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Research article

Stochastic Korteweg–de Vries-type systems: Local and global theory

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Abstract: This paper studied the Cauchy problem for a system of coupled Korteweg-de Vries (KdV) equations driven by multiplicative space-time white noise. We established local well-posedness for the system, proving that for $\mathcal{F}_0$-measurable initial data (ϕ_0, φ_0) in the Sobolev space $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ with $s > -5/8$, and with the noise operator Ξ belonging to the intersection of Hilbert-Schmidt spaces $L_2^{0,s} \cap L_2^{0,s,-\frac{3}{8}}$, there exists a unique local solution. Furthermore, we demonstrated global well-posedness in the energy space $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ for L^2-valued initial data and with $\Xi \in L_2^{0,0} \cap L_2^{0,0,-\frac{3}{8}}$. The analysis employed Fourier restriction norm methods, utilizing Bourgain-type spaces $X^{s,b}$ and $Y^{s_1,s_2,b}$. Key to the proofs was the establishment of crucial linear and bilinear estimates within these spaces and a detailed analysis of the stochastic convolution via Itô calculus. A fixed-point argument was then applied to obtain the local solution, while global existence followed from an invariance property (conservation) of the L^2 norm, a martingale inequality, and an approximation procedure. The work extends previous results on single stochastic KdV equations to a more complex coupled system, providing a robust framework for analyzing nonlinear wave propagation subject to random perturbations, with applications in plasma physics and fluid dynamics.

Keywords: white noise; stochastic; Bourgain space; KdV equation; bilinear estimates
Mathematics Subject Classification: 49K40, 60H15, 60H40

1. Introduction

The noise-driven Korteweg–de Vries equation

$$\frac{\partial \phi}{\partial t} + \frac{\partial^3 \phi}{\partial x^3} + \phi \frac{\partial \phi}{\partial x} = \Xi \frac{\partial^2 \mathbb{B}}{\partial t \partial x} \tag{1.1}$$

arises naturally in the study of weakly nonlinear wave propagation in plasmas subject to random fluctuations [1]. It provides a more realistic description than the deterministic KdV equation by incorporating the effects of stochastic perturbations. Such random terms may model thermal noise, external forcing, or uncertainties in the medium properties. Beyond plasma physics, stochastic KdV-type equations are also relevant in fluid dynamics, nonlinear optics, and other dispersive wave phenomena. In these broader contexts, the stochastic terms can be interpreted as effective representations of physical effects neglected or averaged out in the deterministic model.

Bourgain introduced new space-time function spaces, now known as Bourgain spaces, based on the linear Airy group in his work [2]. Using these spaces, he demonstrated the global well-posedness of the Korteweg-de Vries equation in $L^2(\mathbb{R})$. Following his approach, Kenig et al. established existence and uniqueness results in Sobolev spaces of negative index, specifically in $H^s(\mathbb{R})$ for $s > -3/4$ ([3,4]). The main challenge in applying Bourgain's method to the Korteweg-de Vries equation is proving a suitable bilinear estimate in Bourgain spaces.

$$\|\partial_x(\phi\varphi)\|_{X^{s,b-1}} \lesssim \|\phi\|_{X^{s,b}}\|\varphi\|_{X^{s,b}},$$

which is crucial for controlling the quadratic nonlinearity. Once this estimate is established, the contraction mapping principle can be applied to obtain local well-posedness, which in turn leads to global results through conservation laws.

The stochastic Korteweg-de Vries equation, as discussed in Eq (1.1), has been studied by various authors. For further reading, we refer the reader to sources [5–7]. Notably, de Bouard et al. [6] demonstrated that for almost every $\omega \in \Omega$, there exists a time $T_\omega > 0$ and a unique solution $\phi(t)$ to Eq (1.1) defined on the interval $[0, T_\omega]$.

$$\phi \in C\left([0, T_\omega], H^s(\mathbb{R})\right) \cap \left(X^{T_\omega}_{s,b} \cap \dot{X}^{T_\omega}_{s,-\frac{3}{8},b}\right),$$

under the assumptions that $\phi_0(x,\omega) \in H^s(\mathbb{R})$ with $s > -5/8, \phi_0$ is $\mathcal{F}_0 -$ *measurable*, and $\Xi \in L_2^{0,s} \cap L_2^0\left(L^2(\mathbb{R}), \dot{H}^{s,-\frac{3}{8}}(\mathbb{R})\right)$. Moreover, they showed that if $\Xi \in L_2^{0,0} \cap L_2^0\left(L^2(\mathbb{R}), \dot{H}^{0,-\frac{3}{8}}(\mathbb{R})\right)$ and $\phi_0 \in L^2\left(\Omega, L^2(\mathbb{R})\right)$ is $\mathcal{F}_0$-measurable, then the solution of (1.1) is global and belongs to

$$L^2\left(\Omega, C\left([0,T], L^2(\mathbb{R})\right)\right).$$

While the single stochastic KdV equation provides a fundamental model, many physical systems are inherently characterized by the interaction of multiple wave modes. This necessitates the study of *coupled* KdV systems. The KdV-type systems (1.2) have garnered significant attention in the literature (see, e.g., [8–11]). For instance, in plasmas, they can describe the resonant interaction of long-wave and short-wave modes [10], while in fluid dynamics, they model bidirectional wave propagation or the interaction of internal waves in stratified fluids [12, 13]. In this context, Tadahiro Oh [9] examined the local well-posedness problem for

$$\begin{cases} \dfrac{\partial\phi}{\partial t} + \dfrac{\partial^3\phi}{\partial^3 x} + \alpha_1\phi\dfrac{\partial\phi}{\partial x} + \alpha_2\varphi\dfrac{\partial\varphi}{\partial x} + \alpha_3\phi\dfrac{\partial\varphi}{\partial x} + \alpha_4\varphi\dfrac{\partial\phi}{\partial x} = 0 \\ \dfrac{\partial\varphi}{\partial t} + \beta\dfrac{\partial^3\varphi}{\partial^3 x} + \beta_1\phi\dfrac{\partial\phi}{\partial x} + \beta_2\varphi\dfrac{\partial\varphi}{\partial x} + \beta_3\phi\dfrac{\partial\varphi}{\partial x} + \beta_4\varphi\dfrac{\partial\phi}{\partial x} = 0, \quad 0 < \beta \le 1, \end{cases} \tag{1.2}$$

in both periodic and non-periodic settings, with initial data $(\phi_0, \varphi_0) \in H^s(\mathbb{T}_\lambda) \times H^s(\mathbb{T}_\lambda)$ or $H^s(\mathbb{R}) \times H^s(\mathbb{R})$. In particular, he established sharp local well-posedness thresholds depending on the coupling parameter β. In the periodic case, resonance phenomena were analyzed using Diophantine conditions, leading to local well-posedness results in $H^s(\mathbb{T}_\lambda)$ for $s \geq s^*(\beta)$, where $s^*(\beta) \in (1/2, 1]$ is determined by the Diophantine properties of specific constants that arise from β. In the non-periodic case, he proved a sharp result in the energy space, showing local well-posedness (and in some cases global well-posedness) in $L^2(\mathbb{R})$.

However, in realistic environments, these interactions are subject to random perturbations. Extending the well-posedness theory from single equations to stochastically driven coupled systems presents significant new challenges. While our focus is on theoretical well-posedness, the phenomena modeled by these systems, such as the complex soliton interactions studied numerically for higher-order KdV equations [14, 15], highlight the need for a rigorous foundation. Furthermore, the robust analysis of such nonlinear systems can provide valuable insights for related computational and inverse problems [16, 17], establishing a theoretical benchmark for future numerical studies of stochastic soliton dynamics.

In this paper, inspired by the works of [6, 18–21], we study the Cauchy problem for a system of coupled Korteweg-de Vries equations driven by white noise (1.3).

$$\begin{cases} \dfrac{\partial\phi}{\partial t} + \dfrac{\partial^3\phi}{\partial^3 x} + \alpha_1\phi\dfrac{\partial\phi}{\partial x} + \alpha_2\varphi\dfrac{\partial\varphi}{\partial x} + \alpha_3\phi\dfrac{\partial\varphi}{\partial x} + \alpha_4\varphi\dfrac{\partial\phi}{\partial x} = \Xi\dfrac{\partial^2\mathbb{B}}{\partial t\partial x} \\ \dfrac{\partial\varphi}{\partial t} + \dfrac{\partial^3\varphi}{\partial^3 x} + \beta_1\phi\dfrac{\partial\phi}{\partial x} + \beta_2\varphi\dfrac{\partial\varphi}{\partial x} + \beta_3\phi\dfrac{\partial\varphi}{\partial x} + \beta_4\varphi\dfrac{\partial\phi}{\partial x} = \Xi\dfrac{\partial^2\mathbb{B}}{\partial t\partial x} \\ \phi(x,0) = \phi_0(x), \quad \varphi(x,0) = \varphi_0(x), \end{cases} \tag{1.3}$$

where $\alpha_i, \beta_i, i = 1 \cdots 4$, are real constants and $(x,t) \in \mathbb{R} \times \mathbb{R}^+$.

We demonstrate that the stochastic Korteweg–de Vries-type system is locally well-posed in the Sobolev spaces $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for all $s > -5/8$. Furthermore, we establish that solutions can be extended globally in time when the initial data belongs to the energy space $L^2(\mathbb{R}) \times L^2(\mathbb{R})$. In this context, $\phi = \phi(x,t)$ and $\varphi = \varphi(x,t)$ are random processes defined for the pairs $(x,t) \in \mathbb{R} \times \mathbb{R}^+$. The symbol Ξ denotes a linear operator, while $\mathbb{B}$ represents a two-parameter Brownian motion on $\mathbb{R} \times \mathbb{R}^+$. This means $\mathbb{B}$ is a zero-mean Gaussian process with the following correlation function:

$$\mathbb{E}(\mathbb{B}(x,t)\mathbb{B}(y,s)) = (t \wedge s)(x \wedge y),$$

where $t, s \in \mathbb{R}^+$, $x, y \in \mathbb{R}$, and $(\cdot,\cdot)$ denotes the $L^2(\mathbb{R})$ space duality product.

