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Research article

Local and global stability analysis of an SVEIAR model with age-structure

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Abstract: In this paper, we consider an efficient way to investigate infectious diseases by constructing and analyzing differential equations. Some infectious diseases such as COVID-19, AIDS, and hepatitis B, involve both symptomatic patients and asymptomatic carriers. These asymptomatic carriers are contagious. However, only a small amount of works consider asymptomatic carriers in their models. Other factors such as vaccination and ages of infection are also important to the spread of infectious diseases. Therefore, we incorporate these factors together into an SVEIR model that includes continuous ages of infection for both symptomatic patients and asymptomatic carriers in this work. The conditions for existence and local stability of disease-free and endemic steady states together with the basic reproduction number R_B are presented. Furthermore, the global stability of disease-free and endemic steady states is considered. Several examples by simulations are presented to demonstrate the obtained theoretical results. The importance of asymptomatic carriers in the infected population is also shown in simulations.

Keywords: SVEIAR epidemic model; infection ages; stability; Lyapunov functional
Mathematics Subject Classification: 35Q92, 37N25, 92D30

1. Introduction

In recent years, infectious diseases have greatly affected our lives. It is important to investigate how these diseases spread so that it can be controlled as quickly as possible. Modeling has been an effective way to investigate infectious diseases. Some infectious diseases have asymptomatic carriers, and these asymptomatic carriers are contagious. For example, a large number of asymptomatic carriers emerged during the COVID-19 pandemic [1]. These asymptomatic carriers had no obvious symptoms such as fever or cough, but their nucleic acid or antigen tests were positive. They unknowingly spread the virus to others when they were amongst others. It was reported that asymptomatic carriers participated in activities and led to the infection of many people. Take AIDS as another example.

Some people infected with the HIV virus have no obvious symptoms during the acute phase and the asymptomatic phase [2]. However, the virus exists in their bodies and can spread the virus to others through sexual contact, blood and other means. Moreover, some hepatitis B virus carriers have no symptoms of hepatitis and their liver functions are normal [3]. However, the hepatitis B virus is contained in their blood, and the virus can be transmitted to others during mother-to-child transmission, blood transmission, or sexual contact. Therefore, it is important to take asymptomatic carriers into consideration to investigate infectious diseases by building mathematical models. Recently, researchers have realized the importance of asymptomatic carriers, which was ignored before, but models with asymptomatic individuals are still a small part of infectious models and its dynamics is not fully understood. In the present paper, we will investigate and predict the spread of the infectious diseases mentioned above by considering asymptomatic carriers as well as symptomatic carriers in our model. How the symptomatic, the asymptomatic, and their recovered rates influence the spread of the disease will be investigated in this work.

In real-world scenarios, the age of infection and vaccination are incorporated into the models because they play crucial roles. In [6, 7, 9], they study the dynamical behavior of age-structured SIRS model, SEI model, and SEIR model, respectively; in [8, 10] they studied the age-structured population model with diffusion and imperfect vaccination; in [4, 11], the authors mainly study the global stability of the equilibria of the age-structured model; [5] is a monograph of age-structured population dynamics. Vaccination is one of the most powerful preventive measures against infectious diseases. It can effectively decrease the rate of susceptible individuals and alter the transmission dynamics within a population. By vaccinating a large group of the population, the spread of the disease can be effectively controlled and even prevented from reaching epidemic proportions. Different vaccination strategies, such as mass vaccination campaigns, targeted vaccination of specific age groups or high-risk populations, and booster doses, can have diverse impacts on the disease dynamics. The age of the infected is another important factor. The susceptibility, infectiousness, and the course of the disease can vary depending on the age of the infected individual. For example, younger individuals may have a different immune response compared to older ones, and the probability of developing severe symptoms or complications might be age-dependent. Moreover, the social mixing patterns and contact rates also change with age, which further influences the spread of the infection.

Some researches have focused on similar models with related factors. For instance, Rodrigues et al. investigated an SVEIR epidemic model with age-dependent vaccination, latency, and infection, and obtained the global stability of the equilibria [12]. Yang et al. built a dynamic model to show that giving priority to vaccinating different age groups can have different effects on controlling COVID-19 spread [13]. Wang et al. analyzed stability of an age-structured model for vector-bone disease [14]. Some works such as [12] considered dynamics of equations with age-structure.

Recently, several studies analyzed the spread of COVID-19 by building suitable mathematical models. In [24, 26, 29], they applied SIRD model, SVIRD model, and SEIVR model to study the COVID 19, respectively. In [25, 27, 28], they assessed different strategies on the impact on the COVID-19 by building different models. In [15], the following non-linear differential equations were analyzed:

$$\frac{dS_h}{dt} = \pi + \varphi V_h - \lambda S_h - (\mu + \theta) S_h,$$

$$
\begin{aligned}
\frac{dV_h}{dt} &= \theta S_h - (1-\sigma)\lambda V_h - (\mu+\varphi)V_h,\\
\frac{dE_h}{dt} &= \lambda S_h + (1-\sigma)\lambda V_h - (\mu+\omega)E_h,\\
\frac{dI_{hA}}{dt} &= (1-\epsilon)\omega E_h - (\mu+\alpha)I_{hA},\\
\frac{dI_h}{dt} &= \epsilon\omega E_h + (1-\kappa)\alpha I_{hA} - (\mu+\gamma_1+\gamma_2+\delta_1)I_h,\\
\frac{dJ_h}{dt} &= \gamma_1 I_h - (\mu+\gamma_3+\delta_2)J_h,\\
\frac{dR_h}{dt} &= \gamma_2 I_h + \alpha\kappa I_{hA} + \gamma_3 J_h - \mu R_h. \qquad (1.1)
\end{aligned}
$$

The whole population is denoted as $N_h(t)$. S_h, V_h, E_h, I_{hA}, I_h, J_h and R_h respectively represent the susceptible, vaccinated, exposed, asymptomatic, symptomatic, isolated, and recovered compartments. π denotes rate of individuals entering the susceptible compartment. σ is the efficacy of the COVID-19 vaccine, which is administered to the susceptible at the rate of θ and wanes at the rate φ. μ is the natural death rate. The force of infection is represented by $\lambda = \frac{\beta(\eta I_{hA}+I_h)}{N_h-J_h}$. β is the transmission rate. η represents a modification factor for asymptomatic population. ϵ is the proportion of the exposed persons becoming symptomatic. γ_1, γ_2, and γ_3 respectively denote rate of isolation of the symptomatic class, and the recovery rate of symptomatic and isolated individuals. Other parameters can be seen from [15, Table 2].

Inspired by [15], we extend the model in [15] by considering the age of the infected for symptomatic and asymptomatic individuals and taking a general incidence rate. We study the dynamical behavior of the SEIAR model with vaccination and age of infection for both symptomatic patients and asymptomatic patients as follows:

$$
\begin{aligned}
\dot{S}(t) =& -(\zeta+\phi)S - f_1(S)\int_0^\infty \beta_1(a)g_1(i(t,a))da\\
&- f_2(S)\int_0^\infty \beta_2(a)g_2(A(t,a))da, \qquad t\ge 0, \qquad (1.2a)\\
\dot{V}(t) =& B + \phi S - \zeta V - f_3(V)\int_0^\infty \beta_3(a)g_3(i(t,a))da\\
&- f_4(V)\int_0^\infty \beta_4(a)g_4(A(t,a))da, \qquad t\ge 0, \qquad (1.2b)\\
\dot{E}(t) =& f_1(S)\int_0^\infty \beta_1(a)g_1(i(t,a))da + f_2(S)\int_0^\infty \beta_2(a)g_2(A(t,a))da\\
&+ f_3(V)\int_0^\infty \beta_3(a)g_3(i(a,t))da + f_4(V)\int_0^\infty \beta_4(a)g_4(A(t,a))da\\
&- \zeta E - \omega E, \qquad t\ge 0, \qquad (1.2c)\\
\frac{\partial i(t,a)}{\partial t}&+\frac{\partial i(t,a)}{\partial a} = -(\zeta+\gamma_1(a)+\rho(a))i(t,a), \qquad t\ge 0,\ \ a\ge 0, \qquad (1.2d)\\
\frac{\partial A(t,a)}{\partial t}&+\frac{\partial A(t,a)}{\partial a} = -(\zeta+\gamma_2(a))A(t,a), \qquad t\ge 0,\ \ a\ge 0, \qquad (1.2e)
\end{aligned}
$$

$$\dot{R}(t) = \int_0^\infty \gamma_1(a)i(t,a)da + \int_0^\infty \gamma_2(a)A(t,a)da - (\zeta+\delta)R(t),\ t \geq 0, \tag{1.2f}$$

$$i(t,0) = \xi_1\omega E(t) + \xi_2\delta R(t), \qquad t \geq 0, \tag{1.2g}$$

$$A(t,0) = (1-\xi_1)\omega E(t) + (1-\xi_2)\delta R(t) \qquad t \geq 0. \tag{1.2h}$$

The initial conditions are

$$S(0) > 0,\ V(0) \geq 0,\ E(0) \geq 0,\ i(0,a) \in \mathbf{L}^1_+(0,\infty),$$
$$A(0,a) \in \mathbf{L}^1_+(0,\infty),\ R(0) \geq 0,$$

where $S(t)$, $V(t)$, $E(t)$, and $R(t)$ denote the numbers of susceptible, vaccinated, exposed, and recovered individuals at time t, and $i(t,a)$ and $A(t,a)$ are the density of infected symptomatic and asymptomatic individuals with infection age a at time t, respectively. Suppose all the newly born individuals and individuals moving from other places into this compartment are vaccinated. The parameter B is the influx rate of vaccinated individuals. The natural mortality of individuals in each compartment is expressed by ζ. As the transmission rate of infected individuals changes as time goes on, the transmission rate of susceptible individuals of infected symptomatic and asymptomatic individuals with age a are $\beta_1(a)$ and $\beta_2(a)$, respectively. The imperfect efficiency of vaccination may cause vaccinated persons to revert to the infected. Therefore we assume that vaccinated individuals are infected by infected symptomatic and asymptomatic individuals with age a at rate $\beta_3(a)$ and $\beta_4(a)$, respectively. Exposed individuals are transmitted into infected symptomatic and asymptomatic individuals at constant rates of $\xi_1\omega$ and $(1-\xi_1)\omega$, respectively. Infected symptomatic and asymptomatic individuals with age a recover at rates $\gamma_1(a)$ and $\gamma_2(a)$, respectively. Infected revert to recovered at a constant rate δ in which there are symptomatic individuals with proportion ξ_2 and asymptomatic individuals with ratio $1-\xi_2$. The mortality caused by the disease for infected symptomatic individuals is represented by $\rho(a)$. Susceptible individuals are vaccinated at a rate of ϕ. The functional space of model (1.2) is

$$X^+ = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbf{L}^1_+(0,\infty) \times \mathbf{L}^1_+(0,\infty) \times \mathbb{R}^+.$$

Generally, transmission rates, recovery rates, and mortality rates are positive and less than 1, and their rates of change are not too large for different age groups. Therefore, we make the following assumptions on the functions $\beta_i(\cdot), \gamma_j(\cdot)$, and $\rho(\cdot), i \in \{1,2,3,4\}, j \in \{1,2\}$.

Assumption 1.1. The functions $\beta_i(\cdot), \gamma_j(\cdot)$, and $\rho(\cdot), i \in \{1,2,3,4\}, j \in \{1,2\}$ satisfy the following properties:

(i) $\mid \beta_i(x) - \beta_i(y) \mid \leq L_{\beta_i} \mid x - y \mid$ for any x, y.

(ii) There exist constants $\hat{\beta}_i,\ \hat{\gamma}_j$, and $\hat{\rho}$ such that $0 \leq \beta_i(\cdot) \leq \hat{\beta}_i,\ 0 \leq \gamma_j(\cdot) \leq \hat{\gamma}_j,\ 0 \leq \rho(\cdot) \leq \hat{\rho}$.

Based on the following two facts, we can make the following assumption. 1) The number of susceptible individuals infected increases as more susceptible persons are exposed to an infected individual or more people are infected. 2) Since the number of people that one person is exposed to is limited, the number of susceptible (infected) persons are exposed to an infected (susceptible) individual increases slowly when the number of susceptible (infected) persons are large.

Assumption 1.2. Assume that for all $i \in \{1,2,3,4\}$,

(i) $f_i(x) \geq 0$, $g_i(x) \geq 0$ in which the equation holds if and only if $x = 0$.

(ii) $f_i(x)$ and $g_i(x)$ are increasing as x increases and $f_i^{'}(x)$ and $g_i^{'}(x)$ are decreasing as x increases.

Our model (1.2) is complicated and a generalization of some recent works [11,29–33] . Its analysis is difficult especially in local stability analysis and constructing suitable Lyapunov functionals. Our results will be useful for researchers to study related problems.

This paper is organized as follows. In Section 2, the dissipativeness and positivity of the solutions of the model are shown. In Section 3, the existence and local stability of the disease-free and endemic steady states and the basic reproduction number is considered. In Section 4, the global stability of equilibria will be presented. Examples by numerical simulations are given to verify the validity of the theoretical results in Section 5. Finally, in Section 6 a brief conclusion is given to summarize this work.

2. Preliminaries

Let $(S(t), V(t), E(t), i(t,a), A(t,a), R(t))$ be a solution of system (1.2) satisfying the initial condition (1.3). For convenience, denote

$$\psi_1(a) = e^{-\int_0^a(\zeta+\gamma_1(u)+\rho(u))du}, \tag{2.1a}$$

$$\psi_2(a) = e^{-\int_0^a(\zeta+\gamma_2(u))du}. \tag{2.1b}$$

Solving the fourth and fifth equations of system (1.2) by integrating it along the characteristic lines $t - a = const$, we have

$$i(t,a) = \begin{cases} (\xi_1\omega E(t-a) + \xi_2\delta R(t-a))\psi_1(a), & 0 \le a < t, \\ i(0, a-t)\frac{\psi_1(a)}{\psi_1(a-t)}, & 0 \le t \le a, \end{cases} \tag{2.2}$$

and

$$A(t,a) = \begin{cases} ((1-\xi_1)\omega E(t-a) + (1-\xi_2)\delta R(t-a))\psi_2(a), & 0 \le a < t, \\ A(0, a-t)\frac{\psi_2(a)}{\psi_2(a-t)}, & 0 \le t \le a. \end{cases} \tag{2.3}$$

2.1. Positivity and boundedness of solutions

From the first equation of system (1.2), $S(t)$ remains nonnegative for all $t \ge 0$ since $\dot{S}(t^*) = 0$ for all t^*, satisfying $S(t^*) = 0$. Similarly, $V(t) \ge 0$ for all $t \ge 0$ because $\dot{V}(t^*) > 0$ for all t^* satisfying $V(t^*) = 0$.

Denote t^{**} as the first time when $i(t,a)A(t,a)R(t)E(t) = 0$ holds true. It means that

$$i(t,a) > 0,\ A(t,a) > 0,\ R(t) > 0,\ E(t) > 0$$

for all $t < t^{**}$. From Eqs (2.2) and (2.3), $i(t^{**},a) > 0$ and $A(t^{**},a) > 0$ for all $a > 0$. Therefore, $R(t^{**}) = 0$ or $E(t^{**}) = 0$. By calculation of the sixth equation of system (1.2), $\dot{R}(t^*) \ge 0$ if $R(t^{**}) = 0$. Hence, $R(t)$ is nonnegative for all $t \ge 0$. Similarly, $E(t)$ is nonnegative for all $t \ge 0$.

Define

$$\Omega = \{(x,\ v,\ y,\ z,\ u,\ \psi) \in X^+ |\ \| (x,\ v,\ y,\ z,\ u,\ \psi) \| \le \max\{\frac{B}{\zeta},\ \| x_0 \|\}\}.$$

Using the standard theory in [22], system (1.2) with (1.3) has a unique nonnegative solution on $\mathbb{R}^+$. Define a continuous semiflow, denoting $\Phi :\ \mathbb{R}^+ \times X^+ \to X^+$ as

$$\Phi(t,\ x_0) = (S(t),\ V(t),\ E(t),\ i(\cdot,\ t),\ A(\cdot,\ t),\ R(t)),\ t \in \mathbb{R}^+,\ x_0 \in X^+.$$

Then, we have

$$\begin{aligned}
& \| \Phi(t,\ x_0) \| \\
=& \| (S, V, E, i, A, R) \| \\
=& S(t) + V(t) + E(t) + \int_0^\infty i(t,a)da + \int_0^\infty A(t,a)da + R(t).
\end{aligned} \tag{2.4}$$

Proposition 2.1. *For system (1.2), the following statements hold true:*
(a) Ω is positively invariant for Φ, i.e., $\Phi(t,\ x_0) \in \Omega$ for all $t \geq 0,\ x_0 \in \Omega$;
(b) Φ is point dissipative and Ω attracts all points in X^+.

Proof. From Eq (2.4), we have

$$\begin{aligned}
& \frac{d}{dt} \| \Phi(t,\ x_0) \| \\
=& \frac{dS(t)}{dt} + \frac{dV(t)}{dt} + \frac{dE(t)}{dt} + \frac{d}{dt}\int_0^\infty i(t,a)da \\
& + \frac{d}{dt}\int_0^\infty A(t,a)da + \frac{dR(t)}{dt} \\
=& \frac{dS(t)}{dt} + \frac{dV(t)}{dt} + \frac{dE(t)}{dt} + \int_0^\infty \frac{\partial i(t,a)}{\partial t}da \\
& + \int_0^\infty \frac{\partial A(t,a)}{\partial t}da + \frac{dR(t)}{dt}.
\end{aligned} \tag{2.5}$$

It follows from system (1.2) that

$$\begin{aligned}
& \frac{d}{dt} \| \Phi(t,\ x_0) \| \\
=& \frac{dS(t)}{dt} + \frac{dV(t)}{dt} + \frac{dE(t)}{dt} + \frac{dR(t)}{dt} \\
& + \int_0^\infty [-(\zeta + \gamma_1(a) + \rho(a))i(t,a) - \frac{\partial i(t,a)}{\partial a}]da \\
& + \int_0^\infty [-(\zeta + \gamma_2(a))A(t,a) - \frac{\partial A(t,a)}{\partial a}]da \\
=& \frac{dS(t)}{dt} + \frac{dV(t)}{dt} + \frac{dE(t)}{dt} + \frac{dR(t)}{dt} \\
& - \int_0^\infty (\zeta + \gamma_1(a) + \rho(a))i(t,a)da - i(t,a)|_0^\infty \\
& - \int_0^\infty (\zeta + \gamma_2(a))A(t,a)da - A(t,a)|_0^\infty \\
=& B - \zeta(S(t) + V(t) + E(t) + R(t)) - \omega E(t) - \int_0^\infty \rho(a)i(t,a)da \\
& - \int_0^\infty \zeta(i(t,a) + A(t,a))da - \delta R(t) - i(t,a)|_0^\infty - A(t,a)|_0^\infty.
\end{aligned} \tag{2.6}$$

Substituting the last two equations of system (1.2) into Eq (2.6), we have

$$\begin{aligned}
& \frac{d}{dt} \parallel \Phi(t,\ x_0) \parallel \\
= \ & B - \zeta S(t) - \zeta V(t) - \zeta E(t) - \zeta R(t) \\
& - \int_0^\infty \zeta(i(t,a) + A(t,a))da - \int_0^\infty \rho(a) i(t,a) da \\
\leq \ & B - \zeta \parallel \Phi(t,\ x_0) \parallel .
\end{aligned} \tag{2.7}$$

By the variation of constants formula, we have

$$\parallel \Phi(t,\ x_0) \parallel \leq \frac{B}{\zeta} - e^{-\zeta t}\left(\frac{B}{\zeta} - \parallel x_0 \parallel\right),$$

which implies

$$\parallel \Phi(t,\ x_0) \parallel \leq \max\{\frac{B}{\zeta},\ \parallel x_0 \parallel\}$$

for all $t \geq 0$. This completes the proof. □

From Proposition 2.1, we have the following results.

