepsilon_k \to 0$ set $w^{(k)} = w^{(0)} + \varepsilon_k d$ and denote by $u_h^{(k)}$ the corresponding discrete solutions. Define

$$\Delta w_h^{(k)} := \sum_{i=1}^{n} \|\beta_i\|_{L_h^{s_i}} \|w_i^{(k)} - w_i^{(0)}\|_{L_h^{q_i}}, \qquad \Delta u_h^{(k)} := \|u_h^{(k)} - u_h^{(0)}\|_{E,h},$$

On a log–log plot of $\Delta u_h^{(k)}$ versus $\Delta w_h^{(k)}$ we observe an approximately linear trend with slope close to 1 as k increases and h decreases, which confirms the continuity of S_f under simultaneous perturbations of all parameters (see Figure 1).

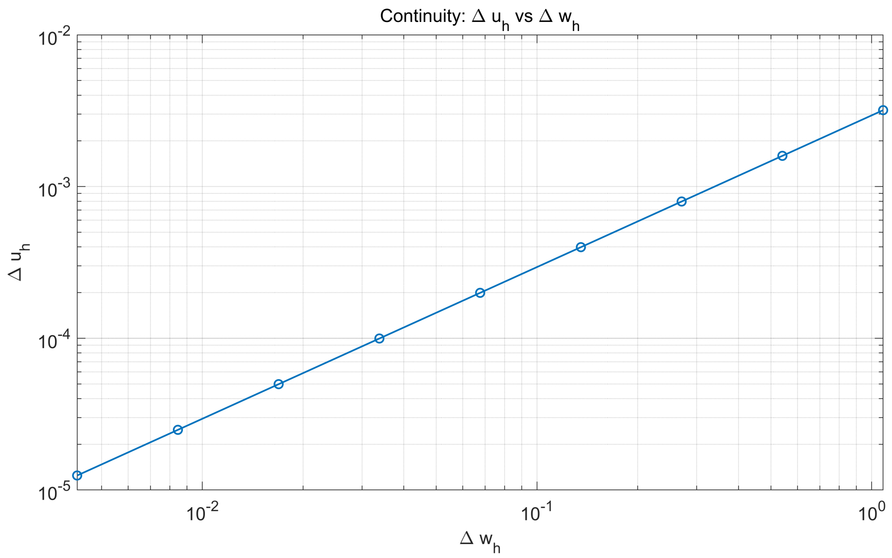

Figure 1. Continuity test: Δu_h versus Δw_h (log–log).

Figure 1 demonstrates the continuity of the solution operator S_f with respect to simultaneous perturbations of all parameters. As the total parameter perturbation Δw_h becomes small, the change in the discrete solution Δu_h scales linearly, with a slope close to one on a log–log plot. This empirical result supports the theoretical continuity result and shows that the multi-parameter system is robust to small changes in all coefficients at once.

5.2. Verification of global Lipschitz estimate

In this experiment, we use the same stationary fractional boundary value problem as in Example 1 and Section 5.1. For the numerical tests reported in Figure 2, we fix $\Omega = (0, 1)^2$ and fractional order $s = 0.5$, and consider uniform grids with $m \in \{16, 32, 64\}$ interior nodes per coordinate direction. Thus, the mesh size is $h = 1/(m + 1)$, and the number of interior nodes is $M = m^2$. Let $\{x_i\}_{i=1}^{M}$ denote the interior grid points. The nonlocal operator $-\mathcal{L}_K$ with kernel $K(y) = |y|^{-(N+2s)}$ $(N = 2)$ is discretized in the symmetric interaction form by midpoint quadrature, leading to a stiffness matrix $-A_h \in \mathbb{R}^{M\times M}$ with entries

$$(A_h)_{ij} = \begin{cases} \sum_{k\neq i} \omega_{ik}, & i = j, \\ -\omega_{ij}, & i \neq j, \end{cases} \qquad \omega_{ij} = h^2 K(x_i - x_j),$$

so that the discrete energy norm satisfies $\|u_h\|^2_{E,h} = \frac{1}{2} u_h^\top(-A_h)u_h$. The discrete L^2 inner product is implemented by the diagonal mass matrix $M_h = h^2 I_M$, i.e., $\|v_h\|^2_{L^2_h} = v_h^\top M_h v_h$. All linear systems are solved by the preconditioned conjugate gradient method with relative residual tolerance 10^{-10}.