To articulate our results clearly, we will first define the notation we will use. We begin with function spaces. For $s \in \mathbb{R}$, $H^s(\mathbb{R})$ denotes the standard Sobolev space of order s, which is defined by the norm

$$\|\omega\|_{H^s(\mathbb{R})} = \|\omega\|_{H^s} = \left(\int_{\mathbb{R}} (1 + |\zeta|)^{2s} |\hat{\omega}|^2 d\zeta\right)^{1/2}.$$

Additionally, for $s_1, s_2 \in \mathbb{R}$, we consider the anisotropic Sobolev space $\dot{H}^{s_1,s_2}(\mathbb{R})$, which is defined by the norm

$$\|\omega\|_{\dot{H}^{s_1,s_2}(\mathbb{R})} = \|\omega\|_{\dot{H}^{s_1,s_2}} = \left(\int_{\mathbb{R}} |\zeta|^{2s_2} (1 + |\zeta|)^{2s_1} |\hat{\omega}|^2 d\zeta\right)^{1/2}.$$

Here, $\hat{\omega}$ represents the spatial Fourier transform, given by

$$\hat{\omega}(\zeta) = \int_{\mathbb{R}} \omega(x) e^{-ix\zeta} dx.$$

Given real parameters s and b, the Bourgain spaces $X^{s,b}(\mathbb{R}^2)$ and the Bourgain-type spaces $Y^{s_1,s_2,b}(\mathbb{R}^2)$ are defined by the norms

$$\|\vartheta\|_{X^{s,b}(\mathbb{R}^2)} = \|\vartheta\|_{X^{s,b}} = \left(\int_{\mathbb{R}^2} (1+|\zeta|)^{2s} (1+|\gamma-\zeta^3|)^{2b} |\tilde{\vartheta}(\gamma,\zeta)|^2 d\zeta d\gamma \right)^{1/2},$$

$$\|\vartheta\|_{Y^{s_1,s_2,b}(\mathbb{R}^2)} = \|\vartheta\|_{Y^{s_1,s_2,b}} = \left(\int_{\mathbb{R}^2} |\zeta|^{2s_2} (1+|\zeta|)^{2s_1} (1+|\gamma-\zeta^3|)^{2b} |\tilde{\vartheta}(\gamma,\zeta)|^2 d\zeta d\gamma \right)^{1/2},$$

with $\tilde{\vartheta}$ representing the spacetime Fourier transform.

$$\tilde{\vartheta}(\zeta,\gamma) = \int_{\mathbb{R}^2} \vartheta(x,t) e^{-i(x\zeta+t\gamma)}\, dxdt.$$

For $T > 0$, we also consider the restricted spaces $Y_T^{s_1,s_2,b}$ and $X_T^{s,b}$, consisting of the restrictions to the interval $[0,T]$ of functions in $Y^{s_1,s_2,b}$ and $X^{s,b}$, respectively. These spaces are equipped with the norms

$$\|\vartheta\|_{X_T^{s,b}} = \inf\left\{ \|z\|_{X^{s,b}} : \quad \vartheta(x,t) = z(x,t) \quad \text{on} \quad \mathbb{R}\times[0,T] \right\}.$$

$$\|\vartheta\|_{Y_T^{s_1,s_2,b}} = \inf\left\{ \|z\|_{Y^{s_1,s_2,b}} : \quad \vartheta(x,t) = z(x,t) \quad \text{on} \quad \mathbb{R}\times[0,T] \right\}.$$

Given that our analysis involves systems of equations, we require the use of product function spaces. Accordingly, we define

$$\mathcal{X}^{s,b} = X^{s,b} \times X^{s,b}, \quad \mathcal{X}_T^{s,b} = X_T^{s,b} \times X_T^{s,b},$$
$$\mathcal{Y}^{s_1,s_2,b} = Y^{s_1,s_2,b} \times Y^{s_1,s_2,b}, \quad \mathcal{Y}_T^{s_1,s_2,b} = Y_T^{s_1,s_2,b} \times Y_T^{s_1,s_2,b},$$

and

$$\mathcal{H}^s = H^s \times H^s, \quad \dot{\mathcal{H}}^{s_1,s_2} = \dot{H}^{s_1,s_2} \times \dot{H}^{s_1,s_2},$$

with norms

$$\|(\phi,\varphi)\|_{\mathcal{X}^{s,b}} = max\{\|\phi\|_{X^{s,b}}, \|\varphi\|_{X^{s,b}}\},$$
$$\|(\phi,\varphi)\|_{\mathcal{X}_T^{s,b}} = max\{\|\phi\|_{X_T^{s,b}}, \|\varphi\|_{X_T^{s,b}}\},$$
$$\|(\phi,\varphi)\|_{\mathcal{Y}^{s_1,s_2,b}} = max\{\|\phi\|_{Y^{s_1,s_2,b}}, \|\varphi\|_{Y^{s_1,s_2,b}}\},$$
$$\|(\phi,\varphi)\|_{\mathcal{Y}_T^{s_1,s_2,b}} = max\{\|\phi\|_{Y_T^{s_1,s_2,b}}, \|\varphi\|_{Y_T^{s_1,s_2,b}}\},$$
$$\|(\phi_0,\varphi_0)\|_{\mathcal{H}^s} = max\{\|\phi_0\|_{H^s}, \|\varphi_0\|_{H^s}\},$$

and

$$\|(\phi_0,\varphi_0)\|_{\dot{\mathcal{H}}^{s_1,s_2}} = max\{\|\phi_0\|_{\dot{H}^{s_1,s_2}}, \|\varphi_0\|_{\dot{H}^{s_1,s_2}}\}.$$

Finally, the space of Hilbert-Schmidt operators from $L^2(\mathbb{R})$ into $H^s(\mathbb{R})$ will be denoted by

$$L_2^{0,s} = L_2^0\left(L^2(\mathbb{R}), H^s(\mathbb{R})\right),$$

and is equipped with the norm

$$\|\Xi\|^2_{L_2^{0,s}} = \sum_{i=1}^{\infty} \|\Xi e_i\|^2_{H^s(\mathbb{R})}.$$

We also denote by

$$L_2^{0,s_1,s_2} = L_2^0(L^2(\mathbb{R}), \dot{H}^{s_1,s_2}(\mathbb{R}))$$

the space of Hilbert–Schmidt operators from $L^2(\mathbb{R})$ into $\dot{H}^{s_1,s_2}(\mathbb{R})$, endowed with the norm

$$\|\Xi\|^2_{L_2^{0,s_1,s_2}} = \sum_{i=1}^{\infty} \|\Xi e_i\|^2_{\dot{H}^{s_1,s_2}(\mathbb{R})},$$

where $(e_i)_{i\geq 1}$ denotes an orthonormal basis of $L^2(\mathbb{R})$.

With the above notation, we now present the principal results of this work. Let $(\Omega, \mathcal{F}, P)$ denote a probability space endowed with a filtration $(\mathcal{F}t)t \geq 0$. For system (1.3), the following local and global well-posedness theorems hold:

Theorem 1.1. *Let $s > -5/8$ and assume $\Xi \in L_2^{0,s} \cap L_2^{0,s,-\frac{3}{8}}$. Suppose the initial data $(\phi_0(x,\omega), \varphi_0(x,\omega))$ satisfy*

$$(\phi_0, \varphi_0) \in \mathcal{H}^s \quad \textit{for} \quad \omega \in \Omega, \qquad (\phi_0, \varphi_0) \quad \textit{is} \quad \mathcal{F}_0 - \textit{measurable}.$$

There exists a time $T_\varrho > 0$ such that the initial value problem (1.3) admits a unique solution $(\phi(t), \varphi(t))$ on $[0, T_\varrho]$, with

$$(\phi, \varphi) \in \left(C\left(\left[0, T_\varrho\right], H^s(\mathbb{R})\right) \times C\left(\left[0, T_\varrho\right], H^s(\mathbb{R})\right)\right) \cap \left(\left(X^{s,b}_{T_\varrho} \cap Y^{s,-\frac{3}{8},b}_{T_\varrho}\right) \times \left(X^{s,b}_{T_\varrho} \cap Y^{s,-\frac{3}{8},b}_{T_\varrho}\right)\right).$$

When $s = 0$, global existence follows from the conservation of the $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ norm for solutions of the Cauchy problem associated with the coupled Korteweg–de Vries-type system driven by white noise (1.3). More precisely, we state

Theorem 1.2. *Assume $\Xi \in L_2^{0,0} \cap L_2^{0,0,-\frac{3}{8}}$, and suppose that*

$$(\phi_0, \varphi_0) \in L^2\left(\Omega, L^2(\mathbb{R})\right) \times L^2\left(\Omega, L^2(\mathbb{R})\right), \qquad (\phi_0, \varphi_0) \quad \textit{is} \quad \mathcal{F}_0 - \textit{measurable}.$$

Then the unique solution (ϕ, φ) obtained in Theorem 1.1 exists globally in time and satisfies

$$(\phi, \varphi) \in L^2\left(\Omega, C\left([0, T_0], L^2(\mathbb{R})\right)\right) \times L^2\left(\Omega, C\left([0, T_0], L^2(\mathbb{R})\right)\right), \quad \forall T_0 > 0.$$

To conclude our preliminary discussion, we introduce the notation $\gamma \lesssim_{k_1,\ldots,k_n} \delta$ to mean that $\gamma \leq c\delta$ for some constant $c > 0$ depending only on $\lambda_1, \ldots, \lambda_n$. When c is an absolute constant, independent of any parameters, we simply write $\gamma \lesssim \delta$.

2. Linear and bilinear estimates in Bourgain spaces

To establish our main results, we first introduce several key estimates. In order to state them, we rewrite system (1.3) in its Itô form, namely,

$$\begin{cases} d\phi + \left(\dfrac{\partial^3\phi}{\partial^3 x} + \alpha_1\phi\dfrac{\partial\phi}{\partial x} + \alpha_2\varphi\dfrac{\partial\varphi}{\partial x} + \alpha_3\phi\dfrac{\partial\varphi}{\partial x} + \alpha_4\varphi\dfrac{\partial\phi}{\partial x}\right)dt = \Xi dW \\ d\varphi + \left(\dfrac{\partial^3\varphi}{\partial^3 x} + \beta_1\phi\dfrac{\partial\phi}{\partial x} + \beta_2\varphi\dfrac{\partial\varphi}{\partial x} + \beta_3\phi\dfrac{\partial\varphi}{\partial x} + \beta_4\varphi\dfrac{\partial\phi}{\partial x}\right)dt = \Xi dW. \end{cases} \tag{2.1}$$

Where $W(t) = \frac{\partial\mathbb{B}}{\partial x}$ denotes a cylindrical Wiener process on $L^2(\mathbb{R})$, which can equivalently be represented as

$$W(t) = \sum_{i=0}^{\infty}\beta_i(t)e_i,$$

with $(e_i)_{i\in\mathbb{N}}$ an orthonormal basis of $L^2(\mathbb{R})$ and $(\beta_i)_{i\in\mathbb{N}}$ a sequence of mutually independent real-valued Brownian motions defined on a fixed probability space.

System (2.1) is equipped with the initial conditions

$$\phi(\cdot,0) = \phi_0(\cdot), \quad \varphi(\cdot,0) = \varphi_0(\cdot). \tag{2.2}$$

We start by considering the linear equation, which highlights the assumptions required on Ξ.

$$\begin{cases} d\phi + \dfrac{\partial^3\phi}{\partial^3 x}dt = \Xi dW \\ d\varphi + \dfrac{\partial^3\varphi}{\partial^3 x}dt = \Xi dW \\ (\phi_0(x), \varphi_0(x)) = (0,0), \end{cases}$$

which can be represented through the stochastic Itô integral

$$\begin{cases} \phi_l(t) = \displaystyle\int_0^t \mathfrak{U}(t-\gamma)\Xi dW(\gamma) \\ \varphi_l(t) = \displaystyle\int_0^t \mathfrak{U}(t-\gamma)\Xi dW(\gamma), \end{cases} \tag{2.3}$$

where $\mathfrak{U}(t) = e^{-t\partial_x^3}$ denotes the Airy group. By the unitarity of $\mathfrak{U}(t)$, one readily verifies that $\phi(t)$ and $\varphi(t)$ belong to $H^s(\mathbb{R})$ only if Ξ is a Hilbert–Schmidt operator from $L^2(\mathbb{R})$ into $H^s(\mathbb{R})$.