Proposition 2.2. *There exists a constant $M > \frac{B}{\zeta}$ such that*

$$\begin{aligned}
S(t) \leq M, \quad V(t) \leq M, \quad E(t) \leq M, \\
\int_0^\infty i(t,a)da \leq M, \quad \int_0^\infty A(t,a)da \leq M, \quad R(t) \leq M
\end{aligned} \tag{2.8}$$

hold true for all $t \geq 0$ if $x_0 \in X^+$ and $\parallel x_0 \parallel \leq M$.

Proposition 2.3. *Define a bounded set $C \subset X^+$. Then:*
(i) $\Phi_t(C)$ is bounded;
(ii) Φ_t is eventually bounded on C.

The asymptotic smoothness of the semi-flow Φ generated by system (1.2) is shown in the appendix.

3. Steady states and its local stability

3.1. Steady states

Clearly, system (1.2) always has a disease-free steady state $E_1 = (0,\ \frac{B}{\zeta},\ 0,\ 0,\ 0,\ 0)$. An endemic steady state $E_2 = (0, V^*, E^*, i^*(a), A^*(a), R^*)$ satisfies

$$\begin{aligned}
& f_3(V^*) \int_0^\infty \beta_3(a) g_3(i^*(a))da + f_4(V^*) \int_0^\infty \beta_4(a) g_4(A^*(a))da = B - \zeta V^*, \\
& f_3(V^*) \int_0^\infty \beta_3(a) g_3(i^*(a))da + f_4(V^*) \int_0^\infty \beta_4(a) g_4(A^*(a))da = \zeta E^* + \omega E^*, \\
& \frac{di^*(a)}{da} = -(\zeta + \gamma_1(a) + \rho(a)) i^*(a), \\
& \frac{dA^*(a)}{da} = -(\zeta + \gamma_2(a)) A^*(a),
\end{aligned}$$

$$\int_0^\infty \gamma_1(a)i^*(a)da + \int_0^\infty \gamma_2(a)A^*(a)da = (\zeta+\delta)R^*,$$
$$i^*(0) = \xi_1\omega E^* + \xi_2\delta R^*,$$
$$A^*(0) = (1-\xi_1)\omega E^* + (1-\xi_2)\delta R^*. \tag{3.1}$$

Solving the third equation and fourth equation of model (3.1) gives

$$i^*(a) = i^*(0)\psi_1(a), \tag{3.2a}$$
$$A^*(a) = A^*(0)\psi_2(a). \tag{3.2b}$$

Substituting Eq (3.2) into the fifth equation of model (3.1), together with sixth and seventh equations, gives

$$i^*(0) = \frac{\frac{(\zeta+\delta)\xi_1}{(\xi_2-\xi_1)\delta} + \int_0^\infty \gamma_2(a)\psi_2(a)da}{\frac{(\zeta+\delta)(1-\xi_1)}{(\xi_2-\xi_1)\delta} - \int_0^\infty \gamma_1(a)\psi_1(a)da} A^*(0) := QA^*(0), \tag{3.3a}$$
$$E^* = \frac{(1-\xi_2)Q-\xi_2}{\omega(\xi_1-\xi_2)}A^*(0), \tag{3.3b}$$
$$R^* = \frac{(1-\xi_1)Q-\xi_1}{\delta(\xi_2-\xi_1)}A^*(0). \tag{3.3c}$$

From the first two equations of models (3.1) and (3.3), it follows that

$$E^* = \frac{B-\zeta V^*}{\zeta+\omega}, \tag{3.4a}$$
$$A^*(0) = \frac{B-\zeta V^*}{\zeta+\omega}\frac{\omega(\xi_1-\xi_2)}{(1-\xi_2)Q-\xi_2}, \tag{3.4b}$$
$$i^*(0) = \frac{B-\zeta V^*}{\zeta+\omega}\frac{\omega(\xi_1-\xi_2)Q}{(1-\xi_2)Q-\xi_2}. \tag{3.4c}$$

Substituting Eqs (3.2) and (3.4) into the first equation of model (3.1), we have

$$f_3(V^*)\int_0^\infty \beta_3(a)g_3\Big(\frac{B-\zeta V^*}{\zeta+\omega}\frac{\omega(\xi_1-\xi_2)Q}{(1-\xi_2)Q-\xi_2}\psi_1(a)\Big)da + f_4(V^*)$$
$$\times\int_0^\infty \beta_4(a)g_4\Big(\frac{B-\zeta V^*}{\zeta+\omega}\frac{\omega(\xi_1-\xi_2)}{(1-\xi_2)Q-\xi_2}\psi_2(a)\Big)da = B-\zeta V^*. \tag{3.5}$$

Proposition 3.1. *System (1.2) has one unique endemic equilibrium E_2 if*

$$\Re_0 \equiv f_3\Big(\frac{B}{\zeta}\Big)\int_0^\infty \beta_3(a)\psi_1(a)da\frac{(\xi_1-\xi_2)Q\omega}{(1-\xi_2)Q-\xi_2}\frac{g_3'(0)}{\zeta+\omega}$$
$$+f_4\Big(\frac{B}{\zeta}\Big)\int_0^\infty \beta_4(a)\psi_2(a)da\frac{(\xi_1-\xi_2)\omega}{(1-\xi_2)Q-\xi_2}\frac{g_4'(0)}{\zeta+\omega} > 1. \tag{3.6}$$

Otherwise, system (1.2) has no endemic equilibrium.

Proof. From Eq (3.4), $\frac{i^*(0)}{A^*(0)} = Q$. System (1.2) has an endemic equilibrium only if $Q > 0$ and $\frac{(\xi_1-\xi_2)}{(1-\xi_2)Q-\xi_2} > 0$ because of the positivity of solutions. Since $0 < \xi_1,\ \xi_2 < 1$, we have

$$\begin{aligned}
& (1-\xi_2)\int_0^\infty \gamma_2(a)\psi_2(a)da + \xi_2\int_0^\infty \gamma_1(a)\psi_1(a)da \\
\leq\ & (1-\xi_2)\int_0^\infty \gamma_2(a)e^{-\int_0^a \gamma_2(u)du}da + \xi_2\int_0^\infty \gamma_1(a)e^{-\int_0^a \gamma_1(u)du}da \\
=\ & (1-\xi_2)(1-e^{-\int_0^\infty \gamma_2(u)du}) + \xi_2(1-e^{-\int_0^\infty \gamma_1(u)du}) \\
\leq\ & 1 \\
<\ & \frac{\zeta+\delta}{\delta}.
\end{aligned} \tag{3.7}$$

Similarly, we have

$$\frac{(\zeta+\delta)(1-\xi_1)}{(\xi_2-\xi_1)\delta} - \int_0^\infty \gamma_1(a)\psi_1(a)da > \frac{(\zeta+\delta)(1-\xi_1)}{(\xi_2-\xi_1)\delta} - 1 > 0, \tag{3.8}$$

if $\xi_2 > \xi_1$ and

$$\frac{(\zeta+\delta)\xi_1}{(\xi_2-\xi_1)\delta} + \int_0^\infty \gamma_2(a)\psi_2(a)da < \frac{(\zeta+\delta)\xi_1}{(\xi_2-\xi_1)\delta} + 1 < 0, \tag{3.9}$$

if $\xi_2 < \xi_1$. Following from Eqs (3.7)–(3.9), we get $Q > 0$ and

$$\frac{\xi_1-\xi_2}{(1-\xi_2)Q-\xi_2} = \frac{\xi_1-\xi_2}{\frac{(1-\xi_2)\int_0^\infty \gamma_2(a)\psi_2(a)da+\xi_2\int_0^\infty \gamma_1(a)\psi_1(a)da-\frac{\zeta+\delta}{\delta}}{\frac{(\zeta+\delta)(1-\xi_1)}{(\xi_2-\xi_1)\delta}-\int_0^\infty \gamma_1(a)\psi_1(a)da}} > 0, \tag{3.10}$$

for all $0 < \xi_1,\ \xi_2 < 1$. From Eq (3.5), if V^* exists, it should be a zero root of the function H in $(0, \frac{B}{\zeta})$, where

$$\begin{aligned}
H(V) = \quad & f_3(V)\int_0^\infty \beta_3(a)g_3(\frac{B-\zeta V}{\zeta+\omega}\frac{\omega(\xi_1-\xi_2)Q}{(1-\xi_2)Q-\xi_2}\psi_1(a))da - B + \zeta V \\
& + f_4(V)\int_0^\infty \beta_4(a)g_4(\frac{B-\zeta V}{\zeta+\omega}\frac{\omega(\xi_1-\xi_2)}{(1-\xi_2)Q-\xi_2}\psi_2(a))da.
\end{aligned} \tag{3.11}$$

For simplicity, denote $\mu = \frac{\omega(\xi_1-\xi_2)}{(1-\xi_2)Q-\xi_2}$. Clearly, we have

$$\begin{aligned}
& H^{'}(V) \\
=\ & \zeta + f_3^{'}(V)\int_0^\infty \beta_3(a)g_3(\frac{(B-\zeta V)Q\mu\psi_1(a)}{\zeta+\omega})da - \int_0^\infty \frac{f_3(V)\beta_3(a)\zeta Q\mu}{\zeta+\omega} \\
& \times\psi_1(a)g_3^{'}(\frac{(B-\zeta V)Q\mu\psi_1(a)}{\zeta+\omega})da + f_4^{'}(V)\int_0^\infty g_4(\frac{(B-\zeta V)\mu\psi_2(a)}{\zeta+\omega}) \\
& \times\beta_4(a)da - f_4(V)\int_0^\infty \frac{\beta_4(a)\zeta\mu\psi_2(a)}{\zeta+\omega}g_4^{'}(\frac{(B-\zeta V)\mu\psi_2(a)}{\zeta+\omega})da,
\end{aligned} \tag{3.12}$$

and

$$H^{''}(V)$$

$$
\begin{aligned}
&= f_3^{''}(V)\int_0^\infty \beta_3(a)g_3(\frac{(B-\zeta V)Q\mu\psi_1(a)}{\zeta+\omega})da - 2f_3^{'}(V)\int_0^\infty \beta_3(a)\\
&\quad\times g_3^{'}(\frac{(B-\zeta V)Q\mu\psi_1(a)}{\zeta+\omega})\frac{\zeta Q\mu\psi_1(a)}{\zeta+\omega}da + f_3(V)\int_0^\infty \beta_3(a)\\
&\quad\times g_3^{''}(\frac{(B-\zeta V)Q\mu\psi_1(a)}{\zeta+\omega})(\frac{\zeta Q\mu\psi_1(a)}{\zeta+\omega})^2 da + f_4^{''}(V)\int_0^\infty \beta_4(a)\\
&\quad\times g_4(\frac{(B-\zeta V)\mu\psi_2(a)}{\zeta+\omega})da - 2f_4^{'}(V)\int_0^\infty \beta_4(a)\frac{\zeta\mu\psi_2(a)}{\zeta+\omega}\\
&\quad\times g_4^{'}(\frac{(B-\zeta V)\mu\psi_2(a)}{\zeta+\omega})da + f_4(V)\int_0^\infty \beta_4(a)(\frac{\zeta\mu\psi_2(a)}{\zeta+\omega})^2\\
&\quad\times g_4^{''}(\frac{(B-\zeta V)\mu\psi_2(a)}{\zeta+\omega})da. \qquad (3.13)
\end{aligned}
$$

From Assumption 1.2, $H^{''}(V) < 0$ for $V \in (0, \frac{B}{\zeta})$. Therefore, $H^{'}(V)$ is decreasing as $V \in [0, \frac{B}{\zeta}]$ increases. From Eq (3.12), $H^{'}(0) > 0$. According to the sign of $H^{'}(\frac{B}{\zeta})$, there are two conditions as follows:

(i) If $H^{'}(\frac{B}{\zeta}) \geq 0$, $H^{'}(V) > 0$ hold for all $V \in (0, \frac{B}{\zeta})$, it follows that $H(V)$ is increasing as V increases in $[0, \frac{B}{\zeta}]$. Since $H(0) < 0$ and $H(\frac{B}{\zeta}) = 0$, there are no roots for $H(V) = 0$ in $(0, \frac{B}{\zeta})$. By direct computation of Eq (3.12),

$$
\begin{aligned}
H^{'}(\frac{B}{\zeta}) &= \zeta - f_3(\frac{B}{\zeta})\int_0^\infty \beta_3(a)g_3^{'}(0)\frac{\zeta}{\zeta+\omega}\frac{\omega(\xi_1-\xi_2)Q}{(1-\xi_2)Q-\xi_2}\psi_1(a)da\\
&\quad - f_4(\frac{B}{\zeta})\int_0^\infty \beta_4(a)g_4^{'}(0)\frac{\zeta}{\zeta+\omega}\frac{\omega(\xi_1-\xi_2)}{(1-\xi_2)Q-\xi_2}\psi_2(a)da\\
&= \zeta(1-\Re_0). \qquad (3.14)
\end{aligned}
$$

Therefore, system (1.2) has no endemic equilibrium if $\Re_0 \leq 1$.

(ii) If $H^{'}(\frac{B}{\zeta}) < 0$, there must exist a $V_0 \in (0, \frac{B}{\zeta})$ such that $H^{'}(V) > 0$ for all $V \in (0, V_0)$ and $H^{'}(V) < 0$ for all $V \in (V_0, \frac{B}{\zeta})$. Hence, $H(V)$ is increasing as V increases in $[0, V_0]$ and $H(V)$ is decreasing as V increases in $[V_0, \frac{B}{\zeta}]$. There must exist one unique zero of the function $H(V)$ in $(0, \frac{B}{\zeta})$ as $H(0) < 0$ and $H(\frac{B}{\zeta}) = 0$. Therefore, system (1.2) has one unique endemic equilibrium E_2 if $\Re_0 > 1$. □

The basic reproduction number for model (1.2) is defined as the mean number of infected persons produced by one infectious individual during its time of infectiousness in a completely susceptible population at the beginning of the disease. In system (1.2), infectious persons refer to people in classes i or A. Therefore the total number of infected individuals is calculated as $i(t,0) + A(t,0)$ which equals $\omega E(t) + \delta R(t)$, here t is very small. From system (1.2), the total number of exposed individuals $E(t)$ is computed as $\frac{f_3(\frac{B}{\zeta})\int_0^\infty \beta_3(a)g_3^{'}(0)\psi_1(a)i(t,0)da + f_4(\frac{B}{\zeta})\int_0^\infty \beta_4(a)g_4^{'}(0)\psi_2(a)A(t,0)da}{\zeta+\omega}$, where $\frac{1}{\zeta+\omega}$ is the period of exposedness. Similarly, the number of recovered persons $R(t)$ is $\frac{\int_0^\infty \gamma_1(a)\psi_1(a)i(t,0)da + \int_0^\infty \gamma_2(a)\psi_2(a)A(t,0)da}{\zeta+\delta}$, in which $\frac{1}{\zeta+\delta}$ is the period of recovery. Hence, the total number of infected persons is

$$
\omega\frac{f_3(\frac{B}{\zeta})\int_0^\infty \beta_3(a)g_3^{'}(0)\psi_1(a)i(t,0)da + f_4(\frac{B}{\zeta})\int_0^\infty \beta_4(a)g_4^{'}(0)\psi_2(a)A(t,0)da}{\zeta+\omega}
$$

$$+\delta\frac{\int_0^\infty \gamma_1(a)\psi_1(a)i(t,0)da + \int_0^\infty \gamma_2(a)\psi_2(a)A(t,0)da}{\zeta+\delta}. \tag{3.15}$$

The mean number of infected persons produced by one infectious individual is

$$\begin{aligned}&\frac{\omega f_3(\frac{B}{\zeta})\int_0^\infty \beta_3(a)g_3'(0)\psi_1(a)i(t,0)da + \omega f_4(\frac{B}{\zeta})\int_0^\infty \beta_4(a)g_4'(0)\psi_2(a)A(t,0)da}{(\zeta+\omega)(i(t,0)+A(t,0))}\\ &+\frac{\int_0^\infty \delta\gamma_1(a)\psi_1(a)i(t,0)da + \int_0^\infty \delta\gamma_2(a)\psi_2(a)A(t,0)da}{(\zeta+\delta)(i(t,0)+A(t,0))}.\end{aligned} \tag{3.16}$$

From the last two equations of system (1.2) and the above calculation of E and R, $i(t,0) = QA(t,0)$. Substituting it into the above equation, the basic reproduction number for model (1.2) is denoted as R_B, satisfying

$$\begin{aligned}R_B = \quad &\frac{f_3(\frac{B}{\zeta})\int_0^\infty \omega\beta_3(a)g_3'(0)\psi_1(a)Qda + f_4(\frac{B}{\zeta})\int_0^\infty \omega\beta_4(a)g_4'(0)\psi_2(a)da}{(\zeta+\omega)(Q+1)}\\ &+\frac{\int_0^\infty \delta\gamma_1(a)\psi_1(a)Qda + \int_0^\infty \gamma_2(a)\psi_2(a)da}{(\zeta+\delta)(Q+1)}.\end{aligned} \tag{3.17}$$

From direct computation, $R_B > 1$ equals $\Re_0 > 1$.

3.2. Local stability

We need to prove the following proposition to get the local stability of $E_1(0, \frac{B}{\zeta}, 0, 0, 0, 0)$ and E_2.

Proposition 3.2. *Define* $\bar{Q} = \frac{\frac{\lambda_0\xi_1}{(\xi_2-\xi_1)\delta} + \frac{(\zeta+\delta)\xi_1}{(\xi_2-\xi_1)\delta} + \int_0^\infty \gamma_2(a)\psi_2(a)e^{-\lambda_0 a}da}{\frac{\lambda_0(1-\xi_1)}{(\xi_2-\xi_1)\delta} + \frac{(\zeta+\delta)(1-\xi_1)}{(\xi_2-\xi_1)\delta} - \int_0^\infty \gamma_1(a)\psi_1(a)e^{-\lambda_0 a}da}$. $\bar{Q} > 0$ *and*

$$\begin{aligned}\left|\frac{(\xi_1-\xi_2)\omega}{\bar{Q}(1-\xi_2)-\xi_2}\right| &\leq \frac{(\xi_1-\xi_2)\omega}{Q(1-\xi_2)-\xi_2},\\ \left|\frac{(\xi_1-\xi_2)\omega\bar{Q}}{\bar{Q}(1-\xi_2)-\xi_2}\right| &\leq \frac{(\xi_1-\xi_2)Q\omega}{Q(1-\xi_2)-\xi_2}\end{aligned} \tag{3.18}$$

hold true, if $\lambda_0 = a_0 + b_0 i,\ a_0 > 0$.