Throughout Sections 5.2 and 5.3, we use the affine forcing (5.1) with $n = 3$ parameter fields and concrete smooth coefficients

$$\begin{aligned} a(x) &= a_0 + \alpha_a \sin(2\pi x_1)\sin(2\pi x_2), \\ \beta_i(x) &= b_{0,i} + \alpha_{b,i}\cos(2\pi x_1)\cos(2\pi x_2), \qquad i = 1, 2, 3, \\ \eta(x) &\equiv 0, \end{aligned}$$

where, for reproducibility, we fix

$$a_0 = 0.3, \;\; \alpha_a = 0.2, \;\; (b_{0,1}, \alpha_{b,1}) = (0.4, 0.3), \;\; (b_{0,2}, \alpha_{b,2}) = (0.25, 0.25), \;\; (b_{0,3}, \alpha_{b,3}) = (-0.2, 0.2).$$

These choices are compatible with the structural hypotheses (F’), (A’), and (W) and are kept fixed in all experiments shown in Figures 2 and 3. For any parameter tuple $w = (w_1, w_2, w_3)$ the discrete solution $u_h(w)$ is computed from the linear system

$$((-A_h) - M_h \operatorname{diag}(a))u_h = M_h\Big(\eta + \sum_{i=1}^{3} \beta_i \odot w_i\Big),$$

where $\odot$ denotes the pointwise product on the grid.

To test the global Lipschitz estimate, we fix a baseline parameter vector $w^{(0)} = (w_1^{(0)}, w_2^{(0)}, w_3^{(0)})$ given by

$$w_1^{(0)}(x) = \sin(\pi x_1)\sin(\pi x_2), \quad w_2^{(0)}(x) = \sin(2\pi x_1)\sin(\pi x_2), \quad w_3^{(0)}(x) = \sin(\pi x_1)\sin(2\pi x_2),$$

and choose a direction $h = (h_1, \ldots, h_3)$ of the same trigonometric form. On each grid, we normalize h with respect to the discrete norms so that

$$\sum_{i=1}^{3} \|\beta_i\|_{L_h^{s_i}} \|h_i\|_{L_h^{q_i}} = 1 \text{ with } q_i = s_i = 4, \; r = 2,$$

where the discrete norms are computed using the mass matrix M_h. For amplitudes $\delta \to 0$, we set $w^{(\delta)} = w^{(0)} + \delta h$ and compute the empirical ratio

$$L_h(\delta) := \frac{\|u_h^{(\delta)} - u_h^{(0)}\|_{E,h}}{\sum_{i=1}^{3} \|\beta_i\|_{L_h^{s_i}} \|w_i^{(\delta)} - w_i^{(0)}\|_{L_h^{q_i}}} = \frac{\|u_h^{(\delta)} - u_h^{(0)}\|_{E,h}}{\delta},$$

in accordance with the theoretical bound (4.1).

In addition, we evaluate the discrete upper bound

$$C_h := \frac{c_{r,h}}{1 - c_{2,h}^{-2} \|a_+\|_{L_h^\infty}} \qquad c_{2,h}^2 := \tfrac{1}{2}\,\lambda_{\min}(\,-A_h,\ M_h), \quad c_{r,h} := \inf_{u_h \neq 0} \frac{\|u_h\|_{E,h}}{\|u_h\|_{L_h^r}}, \quad r = 2,$$

where $\lambda_{\min}((-A_h), M_h)$ denotes the smallest generalized eigenvalue of $(-A_h)u = \lambda\, M_h u$. On all tested grids $m \in \{16, 32, 64\}$, the quantity $L_h(\delta)$ remains bounded as $\delta \to 0$ and lies strictly below the constant C_h. Here, M_h implements the discrete L^2 inner product. The ratio $L_h(\delta)$ represents the relative rate of change of the discrete solution under perturbations of all parameters, while C_h is the computable upper bound for the Lipschitz constant of the discrete system, used for comparison with the theoretical value in Theorem 4.1 (see Figure 2).

