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Research article

Flag-transitive point-primitive 2-(v, k, λ) designs with A_n $(n = 6, 7, 8)$ socle

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Abstract: This paper classifies all flag-transitive point-primitive nontrivial 2-(v, k, λ) designs whose automorphism groups have socle A_n for $n = 6, 7, 8$. Up to isomorphism, there are exactly 50 such designs, including 11 full designs and 3 symmetric ones.

Keywords: flag-transitive; primitive; block design; alternating group
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1. Introduction

A balanced incomplete block design (BIBD) $\mathcal{D}$ is typically characterized as a configuration in which v distinct points are distributed across b blocks such that every block comprises exactly k distinct points, each point appears in exactly r blocks, and every pair of distinct points is simultaneously contained in exactly λ blocks, where $k, r, \lambda > 0$.

The five parameters v, b, r, k, and λ of a design satisfy the following three relationships:

$$vr = bk, \quad \lambda(v-1) = r(k-1), \quad b \geq v \ \text{(Fisher's inequality)}.$$

In a combinatorial design, a *flag* (α, B) refers to a pair consisting of a point α and a block B such that α is incident with B. A design $\mathcal{D}$ is said to be *symmetric* if and only if the number of blocks equals the number of points, that is, $b = v$; otherwise, it is termed a *nonsymmetric design*. A balanced incomplete block design $\mathcal{D}$ is often denoted as a 2-(v, k, λ) design. The set of points is denoted as $\mathcal{P}$, and the set of blocks is denoted as $\mathcal{B}$.

If the parameters satisfy $2 < k < v - 1$, the design is considered *nontrivial*. A 2-(v, k, λ) design that includes every k-subset of the point set as a block is referred to as a *full design*.

A group G is *almost simple* if it satisfies $T \leq G \leq Aut(T)$ where T is a simple group called *the socle* of G. An automorphism of a design $\mathcal{D}$ is a permutation g of the point set $\mathcal{P}$ that induces a permutation on the set of blocks $\mathcal{B}$. The full automorphism group of $\mathcal{D}$, denoted by Aut($\mathcal{D}$), consists of all such automorphisms. Any subgroup of Aut($\mathcal{D}$) is considered an automorphism group of $\mathcal{D}$. The design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is said to be flag-transitive under a group $G \leq$ Aut($\mathcal{D}$) if G acts transitively on the flags of $\mathcal{D}$, and point-primitive (respectively block-primitive) if G acts primitively on $\mathcal{P}$ (respectively on $\mathcal{B}$). For further standard notation and definitions, the reader is referred to sources such as [1–4].

In 2013, by using the O'Nan-Scott theorem, Tian and Zhou [5] proved that if $\mathcal{D}$ is a symmetric 2-(v, k, λ) design with $\lambda \leq 100$ admitting a flag-transitive point-primitive automorphism group G, then G must be an almost simple or an affine group. They subsequently classified all flag-transitive point-primitive symmetric 2-(v, k, λ) designs with sporadic socle [6]. In 2022, Alavi et al. [7] presented a classification of 2-designs with $gcd(r, \lambda) = 1$ admitting flag-transitive automorphism groups. Montinaro et al. [8–10] have recently classified flag-transitive 2-designs for special values of λ. It is meaningful to consider the classification of designs with an almost simple group as the socle.

In the research on the classification of the 2-(v, k, λ) designs of the flag-transitive point-primitive group with alternating A_n as socle, Dong, Zhu et al. [11–13] gave the classification of symmetric designs with $\lambda \leq 100$, and Wang et al. [14,15] gave the classification of 2-designs under $\lambda \geq gcd(r, \lambda)^2$. Recently, Zhang, Shen et al. discussed that λ is a prime number [16] and r is a prime square p^2 [17]. In 2023, Zhang and Zhou [18] studied 2-designs with $gcd(r - \lambda, k) = 1$.

All the above work is based on the classification of 2-designs by limiting parameters. We take a different approach and consider some simple groups as the socle of the automorphism groups to complete the classification of 2-designs.

In 2020, Tian [19] completely classified flag-transitive point-primitive 2-designs with socle M_{11} and discovered exactly 14 nonisomorphic 2-designs. In 2025, the classification of 2-design with socle M_{12} was published in [20].