Proof. Since $Q > 0$, by direct computation, we have $\bar{Q} > 0$ for each case of (i) $\xi_1 > \xi_2$ and (ii) $\xi_1 < \xi_2$. We will prove Eq (3.18) in the following two cases (i) $\xi_1 > \xi_2$ and (ii) $\xi_1 < \xi_2$.

(i) If $\xi_1 > \xi_2$, we have

$$\begin{aligned}&\left|\frac{(\xi_1-\xi_2)\omega}{\bar{Q}(1-\xi_2)-\xi_2}\right|\\ = \quad &\left|\frac{\frac{(1-\xi_1)(1-\xi_2)}{\xi_1-\xi_2}\int_0^\infty \gamma_2(a)\psi_2(a)e^{-\lambda_0 a}da + \frac{\xi_1(1-\xi_2)}{\xi_1-\xi_2}\int_0^\infty \gamma_1(a)\psi_1(a)e^{-\lambda_0 a}da}{\frac{\lambda_0+\zeta+\delta}{\delta} - \int_0^\infty((1-\xi_2)\gamma_2(a)\psi_2(a)+\xi_2\gamma_1(a)\psi_1(a))e^{-\lambda_0 a}da}\right.\\ &\left.+\frac{1-\xi_1}{\xi_1-\xi_2}\right|(\xi_1-\xi_2)\omega\end{aligned}$$

$$\leq \quad \frac{(\xi_1-\xi_2)\omega}{Q(1-\xi_2)-\xi_2}. \tag{3.19}$$

Similarly, we have

$$\begin{aligned}
& \mid \frac{(\xi_1-\xi_2)\omega\bar{Q}}{\bar{Q}(1-\xi_2)-\xi_2} \mid \\
= \quad & \mid \frac{\frac{(1-\xi_1)\xi_2}{\xi_1-\xi_2}\int_0^\infty \gamma_2(a)\psi_2(a)e^{-\lambda_0 a}da + \frac{\xi_1\xi_2}{\xi_1-\xi_2}\int_0^\infty \gamma_1(a)\psi_1(a)e^{-\lambda_0 a}da}{\frac{\lambda_0+\zeta+\delta}{\delta} - \int_0^\infty((1-\xi_2)\gamma_2(a)\psi_2(a)+\xi_2\gamma_1(a)\psi_1(a))e^{-\lambda_0 a}da} \\
& + \frac{\xi_1}{\xi_1-\xi_2} \mid (\xi_1-\xi_2)\omega \\
\leq \quad & \frac{(\xi_1-\xi_2)Q\omega}{Q(1-\xi_2)-\xi_2}.
\end{aligned} \tag{3.20}$$

(ii) If $\xi_1 < \xi_2$,

$$\begin{aligned}
& \mid \frac{(\xi_1-\xi_2)\omega}{\bar{Q}(1-\xi_2)-\xi_2} \mid \\
= \quad & \mid \frac{\frac{(1-\xi_1)(1-\xi_2)}{\xi_2-\xi_1}\int_0^\infty \gamma_2(a)\psi_2(a)e^{-\lambda_0 a}da + \frac{\xi_1(1-\xi_2)}{\xi_2-\xi_1}\int_0^\infty \gamma_1(a)\psi_1(a)e^{-\lambda_0 a}da}{\frac{\lambda_0}{\delta}+\frac{\zeta+\delta}{\delta} - (1-\xi_2)\int_0^\infty \gamma_2(a)\psi_2(a)e^{-\lambda_0 a}da - \xi_2\int_0^\infty \gamma_1(a)\psi_1(a)e^{-\lambda_0 a}da} \\
& + \frac{1-\xi_1}{\xi_2-\xi_1} \mid (\xi_2-\xi_1)\omega \\
\leq \quad & (\xi_2-\xi_1)\omega(\frac{1-\xi_1}{\xi_2-\xi_1} \\
& + \frac{\frac{(1-\xi_1)(1-\xi_2)}{\xi_2-\xi_1}\int_0^\infty \gamma_2(a)\psi_2(a)da + \frac{\xi_1(1-\xi_2)}{\xi_2-\xi_1}\int_0^\infty \gamma_1(a)\psi_1(a)da}{\frac{\zeta+\delta}{\delta} - (1-\xi_2)\int_0^\infty \gamma_2(a)\psi_2(a)da - \xi_2\int_0^\infty \gamma_1(a)\psi_1(a)da}) \\
= \quad & \frac{(\xi_1-\xi_2)\omega}{Q(1-\xi_2)-\xi_2}.
\end{aligned} \tag{3.21}$$

Similarly,

$$\begin{aligned}
& \mid \frac{(\xi_1-\xi_2)\omega\bar{Q}}{\bar{Q}(1-\xi_2)-\xi_2} \mid \\
= \quad & \mid \frac{\frac{(1-\xi_1)\xi_2}{\xi_2-\xi_1}\int_0^\infty \gamma_2(a)\psi_2(a)e^{-\lambda_0 a}da + \frac{\xi_1\xi_2}{\xi_2-\xi_1}\int_0^\infty \gamma_1(a)\psi_1(a)e^{-\lambda_0 a}da}{\frac{\lambda_0}{\delta}+\frac{\zeta+\delta}{\delta} - (1-\xi_2)\int_0^\infty \gamma_2(a)\psi_2(a)e^{-\lambda_0 a}da - \xi_2\int_0^\infty \gamma_1(a)\psi_1(a)e^{-\lambda_0 a}da} \\
& + \frac{\xi_1}{\xi_2-\xi_1} \mid (\xi_2-\xi_1)\omega \\
\leq \quad & \frac{(\xi_1-\xi_2)Q\omega}{Q(1-\xi_2)-\xi_2}.
\end{aligned} \tag{3.22}$$

□

Theorem 3.1. *The disease free equilibrium point E_1 is unstable if $\Re_0 > 1$. E_1 is locally asymptotically stable if $\Re_0 < 1$.*

Proof. Linearizing system (1.2) at equilibrium E_1 and setting $S = y_1e^{\lambda t}$, $V = y_2e^{\lambda t} + \frac{B}{\zeta}$, $E = y_3e^{\lambda t}$, $i(t,a) = y_4(a)e^{\lambda t}$, $A(t,a) = y_5(a)e^{\lambda t}$ and $R = y_6e^{\lambda t}$, we have

$$\begin{aligned}&\frac{\omega\delta(\xi_1-\xi_2)}{(\lambda+\zeta+\omega)(\lambda+\zeta+\delta)}(f_4(\frac{B}{\zeta})g_4^{'}(0)\int_0^\infty \beta_4(a)\psi_2(a)e^{-\lambda a}da\int_0^\infty \gamma_1(a)\psi_1(a)\\ &\cdot e^{-\lambda a}da - f_3(\frac{B}{\zeta})g_3^{'}(0)\int_0^\infty \beta_3(a)\psi_1(a)e^{-\lambda a}da\int_0^\infty \gamma_2(a)\psi_2(a)e^{-\lambda a}da)\\ &+\frac{\omega}{\lambda+\zeta+\omega}((1-\xi_1)f_4(\frac{B}{\zeta})g_4^{'}(0)\int_0^\infty \beta_4(a)\psi_2(a)e^{-\lambda a}da+\xi_1 f_3(\frac{B}{\zeta})g_3^{'}(0)\\ &\cdot\int_0^\infty \beta_3(a)\psi_1(a)e^{-\lambda a}da)+\frac{\delta}{\lambda+\zeta+\delta}((1-\xi_2)\int_0^\infty \gamma_2(a)\psi_2(a)e^{-\lambda a}da\\ &+\xi_2\int_0^\infty \gamma_1(a)\psi_1(a)e^{-\lambda a}da) = 1.\end{aligned} \tag{3.23}$$

Define

$$\begin{aligned}F(\lambda) = \quad &\frac{\omega\delta(\xi_1-\xi_2)}{(\lambda+\zeta+\omega)(\lambda+\zeta+\delta)}(f_4(\frac{B}{\zeta})g_4^{'}(0)\int_0^\infty \beta_4(a)\psi_2(a)e^{-\lambda a}da\\ &\cdot\int_0^\infty \gamma_1(a)\psi_1(a)e^{-\lambda a}da - f_3(\frac{B}{\zeta})g_3^{'}(0)\int_0^\infty \beta_3(a)\psi_1(a)e^{-\lambda a}da\\ &\cdot\int_0^\infty \gamma_2(a)\psi_2(a)e^{-\lambda a}da)+\frac{\omega}{\lambda+\zeta+\omega}((1-\xi_1)f_4(\frac{B}{\zeta})g_4^{'}(0)\\ &\cdot\int_0^\infty \beta_4(a)\psi_2(a)e^{-\lambda a}da+\xi_1 f_3(\frac{B}{\zeta})g_3^{'}(0)\int_0^\infty \beta_3(a)\psi_1(a)e^{-\lambda a}da)\\ &+\frac{\delta}{\lambda+\zeta+\delta}((1-\xi_2)\int_0^\infty \gamma_2(a)\psi_2(a)e^{-\lambda a}da\\ &+\xi_2\int_0^\infty \gamma_1(a)\psi_1(a)e^{-\lambda a}da).\end{aligned}$$

Substituting $Q = \frac{\frac{(\zeta+\delta)\xi_1}{(\xi_2-\xi_1)\delta}+\int_0^\infty \gamma_2(a)\psi_2(a)da}{\frac{(\zeta+\delta)(1-\xi_1)}{(\xi_2-\xi_1)\delta}-\int_0^\infty \gamma_1(a)\psi_1(a)da}$ of (3.3) into Eq (3.6), we have

$$\begin{aligned}&\Re_0 - 1\\ = \quad &f_3(\frac{B}{\zeta})\int_0^\infty \beta_3(a)\psi_1(a)da\frac{(\xi_1-\xi_2)Q\omega}{(1-\xi_2)Q-\xi_2}\frac{g_3^{'}(0)}{\zeta+\omega}\\ &+f_4(\frac{B}{\zeta})\int_0^\infty \beta_4(a)\psi_2(a)da\frac{(\xi_1-\xi_2)\omega}{(1-\xi_2)Q-\xi_2}\frac{g_4^{'}(0)}{\zeta+\omega}-1\\ = \quad &\frac{\frac{\zeta+\delta}{\delta}(F(0)-1)}{\frac{\zeta+\delta}{\delta}-(1-\xi_2)\int_0^\infty \gamma_2(a)\psi_2(a)da-\xi_2\int_0^\infty \gamma_1(a)\psi_1(a)da} > 0.\end{aligned} \tag{3.24}$$

From $\Re_0 > 1$, it follows that $F(0) > 1$. Clearly, $\lim_{\lambda\to+\infty} F(\lambda) = 0$, therefore $F(\lambda) = 1$ has at least one positive root if $F(0) > 1$. So, the equilibrium E_1 is unstable if $\Re_0 > 1$.

Suppose E_1 is unstable when $\Re_0 < 1$. Therefore, there is at least one root $\lambda_1 = a_1 + ib_1$ of Eq (3.23) satisfying $a_1 \geq 0$. Let

$$\hat{Q} = \frac{\frac{\lambda_1\xi_1}{(\xi_2-\xi_1)\delta} + \frac{(\zeta+\delta)\xi_1}{(\xi_2-\xi_1)\delta} + \int_0^\infty \gamma_2(a)\psi_2(a)e^{-\lambda_1 a}da}{\frac{\lambda_1(1-\xi_1)}{(\xi_2-\xi_1)\delta} + \frac{(\zeta+\delta)(1-\xi_1)}{(\xi_2-\xi_1)\delta} - \int_0^\infty \gamma_1(a)\psi_1(a)e^{-\lambda_1 a}da}.$$

Equation (3.23) can be written as

$$\begin{aligned} & f_4(\frac{B}{\zeta})g_4^{'}(0)\int_0^\infty \beta_4(a)\psi_2(a)e^{-\lambda_1 a}da\frac{(\xi_1-\xi_2)\omega}{(1-\xi_2)\hat{Q}-\xi_2}\frac{1}{\lambda_1+\zeta+\omega} \\ & +f_3(\frac{B}{\zeta})\int_0^\infty \beta_3(a)\psi_1(a)e^{-\lambda a}da\frac{(\xi_1-\xi_2)\hat{Q}\omega}{(1-\xi_2)\hat{Q}-\xi_2}\frac{g_3^{'}(0)}{\lambda_1+\zeta+\omega} = 1. \end{aligned} \tag{3.25}$$

From Proposition 3.2 and Eq (3.25), we have

$$\begin{aligned} & |f_4(\frac{B}{\zeta})g_4^{'}(0)\int_0^\infty \beta_4(a)\psi_2(a)e^{-\lambda_1 a}da\frac{(\xi_1-\xi_2)\omega}{(1-\xi_2)\hat{Q}-\xi_2}\frac{1}{\lambda_1+\zeta+\omega} \\ & +f_3(\frac{B}{\zeta})\int_0^\infty \beta_3(a)\psi_1(a)e^{-\lambda_1 a}da\frac{(\xi_1-\xi_2)\hat{Q}\omega}{(1-\xi_2)\hat{Q}-\xi_2}\frac{g_3^{'}(0)}{\lambda_1+\zeta+\omega}| \\ & \leq f_3(\frac{B}{\zeta})\int_0^\infty \beta_3(a)\psi_1(a)da\frac{(\xi_1-\xi_2)Q\omega}{(1-\xi_2)Q-\xi_2}\frac{g_3^{'}(0)}{\zeta+\omega} \\ & +f_4(\frac{B}{\zeta})\int_0^\infty \beta_4(a)\psi_2(a)da\frac{(\xi_1-\xi_2)\omega}{(1-\xi_2)Q-\xi_2}\frac{g_4^{'}(0)}{\zeta+\omega} = \Re_0 < 1, \end{aligned} \tag{3.26}$$

which contradicts to Eq (3.25). Hence, the equilibrium E_1 is stable if $\Re_0 < 1$. □

Theorem 3.2. *E_2 is locally asymptotically stable if $\Re_0 > 1$ holds true.*

Proof. Set

$$\begin{aligned} & m_1e^{\lambda t} = S,\ \ m_2e^{\lambda t} = V - V^*,\ \ m_3e^{\lambda t} = E - E^*,\ \ m_6e^{\lambda t} = R - R^*, \\ & m_4(a)e^{\lambda t} = i(t,a) - i^*(a), \quad m_5(a)e^{\lambda t} = A(t,a) - A^*(a), \end{aligned} \tag{3.27}$$

and linearizing system (1.2) at equilibrium E_2, we have the following linear eigenvalue problem:

$$\begin{cases} (\lambda+\zeta+\phi+f_1^{'}(0)\int_0^\infty \beta_1(a)g_1(i^*(a))da + f_2^{'}(0)\int_0^\infty \beta_2(a)g_2(A^*(a))da)m_1 = 0, \\ (\lambda+\zeta+f_3^{'}(V^*)\int_0^\infty \beta_3(a)g_3(i^*(a))da + f_4^{'}(V^*)\int_0^\infty \beta_4(a)g_4(A^*(a))da)m_2 \\ +f_3(V^*)\int_0^\infty \beta_3(a)g_3^{'}(i^*(a))m_4(a)da + f_4(V^*)\int_0^\infty \beta_4(a)g_4^{'}(A^*(a))m_5(a)da = \phi m_1, \\ (f_1^{'}(0)\int_0^\infty \beta_1(a)g_1(i^*(a))da + f_2^{'}(0)\int_0^\infty \beta_2(a)g_2(A^*(a))da)m_1 \\ +(f_3^{'}(V^*)\int_0^\infty \beta_3(a)g_3(i^*(a))da + f_4^{'}(V^*)\int_0^\infty \beta_4(a)g_4(A^*(a))da)m_2 \\ +f_3(V^*)\int_0^\infty \beta_3(a)g_3^{'}(i^*(a))m_4(a)da + f_4(V^*)\int_0^\infty \beta_4(a)g_4^{'}(A^*(a))m_5(a)da \\ = (\zeta+\omega+\lambda)m_3, \\ m_4(a) = m_4(0)e^{-\int_0^a(\zeta+\gamma_1(s)+\rho_1(s)+\lambda)ds}, \\ m_5(a) = m_5(0)e^{-\int_0^a(\zeta+\gamma_2(s)+\lambda)ds}, \\ (\lambda+\zeta+\delta)m_6 = \int_0^\infty \gamma_1(a)m_4(a)da + \int_0^\infty \gamma_2(a)m_5(a)da, \\ m_4(0) = \xi_1\omega m_3 + \xi_2\delta m_6, \\ m_5(0) = (1-\xi_1)\omega m_3 + (1-\xi_2)\delta m_6. \end{cases} \tag{3.28}$$

Computation of Eq (3.28) yields the characteristic equation of system (1.2) at the equilibrium E_2,

$$P(\lambda) = 1, \tag{3.29}$$

where

$$\begin{aligned} & P(\lambda) \\ = & \frac{f_3(V^*)\int_0^\infty \beta_3(a)g_3'(i^*(a))\tilde{Q}\psi_1(a)e^{-\lambda a}da + f_4(V^*)\int_0^\infty \beta_4(a)g_4'(A^*(a))\psi_2(a)e^{-\lambda a}da}{\lambda+\zeta+f_3'(V^*)\int_0^\infty \beta_3(a)g_3(i^*(a))da + f_4'(V^*)\int_0^\infty \beta_4(a)g_4(A^*(a))da} \\ & \cdot\frac{\lambda+\zeta}{\lambda+\zeta+\omega}\frac{(\xi_1-\xi_2)\omega}{\tilde{Q}(1-\xi_2)-\xi_2}, \\ \tilde{Q} = & \frac{\frac{\lambda\xi_1}{(\xi_2-\xi_1)\delta}+\frac{(\zeta+\delta)\xi_1}{(\xi_2-\xi_1)\delta}+\int_0^\infty \gamma_2(a)\psi_2(a)e^{-\lambda a}da}{\frac{\lambda(1-\xi_1)}{(\xi_2-\xi_1)\delta}+\frac{(\zeta+\delta)(1-\xi_1)}{(\xi_2-\xi_1)\delta}-\int_0^\infty \gamma_1(a)\psi_1(a)e^{-\lambda a}da}. \end{aligned} \tag{3.30}$$

By the method of contradiction, we assume that Eq (3.29) has one eigenvalue $\lambda_2 = a_2 + b_2 i$ satisfying $a_2 \geq 0$. Then, we have

$$\begin{aligned} & \mid P(\lambda_2)\mid \\ \leq & \frac{\mid f_3(V^*)\int_0^\infty \beta_3(a)g_3'(i^*(a))\check{Q}\psi_1(a)da + f_4(V^*)\int_0^\infty \beta_4(a)g_4'(A^*(a))\psi_2(a)da\mid}{\mid a_2+b_2i+\zeta+\omega\mid} \\ & \cdot\frac{\mid a_2+b_2i+\zeta\mid}{\mid a_2+b_2i+\zeta+f_3'(V^*)\int_0^\infty \beta_3(a)g_3(i^*(a))da + f_4'(V^*)\int_0^\infty \beta_4(a)g_4(A^*(a))da\mid} \\ & \cdot\frac{\mid(\xi_1-\xi_2)\omega\mid}{\mid \check{Q}(1-\xi_2)-\xi_2\mid}, \end{aligned} \tag{3.31}$$

where $\check{Q} = \frac{\frac{\lambda_2\xi_1}{(\xi_2-\xi_1)\delta}+\frac{(\zeta+\delta)\xi_1}{(\xi_2-\xi_1)\delta}+\int_0^\infty \gamma_2(a)\psi_2(a)e^{-\lambda_2 a}da}{\frac{\lambda_2(1-\xi_1)}{(\xi_2-\xi_1)\delta}+\frac{(\zeta+\delta)(1-\xi_1)}{(\xi_2-\xi_1)\delta}-\int_0^\infty \gamma_1(a)\psi_1(a)e^{-\lambda_2 a}da}$. Clearly,

$$\frac{\mid a_2+b_2i+\zeta\mid}{\mid a_2+b_2i+\zeta+f_3'(V^*)\int_0^\infty \beta_3(a)g_3(i^*(a))da + f_4'(V^*)\int_0^\infty \beta_4(a)g_4(A^*(a))da\mid} < 1.$$