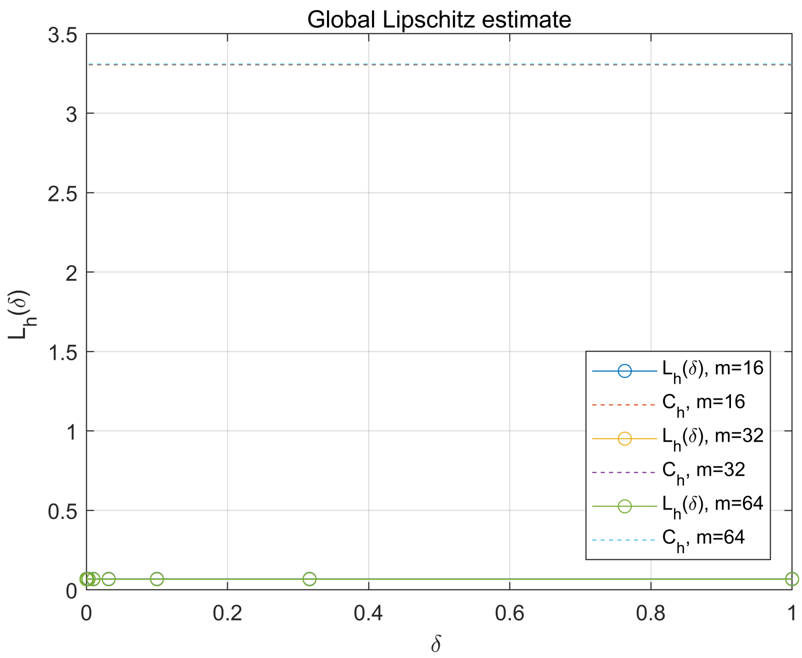

Figure 2. Global Lipschitz estimate on $\Omega = (0, 1)^2$ with $s = 0.5$: $L_h(\delta)$ vs. δ for $m \in \{16, 32, 64\}$, together with the discrete bound C_h. The coefficients use the concrete values $a_0 = 0.3$, $\alpha_a = 0.2$, $(b_{0,i}, \alpha_{b,i})$ as specified in Section 5.2.

Figure 2 presents a quantitative test of the global Lipschitz bound for the parameter-to-solution map. The observed Lipschitz ratio $L_h(\delta)$ remains uniformly bounded across grid sizes and parameter amplitudes, and always stays below the theoretically predicted discrete bound C_h. This directly confirms the global Lipschitz stability established in Theorem 4.1 and highlights the practical advantage of having a computable stability constant for sensitivity analysis and uncertainty quantification.

5.3. Verification of Gâteaux differentiability and linearization

We now verify the Gâteaux differentiability of the parameter-to-solution map and the correctness of the associated linearized problem. We keep the same spatial discretization, kernel $K(y) = |y|^{-(N+2s)}$ with $s = 0.5$, and the same concrete coefficients a, β_i and η as specified in Section 5.2, together with the domain $\Omega = (0,1)^2$ and the mass matrix $M_h = h^2 I_M$. For the directional derivative test in Figure 3, we work on the grid with $m = 64$ (so $h = 1/65$ and $M = 64^2$) and reuse the baseline parameters $w^{(0)}$ and direction h defined in the previous subsection.

Fix $w^{(0)}$ and a direction $h = (h_1, \ldots, h_n)$. For a geometric sequence $\tau \to 0$ (here, we take $\tau_k = 10^{-k}$, $k = 1, \ldots, 5$) consider the difference quotients

$$z_{\tau,h} := \frac{u_h(w^{(0)} + \tau h) - u_h(w^{(0)})}{\tau}.$$

Independently, solve the discrete linearized problem associated with (5.1):

$$((-A_h) - M_h \operatorname{diag}(a))\, z_h = M_h \sum_{i=1}^{n} \beta_i \odot h_i,$$

which is the discrete counterpart of the linearized fractional PDE (4.2). We then define the error

$$e_h(\tau) := \|z_{\tau,h} - z_h\|_{E,h}.$$

As $\tau \to 0$, the sequence $e_h(\tau)$ is expected to decay to zero and, on sufficiently fine grids, to level off at the linear solver tolerance, as predicted by the Gâteaux differentiability theory. This behavior is precisely what we observe in our computations (see Figure 3).