The classification of flag-transitive point-primitive 2-designs with alternating group A_n as socle began in 2019, and the case of $Soc(G) = A_5$ was relatively simple [21]. While the classifications of 2-designs with $Soc(G) = A_6$ and A_7 were established in 2020 by us, the case for A_8 has only recently been resolved. This paper now provides a comprehensive classification by integrating all three results (for A_6, A_7, and A_8) into a unified main theorem.

The main conclusion of this paper is as follows.

Theorem 1. *Let $\mathcal{D}$ be a nontrivial 2-(v, k, λ) design and G be a flag-transitive, point-primitive automorphism group of almost simple type. If the socle is A_n $(n = 6, 7, 8)$, then up to isomorphism, there exist 50 designs listed in Table 1.*

Remark 1. (1) There are 50 non-isomorphic 2-designs, including 47 nonsymmetric designs and 3 symmetric designs: $(15, 7, 3)$, $(15, 8, 4)$, and $(35, 18, 9)$.

(2) The *complement* of a design has parameters $(v, b, b - r, v - k, b - 2r + \lambda)$. The parameters of the two designs, which are mutually complementary, are as follows: $(6, 3, 4)$ and itself; $(7, 3, 5)$ and $(7, 4, 10)$; $(8, 3, 6)$ and $(8, 5, 20)$; $(8, 4, 15)$ and itself; $(10, 4, 2)$ and $(10, 6, 5)$; $(10, 4, 4)$ and $(10, 6, 10)$; $(10, 5, 8)$ and itself; $(10, 5, 16)$ and itself; $(15, 3, 1)$ and $(15, 12, 22)$; $(15, 5, 4)$ and $(15, 10, 18)$; $(15, 5, 12)$ and $(15, 10, 54)$; $(15, 5, 16)$ and $(15, 10, 72)$;
$(15, 6, 10)$ and $(15, 9, 24)$; $(15, 6, 40)$ and $(15, 9, 96)$; $(15, 8, 4)$ and $(15, 7, 3)$.

(3) There are 11 full designs, which are the 2-designs consisting of all k-subsets. Their parameters (v, k, λ) are as follows: $(6, 3, 4)$, $(6, 4, 6)$, $(7, 3, 5)$, $(7, 5, 10)$, $(7, 4, 10)$, $(8, 3, 6)$, $(8, 6, 15)$, $(8, 4, 15)$, $(8, 5, 20)$, $(10, 3, 8)$, and $(10, 8, 28)$.

Table 1. Designs with alternating A_n $(n = 6, 7, 8)$ socle.