Therefore, Eq (3.31) can be rewritten as

$$\begin{aligned} & \mid P(\lambda_2)\mid \\ < & \mid \frac{f_3(V^*)\int_0^\infty \beta_3(a)g_3'(i^*(a))\check{Q}\psi_1(a)da}{\zeta+\omega} \\ & +\frac{f_4(V^*)\int_0^\infty \beta_4(a)g_4'(A^*(a))\psi_2(a)da}{\zeta+\omega}\mid \\ & \mid\frac{(\xi_1-\xi_2)\omega}{\check{Q}(1-\xi_2)-\xi_2}\mid \\ \leq & \frac{f_3(V^*)\int_0^\infty \beta_3(a)g_3'(i^*(a))\psi_1(a)da}{\zeta+\omega}\frac{\mid(\xi_1-\xi_2)\omega\check{Q}\mid}{\mid\check{Q}(1-\xi_2)-\xi_2\mid} \end{aligned}$$

$$+\frac{f_4(V^*)\int_0^\infty \beta_4(a)g_4'(A^*(a))\psi_2(a)da}{\zeta+\omega}\frac{|(\xi_1-\xi_2)\omega|}{|\check{Q}(1-\xi_2)-\xi_2|}. \tag{3.32}$$

From Eqs (3.19)–(3.22), Eq (3.32) can be rewritten as

$$\begin{aligned}&|P(\lambda_2)|\\ <\ &\frac{f_3(V^*)\int_0^\infty \beta_3(a)g_3'(i^*(a))\psi_1(a)da}{\zeta+\omega}\frac{(\xi_1-\xi_2)\omega Q}{Q(1-\xi_2)-\xi_2}\\ &+\frac{f_4(V^*)\int_0^\infty \beta_4(a)g_4'(A^*(a))\psi_2(a)da}{\zeta+\omega}\frac{(\xi_1-\xi_2)\omega}{Q(1-\xi_2)-\xi_2}.\end{aligned} \tag{3.33}$$

Substituting the first and second equations of model (3.3) into Eq (3.33), we have

$$\begin{aligned}&|P(\lambda_2)|\\ <\ &\frac{f_3(V^*)\int_0^\infty \beta_3(a)g_3'(i^*(a))\psi_1(a)da}{\zeta+\omega}\frac{i^*(0)}{E^*}\\ &+\frac{f_4(V^*)\int_0^\infty \beta_4(a)g_4'(A^*(a))\psi_2(a)da}{\zeta+\omega}\frac{A^*(0)}{E^*}.\end{aligned} \tag{3.34}$$

By Assumption 1.2,

$$g_3(i^*(a)) = g_3(i^*(a)) - g_3(0) \geq g_3'(i^*(a))i^*(a) \tag{3.35}$$

and

$$g_4(A^*(a)) = g_4(A^*(a)) - g_4(0) \geq g_4'(A^*(a))A^*(a). \tag{3.36}$$

Therefore,

$$\int_0^\infty \beta_3(a)g_3'(i^*(a))\psi_1(a)da \leq \frac{\int_0^\infty \beta_3(a)g_3(i^*(a))da}{i^*(0)} \tag{3.37}$$

and

$$\int_0^\infty \beta_4(a)g_4'(A^*(a))\psi_2(a)da \leq \frac{\int_0^\infty \beta_4(a)g_4(A^*(a))da}{A^*(0)}. \tag{3.38}$$

From Eqs (3.37) and (3.38) and the second equation of model (3.1), Eq (3.34) can be rewritten as

$$|P(\lambda_2)| < \frac{f_3(V^*)\int_0^\infty \beta_3(a)g_3(i^*(a))da + f_4(V^*)\int_0^\infty \beta_4(a)g_4(A^*(a))da}{(\zeta+\omega)E^*} = 1,$$

which is a contradiction to Eq (3.29). □

4. Global stability

Theorem 4.1. *E_1 is globally asymptotically stable if $\Re_0 < 1$.*

Proof. From model (1.2), we have

$$\frac{dS(t)}{dt} \leq -(\zeta+\phi)S(t). \tag{4.1}$$

By direct computation,

$$\limsup_{t\to+\infty} S(t) \le 0. \tag{4.2}$$

Hence, $S(t) \to 0$ when $t \to +\infty$. It follows from model (1.2) that

$$\frac{dV(t)}{dt} \le B + \phi S(t) - \zeta V(t). \tag{4.3}$$

Clearly,

$$\limsup_{t\to+\infty} V(t) \le \limsup_{t\to+\infty} \frac{B + \phi S(t)}{\zeta} \le \frac{B}{\zeta}. \tag{4.4}$$

From Assumption 1.2,

$$g_i(x) \le g_i^{'}(0)x, \qquad i = 1, 2, 3, 4.$$

From model (1.2) and Assumption 1.1, for any $\epsilon > 0$ as $t \to +\infty$, we have

$$\begin{aligned}
E(t) \le\ & \frac{1}{\zeta+\omega}\int_0^\infty (f_1(S(t))\beta_1(a)g_1(i(t,a)) + \beta_2(a)f_2(S(t))g_2(A(t,a)) \\
& + f_3(V(t))\beta_3(a)g_3(i(t,a)) + f_4(V(t))\beta_4(a)g_4(A(t,a)))da + \epsilon \\
\le\ & \frac{f_3(\frac{B}{\zeta})g_3^{'}(0)(\int_0^t \beta_3(a)i(t,a)da + \int_t^\infty \beta_3(a)i(t,a)da)}{\zeta+\omega} \\
& + \frac{f_4(\frac{B}{\zeta})g_4^{'}(0)(\int_0^t \beta_4(a)A(t,a)da + \int_t^\infty \beta_4(a)A(t,a)da)}{\zeta+\omega} + \epsilon \\
\le\ & \frac{f_3(\frac{B}{\zeta})g_3^{'}(0)\int_0^\infty \beta_3(a)\psi_1(a)da}{\zeta+\omega}(\xi_1\omega E(t) + \xi_2\delta\cdot R(t)) + \epsilon \\
& + \frac{f_4(\frac{B}{\zeta})g_4^{'}(0)\int_0^\infty \beta_4(a)\psi_2(a)da}{\zeta+\omega}((1-\xi_1)\cdot\omega E(t) + (1-\xi_2)\delta R(t)).
\end{aligned} \tag{4.5}$$

Similarly, by model (1.2), for any $\epsilon > 0$ when $t \to +\infty$, we have

$$\begin{aligned}
R(t)\ \le\ & \frac{\int_0^\infty (\xi_1\gamma_1(a)\psi_1(a) + (1-\xi_1)\gamma_2(a)\psi_2(a))da}{\zeta+\delta}\omega E(t) \\
& + \frac{\int_0^\infty (\xi_2\gamma_1(a)\psi_1(a) + (1-\xi_2)\gamma_2(a)\psi_2(a))da}{\zeta+\delta}\delta R(t) + \epsilon.
\end{aligned} \tag{4.6}$$

By direct computation, from Eqs (4.5) and (4.6) together with (3.24), when $t \to +\infty$,

$$\begin{aligned}
& E(t)R(t)(F(0)-1) \\
=\ & \frac{\frac{\zeta+\delta}{\delta} - (1-\xi_2)\int_0^\infty \gamma_2(a)\psi_2(a)da - \xi_2\int_0^\infty \gamma_1(a)\psi_1(a)da}{\frac{\zeta+\delta}{\delta}}\cdot E(t)R(t)(\Re_0 - 1) \\
\ge\ & 0.
\end{aligned} \tag{4.7}$$

As $\Re_0 < 1$, when $t \to +\infty$, the following equation can be derived from Eq (4.7):

$$E(t)R(t) \to 0, \tag{4.8}$$

which implies that

$$E(t) \to 0, \tag{4.9}$$

or

$$R(t) \to 0. \tag{4.10}$$

We prove that both Eqs (4.9) and (4.10) hold in the following.

(i) If Eq (4.9) holds true, Eq (4.6) can be written as

$$R(t)(1 - \frac{\int_0^\infty (\xi_2\gamma_1(a)\psi_1(a) + (1-\xi_2)\gamma_2(a)\psi_2(a))da}{\zeta + \delta}\delta) \leq \epsilon. \tag{4.11}$$

From Eqs (3.7) and (4.11), Eq (4.10) holds true.

(ii) If Eq (4.10) holds true, when $t \to +\infty$, Eq (4.5) can be written as

$$\begin{aligned} E(t) \leq \quad & E(t)(\frac{f_3(\frac{B}{\zeta})g_3^{'}(0)\omega\int_0^\infty \beta_3(a)\psi_1(a)da}{\zeta+\omega}\xi_1 \\ & + \frac{f_4(\frac{B}{\zeta})g_4^{'}(0)\omega\int_0^\infty \beta_4(a)\psi_2(a)da}{\zeta+\omega}(1-\xi_1)) + \epsilon. \end{aligned} \tag{4.12}$$

By direct computation, we have

$$\xi_1 < \frac{Q(\xi_1 - \xi_2)}{Q(1-\xi_2) - \xi_2} \tag{4.13}$$

and

$$1 - \xi_1 < \frac{\xi_1 - \xi_2}{Q(1-\xi_2) - \xi_2}. \tag{4.14}$$

By substituting Eqs (4.13) and (4.14) into Eq (4.12), we have

$$\begin{aligned} E(t) \leq \quad & E(t)(\frac{f_3(\frac{B}{\zeta})g_3^{'}(0)\omega\int_0^\infty \beta_3(a)\psi_1(a)da}{\zeta+\omega}\xi_1 \\ & + \frac{f_4(\frac{B}{\zeta})g_4^{'}(0)\omega\int_0^\infty \beta_4(a)\psi_2(a)da}{\zeta+\omega}(1-\xi_1)) + \epsilon \\ \leq \quad & E(t)R_0 + \epsilon. \end{aligned} \tag{4.15}$$

Since $\Re_0 < 1$, from Eq (4.15), Eq (4.9) holds true. Therefore, both Eqs (4.9) and (4.10) hold true. It follows from Eqs (2.2), (2.3), (4.9) and (4.10) that $i(t,a) \to 0$ and $A(t,a) \to 0$ when $t \to +\infty$. By Eq (4.2) and the second equation of (1.2), for any $\epsilon > 0$, when $t \to +\infty$, we obtain

$$\frac{dV(t)}{dt} \quad \geq \quad B + \phi S(t) - \zeta V(t) - f_3(\frac{B}{\zeta})\int_0^\infty g_3^{'}(0)i(t,a)\beta_3(a)da$$

$$
\begin{aligned}
& -f_4(\frac{B}{\zeta})\int_0^{\infty} A(t,a)\beta_4(a)g_4^{'}(0)da-\epsilon \\
\geq \quad & B+\phi S(t)-\zeta V(t)-f_3(\frac{B}{\zeta})g_3^{'}(0)\int_0^{\infty}\beta_3(a)\psi_1(a)da \\
& \times(\xi_1\omega E(t)+\xi_2\delta R(t))-f_4(\frac{B}{\zeta})g_4^{'}(0)\int_0^{\infty}\beta_4(a)\psi_2(a)da \\
& \times((1-\xi_1)\omega E(t)+(1-\xi_2)\delta R(t))-\epsilon. \qquad (4.16)
\end{aligned}
$$

From Eqs (4.9) and (4.10), Eq (4.16) can be rewritten as

$$
V(t) \quad \geq \quad \frac{B}{\zeta}. \qquad (4.17)
$$

Therefore, $\lim_{t\to+\infty} V(t) = \frac{B}{\zeta}$. From Theorem 3.1, E_1 is globally asymptotically stable from LaSalle's invariance principle. □

Assumption 4.1 Assume that

$$
\begin{aligned}
\frac{g_3(i(t,a))}{g_3(i^*(a))} \leq \frac{f_3(V(t))g_3(i(t,a))V^*}{f_3(V^*)g_3(i^*(a))V(t)} \leq 1, &\quad 0 < g_3(i(t,a)) \leq g_3(i^*(a)), \\
1 \leq \frac{f_3(V(t))g_3(i(t,a))V^*}{f_3(V^*)g_3(i^*(a))V(t)} \leq \frac{g_3(i(t,a))}{g_3(i^*(a))}, &\quad 0 < g_3(i^*(a)) \leq g_3(i(t,a)). \\
\frac{g_4(A(t,a))}{g_4(A^*(a))} \leq \frac{f_4(V(t))g_4(A(t,a))V^*}{f_4(V^*)g_4(A^*(a))V(t)} \leq 1, &\quad 0 < g_4(A(t,a)) \leq g_4(A^*(a)), \\
1 \leq \frac{f_4(V(t))g_4(A(t,a))V^*}{f_4(V^*)g_4(A^*(a))V(t)} \leq \frac{g_4(A(t,a))}{g_4(A^*(a))}, &\quad 0 < g_4(A^*(a)) \leq g_4(A(t,a)).
\end{aligned}
$$

Theorem 4.2. *The infectious equilibrium E_2 is globally asymptotically stable if $\Re_0 > 1$, Assumption 4.1, and $\frac{(1-\xi_1)(1-\xi_2)Q^2-\xi_1\xi_2}{\xi_1-\xi_2} > 0$ hold true.*

Proof. Define the function $M(x) = x - 1 - \ln x \geq 0$ for all $x > 0$. Define a Lyapunov function

$$
\begin{aligned}
L(t) = \quad & L_1(t) + L_2(t), \\
L_1(t) = \quad & C_1 S(t) + C_2 V^* M(\frac{V(t)}{V^*}) + C_3 E^* M(\frac{E(t)}{E^*}) + C_4 R^* M(\frac{R(t)}{R^*}), \\
L_2(t) = \quad & C_5\int_0^{\infty} i^*(a)F_1(a)M(\frac{i(t,a)}{i^*(a)})da + C_6\int_0^{\infty} A^*(a)F_2(a)M(\frac{A(t,a)}{A^*(a)})da \\
& +C_7\int_0^{\infty} i^*(a)F_3(a)M(\frac{i(t,a)}{i^*(a)})da + C_8\int_0^{\infty} A^*(a)F_4(a)M(\frac{A(t,a)}{A^*(a)})da, \\
F_1(a) = \quad & \int_a^{\infty}\frac{f_3(V^*)\beta_3(v)g_3(i^*(v)) + f_4(V^*)\beta_4(v)g_4(A^*(v))}{i^*(a)}dv, \\
F_2(a) = \quad & \int_a^{\infty}\frac{f_3(V^*)\beta_3(v)g_3(i^*(v)) + f_4(V^*)\beta_4(v)g_4(A^*(v))}{A^*(a)}dv, \\
F_3(a) = \quad & \int_a^{\infty}\frac{\gamma_1(v)i^*(v) + \gamma_2(v)A^*(v)}{i^*(a)}dv,
\end{aligned}
$$

$$F_4(a) = \int_a^\infty \frac{\gamma_1(v)i^*(v) + \gamma_2(v)A^*(v)}{A^*(a)} dv.$$

Calculating the time derivative of L_1, we have

$$\begin{aligned}
& L_1'(t) \\
\leq\ & (C_2 - C_1)\phi S + (C_3 - C_1) f_1(S(t)) \int_0^\infty \beta_1(a) g_1(i(t,a)) da \\
& +(C_3 - C_1) f_2(S(t)) \int_0^\infty \beta_2(a) g_2(A(t,a)) da - f_3(V^*) \\
& \cdot \int_0^\infty \beta_3(a) g_3(i^*(a)) (C_2 M(\frac{V^*}{V(t)}) + C_3 M(\frac{f_3(V(t)) g_3(i(t,a)) E^*}{f_3(V^*) g_3(i^*(a)) E(t)}) \\
& -C_2 M(\frac{f_3(V(t)) g_3(i(t,a)) V^*}{f_3(V^*) g_3(i^*(a)) V(t)}) + C_3 \ln \frac{E^*}{E(t)}) da - f_4(V^*) \\
& \cdot \int_0^\infty \beta_4(a) g_4(A^*(a)) (C_2 M(\frac{V^*}{V(t)}) + C_3 M(\frac{f_4(V(t)) g_4(A(t,a)) E^*}{f_4(V^*) g_4(A^*(a)) E(t)}) \\
& -C_2 M(\frac{f_4(V(t)) g_4(A(t,a)) V^*}{f_4(V^*) g_4(A^*(a)) V(t)}) + C_3 \ln \frac{E^*}{E(t)}) da \\
& +C_4 \int_0^\infty \gamma_1(a) i^*(a) (M(\frac{i(t,a)}{i^*(a)}) - M(\frac{i(t,a) R^*}{i^*(a) R(t)}) - \ln \frac{R^*}{R(t)}) da \\
& +C_4 \int_0^\infty \gamma_2(a) A^*(a) (M(\frac{A(t,a)}{A^*(a)}) - M(\frac{A(t,a) R^*}{A^*(a) R(t)}) - \ln \frac{R^*}{R(t)}) da \\
& -C_3(\zeta + \omega)(E(t) - E^*) - C_4(\zeta + \delta)(R(t) - R^*).
\end{aligned} \tag{4.18}$$

Since Assumption 4.1 and [7, Proposition A.1], $M(\frac{g_3(i(t,a))}{g_3(i^*(a))}) \leq M(\frac{i(t,a)}{i^*(a)})$, and $M(\frac{g_4(A(t,a))}{g_4(A^*(a))}) \leq M(\frac{A(t,a)}{A^*(a)})$, we have

$$M(\frac{f_3(V(t)) g_3(i(t,a)) V^*}{f_3(V^*) g_3(i^*(a)) V(t)}) \leq M(\frac{g_3(i(t,a))}{g_3(i^*(a))}) \leq M(\frac{i(t,a)}{i^*(a)}) \tag{4.19}$$

and

$$M(\frac{f_4(V(t)) g_4(A(t,a)) V^*}{f_4(V^*) g_4(A^*(a)) V(t)}) \leq M(\frac{g_4(A(t,a))}{g_4(A^*(a))}) \leq M(\frac{A(t,a)}{A^*(a)}). \tag{4.20}$$

Hence, $\frac{dL_1(t)}{dt}$ can be rewritten as

$$\begin{aligned}
& \frac{dL_1(t)}{dt} \\
\leq\ & (C_2 - C_1)\phi S + (C_3 - C_1) \int_0^\infty f_1(S(t)) \beta_1(a) g_1(i(t,a)) da + (C_3 \\
& -C_1) \int_0^\infty f_2(S(t)) \beta_2(a) g_2(A(t,a)) da + C_2 \int_0^\infty f_3(V^*) \beta_3(a) \\
& \cdot g_3(i^*(a)) M(\frac{i(t,a)}{i^*(a)}) da - C_3(\zeta + \omega)(E(t) - E^*) + C_3 \ln \frac{E(t)}{E^*}
\end{aligned}$$

$$
\begin{aligned}
&\cdot(\zeta+\omega)E^* + C_2 f_4(V^*)\int_0^\infty \beta_4(a) g_4(A^*(a)) M(\frac{A(t,a)}{A^*(a)})da\\
&+C_4\int_0^\infty \gamma_1(a) i^*(a) M(\frac{i(t,a)}{i^*(a)})da + C_4(\zeta+\delta)R^* \ln\frac{R(t)}{R^*}\\
&+C_4\int_0^\infty \gamma_2(a) A^*(a) M(\frac{A(t,a)}{A^*(a)})da - C_4(\zeta+\delta)(R(t)-R^*). \qquad (4.21)
\end{aligned}
$$