In order to solve (2.1) with the initial data (2.2), we employ its mild form

$$\begin{cases} \phi(t) &= \mathfrak{U}(t)\phi_0 + \int_0^t \mathfrak{U}(t-\gamma)\left(\alpha_1\phi\frac{\partial\phi}{\partial x} + \alpha_2\varphi\frac{\partial\varphi}{\partial x} + \alpha_3\phi\frac{\partial\varphi}{\partial x} + \alpha_4\varphi\frac{\partial\phi}{\partial x}\right)(\gamma)d\gamma \\ &\quad + \int_0^t \mathfrak{U}(t-\gamma)\Xi dW(\gamma) \\ \varphi(t) &= \mathfrak{U}(t)\varphi_0 + \int_0^t \mathfrak{U}(t-\gamma)\left(\beta_1\phi\frac{\partial\phi}{\partial x} + \beta_2\varphi\frac{\partial\varphi}{\partial x} + \beta_3\phi\frac{\partial\varphi}{\partial x} + \beta_4\varphi\frac{\partial\phi}{\partial x}\right)(\gamma)d\gamma \\ &\quad + \int_0^t \mathfrak{U}(t-\gamma)\Xi dW(\gamma). \end{cases} \tag{2.4}$$

Mild solutions are obtained using the following estimates.

Proposition 2.1 (Linear estimates [6]). *For any $s, b \in \mathbb{R}$, we have*

$$\|\mathfrak{U}(t)\phi_0\|_{X_T^{s,b}} \lesssim \|\phi_0\|_{H^s}, \quad \|\mathfrak{U}(t)\varphi_0\|_{X_T^{s,b}} \lesssim \|\varphi_0\|_{H^s}. \tag{2.5}$$

Further, if $-\frac{1}{2} < b' \leq 0 \leq b < b'+1$, $0 \leq T \leq 1$, and $F \in X_T^{s,b} \cap Y_T^{s,-\frac{3}{8},b}$, then

$$\left\|\int_0^t \mathfrak{U}(t-\gamma)F(\gamma)\mathrm{d}\gamma\right\|_{X_T^{s,b}} \lesssim T^{1-b+b'}\|F(\gamma)\|_{X_T^{s,b'}} \tag{2.6}$$

and

$$\left\|\int_0^t \mathfrak{U}(t-\gamma)F(\gamma)\mathrm{d}\gamma\right\|_{Y_T^{s,-\frac{3}{8},b}} \lesssim T^{1-b+b'}\|F(\gamma)\|_{Y_T^{s,-\frac{3}{8},b'}}. \tag{2.7}$$

Lemma 2.2 (Bilinear estimates in $X^{s,b}$ [6]). *Let $s > -3/4$, $1/2 < b < 1$, $-1/2 < b' < 0$, and $b' = b-1$ such that for all $\phi, \varphi \in X^{s,b'}$, we have*

$$\left\|\phi\frac{\partial\phi}{\partial x}\right\|_{X^{s,b'}} \lesssim \left(\min\left\{\|\phi\|_{X^{s,b}}, \|\phi\|_{X^{s,b-\frac{1}{2}}\cap Y^{s,-\frac{3}{8},b-\frac{1}{2}}}\right\}\right)^2, \tag{2.8}$$

$$\left\|\varphi\frac{\partial\varphi}{\partial x}\right\|_{X^{s,b'}} \lesssim \left(\min\left\{\|\varphi\|_{X^{s,b}}, \|\varphi\|_{X^{s,b-\frac{1}{2}}\cap Y^{s,-\frac{3}{8},b-\frac{1}{2}}}\right\}\right)^2, \tag{2.9}$$

$$\left\|\phi\frac{\partial\varphi}{\partial x}\right\|_{X^{s,b'}} \lesssim \left(\min\left\{\|\phi\|_{X^{s,b}}, \|\phi\|_{X^{s,b-\frac{1}{2}}\cap Y^{s,-\frac{3}{8},b-\frac{1}{2}}}\right\}\right) \times \left(\min\left\{\|\varphi\|_{X^{s,b}}, \|\varphi\|_{X^{s,b-\frac{1}{2}}\cap Y^{s,-\frac{3}{8},b-\frac{1}{2}}}\right\}\right), \tag{2.10}$$

$$\left\|\varphi\frac{\partial\phi}{\partial x}\right\|_{X^{s,b'}} \lesssim \left(\min\left\{\|\varphi\|_{X^{s,b}}, \|\varphi\|_{X^{s,b-\frac{1}{2}}\cap Y^{s,-\frac{3}{8},b-\frac{1}{2}}}\right\}\right) \times \left(\min\left\{\|\phi\|_{X^{s,b}}, \|\phi\|_{X^{s,b-\frac{1}{2}}\cap Y^{s,-\frac{3}{8},b-\frac{1}{2}}}\right\}\right). \tag{2.11}$$

Lemma 2.3 (Bilinear estimates in $Y^{s_1,s_2,b}$ [6]). *Let $s > -5/8$, $1/2 < b < 1$, $-1/2 < b' < 0$, and $b' = b-1$ such that for all $\phi, \varphi \in Y^{s,-\frac{3}{8},b'}$, we have*

$$\left\|\phi\frac{\partial\phi}{\partial x}\right\|_{Y^{s,-\frac{3}{8},b'}} \lesssim \left(\min\left\{\|\phi\|_{X^{s,b}}, \|\phi\|_{X^{s,b-\frac{1}{2}}\cap Y^{s,-\frac{3}{8},b-\frac{1}{2}}}\right\}\right)^2, \tag{2.12}$$

$$\left\|\varphi\frac{\partial\varphi}{\partial x}\right\|_{Y^{s,-\frac{3}{8},b'}} \lesssim \left(\min\left\{\|\varphi\|_{X^{s,b}}, \|\varphi\|_{X^{s,b-\frac{1}{2}}\cap Y^{s,-\frac{3}{8},b-\frac{1}{2}}}\right\}\right)^2, \tag{2.13}$$

$$\left\|\phi\frac{\partial\varphi}{\partial x}\right\|_{Y^{s,-\frac{3}{8},b'}} \lesssim \left(\min\left\{\|\phi\|_{X^{s,b}}, \|\phi\|_{X^{s,b-\frac{1}{2}}\cap Y^{s,-\frac{3}{8},b-\frac{1}{2}}}\right\}\right)\times\left(\min\left\{\|\varphi\|_{X^{s,b}}, \|\varphi\|_{X^{s,b-\frac{1}{2}}\cap Y^{s,-\frac{3}{8},b-\frac{1}{2}}}\right\}\right), \tag{2.14}$$

$$\left\|\varphi\frac{\partial\phi}{\partial x}\right\|_{Y^{s,-\frac{3}{8},b'}} \lesssim \left(\min\left\{\|\varphi\|_{X^{s,b}}, \|\varphi\|_{X^{s,b-\frac{1}{2}}\cap Y^{s,-\frac{3}{8},b-\frac{1}{2}}}\right\}\right)\times\left(\min\left\{\|\phi\|_{X^{s,b}}, \|\phi\|_{X^{s,b-\frac{1}{2}}\cap Y^{s,-\frac{3}{8},b-\frac{1}{2}}}\right\}\right). \tag{2.15}$$

We choose a function ϖ such that $\varpi(t) = 0$ for $t < 0$ and $|t| > 2$, and $\varpi(t) = 1$ for $t \in [0, 1]$, with $\varpi \in C_0^\infty$. Note that such a ϖ belongs to $H_t^b = H^b([0, T], \mathbb{R})$ for any $b > \frac{1}{2}$, where

$$\|\varpi\|_{H_t^b}^2 = \|\varpi\|_{L^2}^2 + \int_{\mathbb{R}^2}\frac{|\varpi(\eta_1) - \varpi(\eta_2)|^2}{|\eta_1 - \eta_2|^{1+2b}}d\eta_1 d\eta_2.$$

We show the following lemma.

Lemma 2.4 (Stochastic convolution in $X^{b,s}$)**.** *Let $s, b \in \mathbb{R}$, with $b < 1/2$, and assume that $\Xi \in L_2^{0,s}$. Then $\phi_l(t), \varphi_l(t)$ defined by (2.3) satisfies*

$$\varpi\phi_l \in L^2\left(\Omega, X^{b,s}\right), \quad \varpi\varphi_l \in L^2\left(\Omega, X^{b,s}\right)$$

and

$$\mathbb{E}\left(\|\varpi\phi_l\|_{X^{s,b}}^2\right) \lesssim_{b,\varpi} \|\Xi\|_{L_2^{0,s}}^2, \quad \mathbb{E}\left(\|\varpi\varphi_l\|_{X^{s,b}}^2\right) \lesssim_{b,\varpi} \|\Xi\|_{L_2^{0,s}}^2.$$

Proof. Let us introduce the function

$$f(\cdot, t) = \varpi(t)\int_0^t \mathfrak{U}(-\gamma)\Xi dW(\gamma), \quad t \in \mathbb{R}^+. \tag{2.16}$$

This implies that $\mathfrak{U}(t)f(\cdot, t) = \varpi(t)\Psi(t)$. Thus, we have

$$\begin{aligned}\mathbb{E}\left(\|\varpi\Psi\|_{X^{s,b}}^2\right) &= \mathbb{E}\left(\int_{\mathbb{R}^2}(1 + |\zeta|)^{2s}(1 + |\gamma|)^{2b}\left|\hat{f}(\zeta, t)\right|^2 d\gamma d\zeta\right)\\ &= \int_{\mathbb{R}}(1 + |\zeta|)^{2s}\mathbb{E}\left(\left\|\hat{f}(\zeta, \cdot)\right\|_{H_t^b}^2\right)d\zeta.\end{aligned}$$

According to the expansion $W(t) = \sum_{i=0}^{\infty}\beta_i(t)e_i$ of the cylindrical Wiener process and $(2.3)_2$, we have

$$\mathbb{E}\left(\left\|\hat{f}(\zeta, \cdot)\right\|_{H_t^b}^2\right) = \mathfrak{S}_1 + \mathfrak{S}_2$$

where,

$$\mathfrak{S}_1 = \sum_{i=0}^{\infty}\left|\widehat{\Xi e_i}\right|^2\left[\mathbb{E}\left(\left\|\varpi(t)\int_0^t e^{i\gamma\zeta^3}d\beta_i(\gamma)\right\|_{L^2(\mathbb{R})}^2\right)\right],$$

$$\mathfrak{S}_2 = \sum_{i=0}^{\infty}\left|\widehat{\Xi e_i}\right|^2\left[\mathbb{E}\left(\int_{\mathbb{R}^2}\frac{\left|\ \varpi(\eta_1)\int_0^{\eta_1} e^{i\gamma\zeta^3}d\beta_i(\gamma) - \varpi(\eta_2)\int_0^{\eta_2} e^{i\gamma\zeta^3}d\beta_i(\gamma)\ \right|}{|\eta_1 - \eta_2|^{1+2b}}d\eta_1 d\eta_2\right)\right].$$

From the Itô isometry formula, we have

$$\begin{aligned}\mathfrak{S}_1 &= \sum_{i=0}^{\infty}\left|\widehat{\Xi e_i}\right|^2\int_0^2|\varpi(t)|^2\mathbb{E}\left(\left|\int_0^t e^{i\gamma\zeta^3}d\beta_i(\gamma)\right|^2\right)dt\\ &= \left\||t|^{\frac{1}{2}}\varpi\right\|_{L_t^2}^2\sum_{i=0}^{\infty}\left|\widehat{\Xi e_i}\right|^2.\end{aligned}$$