No.	G	G_x	v	b	r	k	λ	Reference
1	A_6, S_6	A_5, S_5	6	20	10	3	4	$\mathcal{D}_1$ (full design)
2			6	15	10	4	6	$\mathcal{D}_2$ (full design)
3	A_6, S_6	F_{36}, $3^2{:}D_8$	10	15	6	4	2	$\mathcal{D}_3$
4			10	15	9	6	5	$\mathcal{D}_4$
5			10	60	18	3	4	$\mathcal{D}_5$
6	A_6, M_{10}	F_{36}, $3^2{:}Q_8$	10	36	18	5	8	$\mathcal{D}_6$
7	A_6, S_6, A_7, A_8	S_4, $S_4 \times 2$, $L_2(7)$, $2^3{:}L_3(2)$	15	15	8	8	4	$\mathcal{D}_7$ (symmetric design)
8	S_6, $PGL_2(9)$, $P\Gamma L_2(9)$	$3^2{:}D_8$, $3^2{:}8$, $3^2{:}[2^4]$	10	72	36	5	16	$\mathcal{D}_8$
9	M_{10}, $PGL_2(9)$, $P\Gamma L_2(9)$	$3^2{:}Q_8$, $3^2{:}8$, $3^2{:}[2^4]$	10	30	12	4	4	$\mathcal{D}_9$
10			10	30	18	6	10	$\mathcal{D}_{10}$
11			10	120	36	3	8	$\mathcal{D}_{11}$ (full design)
12			10	45	36	8	28	$\mathcal{D}_{12}$ (full design)
13			10	180	72	4	24	$\mathcal{D}_{13}$
14	$P\Gamma L_2(9)$	10:4	36	180	40	8	8	$\mathcal{D}_{14}$
15	A_7, S_7	A_6, S_6	7	35	15	3	5	$\mathcal{D}_{15}$ (full design)
16			7	21	15	5	10	$\mathcal{D}_{16}$ (full design)
17			7	35	20	4	10	$\mathcal{D}_{17}$ (full design)
18	A_7, A_8	$L_2(7)$, $2^3{:}L_3(2)$	15	35	7	3	1	$\mathcal{D}_{18}$
19			15	15	7	7	3	$\mathcal{D}_{19}$ (symmetric design)
20			15	105	28	4	6	$\mathcal{D}_{20}$
21			15	35	28	12	22	$\mathcal{D}_{21}$
22			15	105	42	6	15	$\mathcal{D}_{22}$
23			15	120	56	7	24	$\mathcal{D}_{23}$
24			15	420	84	3	12	$\mathcal{D}_{24}$
25			15	420	168	6	60	$\mathcal{D}_{25}$
26	A_7	$L_2(7)$	15	42	14	5	4	$\mathcal{D}_{26}$
27			15	70	28	6	10	$\mathcal{D}_{27}$
28			15	42	28	10	18	$\mathcal{D}_{28}$
29			15	126	42	5	12	$\mathcal{D}_{29}$
30			15	70	42	9	24	$\mathcal{D}_{30}$
31			15	210	56	4	12	$\mathcal{D}_{31}$
32			15	210	84	6	30	$\mathcal{D}_{32}$
33			15	126	84	10	54	$\mathcal{D}_{33}$
34			15	630	168	4	36	$\mathcal{D}_{34}$
35	A_7, S_7	S_5, $S_5 \times 2$	21	70	30	9	12	$\mathcal{D}_{35}$
36			21	252	60	5	12	$\mathcal{D}_{36}$
37	A_7, S_7, A_8, S_8	$(A_4 \times S_3){:}2$, $S_4 \times S_3$, $2^4{:}(S_3 \times S_3)$, $(S_4 \times S_4){:}2$	35	35	18	18	9	$\mathcal{D}_{37}$ (symmetric design)
38	A_8, S_8	A_7, S_7	8	56	21	3	6	$\mathcal{D}_{38}$ (full design)
39			8	28	21	6	15	$\mathcal{D}_{39}$ (full design)
40			8	70	35	4	15	$\mathcal{D}_{40}$ (full design)
41			8	56	35	5	20	$\mathcal{D}_{41}$ (full design)
42	A_8	$2^3{:}L_3(2)$	15	168	56	5	16	$\mathcal{D}_{42}$
43			15	280	112	6	40	$\mathcal{D}_{43}$
44			15	168	112	10	72	$\mathcal{D}_{44}$
45			15	280	168	9	96	$\mathcal{D}_{45}$
46			15	840	224	4	48	$\mathcal{D}_{46}$
47	A_8	$(A_5 \times 3){:}2$	56	840	180	12	36	$\mathcal{D}_{47}$
48			56	840	180	12	36	$\mathcal{D}_{48}$
49	S_8	$S_5 \times S_3$	56	1680	360	12	72	$\mathcal{D}_{49}$
50			56	1680	360	12	72	$\mathcal{D}_{50}$

2. Preliminaries

In this section, we state some useful results in both group theory and design theory.

Lemma 2.1 ([4]). *Let $x \in \Omega$ and $|\Omega| > 1$. A transitive group G on Ω is primitive if and only if G_x is a maximal subgroup of G.*

Lemma 2.2 ([20]). *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a nontrivial 2-design and $G \leq Aut(\mathcal{D})$. The following three claims are equivalent for any point $x \in \mathcal{P}$ and block $B \in \mathcal{B}$:*
(i) G acts flag-transitively on $\mathcal{D}$;
(ii) G acts point-transitively on $\mathcal{D}$, and G_x acts transitively on $B(x)$, where $B(x)$ denotes the set of all blocks that are incident with x;
(iii) G acts block-transitively on $\mathcal{D}$, and G_B acts transitively on the points of B.