Direct computation gives

$$
\begin{aligned}
&\frac{d\int_0^\infty F_1(a) i^*(a) h(\frac{i(t,a)}{i^*(a)})}{dt}\\
= \quad & -\int_0^\infty F_1(a) i^*(a) \frac{\partial h(\frac{i(t,a)}{i^*(a)})}{\partial a} da\\
= \quad & -F_1(a) i^*(a) h(\frac{i(t,a)}{i^*(a)}) \mid_0^\infty\\
& + \int_0^\infty h(\frac{i(t,a)}{i^*(a)})[F_1^{'}(a) i^*(a) + F_1(a) i^{*'}(a)]da\\
= \quad & -F_1(a) i^*(a) h(\frac{i(t,a)}{i^*(a)}) \mid_0^\infty + \int_0^\infty h(\frac{i(t,a)}{i^*(a)}) i^*(a)\\
& \cdot[F_1^{'}(a) - (\zeta + \gamma_1(a) + \rho(a)) F_1(a)]da. \qquad (4.22)
\end{aligned}
$$

Clearly,

$$
F_1(0) = \frac{(\zeta+\omega)E^*}{i^*(0)} \qquad (4.23)
$$

and

$$
\lim_{a\to+\infty} F_1(a) = 0. \qquad (4.24)
$$

It follows from Eqs (4.22)–(4.24) that

$$
\begin{aligned}
&\frac{d\int_0^\infty F_1(a) i^*(a) h(\frac{i(t,a)}{i^*(a)})}{dt}\\
= \quad & \frac{(\zeta+\omega)E^*}{i^*(0)} i(t,0) - (\zeta+\omega)E^* - (\zeta+\omega)E^* \ln\frac{i(t,0)}{i^*(0)}\\
& -\int_0^\infty g_3(i^*(a)) f_3(V^*) \beta_3(a) h(\frac{i(t,a)}{i^*(a)})da\\
& -\int_0^\infty g_4(A^*(a)) f_4(V^*) \beta_4(a) h(\frac{i(t,a)}{i^*(a)})da. \qquad (4.25)
\end{aligned}
$$

Similarly, we have

$$
\frac{dL_2(t)}{dt} = \quad C_5(\zeta+\omega)E^*(\frac{i(t,0)}{i^*(0)} - 1) - C_5(\zeta+\omega)E^* \ln\frac{i(t,0)}{i^*(0)}
$$

$$
\begin{aligned}
&-C_5\int_0^{\infty} f_3(V^*)\beta_3(a)g_3(i^*(a))M(\frac{i(t,a)}{i^*(a)})da\\
&-C_5\int_0^{\infty} f_4(V^*)\beta_4(a)g_4(A^*(a))M(\frac{i(t,a)}{i^*(a)})da\\
&+C_6\frac{(\zeta+\omega)E^*}{A^*(0)}(A(t,0)-A^*(0))-C_6(\zeta+\omega)E^*\ln\frac{A(t,0)}{A^*(0)}\\
&-C_6\int_0^{\infty} f_3(V^*)\beta_3(a)g_3(i^*(a))M(\frac{A(t,a)}{A^*(a)})da\\
&-C_6\int_0^{\infty} f_4(V^*)\beta_4(a)g_4(A^*(a))M(\frac{A(t,a)}{A^*(a)})da\\
&+C_7\frac{(\zeta+\delta)R^*}{i^*(0)}(i(t,0)-i^*(0))-C_7(\zeta+\delta)R^*\ln\frac{i(t,0)}{i^*(0)}\\
&-C_7\int_0^{\infty}\gamma_1(a)i^*(a)M(\frac{i(t,a)}{i^*(a)})da-C_7\int_0^{\infty}\gamma_2(a)A^*(a)M(\frac{i(t,a)}{i^*(a)})da\\
&+C_8\frac{(\zeta+\delta)R^*}{A^*(0)}(A(t,0)-A^*(0))-C_8(\zeta+\delta)R^*\ln\frac{A(t,0)}{A^*(0)}\\
&-C_8\int_0^{\infty}\gamma_1(a)i^*(a)M(\frac{A(t,a)}{A^*(a)})da\\
&-C_8\int_0^{\infty}\gamma_2(a)A^*(a)M(\frac{A(t,a)}{A^*(a)})da. \qquad (4.26)
\end{aligned}
$$

From the sixth and seventh equations of (3.1), to make the terms with $i(t,0)-i^*(0)$, $A(t,0)-A^*(0)$ in L_2 cancel with $E(t)-E^*$, $R(t)-R^*$ in L_1, we have

$$
C_3(\zeta+\omega)=C_5\frac{(\zeta+\omega)E^*\xi_1\omega}{i^*(0)}+C_6\frac{(\zeta+\omega)E^*(1-\xi_1)\omega}{A^*(0)}+C_7\frac{(\zeta+\delta)R^*\xi_1\omega}{i^*(0)}+C_8\frac{(\zeta+\delta)R^*(1-\xi_1)\omega}{A^*(0)}, \qquad (4.27a)
$$

$$
C_4(\zeta+\delta)=C_5\frac{(\zeta+\omega)E^*\xi_2\delta}{i^*(0)}+C_6\frac{(\zeta+\omega)E^*(1-\xi_2)\delta}{A^*(0)}+C_7\frac{(\zeta+\delta)R^*\xi_2\delta}{i^*(0)}+C_8\frac{(\zeta+\delta)R^*(1-\xi_2)\delta}{A^*(0)}. \qquad (4.27b)
$$

Since for $x>0$, $(\ln x)''=-\frac{1}{x^2}<0$ holds, we have $\vartheta\ln\frac{E}{E^*}+(1-\vartheta)\ln\frac{R}{R^*}\le\ln(\frac{\vartheta E}{E^*}+\frac{(1-\vartheta)R}{R^*})$ for any $0<\vartheta<1$. Therefore,

$$
\begin{aligned}
&\frac{\xi_1\omega E^*}{\xi_1\omega E^*+\xi_2\delta R^*}\ln\frac{E}{E^*}+\frac{\xi_2\delta R^*}{\xi_1\omega E^*+\xi_2\delta R^*}\ln\frac{R}{R^*}\le\ln\frac{i(t,0)}{i^*(0)},\\
&\frac{(1-\xi_1)\omega E^*}{(1-\xi_1)\omega E^*+(1-\xi_2)\delta R^*}\ln\frac{E}{E^*}+\frac{(1-\xi_2)\delta R^*}{(1-\xi_1)\omega E^*+(1-\xi_2)\delta R^*}\cdot\ln\frac{R}{R^*}\le\ln\frac{A(t,0)}{A^*(0)}, \qquad (4.28)
\end{aligned}
$$

which can be rewritten as

$$
\begin{aligned}
&\frac{\xi_1\omega E^*}{i^*(0)}(C_5(\zeta+\omega)E^*+C_7(\zeta+\delta)R^*)\ln\frac{E}{E^*}+\frac{\xi_2\delta R^*}{i^*(0)}(C_5(\zeta+\omega)E^*+C_7(\zeta+\delta)R^*)\ln\frac{R}{R^*}\\
&\qquad\le(C_5(\zeta+\omega)E^*+C_7(\zeta+\delta)R^*)\ln\frac{i(t,0)}{i^*(0)}, \qquad (4.29a)
\end{aligned}
$$

$$\frac{(1-\xi_1)\omega E^*}{A^*(0)}(C_6(\zeta+\omega)E^*+C_8(\zeta+\delta)R^*)\ln\frac{E}{E^*}+\frac{(1-\xi_2)\delta R^*}{A^*(0)}(C_6(\zeta+\omega)E^* \\ +C_8(\zeta+\delta)R^*)\ln\frac{R}{R^*}\le(C_6(\zeta+\omega)E^*+C_8(\zeta+\delta)R^*)\ln\frac{A(t,0)}{A^*(0)}. \tag{4.29b}$$

It follows from adding the two equations of Eq (4.29) together that

$$C_3(\zeta+\omega)E^*\ln\frac{E(t)}{E^*}+C_4(\zeta+\delta)R^*\ln\frac{R(t)}{R^*} \\ \le(C_5(\zeta+\omega)E^*+C_7(\zeta+\delta)R^*)\ln\frac{i(t,0)}{i^*(0)}+(C_6(\zeta+\omega)E^*+C_8(\zeta+\delta)R^*)\ln\frac{A(t,0)}{A^*(0)} \tag{4.30}$$

holds if Eq (4.27) holds. To make Eq (4.27) hold true, we define $C_4 = C_7 = C_8 = 1$, $C_3 = \frac{(\zeta+\delta)\omega}{2\delta(\zeta+\omega)}(\frac{\xi_1}{(1-\xi_2)Q}+\frac{(1-\xi_1)Q}{\xi_2})$, $C_5 = \frac{(\zeta+\delta)A^*(0)}{2(\zeta+\omega)E^*\delta\xi_2}\frac{(1-\xi_1)(1-\xi_2)Q^2-\xi_1\xi_2}{\xi_1-\xi_2}$, and $C_6 = \frac{(\zeta+\delta)A^*(0)}{2(\zeta+\omega)QE^*\delta(1-\xi_2)}\frac{(1-\xi_1)(1-\xi_2)Q^2-\xi_1\xi_2}{\xi_1-\xi_2}$. Furthermore, we define $C_2=\min\{C_5,\ C_6\}$ *and* $C_1=\max\{C_2,\ C_3\}$. Hence, we have

$$\frac{dL(t)}{dt}=\frac{d(L_1(t)+L_2(t))}{dt}\le 0. \tag{4.31}$$

As E_2 is locally asymptotically stable if $\Re_0 > 1$, from LaSalle's invariance principle, E_2 is globally asymptotically stable. □

5. Numerical simulations

In this section, several examples are shown to illustrate the theoretical results by numerical simulations.

Case 1. Stability of model (1.2) when $\Re_0 < 1$.

For simplicity, we assume the contact rate is the bilinear incidence rate, which satisfies Assumptions 1.1, 1.2, and 4.1. Therefore, define $f_1(S(t)) = f_2(S(t)) = S(t)$, $f_3(V(t)) = f_4(V(t)) = V(t)$, $g_1(i(t,a)) = g_3(i(t,a)) = i(t,a)$ and $g_2(A(a,t)) = g_4(A(t,a)) = A(t,a)$. Let $B = 0.01$, $\zeta = 0.017$, $\phi = 0.85$, $\delta = 0.125$, $\gamma_1(a) = 0.7$, $\gamma_2(a) = 0.9$, $\rho(a) = 0.02$, $\omega = 0.4$, $\sigma = 0.1$, $\xi_1 = 0.8$ and $\xi_2 = 0.9$. Set the initial conditions as $(2.5, 0.05, 0.2, 0.02, 0.2, 3)$. Suppose that symptomatic carriers and asymptomatic carriers will be cured, dead, or isolated, so they are not infective to susceptible people when $a > 20$ and $a > 30$, respectively. The transmission coefficient of vaccinated individuals should not be larger than that of susceptible individuals infected by symptomatic or asymptomatic carriers due to the effect of vaccination, so $\beta_3(a) \le \beta_1(a)$ and $\beta_4(a) \le \beta_2(a)$. The smaller the values of $\beta_3(a)$ and $\beta_4(a)$ are, the more effective the vaccination is. If $\beta_3(a) = \beta_4(a) = 0$, people will not get infected once they are vaccinated. Suppose infected individuals will be dead, cured, or quarantined after some time in infected class, so they are not infectious after some time. For simplicity, the transmission coefficients $\beta_1(a)$ and $\beta_2(a)$ are chosen as

$$\beta_1(a) = \begin{cases} 0.15 & a \le 3, \\ 0.1a - 0.15 & 3 < a \le 4, \\ 0.25 & 4 < a \le 8, \\ 1.05 - 0.1a & 8 < a \le 9, \\ 0.15 & 9 < a \le 14, \\ 0.85 - 0.05a & 14 < a \le 15, \\ 0.1 & 15 < a \le 19, \\ 2 - 0.1a & 19 < a \le 20, \\ 0 & a > 20 \end{cases} \tag{5.1}$$

and

$$\beta_2(a) = \begin{cases} 0.5 & a \le 3, \\ 0.25a - 0.25 & 3 < a \le 4, \\ 0.75 & 4 < a \le 8, \\ 2.75 - 0.25a & 8 < a \le 9, \\ 0.5 & 9 < a \le 14, \\ 4.7 - 0.3a & 14 < a \le 15, \\ 0.2 & 15 < a \le 19, \\ 3.05 - 0.15a & 19 < a \le 20, \\ 0.05 & 20 < a \le 29, \\ 1.5 - 0.05a & 29 < a \le 30, \\ 0 & a > 30. \end{cases} \tag{5.2}$$

Let $\beta_3(a) = 0.3 * \beta_1(a)$ and $\beta_4(a) = 0.3 * \beta_2(a)$. By computation of Eq (3.1), (1.2) has the disease-free equilibrium $E_1(0,\ 0.5882,\ 0,\ 0,\ 0,\ 0)$. From Theorem 4.1, E_1 is globally asymptotically stable due to $\Re_0 = 0.2641 < 1$ as shown in Figure 1. The disease will die out when the time is long. The parameters are unchanged unless stated in the following cases.

Case 2. Local stability of model (1.2) when $\Re_0 > 1$.

In this case, let $\xi_1 = 0.1$ and $\xi_2 = 0.05$. Model (1.2) has two equilibria, $E_1(\frac{B}{\zeta}, 0, 0, 0, 0, 0)$ and $E_2(0,\ V^*,\ E^*,\ i^*(a),\ A^*(a),\ R^*)$. By computation, $\Re_0 = 2.1895 > 1$ and $\frac{(1-\xi_1)(1-\xi_2)Q^2-\xi_1\xi_2}{\xi_1-\xi_2} = -0.0376 < 0$. Therefore, from Theorem 3.2, E_2 is locally asymptotically stable, which is shown in Figure 2. The disease might persistently exist for some initial conditions.

Case 3. Global stability of model (1.2) when $\Re_0 > 1$.

To show that the conditions could be achieved in Theorem 4.2, we set $\gamma_1(a) = 0.7$, $\gamma_2(a) = 0.03$, $\xi_1 = 0.2$ and $\xi_2 = 0.1$. By computation, $\Re_0 = 1.9372 > 1$ and $\frac{(1-\xi_1)(1-\xi_2)Q^2-\xi_1\xi_2}{\xi_1-\xi_2} = 0.0271 > 0$. By Theorem 4.2, E_2 is globally asymptotically stable as shown in Figure 3. The disease will persistently exist.

Case 4. The effect of ξ_1, ξ_2, $\gamma_1(a)$, $\gamma_2(a)$.

Since a lot of works have already revealed that bigger transmission rates lead to more infectious individuals, it is not discussed here by setting $\beta_1(a)$, $\beta_2(a)$, $\beta_3(a)$ and $\beta_4(a)$ fixed. In this part, we mainly investigate the effect of the asymptomatic on the spread of diseases. We focus on the influence of the proportion of symptomatic persons $\xi_1,\ \xi_2$ and asymptomatic persons $1-\xi_1,\ 1-\xi_2$. As the recovered rate $\gamma_1(a)$ ($\gamma_2(a)$) of symptomatic (asymptomatic) population is also important to the evolution of infectious diseases, their effects are considered in this part.

Let $\gamma_2(a) = 0.1$. From Figure 4, we get the following results:

(i) Spread of the disease changes greatly, especially for big ξ_1 and small ξ_2 when $\gamma_1(a)$ is small, i.e., $\gamma_1(a) \leq 0.1$. Figure 4(a) shows that the disease can be stopped with very big ξ_1 and very small ξ_2 such as $\xi_1 = 0.9$ and $\xi_2 = 0.1$ when $\gamma_1(a)$ is very small, i.e., $\gamma_1(a) = 0.01$. However, from Figure 4(b), the disease cannot be controlled when $\xi_1 = 0.9$, $\xi_2 = 0.1$, and $\gamma_1(a) = 0.07$. Why will the disease be out of control when recovered rate increases? From our assumption, the transmission rate of one asymptomatic individual is higher than one symptomatic individual, since they can contact more susceptible individuals. Big ξ_1 means that most first-infected individuals transmitting from the exposed class are symptomatic persons who have a big death rate and small contagious rate. Small ξ_2 means that most second-infected (called second here if they are not infected for the first time) individuals who transmit from recovered persons are highly contagious asymptomatic persons with a small death rate. A slight increase of $\gamma_1(a)$ results in more recovered persons (see Figure 5(d)) who are more likely to become asymptomatic persons as shown in Figure 5(c). Therefore, more people will be infected, and the disease may even be out of control when $\gamma_1(a)$ is slightly increased with big ξ_1 and small ξ_2. It may also hold true when ξ_1 is big and ξ_2 is of middle size.

(ii) Figure 4 shows that the number of infected people increases when $\gamma_1(a)$ increases in an area $[0.01, 0.2]$, while it decreases when $\gamma_1(a)$ increases in an area $[0.2, 0.98]$ for small ξ_1 and large ξ_2 such as $\xi_1 = 0.2$, and $\xi_2 = 0.95$. Similar to the above analysis, most of first-infected persons are asymptomatic, and most second-infected persons are symptomatic when ξ_1 is small and ξ_2 is large. When $\gamma_1(a) \in [0.01,\ 0.2]$ is small, a slight increase of the recovered rate of symptomatic cases $\gamma_1(a)$ does not induce symptomatic cases $i(t, 0)$ to decrease a lot. Since it can induce an increase in recovered persons, the number of second-infected persons increase, which results in the increasing of symptomatic persons as ξ_2 is large (shown in Figure 6). The increasing rate of symptomatic cases is larger than the decreasing rate of symptomatic cases as $\gamma_1(a)$ is small. Therefore, the disease is out of control when $\gamma_1(a)$ is small and increases slightly for small ξ_1 and large ξ_2. When $\gamma_1(a)$ is big and increases, the decreasing rate of symptomatic individuals is larger than the increasing rate of symptomatic individuals, so the disease can be controlled.

(iii) Evolution of the disease does not change significantly when $\gamma_1(a)$ is large, i.e., $\gamma_1(a) \geq 0.5$ from Figure 4(d)–(f). When $\gamma_1(a)$ is big, increasing $\gamma_1(a)$ cannot change the spread of the disease, except for small ξ_1 and large ξ_2. Compared with $\gamma_1(a)$, increasing $\gamma_2(a)$ may work better at stopping the disease as an asymptomatic individual is more contagious than a symptomatic individual, which can be seen from Figure 7.

(iv) From Figure 4, the disease will disappear when $k_1\xi_1 + k_2\xi_2 > 1$, where the values of $k_1 \geq 0$ and $k_2 \geq 0$ depend on $\gamma_1(a)$ and $\gamma_2(a)$. If most infected individuals coming from exposed and recovered classes are symptomatic, then the disease may be controlled.