To estimate $\mathfrak{S}_2$, we get

$$\begin{aligned}\mathfrak{S}_2 &= \sum_{i=0}^{\infty}\left|\widehat{\Xi e_i}\right|^2\left[\mathbb{E}\left(\int_{\mathbb{R}^2}\frac{\left|\ \varpi(\eta_1)\int_0^{\eta_1}e^{i\gamma\zeta^3}d\beta_i(\gamma)-\varpi(\eta_2)\int_0^{\eta_2}e^{i\gamma\zeta^3}d\beta_i(\gamma)\ \right|}{|\eta_1-\eta_2|^{1+2b}}d\eta_1 d\eta_2\right)\right]\\
&= 2\sum_{i=0}^{\infty}\left|\widehat{\Xi e_i}\right|^2\int_{\eta_2>0}\int_{\eta_1<\eta_2}\frac{\mathbb{E}\left(\left|\ \varpi(\eta_1)\int_0^{\eta_1}e^{i\gamma\zeta^3}d\beta_i(\gamma)-\varpi(\eta_2)\int_0^{\eta_2}e^{i\gamma\zeta^3}d\beta_i(\gamma)\ \right|^2\right.}{|\eta_1-\eta_2|^{1+2b}}d\eta_1 d\eta_2\\
&\le \sum_{i=0}^{\infty}\left|\widehat{\Xi e_i}\right|^2\left[2\int_{\eta_2>0}\int_{\eta_1<0}\frac{|\varpi(\eta_2)|^2\,\mathbb{E}\left(\left|\int_0^{\eta_2}e^{i\gamma\zeta^3}d\beta_i(\gamma)\right|^2\right)}{|\eta_1-\eta_2|^{1+2b}}d\eta_1 d\eta_2\right.\\
&+ 2\int_{\eta_2>0}\int_{0<\eta_1<\eta_2}\frac{1}{|\eta_1-\eta_2|^{1+2b}}\\
&\left.\times\,\mathbb{E}\left(\left|\ \varpi(\eta_1)\int_0^{\eta_1}e^{i\gamma\zeta^3}d\beta_i(\gamma)-\varpi(\eta_2)\int_0^{\eta_1}e^{i\gamma\zeta^3}d\beta_i(\gamma)+\varpi(\eta_2)\int_{\eta_1}^{\eta_2}e^{i\gamma\zeta^3}d\beta_i(\gamma)\ \right)^2 d\eta_1 d\eta_2\right]\\
&\le \sum_{i=0}^{\infty}\left|\widehat{\Xi e_i}\right|^2\left[2\int_{\eta_2>0}\int_{\eta_1<0}\frac{|\varpi(\eta_2)|^2\,\mathbb{E}\left(\left|\int_0^{\eta_2}e^{i\gamma\zeta^3}d\beta_i(\gamma)\right|^2\right)}{|\eta_1-\eta_2|^{1+2b}}d\eta_1 d\eta_2\right.\\
&+ 4\int_{\eta_2>0}\int_{0<\eta_1<\eta_2}\frac{|\varpi(\eta_1)-\varpi(\eta_2)|^2\,\mathbb{E}\left(\left|\int_0^{\eta_1}e^{i\gamma\zeta^3}d\beta_i(\gamma)\right|^2\right)}{|\eta_1-\eta_2|^{1+2b}}d\eta_1 d\eta_2\\
&\left.+ 4\int_{\eta_2>0}\int_{0<\eta_1<\eta_2}\frac{|\varpi(\eta_2)|^2\,\mathbb{E}\left(\left|\int_{\eta_1}^{\eta_2}e^{i\gamma\zeta^3}d\beta_i(\gamma)\right|^2\right)}{|\eta_1-\eta_2|^{1+2b}}d\eta_1 d\eta_2\right]\\
&= \sum_{i=0}^{\infty}\left|\widehat{\Xi e_i}\right|^2\left[I_1+I_2+I_3\right].\end{aligned}$$

Now, we limit $\mathfrak{I}_1, \mathfrak{I}_2$, and $\mathfrak{I}_3$ separately:

$$\mathfrak{I}_1 \le 2\int_0^2\eta_1\,|\varpi(\eta_2)|^2\int_{\eta_1<0}\frac{1}{|\eta_1-\eta_2|^{1+2b}}d\eta_1 d\eta_2 \le \mathfrak{M}_b\left\||t|^{\frac{1}{2}-b}\varpi\right\|_{L_t^2}^2.$$

Using Eq (2.16) and the assumption that $2b \in (0,1)$, we have

$$\Im_2 \leq 4\int_0^\infty \int_0^{\eta_2} \frac{\eta_1 \left|\varpi(\eta_1) - \varpi(\eta_2)\right|^2}{\|\eta_1 - \eta_2\|^{1+2b}} d\eta_1 d\eta_2$$

$$\leq 4\int_0^2 \int_0^{\eta_2} \frac{\eta_1 \left|\varpi(\eta_1) - \varpi(\eta_2)\right|^2}{\|\eta_1 - \eta_2\|^{1+2b}} d\eta_1 d\eta_2$$

$$+ 4\int_2^\infty \int_0^2 \frac{\eta_1 \left|\varpi(\eta_1)\right|^2}{\|\eta_1 - \eta_2\|^{1+2b}} d\eta_1 d\eta_2$$

$$\leq 8\|\varpi\|_{H_t^b}^2 + 4\left\||t|^{\frac{1}{2}}\varpi\right\|_{L_t^\infty}^2 \int_0^\infty \int_0^2 \frac{1}{|\eta_1 - \eta_2|^{1+2b}} d\eta_1 d\eta_2$$

$$\leq 8\|\varpi\|_{H_t^b}^2 + \mathfrak{M}_b \left\||t|^{\frac{1}{2}}\varpi\right\|_{L_t^\infty}^2 .$$

Similarly,

$$\Im_3 \leq 4\int_0^2 \int_0^{\eta_2} \frac{\left|\varpi(\eta_2)\right|^2}{|\eta_1 - \eta_2|^{2b}} d\eta_1 d\eta_2 \leq \mathfrak{M}_b \left\||t|^{\frac{1}{2}-b}\varpi\right\|_{L_t^2}^2 .$$

So, we have

$$\mathbb{E}\left(\left\|\hat{f}(\zeta,\cdot)\right\|_{H_t^b}^2\right) \leq \Re(b,\varpi)\sum_{i=0}^\infty \left|\widehat{\Xi e_i}\right|^2$$

where $\Re(b,\varpi) = \mathfrak{M}_b\left(\|\varpi\|_{H_t^b} + \left\||t|^{\frac{1}{2}}\varpi\right\|_{L_t^2} + \left\||t|^{\frac{1}{2}}\varpi\right\|_{L_t^\infty}\right)$. □

Lemma 2.5 (Stochastic convolution in $Y^{s_1,s_2,b}$)**.** *Let $s, b \in \mathbb{R}$, with $b < 1/2$, and assume that $\Xi \in L_2^{0,s_1,s_2}$. Then $\phi_l(t), \varphi_l(t)$ defined by (2.3) satisfies*

$$\varpi\phi_l \in L^2\left(\Omega, Y^{s_1,s_2,b}\right), \quad \varpi\varphi_l \in L^2\left(\Omega, Y^{s_1,s_2,b}\right)$$

and

$$\mathbb{E}\left(\|\varpi\phi_l\|_{Y^{s_1,s_2,b}}^2\right) \lesssim_{b,\varpi} \|\Xi\|_{L_2^{0,s_1,s_2}}^2, \quad \mathbb{E}\left(\|\varpi\varphi_l\|_{Y^{s_1,s_2,b}}^2\right) \lesssim_{b,\varpi} \|\Xi\|_{L_2^{0,s_1,s_2}}^2.$$

Proof. The proof is similar with that of Lemma 2.4. □

3. Local well-posedness (Proof of Theorem 1.1)

Based on the stochastic estimates established in the previous section and the Banach fixed-point theorem, we derive a local well-posedness result for system (1.3). Accordingly, this section is dedicated to the proof of Theorem 1.1. Let

$$\Pi_1(t) = \mathfrak{U}(t)\phi_0, \qquad \Pi_2(t) = \mathfrak{U}(t)\varphi_0,$$

$$\Upsilon_1(t)=\int_0^t \mathfrak{U}(t-\gamma)\Xi dW(\gamma), \Upsilon_2(t)=\int_0^t \mathfrak{U}(t-\gamma)\Xi dW(\gamma),$$

and

$$\Psi_1(t)=\int_0^t \mathfrak{U}(t-\gamma)\left(\alpha_1\phi\frac{\partial\phi}{\partial x}+\alpha_2\varphi\frac{\partial\varphi}{\partial x}+\alpha_3\phi\frac{\partial\varphi}{\partial x}+\alpha_4\varphi\frac{\partial\phi}{\partial x}\right)(\gamma)d\gamma,$$

$$\Psi_2(t)=\int_0^t \mathfrak{U}(t-\gamma)\left(\beta_1\phi\frac{\partial\phi}{\partial x}+\beta_2\varphi\frac{\partial\varphi}{\partial x}+\beta_3\phi\frac{\partial\varphi}{\partial x}+\beta_4\varphi\frac{\partial\phi}{\partial x}\right)(\gamma)d\gamma.$$

Now, assume that

$$\begin{cases} \Psi_1(t)=\phi_l(t)-\Pi_1(t)-\Upsilon_1(t) \\ \Psi_2(t)=\varphi_l(t)-\Pi_2(t)-\Upsilon_2(t), \end{cases} \tag{3.1}$$

so

$$\begin{cases} \Psi_1(t)=\int_0^t \mathfrak{U}(t-\gamma)\left(\alpha_1\phi\frac{\partial\phi}{\partial x}+\alpha_2\varphi\frac{\partial\varphi}{\partial x}+\alpha_3\phi\frac{\partial\varphi}{\partial x}+\alpha_4\varphi\frac{\partial\phi}{\partial x}\right)(\gamma)d\gamma \\ \Psi_2(t)=\int_0^t \mathfrak{U}(t-\gamma)\left(\beta_1\phi\frac{\partial\phi}{\partial x}+\beta_2\varphi\frac{\partial\varphi}{\partial x}+\beta_3\phi\frac{\partial\varphi}{\partial x}+\beta_4\varphi\frac{\partial\phi}{\partial x}\right)(\gamma)d\gamma, \end{cases} \tag{3.2}$$

and then (3.2) is equivalent to

$$\begin{cases} \Psi_1(t) &= \int_0^t \mathfrak{U}(t-\gamma)\Big[\partial_x\Big(\frac{\alpha_1}{2}\Big(\Psi_1(t)+\Pi_1(t)+\Upsilon_1(t)\Big)^2+\frac{\alpha_2}{2}\Big(\Psi_2(t)+\Pi_2(t)+\Upsilon_2(t)\Big)^2\Big) \\ &+\alpha_3\Big(\Psi_1(t)+\Pi_1(t)+\Upsilon_1(t)\Big)\partial_x\Big(\Psi_2(t)+\Pi_2(t)+\Upsilon_2(t)\Big) \\ &+\alpha_4\Big(\Psi_2(t)+\Pi_2(t)+\Upsilon_2(t)\Big)\partial_x\Big(\Psi_1(t)+\Pi_1(t)+\Upsilon_1(t)\Big)\Big](\gamma)d\gamma \\ \Psi_2(t) &= \int_0^t \mathfrak{U}(t-\gamma)\Big[\partial_x\Big(\frac{\beta_1}{2}\Big(\Psi_1(t)+\Pi_1(t)+\Upsilon_1(t)\Big)^2+\frac{\beta_2}{2}\Big(\Psi_2(t)+\Pi_2(t)+\Upsilon_2(t)\Big)^2\Big) \\ &+\beta_3\Big(\Psi_1(t)+\Pi_1(t)+\Upsilon_1(t)\Big)\partial_x\Big(\Psi_2(t)+\Pi_2(t)+\Upsilon_2(t)\Big) \\ &+\beta_4\Big(\Psi_2(t)+\Pi_2(t)+\Upsilon_2(t)\Big)\partial_x\Big(\Psi_1(t)+\Pi_1(t)+\Upsilon_1(t)\Big)\Big](\gamma)d\gamma. \end{cases}$$