Lemma 2.3. *Let $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ be a nontrivial flag-transitive 2-(v, k, λ) design and $G \leq Aut(\mathcal{D})$. Then the following statements hold:*
(i) $r > \lambda$, $r \geq k$, $r^2 > \lambda v$;
(ii) $b \mid |G|$, $r \mid |G_x|$, where G_x is any point-stabilizer of G.

Proof. (i) $\mathcal{D}$ is a nontrivial 2-(v, k, λ) design; hence, $2 < k < v - 1$. From equation $\lambda(v - 1) = r(k - 1)$, we get $r > \lambda$. Fisher's inequality $b \geq v$ and $vr = bk$ imply that $r \geq k$, then

$$r^2 \geq rk > rk + (\lambda - r) = r(k - 1) + \lambda = \lambda(v - 1) + \lambda = \lambda v.$$

(ii) According to Lemma 2.2, G_B acts transitively on B, and G_x acts transitively on $B(x)$, so $b \mid |G|$ and $r \mid |G_x|$ holds. □

Lemma 2.4. *Let $\mathcal{D}$ be a nontrivial 2-(v, k, λ) design and G be an automorphism group of almost simple type, and G_x be a maximal subgroup of G. If $r \mid |G_x|$, then $b \mid |G|$, but not vice versa.*

Proof. By Lemma 2.1, the group G is a primitive group.

Firstly, from $r \mid |G_x|$, it can be inferred that $b \mid |G|$ holds. From $[G : G_x] = v$, we can get $v = \frac{|G|}{|G_x|}$. Substitute it into $bk = vr$. By the hypothesis, $r \mid |G_x|$, it follows that $bk(\frac{|G_x|}{r}) = |G|$, and hence $b \mid |G|$ holds.

Secondly, if $b \mid |G|$, it does not necessarily follow that $r \mid |G_x|$. We can give a counterexample. Let $G = A_6$ and $G_x \cong F_{36}$; then $|G| = 360$, $|G_x| = 36$, and $v = 10$. Assuming $k = 4$, $\lambda = 24$, we can calculate $b = 180$ and $r = 72$. Obviously, $b \mid |G|$ holds, but $r \mid |G_x|$ does not. □

Remark 2. (1) This lemma does not conflict with Lemma 2.3(ii), and $b \mid |G|$ and $r \mid |G_x|$ may not be established at the same time without knowing whether the design $\mathcal{D}$ is flag-transitive. According to Lemma 2.4, when looking for possible design parameters, we assume that $r \mid |G_x|$ holds, so it is unnecessary to verify $b \mid |G|$.

(2) The counterexample in the proof of Lemma 2.4 also demonstrates that for $G = A_6$, a design with parameters (10,180,72,4,24) cannot be constructed. However, for $G = M_{10}$, such a design is constructible, as evidenced by design $\mathcal{D}_{13}$ in Table 1.

3. Proof of Theorem 1

In the next two subsections, we will prove Theorem 1.

3.1. Finding possible parameters of 2-designs

In this subsection, our main goal is to find possible 5-tuple parameters (v, b, r, k, λ) of 2-designs.

Lemma 3.1 ([22]). *For $n = 5$ or $n \geq 7$, the automorphism group $Aut(A_n) \cong S_n$, while $Aut(A_6) \cong P\Gamma L_2(9)$.*

Assume that there is a nontrivial 2-design $\mathcal{D}$ admitting a flag-transitive and point-primitive almost simple automorphism group G with socle A_n. According to Lemma 3.1, when $n = 6$, $G = A_6, S_6, M_{10}, PGL_2(9), P\Gamma L_2(9)$, and when $n = 7, 8$, $G = A_n, S_n$.

If M is any maximal subgroup of G, then the permutation action of G on the cosets of M is primitive, so G embeds as a primitive subgroup of S_m, where $m = [G : M]$.

According to Lemma 2.1, if and only if the stabilizer G_x is a maximum subgroup of G, where $x \in \mathcal{P}$, then G is point-primitive on $\mathcal{P}$. Consequently, $v = [G : G_x]$. In the ATLAS, the maximal subgroups of G are listed [2].