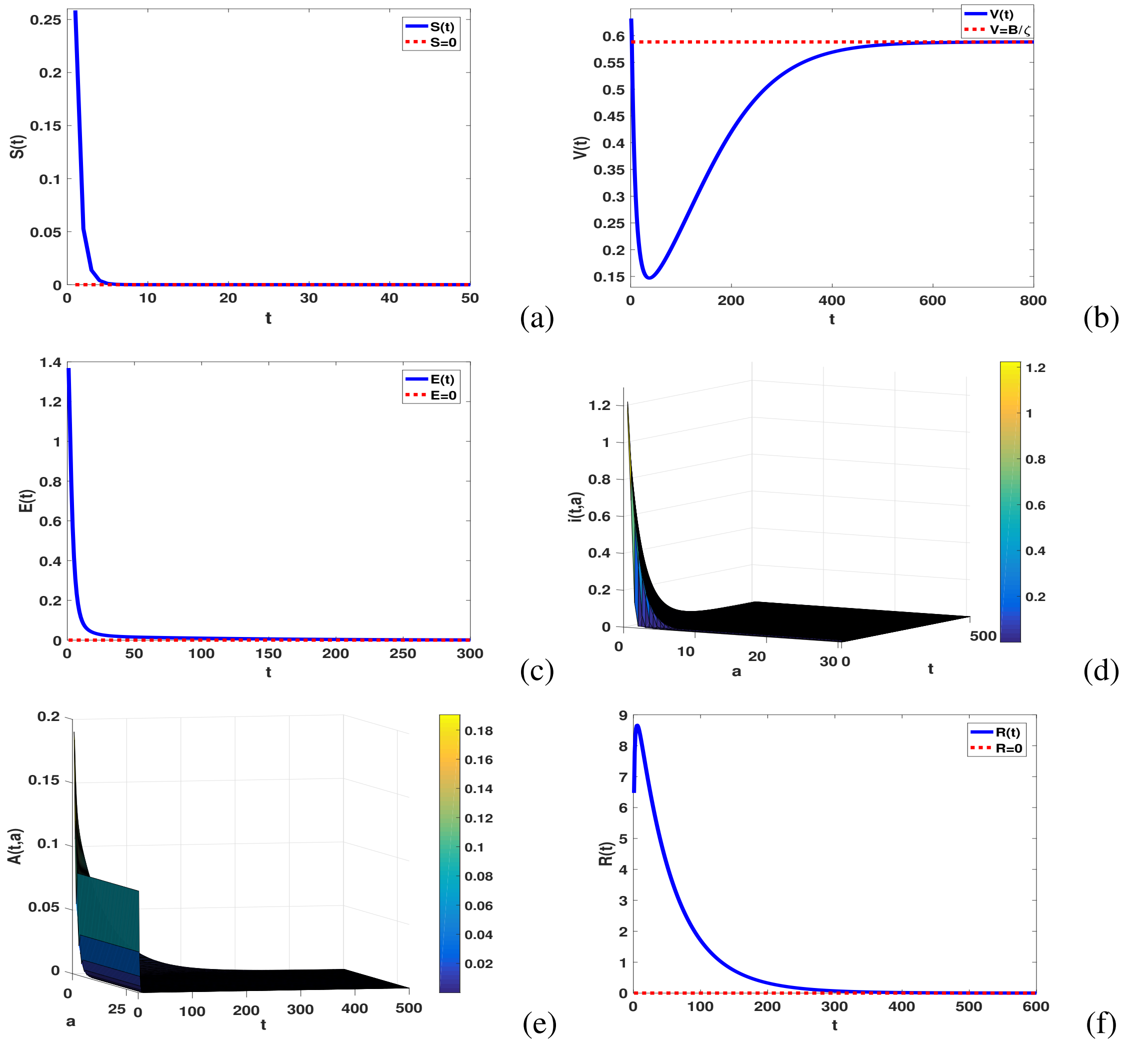

Figure 1. The disease-free equilibrium $E_1(0, 0.5882, 0, 0, 0, 0)$ of model (1.2) in red dashed lines, while the blue lines show solutions with initial conditions $(2.5, 0.05, 0.2, 0.02, 0.2, 3)$. The parameters are $B = 0.01$, $\zeta = 0.017$, $\phi = 0.85$, $\delta = 0.125$, $\gamma_1(a) = 0.7$, $\gamma_2(a) = 0.9$, $\rho(a) = 0.02$, $\omega = 0.4$, $\sigma = 0.1$, $\xi_1 = 0.8$, and $\xi_2 = 0.9$ from direct simulation.

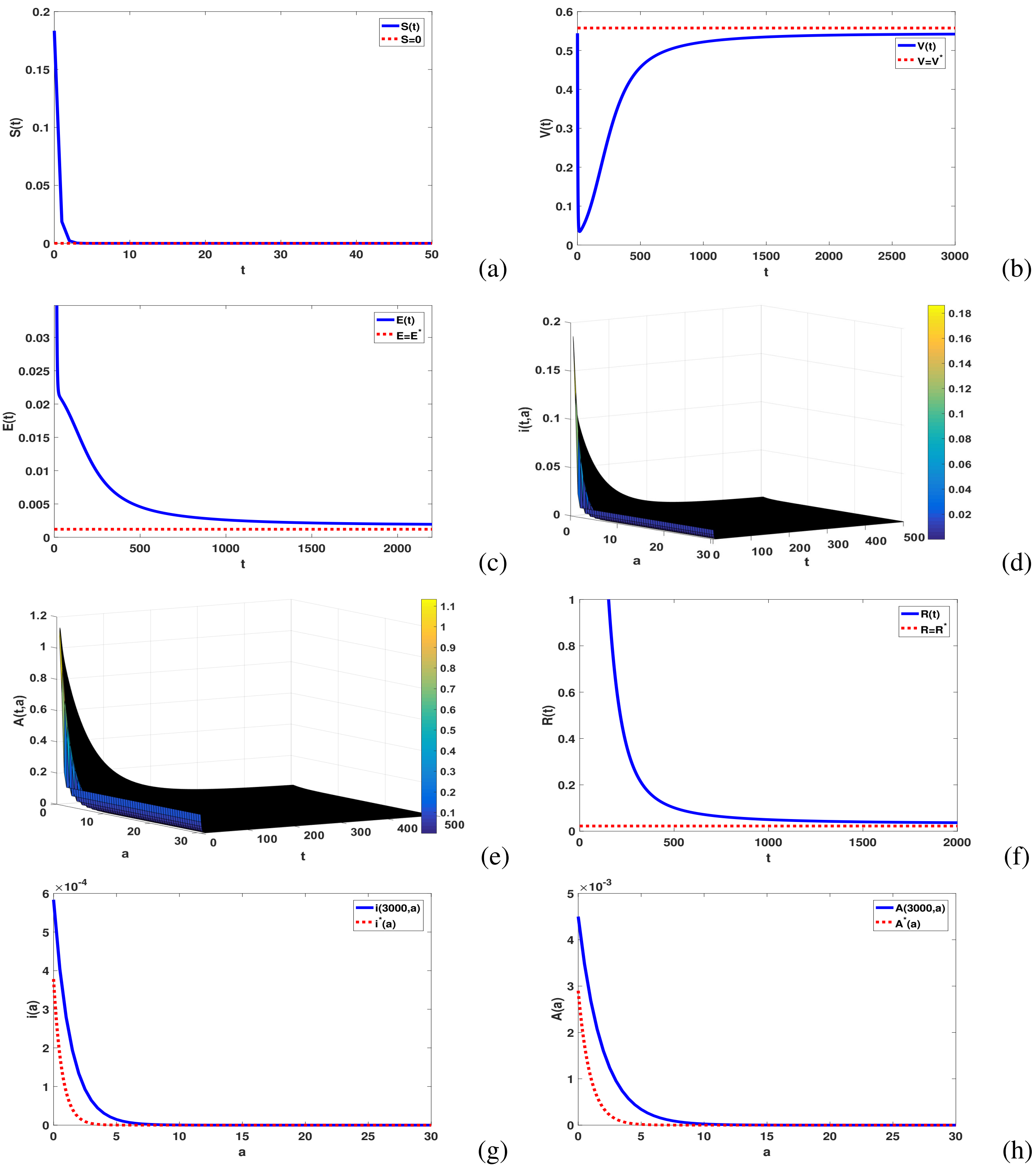

Figure 2. The endemic steady state $E_2(0, 0.5577, 0.0012, 0.000378 \times e^{-0.737a}, 0.0029e^{-0.517a}, 0.0223)$ (the red dashed lines) of model (1.2), and one solution (the blue lines) with $\xi_1 = 0.1$ and $\xi_2 = 0.05$ from direct simulation.

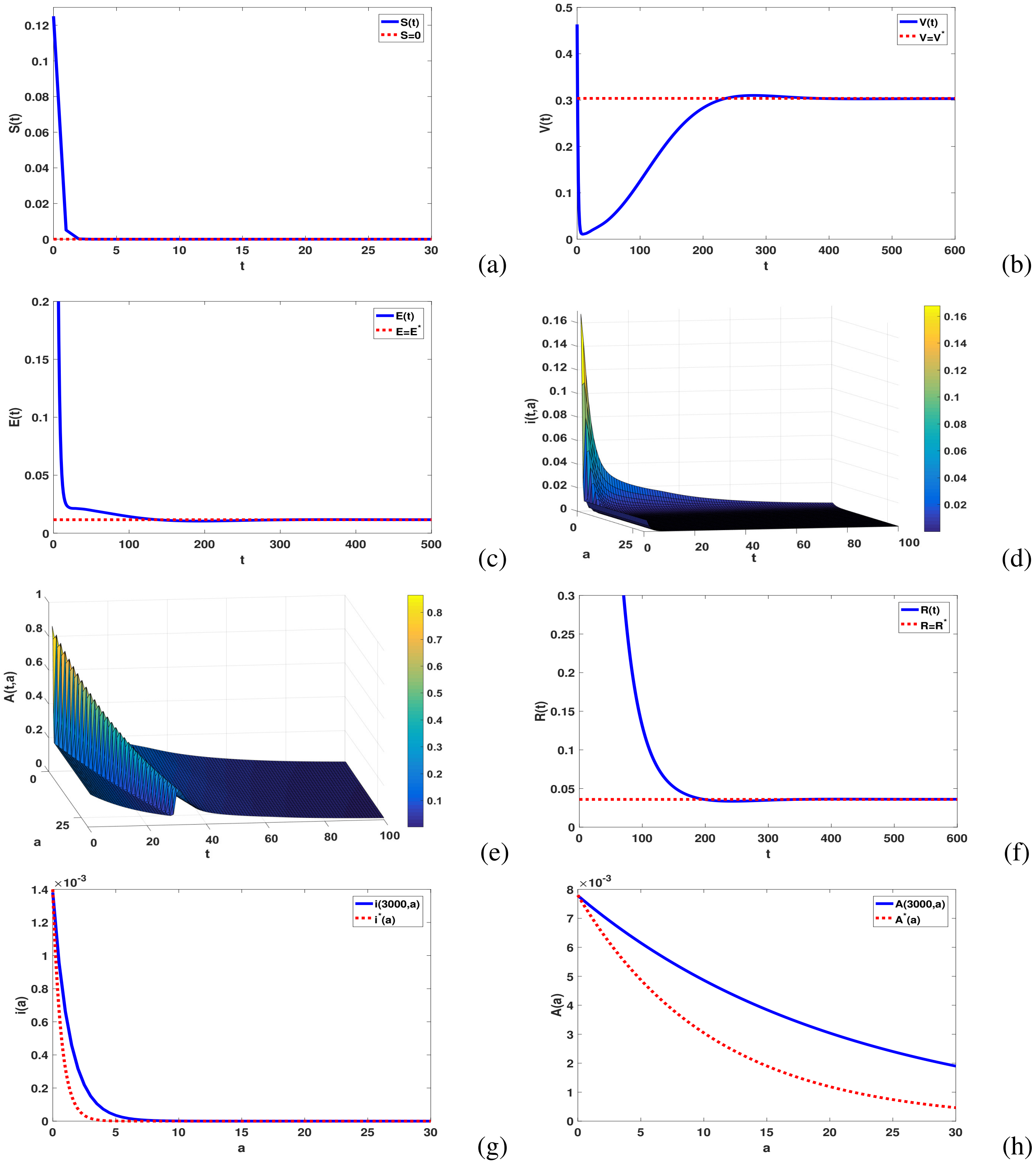

Figure 3. Endemic steady state $E_2(0,\ 0.3037,\ 0.0116,\ 0.0014e^{-0.737a},\ 0.0078 \times e^{-0.047a},\ 0.0359)$ (shown in the red dashed lines) of model (1.2), and one solution (in the blue lines) with $\gamma_1(a) = 0.7,\ \gamma_2(a) = 0.03,\ \xi_1 = 0.2$, and $\xi_2 = 0.1$ from direct simulation.

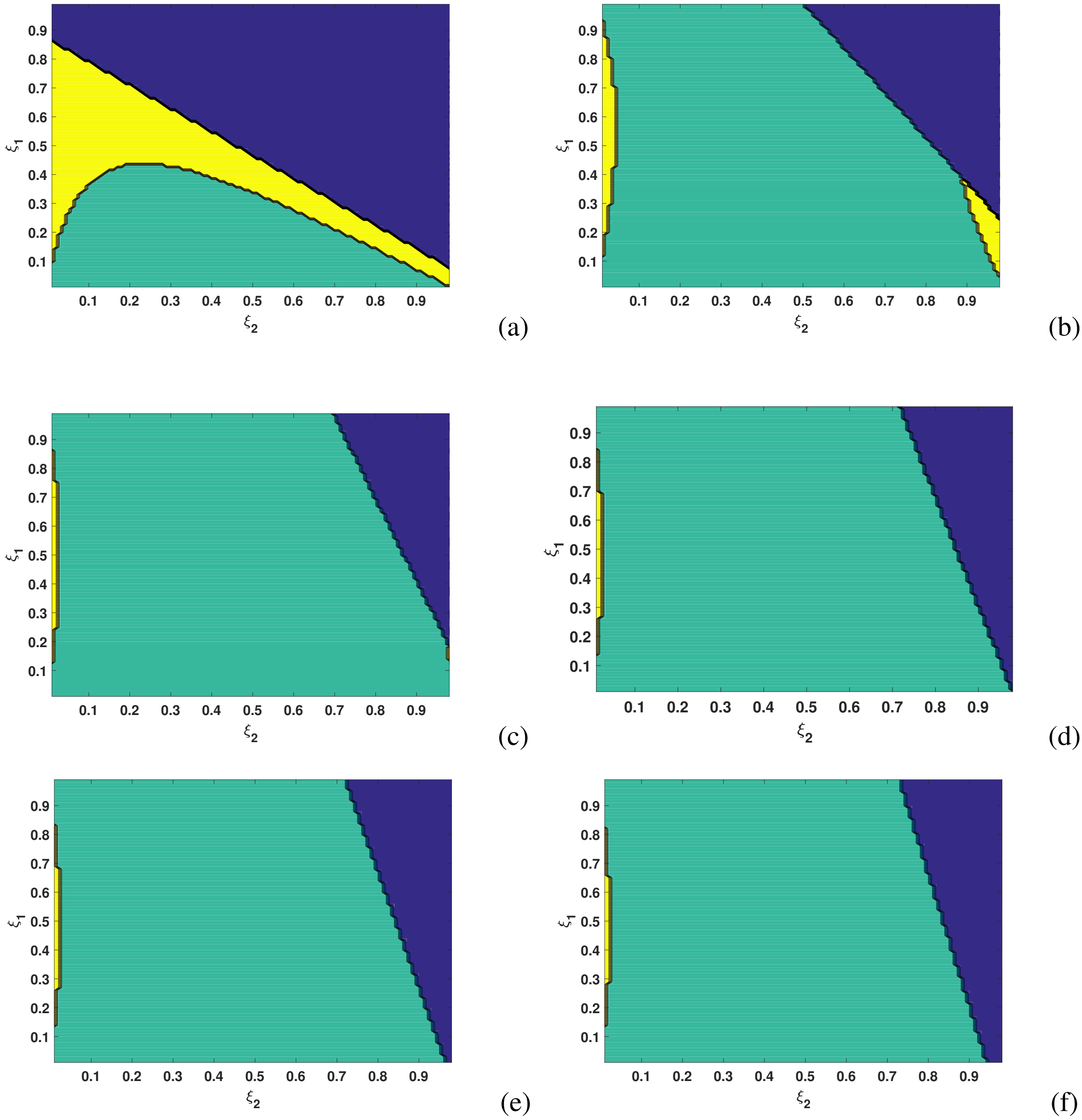

Figure 4. Parameters are set to be $\gamma_2(a) = 0.1$ and (a) $\gamma_1(a) = 0.01$, (b) $\gamma_1(a) = 0.07$, (c) $\gamma_1(a) = 0.2$, (d) $\gamma_1(a) = 0.4$, (e) $\gamma_1(a) = 0.5$, and (f) $\gamma_1(a) = 0.98$. $\Re_0 > 1$, $\frac{(1-\xi_1)(1-\xi_2)Q^2-\xi_1\xi_2}{\xi_1-\xi_2} > 0$ (yellow area), $\Re_0 > 1$, $\frac{(1-\xi_1)(1-\xi_2)Q^2-\xi_1\xi_2}{\xi_1-\xi_2} < 0$ (green area), and $\Re_0 < 1$ (blue area) from calculation are shown.