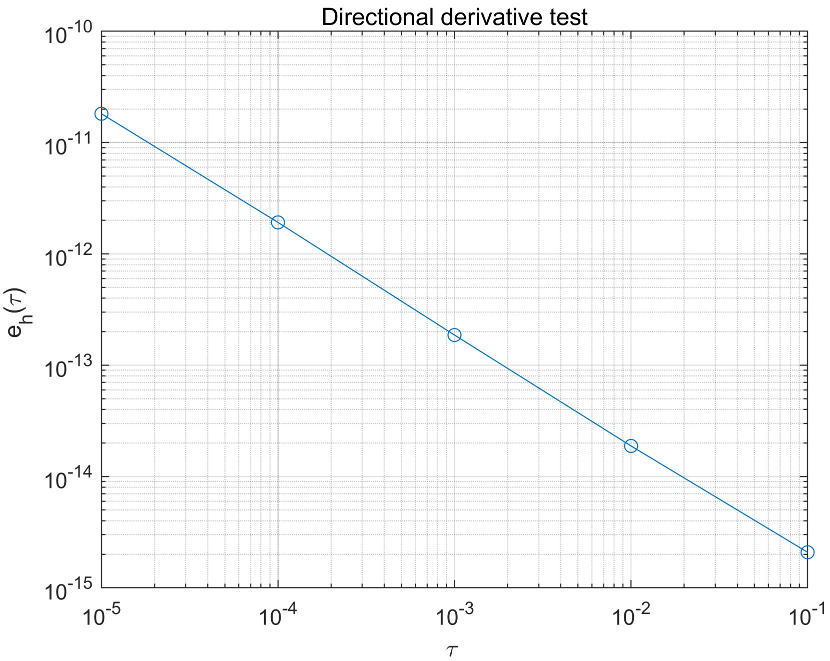

Figure 3. Directional derivative test on $\Omega = (0,1)^2$ with $s = 0.5$ and the coefficients from Section 5.2: Decay of $e_h(\tau) = \|z_{\tau,h} - z_h\|_{E,h}$ versus τ (log–log) on the grid $m = 64$.

Figure 3 shows the convergence of the difference quotient $z_{\tau,h}$ to the discrete linearized solution z_h as $\tau \to 0$. The error $e_h(\tau)$ decreases rapidly with smaller τ, and levels off at the linear solver tolerance, matching the linearization predicted by the Gâteaux differentiability theory (Theorem 4.2). This result demonstrates that the multi-parameter system can indeed be accurately linearized for small parameter changes, enabling efficient gradient-based methods and sensitivity computations in applications.

5.4. *Convergence study and comparisons*

To quantify the empirical convergence behavior, we record the error between consecutive uniform refinements in the energy norm and report the experimental order of convergence (EOC). For two successive levels $h_\ell > h_{\ell+1}$ with errors $E(h_\ell)$ and $E(h_{\ell+1})$, we set

$$\mathrm{EOC}(h_\ell) = \frac{\log(E(h_\ell)/E(h_{\ell+1}))}{\log(h_\ell/h_{\ell+1})}. \tag{5.2}$$

Under dyadic refinement ($h_{\ell+1} = h_\ell/2$) this reduces to $\mathrm{EOC}(h_\ell) = \log_2(E(h_\ell)/E(h_{\ell+1}))$. We compare the observed EOC side-by-side with the benchmark rate reported/predicted in [24] using the same error norm.

Comparison with single-parameter studies. To assess whether the claimed multi-parameter setting alters discretization behavior, we juxtapose our asymptotic trend with single-parameter studies under the same norm and comparable regularity. The researchers in [11, 17] focus on stability/continuous dependence in the single-parameter regime; consistent with standard finite-difference behavior, our asymptotic EOC in Table 1 shows no degradation relative to those single-parameter settings (first-order trend in the energy norm). This indicates that the scheme preserves the expected order even when multiple spatially varying coefficients are present simultaneously.

Table 1. Errors and experimental order of convergence (EOC) with side-by-side comparison to [24] (same energy norm).