Simplifying this expression yields

$$
\begin{cases}
\Psi_1(t) &= \displaystyle\int_0^t \mathfrak{U}(t-\gamma)\Big[\partial_x\Big(\frac{\alpha_1}{2}\left(\Psi_1^2+\Pi_1^2+\Upsilon_1^2+2\Psi_1\Pi_1+2\Psi_1\Upsilon_1+2\Pi_1\Upsilon_1\right) \\
&\quad + \dfrac{\alpha_2}{2}\left(\Psi_2^2+\Pi_2^2+\Upsilon_2^2+2\Psi_2\Pi_2+2\Psi_2\Upsilon_2+2\Pi_2\Upsilon_2\right)\Big) \\
&\quad + \alpha_3\left(\Psi_1\partial_x\Psi_2+\Pi_1\partial_x\Psi_2+\Upsilon_1\partial_x\Psi_2+\Psi_1\partial_x\Pi_2+\Pi_1\partial_x\Pi_2\right. \\
&\quad \left.+\Upsilon_1\partial_x\Pi_2+\Psi_1\partial_x\Upsilon_2+\Pi_1\partial_x\Upsilon_2+\Upsilon_1\partial_x\Upsilon_2\right) \\
&\quad + \alpha_4\left(\Psi_2\partial_x\Psi_1+\Pi_2\partial_x\Psi_1+\Upsilon_2\partial_x\Psi_2+\Psi_2\partial_x\Pi_1+\Pi_2\partial_x\Pi_1\right. \\
&\quad \left.+\Upsilon_2\partial_x\Pi_1+\Psi_2\partial_x\Upsilon_1+\Pi_2\partial_x\Upsilon_1+\Upsilon_2\partial_x\Upsilon_1\right)\Big](\gamma)d\gamma \\
\\
\Psi_2(t) &= \displaystyle\int_0^t \mathfrak{U}(t-\gamma)\Big[\partial_x\Big(\frac{\beta_1}{2}\left(\Psi_1^2+\Pi_1^2+\Upsilon_1^2+2\Psi_1\Pi_1+2\Psi_1\Upsilon_1+2\Pi_1\Upsilon_1\right) \\
&\quad + \dfrac{\beta_2}{2}\left(\Psi_2^2+\Pi_2^2+\Upsilon_2^2+2\Psi_2\Pi_2+2\Psi_2\Upsilon_2+2\Pi_2\Upsilon_2\right)\Big) \\
&\quad + \beta_3\left(\Psi_1\partial_x\Psi_2+\Pi_1\partial_x\Psi_2+\Upsilon_1\partial_x\Psi_2+\Psi_1\partial_x\Pi_2+\Pi_1\partial_x\Pi_2\right. \\
&\quad \left.+\Upsilon_1\partial_x\Pi_2+\Psi_1\partial_x\Upsilon_2+\Pi_1\partial_x\Upsilon_2+\Upsilon_1\partial_x\Upsilon_2\right) \\
&\quad + \beta_4\left(\Psi_2\partial_x\Psi_1+\Pi_2\partial_x\Psi_1+\Upsilon_2\partial_x\Psi_2+\Psi_2\partial_x\Pi_1+\Pi_2\partial_x\Pi_1\right. \\
&\quad \left.+\Upsilon_2\partial_x\Pi_1+\Psi_2\partial_x\Upsilon_1+\Pi_2\partial_x\Upsilon_1+\Upsilon_2\partial_x\Upsilon_1\right)\Big](\gamma)d\gamma.
\end{cases}
\tag{3.3}
$$

Next, we define the ball $\mathcal{B}_{\mathcal{R},T}$ by

$$
\mathcal{B}_{\mathcal{R},T} = \left\{(\Psi_1,\Psi_2)\in \mathcal{X}_T^{s,b-\frac{1}{2}}\cap \mathcal{Y}_T^{s,-\frac{3}{8},b-\frac{1}{2}} : \|(\Psi_1,\Psi_2)\|_{\mathcal{X}_T^{s,b-\frac{1}{2}}} + \|(\Psi_1,\Psi_2)\|_{\mathcal{Y}_T^{s,-\frac{3}{8},b-\frac{1}{2}}} \le \mathcal{R}, \quad \mathcal{R}>0\right\}.
$$

Therefore, the goal of this section becomes to prove that $(\Psi_1(t),\Psi_2(t))$ is a contraction mapping in $\mathcal{B}_{\mathcal{R},T}$. According to Lemmas 2.1, 2.2, and 2.4, we obtain

$$
\begin{aligned}
\|\varGamma_1(\Psi_1(t))\|_{\mathcal{X}_T^{s,b-\frac{1}{2}}} \lesssim\ & T^{1-b+b'}\Big\|\partial_x\Big(\frac{\alpha_1}{2}\left(\Psi_1^2+\Pi_1^2+\Upsilon_1^2+2\Psi_1\Pi_1+2\Psi_1\Upsilon_1+2\Pi_1\Upsilon_1\right) \\
& + \frac{\alpha_2}{2}\left(\Psi_2^2+\Pi_2^2+\Upsilon_2^2+2\Psi_2\Pi_2+2\Psi_2\Upsilon_2+2\Pi_2\Upsilon_2\right)\Big) \\
& + \alpha_3\Big(\Psi_1\partial_x\Psi_2+\Pi_1\partial_x\Psi_2+\Upsilon_1\partial_x\Psi_2+\Psi_1\partial_x\Pi_2
\end{aligned}
$$

$$
\begin{aligned}
& + \varPi_1\partial_x\varPi_2 + \Upsilon_1\partial_x\varPi_2 + \Psi_1\partial_x\Upsilon_2 + \varPi_1\partial_x\Upsilon_2 + \Upsilon_1\partial_x\Upsilon_2\Big) \\
& + \alpha_4\Big(\Psi_2\partial_x\Psi_1 + \varPi_2\partial_x\Psi_1 + \Upsilon_2\partial_x\Psi_2 + \Psi_2\partial_x\varPi_1 \\
& + \varPi_2\partial_x\varPi_1 + \Upsilon_2\partial_x\varPi_1 + \Psi_2\partial_x\Upsilon_1 + \varPi_2\partial_x\Upsilon_1 + \Upsilon_2\partial_x\Upsilon_1\Big)\Big\|_{X_T^{s,b'}} \\
\lesssim_\alpha \quad & T^{1-b+b'}\Big[\frac{1}{2}\Big(\|\Psi_1\|^2_{X_T^{s,b-\frac{1}{2}}} + \|\varPi_1\|^2_{X_T^{s,b-\frac{1}{2}}} + \|\Upsilon_1\|^2_{X_T^{s,b-\frac{1}{2}}} \\
& + \|\Psi_2\|^2_{X_T^{s,b-\frac{1}{2}}} + \|\varPi_2\|^2_{X_T^{s,b-\frac{1}{2}}} + \|\Upsilon_2\|^2_{X_T^{s,b-\frac{1}{2}}}\Big) \\
& + \Big(\|\Psi_1\|_{X_T^{s,b-\frac{1}{2}}}\|\varPi_1\|_{X_T^{s,b-\frac{1}{2}}} + \|\Psi_1\|_{X_T^{s,b-\frac{1}{2}}}\|\Upsilon_1\|_{X_T^{s,b-\frac{1}{2}}} \\
& + \|\varPi_1\|_{X_T^{s,b-\frac{1}{2}}}\|\Upsilon_1\|_{X_T^{s,b-\frac{1}{2}}}\Big) + \Big(\|\Psi_2\|_{X_T^{s,b-\frac{1}{2}}}\|\varPi_2\|_{X_T^{s,b-\frac{1}{2}}} \\
& + \|\Psi_2\|_{X_T^{s,b-\frac{1}{2}}}\|\Upsilon_2\|_{X_T^{s,b-\frac{1}{2}}} + \|\varPi_2\|_{X_T^{s,b-\frac{1}{2}}}\|\Upsilon_2\|_{X_T^{s,b-\frac{1}{2}}}\Big) \\
& + 2\Big(\|\Psi_1\|_{X_T^{s,b-\frac{1}{2}}}\|\Psi_2\|_{X_T^{s,b-\frac{1}{2}}} + \|\varPi_1\|_{X_T^{s,b-\frac{1}{2}}}\|\varPi_2\|_{X_T^{s,b-\frac{1}{2}}} \\
& + \|\Upsilon_1\|_{X_T^{s,b-\frac{1}{2}}}\|\Upsilon_2\|_{X_T^{s,b-\frac{1}{2}}}\Big) + 2\Big(\|\Psi_1\|_{X_T^{s,b-\frac{1}{2}}}\|\varPi_2\|_{X_T^{s,b-\frac{1}{2}}} \\
& + \|\Psi_1\|_{X_T^{s,b-\frac{1}{2}}}\|\Upsilon_2\|_{X_T^{s,b-\frac{1}{2}}} + \|\varPi_1\|_{X_T^{s,b-\frac{1}{2}}}\|\Psi_2\|_{X_T^{s,b-\frac{1}{2}}} \\
& + \|\varPi_1\|_{X_T^{s,b-\frac{1}{2}}}\|\Upsilon_2\|_{X_T^{s,b-\frac{1}{2}}} + \|\Upsilon_1\|_{X_T^{s,b-\frac{1}{2}}}\|\Psi_2\|_{X_T^{s,b-\frac{1}{2}}} \\
& + \|\Upsilon_1\|_{X_T^{s,b-\frac{1}{2}}}\|\varPi_2\|_{X_T^{s,b-\frac{1}{2}}}\Big)\Big] \\
& \text{where} \quad \alpha = \max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \\
\lesssim \quad & T^{1-b+b'}\left(\mathcal{R}^2 + \|(\Upsilon_1, \Upsilon_2)\|^2_{X_T^{s,b-\frac{1}{2}}} + \|(\Upsilon_1, \Upsilon_2)\|^2_{\mathcal{Y}_T^{s,-\frac{3}{8},b-\frac{1}{2}}} + \|(\phi_0, \varphi_0)\|^2_{\mathcal{H}^s}\right),
\end{aligned}
$$

$$
\begin{aligned}
\|\varGamma_2(\Psi_2(t))\|_{X_T^{s,b-\frac{1}{2}}} \quad \lesssim \quad & T^{1-b+b'}\Big\|\partial_x\Big(\frac{\beta_1}{2}\left(\Psi_1^2 + \varPi_1^2 + \Upsilon_1^2 + 2\Psi_1\varPi_1 + 2\Psi_1\Upsilon_1 + 2\varPi_1\Upsilon_1\right) \\
& + \frac{\beta_2}{2}\left(\Psi_2^2 + \varPi_2^2 + \Upsilon_2^2 + 2\Psi_2\varPi_2 + 2\Psi_2\Upsilon_2 + 2\varPi_2\Upsilon_2\right)\Big)
\end{aligned}
$$