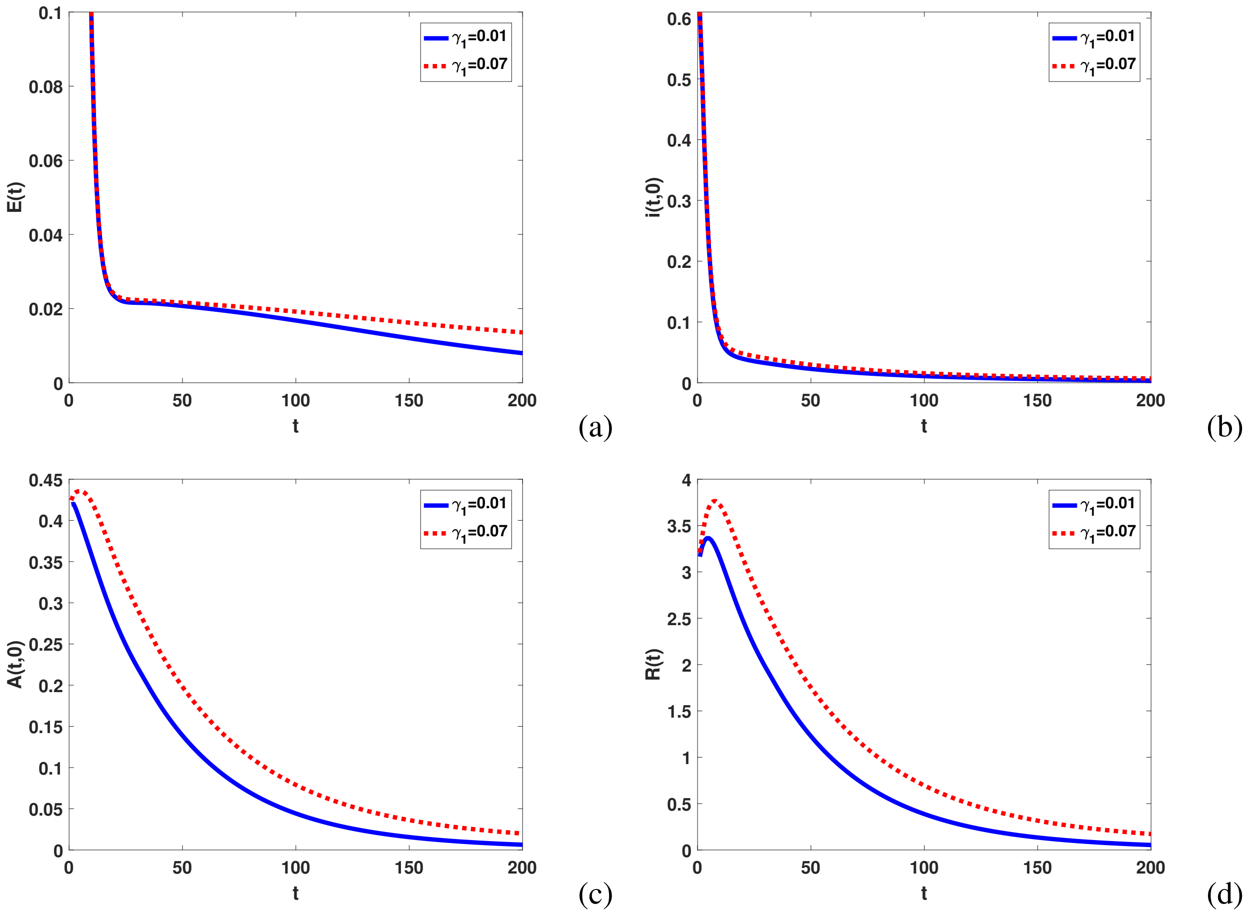

Figure 5. Evolution of $E(t)$, $i(t, 0)$, $A(t, 0)$ and $R(t)$ for different $\gamma_1(a)$. Set $\xi_1 = 0.9$, and $\xi_2 = 0.1$.

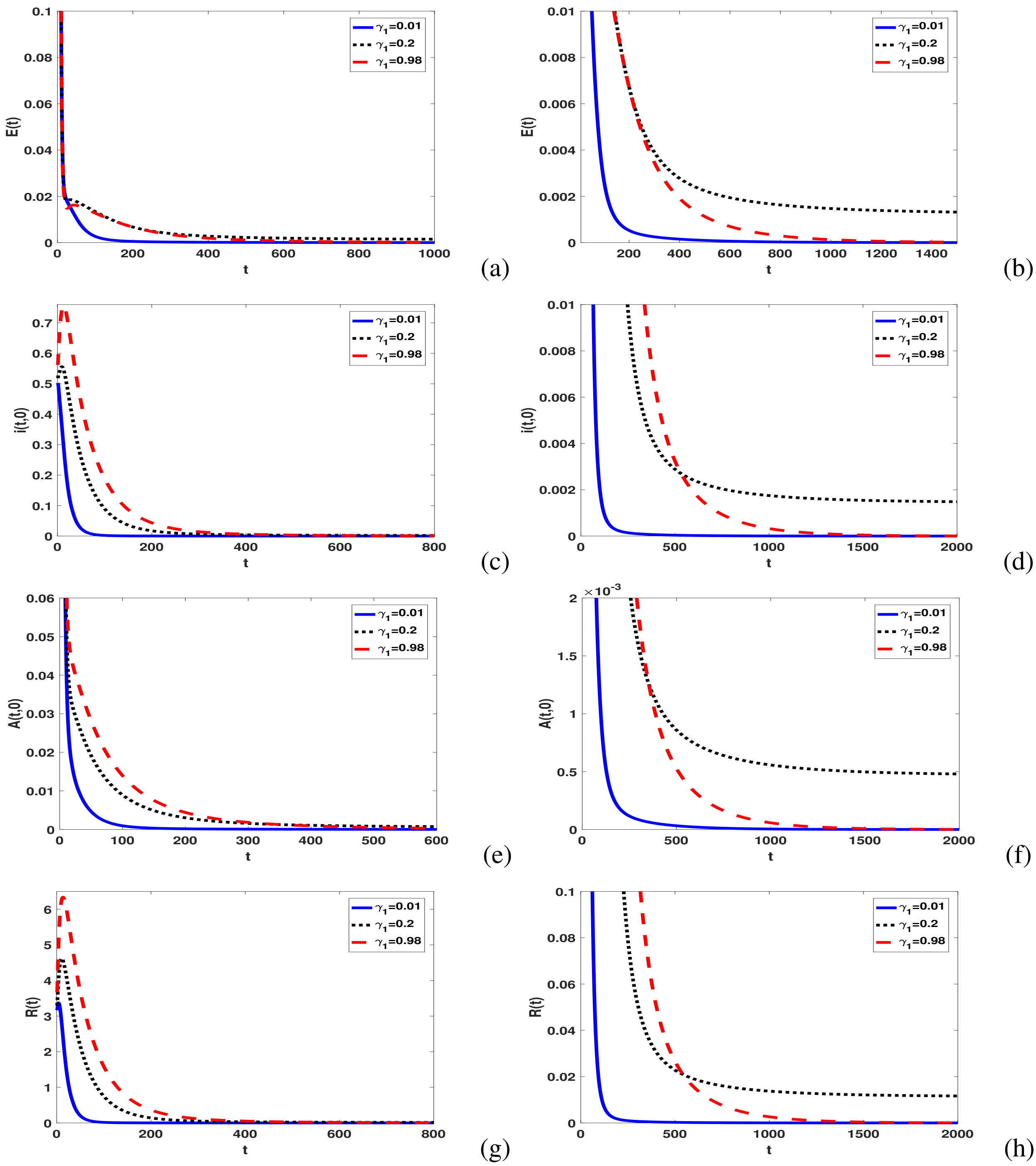

Figure 6. Evolution of $E(t)$, $i(t, 0)$, $A(t, 0)$ and $R(t)$ for different $\gamma_1(a)$. Set $\xi_1 = 0.2$ and $\xi_2 = 0.95$. The right column is a scaled-up version of the left column.

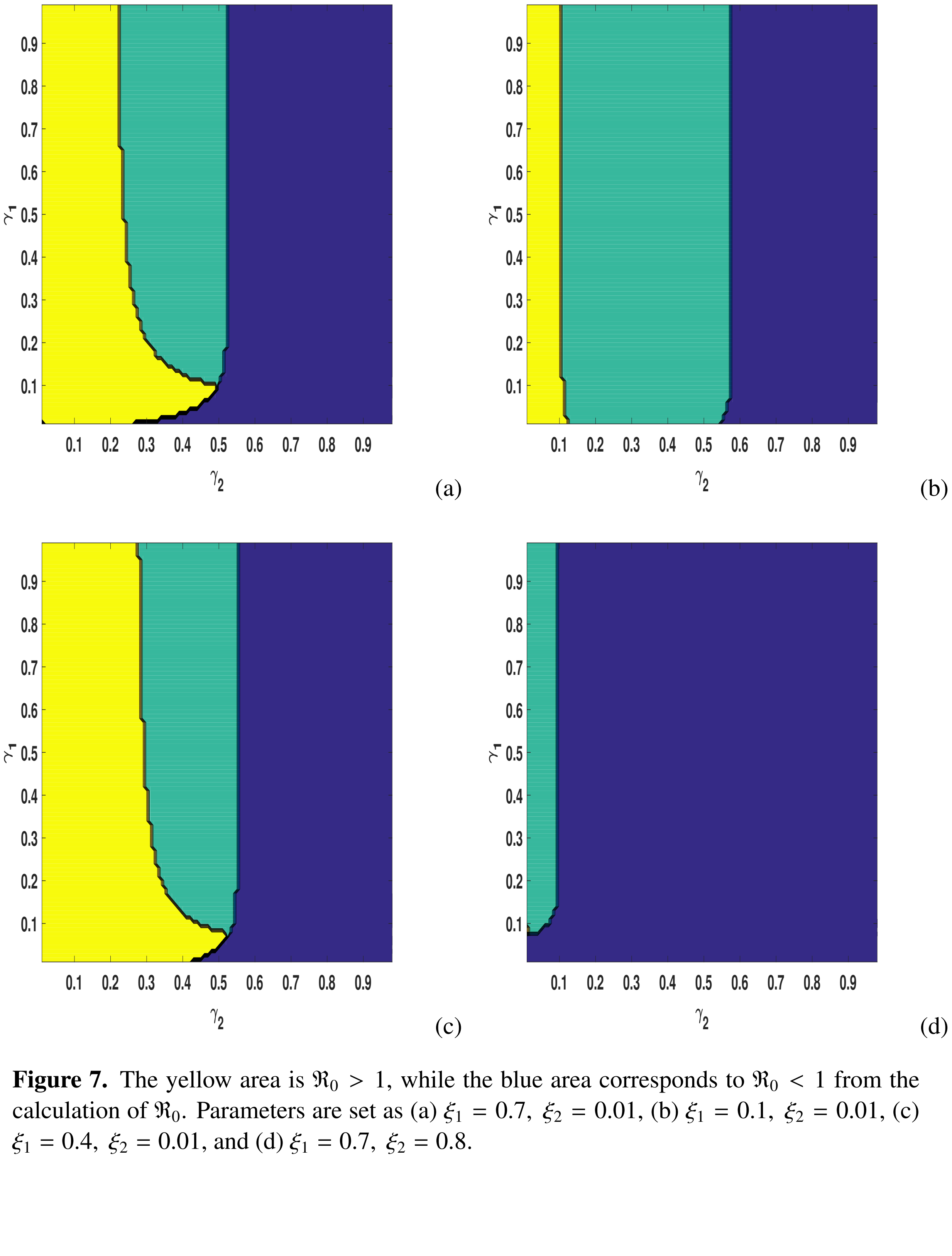

Figure 7. The yellow area is $\Re_0 > 1$, while the blue area corresponds to $\Re_0 < 1$ from the calculation of $\Re_0$. Parameters are set as (a) $\xi_1 = 0.7,\ \xi_2 = 0.01$, (b) $\xi_1 = 0.1,\ \xi_2 = 0.01$, (c) $\xi_1 = 0.4,\ \xi_2 = 0.01$, and (d) $\xi_1 = 0.7,\ \xi_2 = 0.8$.

6. Conclusions

In the present work, we have considered an SEIAR model with vaccination and the age of infection. We have shown the positivity, boundedness, and asymptotic smoothness (see the appendix) of the solutions. From our computation, the basic reproduction number R_B is derived, which is equal to $\Re_0$. The model always has a disease-free steady state E_1, and has one unique endemic equilibrium E_2 if $\Re_0 > 1$. When $\Re_0 < 1$, the equilibrium E_1 is not only locally asymptotically stable, but also globally asymptotically stable. In this case, the disease will be controlled. If $\Re_0 > 1$ holds, the equilibrium E_1 is unstable, but the equilibrium E_2 is locally asymptotically stable, which means that the disease could be controlled for some initial conditions. Sufficient conditions for the equilibrium E_2 to be globally asymptotically stable, i.e. $\Re_0 > 1$, $\frac{(1-\xi_1)(1-\xi_2)Q^2-\xi_1\xi_2}{\xi_1-\xi_2} > 0$, and Assumption 4.1 are given. Examples of (i) $\Re_0 < 1$, (ii) $\Re_0 > 1$ and (iii) $Q > 0$, $\frac{(1-\xi_1)(1-\xi_2)Q^2-\xi_1\xi_2}{\xi_1-\xi_2} > 0$, and Assumption 4.1 are presented to illustrate theoretical results by simulation.

By numerical simulation, we have also shown the effect of ξ_1 and ξ_2 on the spread of infectious diseases. From the investigation of numerical results, we find the following conclusions drawn from the results in Case 4. (1) The disease may become out of control if most of the first-infected persons are symptomatic and most of the second-infected persons are asymptomatic when the recovered rate of the symptomatic is small and slightly increased. (2) When most of first-infected persons are asymptomatic and most of second-infected persons are symptomatic, the disease might become out of control for small recovered rate of the symptomatic, and could be controlled for a large recovered rate of the symptomatic as the recovered rate of the symptomatic increases. (3) The spread of disease cannot be changed only by increasing the recovered rate of the symptomatic when the recovered rate of the symptomatic is large except that most of the first-infected persons are asymptomatic, and most of the second-infected persons are symptomatic. The spread of disease depends largely on the recovered rate of the asymptomatic. (4) If most infected individuals coming from exposed and recovered classes are symptomatic, then the disease may be controlled. These conclusions may be useful to control infectious diseases. In this paper, we have not investigated control strategies of the disease. In the future, we will consider parameter estimation and optimal control strategies to minimize the number of infected and deaths in infectious diseases.

Author contributions

Huaixing Li: Conceptualization, methodology, formal analysis, writing-original draft; Jiaoyan Wang: Formal analysis, methodology, validation, writing-review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

1. S. H. Duan, M. Y. Zhou, W. Zhang, J. W. Shen, R. Qi, X. R. Qin, et al., Seroprevalence and asymptomatic carrier status of SARS-CoV-2 in Wuhan City and other places of China, *PLOS Neglect. Trop. D.*, **15** (2021), e0008975. http://dx.doi.org/10.1371/journal.pntd.0008975
2. A. S. Fauci, S. M. Schnittman, G. Poli, S. Koenig, G. Pantaleo, Immunopathogenic mechanisms in human immunodeficiency virus (HIV) infection, *Ann. Intern. Med.*, **114** (1991), 678–693. https://doi.org/10.7326/0003-4819-114-8-678
3. A. S. F. Lok, B. J. McMahon, Chronic hepatitis B: Update 2009, *Hepatology*, **50** (2009), 661–662. http://dx.doi.org/10.1002/hep.23190
4. P. Magal, C. C. McCluskey, G. G. Webb, Lyapunov functional and global asymptotic stability for an infection-age model, *Appl. Anal.*, **89** (2010), 1109–1140. https://dx.doi.org/10.1080/00036810903208122
5. H. Inaba, *Age-structured population dynamics in demography and epidemiology*, 1 Eds., Singapore: Springer, 2017. https://dx.doi.org/10.1007/978-981-10-0188-8
6. K. Okuwa, H. Inaba, T. Kuniya, Mathematical analysis for an age-structured SIRS epidemic model, *Math. Biosci. Eng.*, **16** (2019), 6071–6102. https://dx.doi.org/10.3934/mbe.2019304
7. R. P. Sigdel, C. C. McCluskey, Global stability for an SEI model of infectious disease with immigration, *Appl. Math. Comput.*, **243** (2014), 684–689. https://dx.doi.org/10.1016/j.amc.2014.06.020
8. H. Kang, S. G. Ruan, X. Yu, Age-structured population dynamics with nonlocal diffusion, *J. Dyn. Differ. Equ.*, **34** (2022), 789–823. https://dx.doi.org/10.1007/s10884-020-09860-5
9. R. Zhang, D. Li, S. Liu, Global analysis of an age-structured SEIR model with immigration of population and nonlinear incidence rate, *J. Appl. Comput.*, **9** (2019), 1470–1492. https://dx.doi.org/10.11948/2156-907X.20180281
10. J. Lin, R. Xu, X. Tian, Global dynamics of an age-structured cholera model with multiple transmissions, saturation incidence and imperfect vaccination, *J. Biol. Dynam.*, **13** (2019), 69–102. https://dx.doi.org/10.1080/17513758.2019.1570362
11. H. X. Li, J. Y. Wang, Global dynamics of an SEIR model with the age of infection and vaccination, *Mathematics*, **9** (2021), 2195. https://doi.org/10.3390/math9182195

12. R. Y. M. Massoukou, S. C. O. Noutchie, R. Guiem, Global dynamics of an SVEIR model with age-dependent vaccination, infection, and latency, *Abstr. Appl. Anal.*, 2018, 8479638. https://doi.org/10.1155/2018/8479638

13. C. Yang, Y. L. Yang, Y. Li, Assessing vaccination priorities for different ages and age-specific vaccination strategies of COVID-19 using an SEIR modelling approach, *Plos One*, **16** (2021), e0261236. https://dx.doi.org/10.1371/journal.pone.0261236

14. X. Wang, Y. Chen, M. Martcheva, L. Rong, Asymptotic analysis of a vector-borne disease model with the age of infection, *J. Biol. Dynam.*, **14** (2020), 332–367. https://dx.doi.org/10.1080/17513758.2020.1745912

15. F. M. Muli, B. Okelo, R. Magwanga, O. Ongati, Mathematical analysis of COVID-19 model incorporating vaccination of susceptible and isolation of symptomatic individuals, *J. Appl. Math. Comput.*, **70** (2024), 461–488. https://doi.org/10.1007/s12190-023-01966-5

16. W. O. Kermack, A. G. McKendrick, A contribution to mathematical theory of epidemics, *Proc. R. Soc. Lond. A*, **115** (1927), 700–721. https://doi.org/10.1098/rspa.1927.0118

17. G. Rost, J. Wu, SEIR epidemiological model with varying infectivity and infinite delay, *Math. Biosci. Eng.*, **5** (2008), 389–402. https://dx.doi.org/10.3934/mbe.2008.5.389

18. W. O. Kermack, A. G. McKendrick, Contributions to the mathematical theory of epidemics II. The problem of endemicity, *B. Math. Biol.*, **53** (1991), 57–87.

19. W. O. Kermack, A. G. McKendrick, Contributions to the mathematical theory of epidemics III. Further studies of the problem of endemicity, *B. Math. Biol.*, **53** (1991), 89–118.

20. H. Inaba, Endemic threshold analysis for the Kermack-McKendrick reinfection model, *Josai Math. Monograph*, **9** (2016), 105–133.

21. A. Perassom Q. Richard, Implication of age-structrue on the dynamics of Lotka-Volterra equations, *Differ. Integral Equ.*, **32** (2019), 91–120. https://dx.doi.org/10.57262/die/1544497287

22. G. F. Webb, *Theory of nonlinear age-dependent population dynamics*, New York: Marcel Dekker, 1985.

23. H. L. Smith, H. R. Thieme, *Dynamical systems and population persistence*, Providence: Amecican Mathematical Society, **118** (2011).