Grid size h	$\|u_h - u_{h/2}\|_{E,h}$	EOC via (5.2)	Relative error $\frac{\|u_h - u_{\mathrm{exact}}\|_{E,h}}{\|u_{\mathrm{exact}}\|_{E,h}}$	Rate in [24]	Match?
1/16	1.77×10^{-2}	–	1.12×10^{-2}	$O(h)$	–
1/32	8.51×10^{-3}	1.06	5.41×10^{-3}	$O(h)$	✓
1/64	4.09×10^{-3}	1.06	2.67×10^{-3}	$O(h)$	✓
1/128	1.98×10^{-3}	1.05	1.33×10^{-3}	$O(h)$	✓

Discussion. On the finest levels, EOC stabilizes at approximately first order, which agrees with [24]. Minor deviations on coarse meshes are pre-asymptotic and do not affect the asymptotic trend. The single-parameter comparison confirms that multi-parameter coupling does not reduce the discretization order in the chosen norm.

6. Conclusions

In this paper, we develop a rigorous stability and sensitivity theory for fractional nonlocal boundary-value problems with multiple spatially varying parameters. Under mild Carathéodory growth and one-sided Lipschitz conditions, existence and uniqueness follow from variational monotonicity. Beyond qualitative continuity, we obtain a global Lipschitz estimate for the parameter-to-solution map and prove its Gâteaux differentiability together with the associated linearized problem; numerical experiments corroborate these results.

A key novelty of our work is that, instead of focusing on just a single parameter or constant, we handle multiple spatially varying coefficients at the same time, providing a much more detailed and practical analysis. Our results, such as the global Lipschitz estimate and the precise directional derivatives, are especially useful for applications like parameter identification, uncertainty analysis, and inverse problems, which are common in physics, engineering, and data science. Importantly, the methods we use remain compatible with the standard assumptions and can be easily connected to known literature.

Looking forward, this theory can be expanded in many directions: For example, by considering non-symmetric or anisotropic kernels, adding time-dependence or system dynamics, enabling even rougher data or more general nonlinearities, or including random effects. The results in this paper also set a strong foundation for developing new computational algorithms, especially those that need accurate sensitivity and stability information for modeling, optimization, and data-driven inverse problems.

Author contributions

Jia Zheng: Conceptualization, methodology, writing–original draft preparation; Xiuling Li: Conceptualization, writing–original draft preparation; Yanni Pang: Methodology, writing–review and editing; Hongying Wang: Methodology, writing–review and editing; Tongchao Wang: Writing–original draft preparation, writing–review and editing; Jiaxuan Sun: Conceptualization, methodology, writing–original draft preparation. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors gratefully acknowledge the financial support from Qianyi Technology (Changchun) Co., Ltd. (Grant Number RES0008506), which facilitated the research activities, data analysis, and experimental resources essential to the study.

Conflict of interest

The authors declare no conflict of interest.

References

1. A. Annadurai, V. Sureshkumar, D. Jaganathan, S. Dhanasekaran, Enhancing medical image quality using fractional order denoising integrated with transfer learning, *Fractal Fract.*, **8** (2024), 511. https://doi.org/10.3390/fractalfract8090511
2. Y. Sire, E. Valdinoci, Fractional Laplacian phase transitions with boundary reactions: A geometric inequality and a symmetry result, *J. Funct. Anal.*, **256** (2009), 1842–1864. https://doi.org/10.1016/j.jfa.2009.01.020