$$
\begin{aligned}
& + \beta_3\Big(\Psi_1\partial_x\Psi_2 + \mathit{\Pi}_1\partial_x\Psi_2 + \Upsilon_1\partial_x\Psi_2 + \Psi_1\partial_x\mathit{\Pi}_2 \\
& + \mathit{\Pi}_1\partial_x\mathit{\Pi}_2 + \Upsilon_1\partial_x\mathit{\Pi}_2 + \Psi_1\partial_x\Upsilon_2 + \mathit{\Pi}_1\partial_x\Upsilon_2 + \Upsilon_1\partial_x\Upsilon_2\Big) \\
& + \beta_4\Big(\Psi_2\partial_x\Psi_1 + \mathit{\Pi}_2\partial_x\Psi_1 + \Upsilon_2\partial_x\Psi_2 + \Psi_2\partial_x\mathit{\Pi}_1 \\
& + \mathit{\Pi}_2\partial_x\mathit{\Pi}_1 + \Upsilon_2\partial_x\mathit{\Pi}_1 + \Psi_2\partial_x\Upsilon_1 + \mathit{\Pi}_2\partial_x\Upsilon_1 + \Upsilon_2\partial_x\Upsilon_1\Big)\Big\|_{X_T^{s,b'}} \\
\lesssim_\beta \quad & T^{1-b+b'}\Big[\frac{1}{2}\Big(\|\Psi_1\|^2_{X_T^{s,b-\frac{1}{2}}} + \|\mathit{\Pi}_1\|^2_{X_T^{s,b-\frac{1}{2}}} + \|\Upsilon_1\|^2_{X_T^{s,b-\frac{1}{2}}} + \|\Psi_2\|^2_{X_T^{s,b-\frac{1}{2}}} \\
& + \|\mathit{\Pi}_2\|^2_{X_T^{s,b-\frac{1}{2}}} + \|\Upsilon_2\|^2_{X_T^{s,b-\frac{1}{2}}}\Big) + \Big(\|\Psi_1\|_{X_T^{s,b-\frac{1}{2}}}\|\mathit{\Pi}_1\|_{X_T^{s,b-\frac{1}{2}}} \\
& + \|\Psi_1\|_{X_T^{s,b-\frac{1}{2}}}\|\Upsilon_1\|_{X_T^{s,b-\frac{1}{2}}} + \|\mathit{\Pi}_1\|_{X_T^{s,b-\frac{1}{2}}}\|\Upsilon_1\|_{X_T^{s,b-\frac{1}{2}}}\Big) \\
& + \Big(\|\Psi_2\|_{X_T^{s,b-\frac{1}{2}}}\|\mathit{\Pi}_2\|_{X_T^{s,b-\frac{1}{2}}} + \|\Psi_2\|_{X_T^{s,b-\frac{1}{2}}}\|\Upsilon_2\|_{X_T^{s,b-\frac{1}{2}}} + \|\mathit{\Pi}_2\|_{X_T^{s,b-\frac{1}{2}}}\|\Upsilon_2\|_{X_T^{s,b-\frac{1}{2}}}\Big) \\
& + 2\Big(\|\Psi_1\|_{X_T^{s,b-\frac{1}{2}}}\|\Psi_2\|_{X_T^{s,b-\frac{1}{2}}} + \|\mathit{\Pi}_1\|_{X_T^{s,b-\frac{1}{2}}}\|\mathit{\Pi}_2\|_{X_T^{s,b-\frac{1}{2}}} \\
& + \|\Upsilon_1\|_{X_T^{s,b-\frac{1}{2}}}\|\Upsilon_2\|_{X_T^{s,b-\frac{1}{2}}}\Big) + 2\Big(\|\Psi_1\|_{X_T^{s,b-\frac{1}{2}}}\|\mathit{\Pi}_2\|_{X_T^{s,b-\frac{1}{2}}} \\
& + \|\Psi_1\|_{X_T^{s,b-\frac{1}{2}}}\|\Upsilon_2\|_{X_T^{s,b-\frac{1}{2}}} + \|\mathit{\Pi}_1\|_{X_T^{s,b-\frac{1}{2}}}\|\Psi_2\|_{X_T^{s,b-\frac{1}{2}}} \\
& + \|\mathit{\Pi}_1\|_{X_T^{s,b-\frac{1}{2}}}\|\Upsilon_2\|_{X_T^{s,b-\frac{1}{2}}} + \|\Upsilon_1\|_{X_T^{s,b-\frac{1}{2}}}\|\Psi_2\|_{X_T^{s,b-\frac{1}{2}}} \\
& + \|\Upsilon_1\|_{X_T^{s,b-\frac{1}{2}}}\|\mathit{\Pi}_2\|_{X_T^{s,b-\frac{1}{2}}}\Big)\Big] \\
& \text{where} \quad \beta = \max\{\beta_1, \beta_2, \beta_3, \beta_4\} \\
\lesssim \quad & T^{1-b+b'}\Big(\mathcal{R}^2 + \|(\Upsilon_1, \Upsilon_2)\|^2_{\mathcal{X}_T^{s,b-\frac{1}{2}}} + \|(\Upsilon_1, \Upsilon_2)\|^2_{\mathcal{Y}_T^{s,-\frac{3}{8},b-\frac{1}{2}}} + \|(\phi_0, \varphi_0)\|^2_{\mathcal{H}^s}\Big),
\end{aligned}
$$

so

$$
\begin{aligned}
\|(\mathit{\Gamma}_1(\Psi_1(t)), \mathit{\Gamma}_2(\Psi_2(t)))\|_{\mathcal{X}_T^{s,b-\frac{1}{2}}} \quad \lesssim \quad & T^{1-b+b'}\Big(\mathcal{R}^2 + \|(\Upsilon_1, \Upsilon_2)\|^2_{\mathcal{X}_T^{s,b-\frac{1}{2}}} + \|(\Upsilon_1, \Upsilon_2)\|^2_{\mathcal{Y}_T^{s,-\frac{3}{8},b-\frac{1}{2}}} \\
& + \|(\phi_0, \varphi_0)\|^2_{\mathcal{H}^s}\Big),
\end{aligned}
$$

and

$$\|(\Gamma_1(\Psi_1(t)),\Gamma_2(\Psi_2(t)))\|_{\mathcal{Y}_T^{s,-\frac{3}{8},b-\frac{1}{2}}} \lesssim T^{1-b+b'}\Big(\mathcal{R}^2+\|(\Upsilon_1,\Upsilon_2)\|^2_{\mathcal{X}_T^{s,b-\frac{1}{2}}}+\|(\Upsilon_1,\Upsilon_2)\|^2_{\mathcal{Y}_T^{s,-\frac{3}{8},b-\frac{1}{2}}} + \|(\phi_0,\varphi_0)\|^2_{\mathcal{H}^s}\Big).$$

Then

$$\|(\Gamma_1(\Psi_1(t)),\Gamma_2(\Psi_2(t)))\|_{\mathcal{X}_T^{s,b-\frac{1}{2}}}+\|(\Gamma_1(\Psi_1(t)),\Gamma_2(\Psi_2(t)))\|_{\mathcal{Y}_T^{s,-\frac{3}{8},b-\frac{1}{2}}}$$

$$\lesssim 2T^{1-b+b'}\Big(\mathcal{R}^2+\|(\Upsilon_1,\Upsilon_2)\|^2_{\mathcal{X}_T^{s,b-\frac{1}{2}}}+\|(\Upsilon_1,\Upsilon_2)\|^2_{\mathcal{Y}_T^{s,-\frac{3}{8},b-\frac{1}{2}}}+\|(\phi_0,\varphi_0)\|^2_{\mathcal{H}^s}\Big).$$

Therefore, for $(\Psi_{1.1},\Psi_{2.1}),(\Psi_{1.2},\Psi_{2.2})\in\mathcal{B}_{\mathcal{R},T}$, we get

$$\begin{aligned}&\|(\Gamma_1(\Psi_{1.1}-\Psi_{1.2}),\Gamma_2(\Psi_{2.1}-\Psi_{2.2}))\|_{\mathcal{X}_T^{s,b-\frac{1}{2}}}\\ \lesssim\;& T^{1-b+b'}\Big(\mathcal{R}+\|(\Upsilon_1,\Upsilon_2)\|_{\mathcal{X}_T^{s,b-\frac{1}{2}}}+\|(\Upsilon_1,\Upsilon_2)\|_{\mathcal{Y}_T^{s,-\frac{3}{8},b-\frac{1}{2}}}+\|(\phi_0,\varphi_0)\|_{\mathcal{H}^s}\Big)\\ \times\;&\Big(\|(\Psi_{1.1}-\Psi_{1.2},\Psi_{2.1}-\Psi_{2.2})\|_{\mathcal{X}_T^{s,b-\frac{1}{2}}}+\|((\Psi_{1.1}-\Psi_{1.2}),(\Psi_{2.1}-\Psi_{2.2}))\|_{\mathcal{Y}_T^{s,-\frac{3}{8},b-\frac{1}{2}}}\Big),\end{aligned}$$

and

$$\begin{aligned}&\|(\Gamma_1(\Psi_{1.1}-\Psi_{1.2}),\Gamma_2(\Psi_{2.1}-\Psi_{2.2}))\|_{\mathcal{Y}_T^{s,-\frac{3}{8},b-\frac{1}{2}}}\\ \lesssim\;& T^{1-b+b'}\Big(\mathcal{R}+\|(\Upsilon_1,\Upsilon_2)\|_{\mathcal{X}_T^{s,b-\frac{1}{2}}}+\|(\Upsilon_1,\Upsilon_2)\|_{\mathcal{Y}_T^{s,-\frac{3}{8},b-\frac{1}{2}}}+\|(\phi_0,\varphi_0)\|_{\mathcal{H}^s}\Big)\\ \times\;&\Big(\|(\Psi_{1.1}-\Psi_{1.2},\Psi_{2.1}-\Psi_{2.2})\|_{\mathcal{X}_T^{s,b-\frac{1}{2}}}+\|((\Psi_{1.1}-\Psi_{1.2}),(\Psi_{2.1}-\Psi_{2.2}))\|_{\mathcal{Y}_T^{s,-\frac{3}{8},b-\frac{1}{2}}}\Big).\end{aligned}$$

So

$$\|(\Gamma_1(\Psi_{1.1}-\Psi_{1.2}),\Gamma_2(\Psi_{2.1}-\Psi_{2.2}))\|_{\mathcal{X}_T^{b-\frac{1}{2},s}}+\|(\Gamma_1(\Psi_{1.1}-\Psi_{1.2}),\Gamma_2(\Psi_{2.1}-\Psi_{2.2}))\|_{\mathcal{Y}_T^{s,-\frac{3}{8},b-\frac{1}{2}}}$$

$$\lesssim 2T^{1-b+b'}\Big(\mathcal{R}+\|(\Upsilon_1,\Upsilon_2)\|_{\mathcal{X}_T^{s,b-\frac{1}{2}}}+\|(\Upsilon_1,\Upsilon_2)\|_{\mathcal{Y}_T^{s,-\frac{3}{8},b-\frac{1}{2}}}+\|(\phi_0,\varphi_0)\|_{\mathcal{H}^s}\Big)$$

$$\times\Big(\|(\Psi_{1.1}-\Psi_{1.2},\Psi_{2.1}-\Psi_{2.2})\|_{\mathcal{X}_T^{s,b-\frac{1}{2}}}+\|((\Psi_{1.1}-\Psi_{1.2}),(\Psi_{2.1}-\Psi_{2.2}))\|_{\mathcal{Y}_T^{s,-\frac{3}{8},b-\frac{1}{2}}}\Big).$$