We calculate all possible parameters (v, b, r, k, λ) that meet the requirements listed below:

(i) $G \in \{A_n, S_n, M_{10}, PGL_2(9), P\Gamma L_2(9)\}$ with $6 \leq n \leq 8$, and G_x is one of its maximal subgroups with the index greater than 2.
(ii) $v = [G : G_x]$, and we list the set of v in Table 2;
(iii) $2 < k < v - 1$;
(iv) $r \mid |G_x|$, $r > \lambda$ and $r^2 > \lambda v$ (Lemma 2.3);
(v) $bk = vr$, $\lambda(v - 1) = r(k - 1)$;
(vi) $v \leq b \leq \binom{v}{k}$.

We obtain 592 5-tuple parameters (v, b, r, k, λ) with the help of the computer algebra system GAP [23]. The number of possible 5-tuple corresponding to group G are listed in Table 2.

Table 2. The number of possible 5-tuple (v, b, r, k, λ).

G	The set of v	Total
A_6	{6,10,15}	19
S_6	{6,10,15}	25
M_{10}	{10,36,45}	22
$PGL_2(9)$	{10,36,45}	22
$P\Gamma L_2(9)$	{10,36,45}	26
A_7	{7,15,21,35}	120
S_7	{7,21,35,120}	101
A_8	{8,15,28,35,56}	157
S_8	{8,28,35,56,105,120}	100

These possible 5-tuple of parameters are verified one by one, and most of them are eliminated in the following three steps:

Step (i) By Lemma 2.2, the group G acts block-transitively on the design $\mathcal{D}$. Hence, the stabilizer G_B of a block $B \in \mathcal{B}$ is a subgroup of G with index b, where $b = [G : G_B]$. To determine whether G contains at least one subgroup with index b, one may use the command

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Subgroups(G : OrderEqual := n)
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in Magma [24], where $n = |G|/b$.

Step (ii) Since G_B acts transitively on the points of B, it must have an orbit O of length k. Moreover, the G-orbit of O must have size b. It is necessary to verify whether such an orbit O indeed exists under the action of G_B.

Step (iii) For any two distinct points $\alpha, \beta \in \mathcal{P}$, the number of blocks containing both must be exactly λ. One must verify that this condition holds uniformly for all pairs of points.

Clearly, it is computationally intensive to eliminate all parameter sets that fail the conditions. Here are some concrete examples of applying the above three steps to exclude parameters:

Example 1. (v, b, r, k, λ)=$(15, 63, 21, 5, 6)$ with $G = A_7$.

There is no subgroup with index 63 of G, so these parameters can be eliminated by Step (i).

Example 2. (v, b, r, k, λ)=$(56, 70, 15, 12, 3)$ with $G = S_8$.

There are 3 conjugacy classes of subgroups with index 70 of G, denoted by H, J, and K as representatives. The orbit lengths of H are 4^2, 24^2, and the orbit lengths of both J and K are 8 and 48. The notation s^t means that the number s appears with multiplicity t. There is no orbit O of G_B with size $k = 12$, so these parameters can be eliminated by Step (ii).

Example 3. (v, b, r, k, λ)=$(45, 60, 16, 12, 4)$ with $G = M_{10}$. There is only one conjugacy class of subgroups with index 60 of G, denoted by H, so $G_B \cong H$.

The generators of G are listed:

g_1= (1, 2, 7)(3, 11, 27)(4, 14, 31)(5, 18, 32)(6, 20, 36)(8, 24, 39)
(9, 25, 28)(10, 26, 42)(12, 15, 16)(13, 30, 40)(17, 19, 21)
(22, 35, 44)(23, 33, 29)(34, 43, 37)(38, 45, 41),
g_2= (1, 5, 7, 13)(2, 9)(3, 6, 22, 23)(4, 16, 14, 19)(8, 10)(11, 29, 44,
20)(12, 15, 17, 21)(18, 34, 40, 41)(24, 36, 39, 33)(25, 37, 28, 38)
(26, 35, 42, 27)(30, 43, 32, 45),
g_3= (1, 4)(3, 12)(5, 19)(6, 21)(7, 14)(8, 10)(11, 20)(13, 16)(15, 23)
(17, 22)(18, 33)(24, 41)(25, 28)(26, 43)(27, 32)(29, 44)(30, 35)
(34, 39)(36, 40)(42, 45),
g_4= (1, 6, 7, 23)(2, 10)(3, 13, 22, 5)(4, 17, 14, 12)(8, 9)(11, 28, 44,
25)(15, 19, 21, 16)(18, 35, 40, 27)(20, 38, 29, 37)(24, 32, 39, 30)
(26, 41, 42, 34)(33, 45, 36, 43).

