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Research article

Exploring prime F−filters of almost distributive lattices

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Abstract: The concept of F−filters is introduced in an almost distributive lattice(ADL) and their properties are studied. A set of equivalent conditions is established for every proper F−filter of an ADL to become a prime F−filter. For any F−filter U of an ADL, $O^F(U)$ is defined, and it is proved that $O^F(U)$ is an F−filter if U is prime. It is also derived that each minimal prime F−filter belonging to $O^F(U)$ is contained in U, and $O^F(U)$ is the intersection of all the minimal prime F−filters contained in U. The concept of F−normal ADL is defined and characterized in terms of the prime F−filters and minimal prime F−filters, as well as relative annihilators with respect to F.

Keywords: almost distributive lattice; F−filter; F−normal; totally ordered set; comaximal
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1. Introduction

The notion of an almost distributive lattice (ADL) was introduced by Swamy and Rao [9] as a unifying algebraic framework that generalizes both Boolean algebras and distributive lattices. Their motivation was to capture the essential distributive features found in these classical structures while extending their applicability to broader algebraic systems, particularly those inspired by ring theory. In their pioneering work, they also formulated the idea of ideals in ADLs, mirroring the role of ideals in distributive lattices, and proved that the set of all principal ideals, denoted by $\Im^{\mathcal{PI}}(A)$, forms a distributive lattice. This significant result laid the groundwork for transferring several key concepts and theorems from traditional lattice theory into the newly defined class of ADLs.

Subsequent research enriched the theory by introducing new structural notions. Cornish [2, 3] developed the ideas of normal lattices and n-normal lattices, contributing to the deeper structural understanding of distributive systems. Building upon these foundational ideas, Rao and Ravi Kumar [7] introduced the concept of minimal prime ideals corresponding to a given ideal in an ADL and explored their algebraic properties. Later, in [8], the same authors proposed the concept of normal ADLs and presented several equivalent conditions for an ADL to be normal, expressed through its annulet structure.

Further progress was made by Kumar et al. [5], who introduced and analyzed the concept of $F-$filters in lattices. They established a series of equivalences characterizing when a proper $F-$filter becomes a prime $F-$filter, offering new insights into filter theory.

Motivated by these advancements, the present study extends the investigation of $F-$filters and their prime counterparts to the class of almost distributive lattices. We introduce the definitions of $F-$filters and prime $F-$filters in ADLs and derive necessary and sufficient conditions for a proper $F-$filter to be prime. It is shown that every maximal $F-$filter in an ADL is prime. Moreover, for any prime $F-$filter U of an ADL A, we demonstrate that $O^F(U) = \{s \in A \mid s \in (p, F) \text{ for some } p \in A \setminus U\}$ is precisely the intersection of all minimal prime $F-$filters contained in U. Finally, the paper presents a characterization of ADLs in terms of prime and minimal prime $F-$filters, as well as relative annihilators with respect to F.

2. Preliminaries

This section compiles essential definitions and key results drawn from [6, 9], which will serve as foundational tools throughout the remainder of this work.

Definition 2.1. *[9] An algebra $(A, \vee, \wedge, 0)$ of type $(2, 2, 0)$ is said to be an almost distributive lattice (ADL) with zero if it satisfies the following axioms:*

(1) $(p \vee q) \wedge r = (p \wedge r) \vee (q \wedge r)$;

(2) $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$;

(3) $(p \vee q) \wedge q = q$;

(4) $(p \vee q) \wedge p = p$;

(5) $p \vee (p \wedge q) = p$;

(6) $0 \wedge p = 0$, *for any $p, q, r \in A$.*

Example 2.2. *Every non-empty set B can be regarded as an ADL as follows. Let $s_0 \in B$. Define the binary operations $\vee, \wedge$ on B by*

$$s \vee t = \begin{cases} s & \text{if } s \neq s_0 \\ t & \text{if } s = s_0 \end{cases} \qquad s \wedge t = \begin{cases} t & \text{if } s \neq s_0 \\ s_0 & \text{if } s = s_0. \end{cases}$$

Then, $(B, \vee, \wedge, s_0)$ is an ADL (where s_0 is the zero) and is called a discrete ADL.

Let us define a partial order $\leq$ on the set A by the condition that also, for all $p, q \in A$, we write $p \leq q$ iff $p = p \wedge q$, or equivalently, $p \vee q = q$.