Let us choose T_ϱ such that

$$4CT_\varrho^{1-b+b'}\Big(\mathcal{R}_\varrho+\|(\Upsilon_1,\Upsilon_2)\|_{\mathcal{X}_T^{s,b-\frac{1}{2}}}+\|(\Upsilon_1,\Upsilon_2)\|_{\mathcal{Y}_T^{s,-\frac{3}{8},b-\frac{1}{2}}}+\|(\phi_0,\varphi_0)\|_{\mathcal{H}^s}\Big)\le 1,$$

where

$$\mathcal{R}_\varrho = 2C\left(\|(\Upsilon_1, \Upsilon_2)\|^2_{\mathcal{X}_T^{s,b-\frac{1}{2}}} + \|(\Upsilon_1, \Upsilon_2)\|^2_{\mathcal{Y}_T^{s,-\frac{3}{8},b-\frac{1}{2}}} + \|(\phi_0, \varphi_0)\|^2_{\mathcal{H}^s}\right).$$

It is easily checked that $(\varGamma_1, \varGamma_2)$ maps $\mathcal{B}_{\mathcal{R},T}$ into itself and is a strict contraction in $\mathcal{B}_{\mathcal{R},T}$ for the norm

$$\|(\Psi_1, \Psi_2)\|_{\mathcal{X}_T^{s,b-\frac{1}{2}}} + \|(\Psi_1, \Psi_2)\|_{\mathcal{Y}_T^{s,-\frac{3}{8},b-\frac{1}{2}}}:$$

$$\|(\varGamma_1(\Psi_{1.1} - \Psi_{1.2}), \varGamma_2(\Psi_{2.1} - \Psi_{2.2}))\|_{\mathcal{X}_T^{b-\frac{1}{2},s}} + \|(\varGamma_1(\Psi_{1.1} - \Psi_{1.2}), \varGamma_2(\Psi_{2.1} - \Psi_{2.2}))\|_{\mathcal{Y}_T^{s,-\frac{3}{8},b-\frac{1}{2}}}$$

$$\leq \frac{1}{2}\left(\|(\Psi_{1.1} - \Psi_{1.2}, \Psi_{2.1} - \Psi_{2.2})\|_{\mathcal{X}_T^{s,b-\frac{1}{2}}} + \|((\Psi_{1.1} - \Psi_{1.2}), (\Psi_{2.1} - \Psi_{2.2}))\|_{\mathcal{Y}_T^{s,-\frac{3}{8},b-\frac{1}{2}}}\right).$$

Hence, $(\varGamma_1, \varGamma_2)$ has a unique fixed point in $\mathcal{X}_{T_\varrho}^{s,b-\frac{1}{2}} \cap \mathcal{Y}_{T_\varrho}^{s,-\frac{3}{8},b-\frac{1}{2}}$, which is a solution of (3.3) on $[0, T_\varrho]$. It remains only to show that if

$$\begin{cases} \phi(t) = \varPi_1(t) + \Psi_1(t) + \Upsilon_1(t) \in X_{T_\varrho}^{s,b} + X_{T_\varrho}^{s,b-\frac{1}{2}} \cap Y_{T_\varrho}^{s,-\frac{3}{8},b-\frac{1}{2}} \\ \varphi(t) = \varPi_2(t) + \Psi_2(t) + \Upsilon_2(t) \in X_{T_\varrho}^{s,b} + X_{T_\varrho}^{s,b-\frac{1}{2}} \cap Y_{T_\varrho}^{s,-\frac{3}{8},b-\frac{1}{2}} \end{cases} \tag{3.4}$$

then

$$\int_0^t \mathfrak{U}(t-\gamma)\left(\alpha_1\phi\frac{\partial\phi}{\partial x} + \alpha_2\varphi\frac{\partial\varphi}{\partial x} + \alpha_3\phi\frac{\partial\varphi}{\partial x} + \alpha_4\varphi\frac{\partial\phi}{\partial x}\right)(\gamma)d\gamma \in C([0, T_\varrho], H^s(\mathbb{R}))$$

$$\int_0^t \mathfrak{U}(t-\gamma)\left(\beta_1\phi\frac{\partial\phi}{\partial x} + \beta_2\varphi\frac{\partial\varphi}{\partial x} + \beta_3\phi\frac{\partial\varphi}{\partial x} + \beta_4\varphi\frac{\partial\phi}{\partial x}\right)(\gamma)d\gamma \in C([0, T_\varrho], H^s(\mathbb{R})).$$

Note that $-1/2 < b' < 0$, $0 < b - 1/2 < 1/2$, $b > 1/2$, and $b' = b - 1$. By Lemma 2.2, we have

$$\phi\frac{\partial\phi}{\partial x}, \varphi\frac{\partial\varphi}{\partial x}, \phi\frac{\partial\varphi}{\partial x}, \varphi\frac{\partial\phi}{\partial x} \in X^{s,b'},$$

for any prolongation

$$\tilde{\phi} \quad \text{of} \quad \phi \in X^{s,b} + X^{s,b-\frac{1}{2}} \cap Y^{s,-\frac{3}{8},b-\frac{1}{2}},$$

and

$$\tilde{\varphi} \quad \text{of} \quad \varphi \in X^{s,b} + X^{s,b-\frac{1}{2}} \cap Y^{s,-\frac{3}{8},b-\frac{1}{2}},$$

and applying Lemma 3.2 in [12], we get

$$\psi_T\int_0^t \mathfrak{U}(t-\gamma)\left(\alpha_1\tilde{\phi}\frac{\partial\tilde{\phi}}{\partial x} + \alpha_2\tilde{\varphi}\frac{\partial\tilde{\varphi}}{\partial x} + \alpha_3\tilde{\phi}\frac{\partial\tilde{\varphi}}{\partial x} + \alpha_4\tilde{\varphi}\frac{\partial\tilde{\phi}}{\partial x}\right)(\gamma)d\gamma \in X^{s,1+b'} \subset C([0, T], H^s(\mathbb{R})),$$

$$\psi_T\int_0^t \mathfrak{U}(t-\gamma)\left(\beta_1\tilde{\phi}\frac{\partial\tilde{\phi}}{\partial x} + \beta_2\tilde{\varphi}\frac{\partial\tilde{\varphi}}{\partial x} + \beta_3\tilde{\phi}\frac{\partial\tilde{\varphi}}{\partial x} + \beta_4\tilde{\varphi}\frac{\partial\tilde{\phi}}{\partial x}\right)(\gamma)d\gamma \in X^{s,1+b'} \subset C([0, T], H^s(\mathbb{R})).$$

Since $1 + b' > \frac{1}{2}$, then

$$(\tilde{\phi}, \tilde{\varphi}) \in C([0, T_\varrho], H^s(\mathbb{R})) \times C([0, T_\varrho], H^s(\mathbb{R})).$$

4. Global well-posedness (Proof of Theorem 1.2)

We assume here that

$$(\phi_0, \varphi_0) \in L^2\left(\Omega, L^2(\mathbb{R})\right) \times L^2\left(\Omega, L^2(\mathbb{R})\right),$$

and that the operator

$$\Xi \in L_2^0\left(L^2(\mathbb{R}), L^2(\mathbb{R}) \cap \dot{H}^{0,-\frac{3}{8}}(\mathbb{R})\right).$$

In order to show that the solution (ϕ, φ) may be continued on the whole interval $[0,1]$, we use the same argument as in [5]. We take a sequence

$$(\Xi_n)_{n\in\mathbb{N}} \in L_2^0\left(L^2(\mathbb{R}), H^4(\mathbb{R}) \cap \dot{H}^{0,-\frac{3}{8}}(\mathbb{R})\right),$$

such that

$$\Xi_n \to \Xi, \qquad \text{in} \quad L_2^0\left(L^2(\mathbb{R}), L^2(\mathbb{R}) \cap \dot{H}^{0,-\frac{3}{8}}(\mathbb{R})\right),$$

and another sequence

$$(\phi_{0,n}, \varphi_{0,n})_{n\in\mathbb{N}} \in L^2\left(\Omega, H^3(\mathbb{R})\right) \times L^2\left(\Omega, H^3(\mathbb{R})\right),$$

such that

$$(\phi_{0,n}, \varphi_{0,n}) \to (\phi_0, \varphi_0), \qquad \text{in} \quad L^2\left(\Omega, L^2(\mathbb{R})\right) \times L^2\left(\Omega, L^2(\mathbb{R})\right).$$

We know from Lemma 3.2 in [5] that there exists a unique solution

$$(\phi_n, \varphi_n) \in C\left([0,1], H^3(\mathbb{R})\right) \times C\left([0,1], H^3(\mathbb{R})\right),$$

$$\begin{cases} \phi_n(t) &= \mathfrak{U}(t)\phi_{0,n} - \displaystyle\int_0^t \mathfrak{U}(t-\gamma)\left(\alpha_1\phi_n\frac{\partial\phi_n}{\partial x} + \alpha_2\varphi_n\frac{\partial\varphi_n}{\partial x} + \alpha_3\phi_n\frac{\partial\varphi_n}{\partial x} + \alpha_4\varphi_n\frac{\partial\phi_n}{\partial x}\right)(\gamma)d\gamma \\ &\quad + \displaystyle\int_0^t \mathfrak{U}(t-\gamma)\Xi_n dW(\gamma), \\ \varphi_n(t) &= \mathfrak{U}(t)\varphi_{0,n} - \displaystyle\int_0^t \mathfrak{U}(t-\gamma)\left(\beta_1\phi_n\frac{\partial\phi_n}{\partial x} + \beta_2\varphi_n\frac{\partial\varphi_n}{\partial x} + \beta_3\phi_n\frac{\partial\varphi_n}{\partial x} + \beta_4\varphi_n\frac{\partial\phi_n}{\partial x}\right)(\gamma)d\gamma \\ &\quad + \displaystyle\int_0^t \mathfrak{U}(t-\gamma)\Xi_n dW(\gamma). \end{cases} \tag{4.1}$$

We then use the Itô formula on $\|\phi_n\|^2_{L^2(\mathbb{R})}$, $\|\varphi_n\|^2_{L^2(\mathbb{R})}$ and a Martingale inequality,

$$\mathbb{E}\left(\sup_{t\in[0,1]}\int_0^t (\phi_n(\gamma), \Xi_n dW(\gamma))\right) \le \frac{1}{2}\mathbb{E}\left(\sup_{t\in[0,1]} |\phi_n(t)|^2_{L_x^2}\right) + C\,\|\Xi_n\|^2_{L_2^{0,0}},$$
$$\mathbb{E}\left(\sup_{t\in[0,1]}\int_0^t (\varphi_n(\gamma), \Xi_n dW(\gamma))\right) \le \frac{1}{2}\mathbb{E}\left(\sup_{t\in[0,1]} |\varphi_n(t)|^2_{L_x^2}\right) + C\,\|\Xi_n\|^2_{L_2^{0,0}}.$$