24. N. A. Babaei, M. Kr*ö*ger, T.*öz*er, Theoretical analysis of a SIRD model with constant amount of alive population and COVID-19 applications, *Appl. Math. Model.*, **127** (2024), 237–258. https://dx.doi.org/10.1016/j.apm.2023.12.006

25. A. Gupta, R. Katarya, A deep-SIQRV epidemic model for COVID-19 to access the impact of prevention and control measures, *Comput. Biol. Chem.*, **107** (2023), 107941. https://dx.doi.org/10.1016/j.compbiolchem.2023.107941

26. Y. J. Zhu, F. Liu, Y. L. Bai, Z. B. Zhao, C. F. Ma, A. D. Wu, et al., Effectiveness analysis of multiple epidemic prevention measures in the context of COVID-19 using the SVIRD model and ensemble Kalman filter, *Heliyon*, **9** (2023), e14231. https://dx.doi.org/10.1016/j.heliyon.2023.e14231

27. A. K. Paul, N. Basak, M. A. Kuddus, Mathematical analysis and simulation of COVID-19 model with booster dose vaccination strategy in Bangladesh, *Results Eng.*, **21** (2024), 101741. https://dx.doi.org/10.1016/j.rineng.2023.101741

28. X. Wang, H. L. Wu, S. Y. Tang, Assessing age-specific vaccination strategies and post-vaccination reopening policies for COVID-19 control using SEIR modeling approach, *Math. Biosci.*, **84** (2022), 108. https://dx.doi.org/10.1007/s11538-022-01064-w
29. M. Shadabfar, M. Mahsuli, A. S. Khoojine, V. R. Hosseini, Time-variant reliability-based prediction of COVID-19 spread using extended SEIVR model and Monte Carlo sampling, *Res. Phys.*, **26** (2021), 104364. https://dx.doi.org/10.1016/j.rinp.2021.104364
30. A. P. Wu, L. Hu, D. J. Li, J. X. Zhu, P. Shang, A Queue-SEIAR model: Revealing the transmission mechanism of epidemics in a metro line from a meso level, *Transport. Res. E-Log.*, **190** (2024), 103716. https://dx.doi.org/10.1016/j.tre.2024.103716
31. F. Cao, X. Lü, Y. X. Zhou, X. Y. Cheng, Modified SEIAR infectious disease model for Omicron variants spread dynamics, *Nonlinear Dynam.*, **111** (2023), 14597–14620. https://dx.doi.org/10.1007/s11071-023-08595-4
32. P. Saha, S. K. Biswas, M. H. A. Biswas, U. Ghosh, An SEQAIHR model to study COVID-19 transmission and optimal control strategies in Hong Kong, *Nonlinear Dynam.*, **111** (2023), 6873–6893. https://dx.doi.org/10.1007/s11071-022-08181-0
33. J. Li, J. Zhong, Y. M. Ji, F. Yang, A new SEIAR model on small-world networks to assess the intervention measures in the COVID-19 pandemics, *Res. Phys.*, **25** (2021), 104283. https://dx.doi.org/10.1016/j.rinp.2021.104283

Appendix: Asymptotic smoothness

Asymptotic smoothness of the semiflow Φ is considered in this section to show the existence of an attractor.

Proposition 5.1. *Defining*

$$\begin{aligned} L_1(t) &= \int_0^\infty \beta_1(a)g_1(i(t,a))da, \qquad L_2(t) = \int_0^\infty \beta_2(a)g_2(A(t,a))da, \\ L_3(t) &= \int_0^\infty \beta_3(a)g_3(i(t,a))da, \qquad L_4(t) = \int_0^\infty \beta_4(a)g_1(A(t,a))da, \\ J_1(t) &= \int_0^\infty \gamma_1(a)i(t,a)da, \qquad J_2(t) = \int_0^\infty \gamma_2(a)A(t,a)da, \end{aligned} \tag{5.1}$$

then the functions $L_1(t),\ L_2(t),\ L_3(t),\ L_4(t)$, $J_1(t)$ and $J_2(t)$ are Lipschitz continuous on $\mathbb{R}^+$.

Proof. For fixed $t \geq 0$ and $h > 0$, we have

$$\begin{aligned} & \mid L_1(t+h) - L_1(t) \mid \\ = & \mid \int_0^\infty \beta_1(a)g_1(i(a,t+h))da - \int_0^\infty \beta_1(a)g_1(i(t,a))da \mid \\ = & \mid \int_0^h \beta_1(a)g_1(i(a,t+h))da + \int_h^\infty \beta_1(a)g_1(i(a,t+h))da \\ & - \int_0^\infty \beta_1(a)g_1(i(t,a))da \mid \end{aligned}$$

$$\leq \mid \int_h^\infty \beta_1(a)g_1(i(a,t+h))da - \int_0^\infty \beta_1(a)g_1(i(t,a))da \mid$$
$$+ \mid \int_0^h \beta_1(a)g_1(i(a,t+h))da \mid . \tag{5.2}$$

From Assumption 1.2, we have

$$\mid g_1(i(a,t+h)) \mid \leq g_1^{'}(0) \mid i(a,t+h) \mid = g_1^{'}(0)i(a,t+h).$$

Then,

$$\begin{aligned}\mid L_1(t+h) - L_1(t) \mid \;\leq\; & \mid \int_h^\infty \beta_1(a)g_1(i(a,t+h))da - \int_0^\infty \beta_1(a)g_1(i(t,a))da \mid \\ & + \mid \int_0^h \beta_1(a)g_1^{'}(0)i(a,t+h)da \mid .\end{aligned} \tag{5.3}$$

Substituting Eq (2.2) into Eq (5.3), we obtain

$$\begin{aligned}& \int_0^h \beta_1(a)g_1^{'}(0)i(a,t+h)da \\ = & \int_0^h \beta_1(a)g_1^{'}(0)e^{-\int_0^a(\zeta+\gamma_1(s)+\rho(s))ds}(\xi_1\omega E(t+h-a)+\xi_2\delta R(t+h-a))da.\end{aligned} \tag{5.4}$$

By Assumption 1.1 and Proposition 2.2, Eq (5.4) can be rewritten as

$$\int_0^h \beta_1(a)g_1^{'}(0)i(a,t+h)da \leq g_1^{'}(0)\hat{\beta}_1 M(\xi_1\omega+\xi_2\delta)h.$$

It then follows from Eq (5.3) that

$$\begin{aligned}& \mid L_1(t+h) - L_1(t) \mid \\ \leq & \mid \int_h^\infty \beta_1(a)g_1(i(a,t+h))da - \int_0^\infty \beta_1(a)g_1(i(t,a))da \mid \\ & + g_1^{'}(0)\hat{\beta}_1 M(\xi_1\omega+\xi_2\delta)h \\ = & \mid \int_0^\infty \beta_1(\sigma+h)g_1(i(\sigma+h,t+h))d\sigma \\ & - \int_0^\infty \beta_1(a)g_1(i(t,a))da \mid + g_1^{'}(0)\hat{\beta}_1 M(\xi_1\omega+\xi_2\delta)h \\ \leq & \mid \int_0^\infty \beta_1(a+h)(g_1(i(a+h,t+h)) - g_1(i(t,a)))da \mid \\ & + \mid \int_0^\infty (\beta_1(a+h) - \beta_1(a))g_1(i(t,a))da \mid \\ & + g_1^{'}(0)\hat{\beta}_1 M(\xi_1\omega+\xi_2\delta)h.\end{aligned} \tag{5.5}$$

By Assumption 1.2, we have

$$\mid g_1(i(a+h,t+h)) - g_1(i(t,a)) \mid \leq g_1^{'}(0) \mid i(a+h,t+h) - i(t,a) \mid . \tag{5.6}$$

From Eq (2.2),

$$i(a+h,t+h) = i(t,a)e^{-\int_a^{a+h}(\zeta+\gamma_1(s)+\rho(s))ds} \tag{5.7}$$

for all $a \geq 0,\ t \geq 0$ and $h \geq 0$. Hence, we have

$$\begin{aligned} \mid g_1(i(a+h,t+h)) - g_1(i(t,a)) \mid &\leq g_1^{'}(0)i(t,a)(1-e^{-\int_a^{a+h}(\zeta+\gamma_1(s)+\rho(s))ds}) \\ &\leq g_1^{'}(0)i(t,a)\int_a^{a+h}(\zeta+\gamma_1(s)+\rho(s))ds \\ &\leq g_1^{'}(0)i(t,a)(\zeta+\hat{\gamma}_1+\hat{\rho})h. \end{aligned} \tag{5.8}$$

From Eq (5.8), the first term in Eq (5.5) can be rewritten as

$$\begin{aligned} &\mid \int_0^\infty \beta_1(a+h)(g_1(i(a+h,t+h)) - g_1(i(t,a)))da \mid \\ \leq\ &\hat{\beta}_1 g_1^{'}(0)M(\zeta+\hat{\gamma}_1+\hat{\rho})h. \end{aligned} \tag{5.9}$$

By Assumption 1.1, $\mid \beta_1(a+h) - \beta_1(a) \mid\leq L_{\beta_1}h$. From Assumption 1.2, $g_1(i(t,a)) \leq g_1^{'}(0)i(t,a)$. Therefore, the second term in Eq (5.5) can be rewritten as

$$\mid \int_0^\infty (\beta_1(a+h)-\beta_1(a))g_1(i(t,a))da \mid\leq L_{\beta_1}g_1^{'}(0)Mh. \tag{5.10}$$

From Eqs (5.9) and (5.10), Eq (5.5) can be rewritten as

$$\mid L_1(t+h) - L_1(t) \mid\leq (\hat{\beta}_1(\zeta+\hat{\gamma}_1+\hat{\rho}) + L_{\beta_1} + \hat{\beta}_1(\xi_1\omega+\xi_2\delta))g_1^{'}(0)Mh.$$

Denote $M_{L_1} = (\hat{\beta}_1(\zeta+\hat{\gamma}_1+\hat{\rho}) + L_{\beta_1} + \hat{\beta}_1(\xi_1\omega+\xi_2\delta))g_1^{'}(0)M$. Then,

$$\mid L_1(t+h) - L_1(t) \mid\leq M_{L_1}h.$$

Similarly, we have that $L_2(t),\ L_3(t),\ L_4(t),\ J_1(t)$ and $J_2(t)$ are Lipschitz continuous on $\mathbb{R}^+$. Then, there exist $M_{L_2},\ M_{L_3},\ M_{L_4},\ L_{J_1},\ L_{J_2} > 0$ such that

$$\begin{aligned} \mid L_2(t+h) - L_2(t) \mid\leq M_{L_2}h, \quad &\mid L_3(t+h) - L_3(t) \mid\leq M_{L_3}h, \\ \mid L_4(t+h) - L_4(t) \mid\leq M_{L_4}h, \quad &\mid J_1(t+h) - J_1(t) \mid\leq L_{J_1}h, \\ &\mid J_2(t+h) - J_2(t) \mid\leq L_{J_2}h. \end{aligned} \tag{5.11}$$

This completes the proof. □

Lemma 5.1. *[23, Theorem 2.46] The semiflow $\Phi : \mathbb{R}^+ \times \mathcal{X}_+ \to \mathcal{X}_+$ is asymptotically smooth if there are maps $\Theta,\ \Psi : \mathbb{R}^+ \times \mathcal{X}_+ \to \mathcal{X}_+$ such that $\Phi(t,X) = \Theta(t,X) + \Psi(t,X)$ and the following conditions hold for any bounded closed set $C \subset \mathcal{X}_+$ that is forward invariant under Φ:*
(1) $\lim_{t\to+\infty} diam\Theta(t,C) = 0$;
(2) there exists $t_C \geq 0$ such that $\Psi(t,C)$ has compact closure for each $t \geq t_C$.

Lemma 5.2. *[23, Theorem B.2] A set $C \in \mathbf{L}^1_+(0,\infty)$ has compact closure if and only if the following conditions hold:*
(i) $\sup_{f\in C}\int_0^\infty \mid f(a)\mid da < \infty$;
(ii) $\lim_{r\to\infty}\int_r^\infty \mid f(a)\mid da \to 0$ *uniformly in* $f \in C$;
(iii) $\lim_{h\to 0^+}\int_0^\infty \mid f(a+h)-f(a)\mid da = 0$ *uniformly in* $f\in C$;
(iv) $\lim_{h\to 0^+}\int_0^h \mid f(a)\mid da = 0$ *uniformly in* $f \in C$.

From the above preparations, we can show the asymptotic smoothness of the semi-flow Φ generated by system (1.2).

Theorem 5.1. *The semi-flow Φ generated by system (1.2) is asymptotically smooth.*

Proof. We first decompose the semi-flow $\Phi = \Psi + \Theta$ into two maps: $\Psi(t,x_0) := (S(t),V(t),E(t),\tilde{i}(\cdot,t),\tilde{A}(\cdot,t),R(t))$ and $\Theta(t,x_0) := (0,0,0,\tilde{\phi}_i(\cdot,t),\tilde{\phi}_A(\cdot,t),0)$, where

$$
\begin{aligned}
\tilde{i}(a,t) &= \begin{cases} (\xi_1\omega E(t-a)+\xi_2\delta R(t-a))\psi_1(a) & 0\le a<t,\\ 0 & 0\le t\le a,\end{cases}\\
\tilde{A}(a,t) &= \begin{cases} ((1-\xi_1)\omega E(t-a)+(1-\xi_2)\delta R(t-a))\psi_2(a) & 0\le a<t,\\ 0 & 0\le t\le a,\end{cases}\\
\tilde{\phi}_i(a,t) &= \begin{cases} 0 & 0\le a<t,\\ i_0(a-t)e^{-\int_{a-t}^a(\zeta+\gamma_1(s)+\rho(s))ds} & 0\le t\le a,\end{cases}\\
\tilde{\phi}_A(a,t) &= \begin{cases} 0 & 0\le a<t,\\ A_0(a-t)e^{-\int_{a-t}^a(\zeta+\gamma_2(s))ds} & 0\le t\le a.\end{cases}
\end{aligned}
\tag{5.12}
$$

Let $C\subset \mathcal{X}_+$ be a closed bounded subset with bound K.

To verify that the conditions of Lemma 5.1 are satisfied, we take two steps. First, condition (1) of Lemma 5.1 is verified in the following. Let $x_0 = (S_0,V_0,E_0,i_0(\cdot),A_0(\cdot),R_0)\in C$,

$$
\begin{aligned}
\| \Theta(t,x_0)\| &= \int_0^\infty \mid \tilde{\phi}_i(a,t)\mid da + \int_0^\infty \mid \tilde{\phi}_A(a,t)\mid da\\
&= \int_t^\infty i_0(a-t)e^{-\int_{a-t}^a(\zeta+\gamma_1(s)+\rho(s))ds}da\\
&\quad + \int_t^\infty A_0(a-t)e^{-\int_{a-t}^a(\zeta+\gamma_2(s))ds}da\\
&= \int_0^\infty i_0(\sigma)e^{-\int_\sigma^{\sigma+t}(\zeta+\gamma_1(a)+\rho(a))da}d\sigma\\
&\quad + \int_0^\infty A_0(\sigma)e^{-\int_\sigma^{\sigma+t}(\zeta+\gamma_2(a))da}d\sigma\\
&\le e^{-\zeta t}\| x_0\|\\
&\le e^{-\zeta t}M.
\end{aligned}
\tag{5.13}
$$

Hence, $\| \Theta(t,x_0)\| \to 0$ as $t\to\infty$, and $\| \Theta(t,x_0)\|$ approaches 0 with uniform exponetial speed. Therefore, $lim_{t\to+\infty} diam\Theta(t,C) = 0$ and condition (1) holds in Lemma 5.1.
Next, we will show that conditions (i)-(iv) in Lemma 5.2 hold. From Proposition 2.2,

$$0\le \tilde{i}(a,t)\le(\xi_1\omega+\xi_2\delta)Me^{-a\zeta},\quad 0\le\tilde{A}(a,t)\le((1-\xi_1)\omega+(1-\xi_2)\delta)Me^{-a\zeta}.$$

Hence, conditions (i)–(iv) of Lemma 5.2 are satisfied. In the following, we will show that condition (iii) holds. Assume sufficiently small $h \in (0, t)$. By computation, we have

$$
\begin{aligned}
& \int_0^\infty |\, \tilde{i}(a+h,t) - \tilde{i}(a,t) \,|\, da \\
= \; & \int_0^{t-h} |\, (\xi_1 \omega E(t-a-h) + \xi_2 \delta R(t-a-h)) e^{-\int_0^{a+h}(\zeta+\gamma_1(s)+\rho(s))ds} \\
& -(\xi_1 \omega E(t-a) + \xi_2 \delta R(t-a)) e^{-\int_0^{a}(\zeta+\gamma_1(s)+\rho(s))ds} \,|\, da \\
& + \int_{t-h}^{t} (\xi_1 \omega E(t-a) + \xi_2 \delta R(t-a)) e^{-\int_0^{a}(\zeta+\gamma_1(s)+\rho(s))ds} da \\
\leq \; & \int_0^{t-h} \xi_1 \omega \,|\, E(t-a-h) - E(t-a) \,|\, e^{-\int_0^{a}(\zeta+\gamma_1(s)+\rho(s))ds} da \\
& + \int_0^{t-h} \xi_2 \delta \,|\, R(t-a-h) - R(t-a) \,|\, e^{-\int_0^{a}(\zeta+\gamma_1(s)+\rho(s))ds} da \\
& + \int_{t-h}^{t} (\xi_1 \omega E(t-a) + \xi_2 \delta R(t-a)) e^{-\int_0^{a}(\zeta+\gamma_1(s)+\rho(s))ds} da \\
& + \int_0^{t-h} (\xi_1 \omega E(t-a-h) + \xi_2 \delta R(t-a-h)) \\
& \times (1 - e^{-\int_a^{a+h}(\zeta+\gamma_1(s)+\rho(s))ds}) e^{-\int_0^{a}(\zeta+\gamma_1(s)+\rho(s))ds} da. \qquad (5.14)
\end{aligned}
$$

From Propositions 2.2, 5.1, and system (1.2), we have

$$
\begin{aligned}
|\, E(t-a-h) - E(t-a) \,| \; & \leq \; (f_1'(0) g_1'(0) M \hat{\beta}_1 + f_2'(0) g_2'(0) M \hat{\beta}_2 + \zeta + \omega \\
& \quad + f_3'(0) g_3'(0) M \hat{\beta}_3 + f_4'(0) g_4'(0) M \hat{\beta}_4) M h := M_1 h, \\
|\, R(t-a-h) - R(t-a) \,| \; & \leq \; (\hat{\gamma}_1 M + \hat{\gamma}_2 M + \zeta M + \delta M) h := M_2 h. \qquad (5.15)
\end{aligned}
$$

From Eq (5.15) and $1 - e^{-x} \leq x$ for all $x \geq 0$, Eq (5.14) can be rewritten as

$$
\begin{aligned}
& \int_0^\infty |\, \tilde{i}(a+h,t) - \tilde{i}(a,t) \,|\, da \\
\leq \; & \frac{\xi_1 \omega M_1 h}{\zeta} + \frac{\xi_2 \delta M_2 h}{\zeta} + (\xi_1 \omega + \xi_2 \delta) M h + \int_0^{t-h} e^{-a\zeta} \\
& \cdot (\xi_1 \omega E(t-a-h) + \xi_2 \delta R(t-a-h)) \int_a^{a+h} (\zeta + \gamma_1(s) + \rho(s)) ds da \\
\leq \; & \frac{\xi_1 \omega M_1 h}{\zeta} + \frac{\xi_2 \delta M_2 h}{\zeta} + (\xi_1 \omega + \xi_2 \delta) M h \\
& + (\xi_1 \omega + \xi_2 \delta)(\zeta + \hat{\gamma}_1 + \hat{\rho}) \frac{Mh}{\zeta}. \qquad (5.16)
\end{aligned}
$$

Similarly, we have

$$
\int_0^\infty |\, \tilde{A}(a+h,t) - \tilde{A}(a,t) \,|\, da
$$

$$\leq \frac{(1-\xi_1)\omega M_1 h}{\zeta} + \frac{(1-\xi_2)\delta M_2 h}{\zeta} + ((1-\xi_1)\omega + (1-\xi_2)\delta)(1 + \frac{\zeta + \hat{\gamma}_1 + \hat{\rho}}{\zeta})Mh. \tag{5.17}$$

Therefore, condition (iii) of Lemma 5.2 holds true. Using Lemma 5.1, the semi-flow Φ of system (1.2) is asymptotically smooth. This completes the proof. □