The orbits of H acting on $\Omega = \{1, 2, \ldots, 45\}$ are

O_1={ 9, 25, 28 }, O_2={ 1, 2, 4, 7, 14, 31 },
O_3={ 3, 11, 13, 17, 19, 21, 23, 27, 29, 30, 33, 40 },
O_4={ 5, 6, 12, 15, 16, 18, 20, 22, 32, 35, 36, 44 },
O_5={ 8, 10, 24, 26, 34, 37, 38, 39, 41, 42, 43, 45 }.

The orbit lengths of H are 3, 6 and 12^3. There exist two orbits, $O_i(i = 3, 4)$ with size 12 and $|O_i^G| = 60$, while orbit O_5 has size 12 and $|O_5^G| = 30$. For $1 \leq j < k \leq 45$, 2-subset $\{j, k\}$ is incident with 0, 3, 6, or 8 blocks in $O_i(i = 3, 4)$, i.e., point pair intersection numbers are {0, 3, 6, 8}, so it is not a fixed number $\lambda = 4$.

Thus, these parameters can be eliminated by Step (iii).

Example 4. (v, b, r, k, λ)=$(120, 2240, 336, 18, 48)$ with $G = S_8$.

The symmetric group S_8 contains 7 conjugacy classes of subgroups with index 2240, and we analyze them one by one.

We list the serial number of the conjugate class of subgroups in the first column in Table 3. The orbital lengths under the action of the conjugate class in the second column. If the orbital length is 18, then the number of elements in the set generated by this orbital under the action of group G is indicated in parentheses. In the third column, the treatment method for this situation is given. If some orbit O has orbital length 18, and $|O^G| = 2240$, then we list the point pair $\{\alpha, \beta\}$ intersection numbers in the fourth column.

Table 3. Analysis of possible 5-tuple $(120, 2240, 336, 18, 48)$.

Conjugacy class	Orbital lengths	Treatment methods	Point pair $\{\alpha, \beta\}$ intersection numbers
1	6^2, $18(560)^2$, $18(1120)^4$	Step(ii)	—
2	3^4, 9^4, $18(1120)^4$	Step(ii)	—
3	6^2, $18(560)^2$, $18(1120)^4$	Step(ii)	—
4	3^2, 6, 9^2, 18(560), $18(1120)^4$	Step(ii)	—
5	6^2, $18(1120)^2$, $18(2240)^4$	Step(ii), Step(iii)	{12, 48, 60, 72}
6	6^2, $18(1120)^2$, $18(2240)^4$	Step(ii), Step(iii)	{0, 48, 60}
7	3^2, 6, 9^6, 18(560), $18(1120)^2$	Step(ii)	—

3.2. Base blocks of 2-designs

After the three-step verification in the last subsection, most of the 5-tuple of parameters are excluded, and the remaining parameters can form 50 different 2-designs up to isomorphism, as shown in Table 1.

The corresponding base blocks are listed in Table 4. Up to isomorphism, if different automorphism groups have the same base block, put them together.

Proof of Theorem 1. Through the discussion of the above two subsections, we get 50 different 2-designs up to isomorphism. According to the parameters, there are 11 full designs and 3 symmetric designs.

This completes the proof of Theorem 1. □

Table 4. Up to isomorphism, the base block of each design.