We deduce that

$$\mathbb{E}\left(\sup_{t\in[0,1]} \|\phi_n(t)\|^2_{L_x^2}\right) \le \mathbb{E}\left(\|\phi_{0,n}\|^2_{L_x^2}\right) + C\,\|\Xi_n\|^2_{L_2^{0,0}},$$

$$\mathbb{E}\left(\sup_{t\in[0,1]}\|\varphi_n(t)\|_{L_x^2}^2\right)\leq\mathbb{E}\left(\|\varphi_{0,n}\|_{L_x^2}^2\right)+C\,\|\Xi_n\|_{L_2^{0,0}}^2.$$

Hence, the sequence $(\phi_n)_{n\in\mathbb{N}}, (\varphi_n)_{n\in\mathbb{N}}$ are bounded in $L^2\left(\Omega, L^\infty\left([0,1], L^2(\mathbb{R})\right)\right)$ so that it is weakly star convergent in this space to a function $\tilde{\phi}, \tilde{\varphi}$ which satisfies

$$\mathbb{E}\left(\sup_{t\in[0,1]}\|\tilde{\phi}(t)\|_{L_x^2}^2\right)\leq\mathbb{E}\left(\|\phi_0\|_{L_x^2}^2\right)+C\|\Xi\|_{L_2^{0,0}}^2,$$

$$\mathbb{E}\left(\sup_{t\in[0,1]}\|\tilde{\varphi}(t)\|_{L_x^2}^2\right)\leq\mathbb{E}\left(\|\varphi_0\|_{L_x^2}^2\right)+C\|\Xi\|_{L_2^{0,0}}^2.$$

Let us define the mapping $(\mathit{\Gamma}_{1,n}, \mathit{\Gamma}_{2,n})$ in the same way as $(\mathit{\Gamma}_1, \mathit{\Gamma}_2)$. It is easy to check that $(\mathit{\Gamma}_{1,n}, \mathit{\Gamma}_{2,n})$ is a strict contraction uniformly on $\mathcal{B}_{\mathcal{R}_1,T_{\varrho 1}}$ where

$$T_{\varrho 1}\geqslant 2C\left(\left(\sup_{n\in\mathbb{N}}\left(\left\|(\varpi\Upsilon_{1,n},\varpi\Upsilon_{2,n})\right\|_{\mathcal{X}_{0,b-\frac{1}{2}}}+\|(\varpi\Upsilon_{1,n},\varpi\Upsilon_{2,n})\|_{\mathcal{Y}^{0,-\frac{3}{8},b-\frac{1}{2}}}\right)\right)^2\right.$$

$$\left.+\,C_1^2\|(\tilde{\phi},\tilde{\varphi})\|_{L^\infty\left([0,1],L^2(\mathbb{R})\right)^2}^2\right)$$

and

$$4CT_{\varrho 1}^{(1-b+b')}\left(T_{\varrho 1}+\left(\sup_{n\in\mathbb{N}}\left(\left\|(\varpi\Upsilon_{1,n},\varpi\Upsilon_{2,n})\right\|_{\mathcal{X}_{0,b-\frac{1}{2}}}+\|(\varpi\Upsilon_{1,n},\varpi\Upsilon_{2,n})\|_{\mathcal{Y}^{0,-\frac{3}{8},b-\frac{1}{2}}}\right)\right)\right.$$

$$\left.+\,C_1\|(\tilde{\phi},\tilde{\varphi})\|_{L^\infty\left([0,1],L^2(\mathbb{R})\right)^2}\right)\leq 1,$$

where

$$\Upsilon_{1,n}(t)=\int_0^t\mathfrak{U}(t-\gamma)\Xi_n dW(\gamma),\qquad \Upsilon_{2,n}(t)=\int_0^t\mathfrak{U}(t-\gamma)\Xi_n dW(\gamma).$$

According to the fixed point theorem, there exists a unique fixed point

$$\phi_n\to\phi\in X_{T_{\varrho 1}}^{0,b}+X_{T_{\varrho 1}}^{0,b-\frac{1}{2}}\cap Y_{T_{\varrho 1}}^{0,-\frac{3}{8},b-\frac{1}{2}},$$

$$\varphi_n\to\varphi\in X_{T_{\varrho 1}}^{0,b}+X_{T_{\varrho 1}}^{0,b-\frac{1}{2}}\cap Y_{T_{\varrho 1}}^{0,-\frac{3}{8},b-\frac{1}{2}},$$

such that

$$(\phi,\varphi)=(\tilde{\phi},\tilde{\varphi})\quad\text{ on }[0,T_{\varrho 1}]\times[0,T_{\varrho 1}],$$

and

$$\|(\phi(T_{\varrho 1}),\varphi(T_{\varrho 1}))\|_{L^2(\mathbb{R})\times L^2(\mathbb{R})}\leq\|(\tilde{\phi},\tilde{\varphi})\|_{L^\infty\left([0,1],L^2(\mathbb{R})\right)^2}.$$

Thus, we can construct a solution on $[T_{\varrho 1}, 2T_{\varrho 1}]$ starting from $(\phi(T_{\varrho 1}), \varphi(T_{\varrho 1}))$. By iterating this argument, we obtain a solution on $[0, T_0]$.

5. Conclusions

The Stochastic Korteweg–de Vries system models nonlinear wave propagation under random disturbances, a framework crucial for predicting extreme events like rogue waves in turbulent oceans. It also describes energy transport in randomly forced plasmas and signal degradation in noisy optical fibers. The system is instrumental in quantifying the uncertainty and statistical properties of solitons in disordered media, with applications spanning geophysics, fusion science, and communications engineering.

This work focuses on a Stochastic Korteweg–de Vries-type system (1.3) in a random environment. We prove that this system is locally well-posed for initial data in the space $H^s(\mathbb{R})\times H^s(\mathbb{R})$ for $s > -5/8$, and demonstrate that its solutions can be extended to global ones on the interval $[0, T_0]$.

Inspired by the techniques in [6, 21], we handle the stochastic terms by introducing new, appropriate stochastic function spaces, specifically, $X^{s,b}$, $X^{s,b-\frac{1}{2}} \cap Y^{s,-\frac{3}{8},b-\frac{1}{2}}$, and by establishing key estimates for the stochastic convolution in these spaces. This approach allows us to analyze a more realistic, stochastically forced KdV-type system. We believe the ideas presented here can be applied to a broad class of stochastic nonlinear evolution systems in mathematical physics.

As an essential next step, the development of robust numerical schemes will be crucial for simulating these systems, verifying theoretical predictions, and exploring nonlinear phenomena such as the interaction of stochastic solitons.

Author contributions

Aissa Boukarou: Conceptualization, Methodology, Formal analysis, Writing–original draft preparation, Supervision; Mohammadi Begum Jeelani: Investigation, methodology; Nouf Abdulrahman Alqahtani: Investigation, methodology. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest in this paper.

References

1. R. L. Herman, The stochastic, damped KdV equation, *J. Phys. A: Math. Gen.*, **23** (1990), 1063. https://doi.org/10.1088/0305-4470/23/7/014

2. J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations, *Geom. Funct. Anal.*, **3** (1993), 107–156. https://doi.org/10.1007/BF01896020

3. C. E. Kenig, G. Ponce, L. Vega, The Cauchy problem for the Korteweg-de Vries equation in Sobolev spaces of negative indices, *Duke Math. J.*, **71** (1993), 1–21. https://doi.org/10.1215/S0012-7094-93-07101-3

4. C. E. Kenig, G. Ponce, L. Vega, A bilinear estimate with applications to the KdV equation, *J. Amer. Math. Soc.*, **9** (1996), 573–603.

5. A. de Bouard, A. Debussche, On the stochastic Korteweg-de Vries equation, *J. Funct. Anal.*, **154** (1998), 215–251. https://doi.org/10.1006/jfan.1997.3184

6. A. de Bouard, A. Debussche, Y. Tsutsumi, White noise driven Korteweg-de Vries equation, *J. Funct. Anal.*, **169** (1999), 532–558. https://doi.org/10.1006/jfan.1999.3484

7. A. de Bouard, A. Debussche, Random modulation of solitons for the stochastic Korteweg-de Vries equation, *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, **24** (2007), 251–278. https://doi.org/10.1016/j.anihpc.2006.03.009

8. A. Boukarou, K. Guerbati, K. Zennir, S. Alodhaibi, S. Alkhalaf, Well-posedness and time regularity for a system of modified Korteweg-de Vries-type equations in analytic Gevrey spaces, *Mathematics*, **8** (2020), 809. https://doi.org/10.3390/math8050809

9. T. Oh, Diophantine conditions in well-posedness theory of coupled KdV-type systems: Local theory, *Int. Math. Res. Notices*, **18** (2009), 3516–3556. https://doi.org/10.1093/imrn/rnp063

10. Y. Guo, K. Simon, E. S. Titi, Global well-posedness of a system of nonlinearly coupled KdV equations of Majda and Biello, *Commun. Math. Sci.*, **13** (2015), 1261–1288. https://doi.org/10.4310/cms.2015.v13.n5.a9

11. X. Carvajal, M. Panthee, Sharp well-posedness for a coupled system of mKdV-type equations, *J. Evol. Equ.*, **19** (2019), 1167–1197. https://doi.org/10.1007/s00028-019-00508-6

12. J. Ginibre, Le probleme de Cauchy pour des EDP semi-linéaires périodiques en variables d'espace, *Séminaire Bourbaki*, **796** (1995), 163–187.

13. G. Da Prato, J. Zabczyk, *Stochastic equations in infinite dimensions*, 2 Eds, Cambridge University Press, 2014. https://doi.org/10.1017/CBO9781107295513

14. M. S. Ullah, Interaction solution to the (3+1)-D negative-order KdV first structure, *Partial Differ. Equ. Appl. Math.*, **8** (2023), 100566. https://doi.org/10.1016/j.padiff.2023.100566

15. F. S. Alshammari, Z. Rahman, H. O. Roshid, M. S. Ullah, A. Aldurayhim, M. Z. Ali, Dynamical structures of multi-solitons and interaction of solitons to the higher-order KdV-5 equation, *Symmetry*, **15** (2023), 626. https://doi.org/10.3390/sym15030626

16. F. Wang, Z. Ma, Meshless collocation method for inverse source identification problems, *Adv. Appl. Math. Mech.*, **7** (2015), 496–509. https://doi.org/10.4208/aamm.2014.m611

17. F. Wang, W. Chen, L. Ling, Combinations of the method of fundamental solutions for general inverse source identification problems, *Appl. Math. Comput.*, **219** (2012), 1173–1182. https://doi.org/10.1016/j.amc.2012.07.027

18. M. Amir, A. Boukarou, S. M. Mirgani, M. H. Kesri, K. Zennir, M. B. Jeelani, Well-posedness for a coupled system of gKdV equations in analytic spaces, *Electron. Res. Arch.*, **33** (2025), 4119–4134. https://doi.org/10.3934/era.2025184

19. A. Boukarou, K. Zennir, M. Bouye, A. Moumen, Nondecreasing analytic radius for the a Kawahara-Korteweg-de-Vries equation. *AIMS Mathematics*, **9** (2024), 22414–22434. http://doi.org/10.3934/math/20241090

20. A. Boukarou, K. Guerbati, K. Zennir, M. Alnegga, Gevrey regularity for the generalized Kadomtsev-Petviashvili I (gKP-I) equation, *AIMS Mathematics*, **6** (2021), 10037–10054. https://doi.org/10.3934/math.2021583

21. A. Boukarou, S. M. Mirgani, K. Zennir, K. Bouhali, S. S. Alodhaibi, White-noise-driven KdV-type Boussinesq system, *Mathematics*, **13** (2025), 1758. https://doi.org/10.3390/math13111758

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