3. R. Cont, P. Tankov, *Financial modelling with jump processes*, Boca Raton: Chapman & Hall/CRC, 2004. https://doi.org/10.1201/9780203485217
4. R. F. Zhang, S. Bilige, Bilinear neural network method to obtain the exact analytical solutions of nonlinear partial differential equations and its application to p-gBKP equation, *Nonlinear Dynam.*, **95** (2019), 3041–3048. https://doi.org/10.1007/s11071-018-04739-z
5. W. Li, J. Sun, Y. Pang, On the study of solutions for a class of third-order semilinear nonhomogeneous delay differential equations, *Mathematics*, **13** (2025), 1926. https://www.mdpi.com/3351396
6. X. R. Xie, R. F. Zhang, Neural network-based symbolic calculation approach for solving the Korteweg–de Vries equation, *Chaos Soliton. Fract.*, **194** (2025), 116232. https://doi.org/10.1016/j.chaos.2025.116232
7. S. Liu, G. Dong, H. Bi, B. Wu, On the solutions to variable-order fractional p-Laplacian evolution equation with L^1-data, *Discrete Cont. Dyn.-B*, **30** (2025), 4832–4857. https://doi.org/10.3934/dcdsb.2025086
8. M. K. Hamdani, L. Mbarki, M. Allaoui, A new class of multiple nonlocal problems with two parameters and variable-order fractional $p(\cdot)$-Laplacian, *Commun. Anal. Mech.*, **15** (2023), 551–574. https://doi.org/10.3934/cam.2023027
9. S. Biagi, S. Dipierro, E. Valdinoci, E. Vecchi, Semilinear elliptic equations involving mixed local and nonlocal operators, *P. Roy. Soc. Edinb. A*, **151** (2021), 1611–1641. https://doi.org/10.1017/prm.2020.75
10. M. d. M. González, R. Monneau, Slow motion of particle systems as a limit of a reaction–diffusion equation with half-Laplacian in dimension one, *Discrete Cont. Dyn. Syst.*, **32** (2012), 1255–1286. https://doi.org/10.3934/dcds.2012.32.1255
11. F. Colasuonno, B. Noris, E. Sovrano, Continuous dependence for p-Laplace equations with varying operators, *Discrete Cont. Dyn.-S*, **18** (2025), 1561–1573. https://doi.org/10.3934/dcdss.2024121
12. I. M. Batiha, A. A. Abubaker, I. H. Jebril, S. B. Aish, High-performance adaptive step size schemes for nonlinear (variable-order) fractional differential equations, *Math. Comput. Simulat.*, **214** (2025), 43–62. https://doi.org/10.1016/j.matcom.2024.10.012
13. H. Gao, M. K. Ng, Uncertainty quantification of physics-informed neural networks using Wasserstein generative adversarial networks, *J. Comput. Phys.*, **468** (2022), 111510. https://doi.org/10.1016/j.jcp.2022.111510
14. X. Fu, M. Ng, Z. Zhang, Physics-informed kernel function neural networks for parametric partial differential equations, *J. Comput. Phys.*, **506** (2024), 112951. https://doi.org/10.1016/j.jcp.2024.112951
15. G. Rigas, C. G. Giannetti, D. Lucas, M. A. Scarselli, Adaptive training of physics-informed Kolmogorov–Arnold networks for multi-fidelity modeling, *J. Comput. Phys.*, **503** (2024), 112938. https://doi.org/10.1016/j.jcp.2024.112938
16. T. Tripura, S. Chakraborty, Wavelet neural operator for solving parametric partial differential equations in computational mechanics problems, *Comput. Method. Appl. M.*, **404** (2023), 115783. https://doi.org/10.1016/j.cma.2022.115783

17. R. Stegliński, Existence of a unique solution to a fractional partial differential equation and its continuous dependence on parameters, *Entropy*, **23** (2021), 851. https://doi.org/10.3390/e23070851

18. R. Courant, D. Hilbert, *Methods of mathematical physics: Partial differential equations*, New York: Interscience, **2** (1962). https://doi.org/10.1002/9783527617234

19. A. Kirsch, *An introduction to the mathematical theory of inverse problems*, 2 Eds., New York: Springer, 2011. https://doi.org/10.1007/978-1-4419-8474-6

20. M. Galewski, *Basic monotonicity methods with some applications*, Compact Textbooks in Mathematics, Birkhäuser/Springer, Cham, 2021. https://doi.org/10.1007/978-3-030-75308-5

21. E. Zeidler, *Nonlinear functional analysis and its applications II/A: Linear monotone operators*, New York: Springer, 1990. https://doi.org/10.1007/978-1-4612-0985-0

22. E. Zeidler, *Nonlinear functional analysis and its applications II/B: Nonlinear monotone operators*, New York: Springer, 1990. https://doi.org/10.1007/978-1-4612-0981-2

23. D. Motreanu, V. V. Motreanu, N. S. Papageorgiou, *Topological and variational methods with applications to nonlinear boundary value problems*, New York: Springer, 2014. https://doi.org/10.1007/978-1-4614-9323-5

24. C. Bucur, E. Valdinoci, *Nonlocal diffusion and applications*, Lecture Notes of the Unione Matematica Italiana, Springer, Cham, **20** (2016). https://doi.org/10.1007/978-3-319-28739-3