No.	G	G_x	(v, k, λ)	Base block	Design
1	A_6, S_6	A_5, S_5	(6, 3, 4)	{ 1, 2, 3 }	$\mathcal{D}_1$
2			(6, 4, 6)	{ 1, 2, 3, 4 }	$\mathcal{D}_2$
3	A_6, S_6	F_{36}, $3^2{:}D_8$	(10, 4, 2)	{ 1, 4, 8, 9 }	$\mathcal{D}_3$
4			(10, 6, 5)	{ 2, 3, 5, 6, 7, 10 }	$\mathcal{D}_4$
5			(10, 3, 4)	{ 1, 2, 3 }	$\mathcal{D}_5$
6	A_6, M_{10}	F_{36}, $3^2{:}Q_8$	(10, 5, 8)	{ 1, 2, 5, 8, 9 }	$\mathcal{D}_6$
7	A_6, S_6, A_7, A_8	S_4, $S_4 \times 2$, $L_2(7)$, $2^3{:}L_3(2)$	(15, 8, 4)	{ 8, 9, 10, 11, 12, 13, 14, 15 }	$\mathcal{D}_7$
8	S_6, $PGL_2(9)$, $P\Gamma L_2(9)$	$3^2{:}D_8$, $3^2{:}8$, $3^2{:}[2^4]$	(10, 5, 16)	{ 2, 3, 7, 8, 10 }	$\mathcal{D}_8$
9	M_{10}, $PGL_2(9)$, $P\Gamma L_2(9)$	$3^2{:}Q_8$, $3^2{:}8$, $3^2{:}[2^4]$	(10, 4, 4)	{ 3, 5, 7, 10 }	$\mathcal{D}_9$
10			(10, 6, 10)	{ 1, 2, 4, 6, 8, 9 }	$\mathcal{D}_{10}$
11			(10, 3, 8)	{ 1, 2, 3 }	$\mathcal{D}_{11}$
12			(10, 8, 28)	{ 1, 2, 3, 4, 5, 6, 7, 8 }	$\mathcal{D}_{12}$
13			(10, 4, 24)	{ 1, 2, 5, 9 }	$\mathcal{D}_{13}$
14	$P\Gamma L_2(9)$	10:4	(36, 8, 8)	{ 4, 5, 17, 22, 27, 31, 32, 35 }	$\mathcal{D}_{14}$
15	A_7, S_7	A_6, S_6	(7, 3, 5)	{ 1, 2, 3 }	$\mathcal{D}_{15}$
16			(7, 5, 10)	{ 1, 2, 3, 4, 5 }	$\mathcal{D}_{16}$
17			(7, 4, 10)	{ 4, 5, 6, 7 }	$\mathcal{D}_{17}$
18	A_7, A_8	$L_2(7)$, $2^3{:}L_3(2)$	(15, 3, 1)	{ 1, 2, 3 }	$\mathcal{D}_{18}$
19			(15, 7, 3)	{ 1, 2, 3, 4, 5, 6, 7 }	$\mathcal{D}_{19}$
20			(15, 4, 6)	{ 2, 3, 8, 9 }	$\mathcal{D}_{20}$
21			(15, 12, 22)	{ 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15 }	$\mathcal{D}_{21}$
22			(15, 6, 15)	{ 2, 5, 9, 11, 12, 14 },	$\mathcal{D}_{22}$
23			(15, 7, 24)	{ 4, 6, 7, 12, 13, 14, 15 }	$\mathcal{D}_{23}$
24			(15, 3, 12)	{ 4, 6, 11 }	$\mathcal{D}_{24}$
25			(15, 6, 60)	{ 2, 3, 6, 11, 14, 15 }	$\mathcal{D}_{25}$
26	A_7	$L_2(7)$	(15, 5, 4)	{ 4, 11, 12, 13, 14 }	$\mathcal{D}_{26}$
27			(15, 6, 10)	{ 2, 3, 9, 10, 13, 15 }	$\mathcal{D}_{27}$
28			(15, 10, 18)	{ 1, 2, 3, 5, 6, 7, 8, 9, 10, 15 }	$\mathcal{D}_{28}$
29			(15, 5, 12)	{ 2, 4, 5, 9, 10 }	$\mathcal{D}_{29}$
30			(15, 9, 24)	{ 1, 4, 5, 6, 7, 8, 11, 12, 14 }	$\mathcal{D}_{30}$
31			(15, 4, 12)	{ 4, 7, 14, 15 }	$\mathcal{D}_{31}$
32			(15, 6, 30)	{ 1, 6, 11, 13, 14, 15 }	$\mathcal{D}_{32}$
33			(15, 10, 54)	{ 1, 3, 6, 7, 8, 11, 12, 13, 14, 15 }	$\mathcal{D}_{33}$
34			(15, 4, 36)	{ 2, 4, 8, 15 }	$\mathcal{D}_{34}$
35	A_7, S_7	S_5, $S_5 \times 2$	(21, 9, 12)	{ 7, 8, 9, 14, 15, 17, 18, 19, 20 }	$\mathcal{D}_{35}$
36			(21, 5, 12)	{ 7, 8, 14, 18, 21 }	$\mathcal{D}_{36}$
37	A_7, S_7, A_8, S_8	$(A_4 \times S_3){:}2$, $S_4 \times S_3$, $2^4{:}(S_3 \times S_3)$, $(S_4 \times S_4){:}2$	(35, 18, 9)	{ 2, 5, 6, 11, 13, 14, 16, 17, 18, 21, 22, 23, 25, 28, 30, 31, 32, 35 }	$\mathcal{D}_{37}$
38	A_8, S_8	A_7, S_7	(8, 3, 6)	{ 1, 2, 3 }	$\mathcal{D}_{38}$
39			(8, 6, 15)	{ 1, 2, 3, 4, 5, 6 }	$\mathcal{D}_{39}$
40			(8, 4, 15)	{ 1, 2, 3, 4 }	$\mathcal{D}_{40}$
41			(8, 5, 20)	{ 4, 5, 6, 7, 8 }	$\mathcal{D}_{41}$
42	A_8	$2^3{:}L_3(2)$	(15, 5, 16)	{ 1, 3, 7, 10, 15 }	$\mathcal{D}_{42}$
43			(15, 6, 40)	{ 2, 5, 6, 7, 10, 12 }	$\mathcal{D}_{43}$
44			(15, 10, 72)	{ 2, 4, 5, 6, 8, 9, 11, 12, 13, 14 }	$\mathcal{D}_{44}$
45			(15, 9, 96)	{ 1, 3, 4, 8, 9, 11, 13, 14, 15 }	$\mathcal{D}_{45}$
46			(15, 4, 48)	{ 1, 8, 12, 14 }	$\mathcal{D}_{46}$
47	A_8	$(A_5 \times 3){:}2$	(56, 12, 36)	{ 1, 2, 3, 10, 19, 23, 34, 37, 41, 44, 48, 52 }	$\mathcal{D}_{47}$
48			(56, 12, 36)	{ 1, 3, 19, 20, 25, 31, 34, 36, 41, 43, 44, 45 }	$\mathcal{D}_{48}$
49	S_8	$S_5 \times S_3$	(56, 12, 72)	{ 1, 17, 18, 19, 21, 26, 39, 40, 41, 42, 45, 50 }	$\mathcal{D}_{49}$
50			(56, 12, 72)	{ 2, 6, 9, 10, 15, 16, 20, 22, 37, 47, 52, 53 }	$\mathcal{D}_{50}$

4. Open questions

In this paper, we studied the classification problem of flag-transitive point-primitive 2-design with alternating group A_n (n=6, 7, 8) as socle. The following open questions are raised.

Problem 1. Can n be extended to a larger value?

Problem 2. Do there exist families of 2-(v, k, λ) designs, with specific characteristic properties, that are invariant under the action of the alternating group A_n?

5. Conclusions

In this paper, we have completed the classification of all nontrivial flag-transitive point-primitive 2-(v, k, λ) designs whose automorphism groups have socle A_n for n=6, 7, 8. Through a systematic search and verification process using group-theoretic methods and computational tools such as GAP and Magma , we identified exactly 50 non-isomorphic designs, including 11 full designs and 3 symmetric ones.

Our approach differs from previous studies that restricted the parameters λ or r; instead, we focused on the structure of the automorphism group itself. By leveraging the maximal subgroups of the almost simple groups with socle A_6, A_7, A_8, we enumerated all possible parameter sets and rigorously verified their realizability through three key steps: subgroup existence, orbit analysis, and pairwise intersection consistency.

The results not only extend the known classifications for smaller alternating groups but also provide a complete and unified treatment for n=6, 7, 8. This work contributes to the broader program of classifying flag-transitive designs with almost simple socle, and opens the door to further investigations for larger alternating groups or other families of simple groups.

Future research may focus on extending this classification to $n \geq 9$, or on identifying infinite families of 2-designs with specific combinatorial or group-theoretic properties preserved under the action of A_n.

Author contributions

Delu Tian: conceptualization, methodology, writing-original draft, software, formal Analysis. Qianfen Liao: data curation, investigation, writing. Zhilin Zhang: supervision, resources, reviewing and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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