This relation clearly satisfies the requirements of a partial order on A. As usual, an element $m \in A$ is called maximal if it is a maximal element in the partially ordered set $(A, \leq)$. That is, for any $p \in A$,

$m \leq p \Rightarrow m = p$. The set of all elements possessing this property is denoted by $\mathcal{M}_{\text{Max.elts}}$. According to Swamy's findings in [9], it can be observed that an ADL A satisfies almost all the properties of a distributive lattice except the right distributivity of $\vee$ over $\wedge$, commutativity of $\vee$, and commutativity of $\wedge$. Any one of these properties make an ADL A a distributive lattice. A subset J of A is defined to be an ideal (or a filter) if it is nonempty and satisfies the following: For all $p, q \in J$ and $s \in A$, the elements $p \vee q$ and $p \wedge s$ (respectively, $p \wedge q$ and $s \vee p$) belong to J. An ideal (or filter) P of A is called prime if it is proper and whenever two elements s and t from A satisfy $s \wedge t \in P$ (respectively, $s \vee t \in P$), then either $s \in P$ or $t \in P$. A proper ideal (filter) U of A is said to be maximal if it is not properly contained in any proper ideal (filter) of A. It can be observed that every maximal ideal (filter) of A is a prime ideal (filter). For any subset G of A, the smallest ideal containing G is given by $(G] := \{(\bigvee_{i=1}^{n} p_i) \wedge s \mid p_i \in G, s \in A, n \in \mathbb{N}\}$. Let $G = \{p\}$ be a singleton subset of A. The ideal generated by p is denoted as $(p]$, which is referred to as a principal ideal. Likewise, for any subset $G \subseteq A$, the smallest filter containing G is given by

$$[G) = \{s \vee (\bigwedge_{i=1}^{n} p_i) \mid p_i \in G,\ s \in A,\ n \in \mathbb{N}\}.$$

When G is a singleton $\{p\}$, this filter is written as $[p)$ and is referred to as the principal filter of A. For elements $p, q \in A$, it holds that $(p] \vee (q] = (p \vee q]$ and $(p] \cap (q] = (p \wedge q]$. Therefore, the structure $(\mathcal{PI}(A), \vee, \cap)$, comprising all principal ideals, forms a sublattice of the complete lattice $(\Im(A), \vee, \cap)$ of all ideals. In parallel, the collection $\mathfrak{F}(A)$ of all filters, under join and meet, forms a bounded distributive lattice. In the setting of ADLs, it was established in [7] that a proper ideal P is prime if and only if its complement $A \setminus P$ is a prime filter. Unless otherwise specified, the notation A will denote an Almost Distributive Lattice that includes maximal elements, and F will refer to a filter within A.

3. On prime $F-$filters

This section presents the notations of $F-$filters, prime $F-$filters within an ADL, and examines their properties. It establishes a set of conditions under which each proper $F-$filter of an ADL becomes as prime. Furthermore, it demonstrates that every maximal $F-$filter in an ADL qualifies as prime.

Definition 3.1. *A filter J of A is referred to as an $F-$filter of A if F is contained in J.*

Let us now look at an example of an $F-$filter in the context of an ADL.

Example 3.2. *Consider the set $A = \{0, p, q, r, d, e, g, f\}$, with the operations $\vee$ (join) and $\wedge$ (meet) defined on A as follows:*

∧	0	p	q	r	d	e	g	f
0	0	0	0	0	0	0	0	0
p	0	p	q	r	d	e	g	f
q	0	p	q	r	d	e	g	f
r	0	r	r	r	0	0	r	0
d	0	d	e	0	d	e	f	f
e	0	d	e	0	d	e	f	f
g	0	g	g	r	f	f	g	f
f	0	f	f	0	f	f	f	f

∨	0	p	q	r	d	e	g	f
0	0	p	q	r	d	e	g	f
p	p	p	p	p	p	p	p	p
q	q	q	q	q	q	q	q	q
r	r	p	q	r	p	q	g	g
d	d	p	p	p	d	d	p	d
e	e	q	q	q	e	e	q	e
g	g	p	q	g	p	q	g	g
f	f	p	q	g	d	e	g	f

Thus, $(A, \vee, \wedge)$ *is an ADL. It is evident that* $F = \{p, q, g\}$ *and* $J = \{p, q, r, g\}$ *are filters of A with* $F \subseteq J$*. Hence, J qualifies as an F−filter of A.*

The validity of the following result can be readily verified.

Lemma 3.3. *For each non-void subset G of A, the expression* $[G) \vee F$ *represents the smallest F−filter of A that includes G.*

We represent $[G) \vee F$ as G^F, meaning $G^F = [G) \vee F$. For the case when $G = \{p\}$, we simply denote this as $(p)^F$ instead of $\{p\}^F$. It is evident that $(p)^F$ is the smallest F-filter that contains p, which is referred to as the principal F−filter generated by p.

Lemma 3.4. *For any pair of elements s and t in A, the following is true:*
(1) $(0)^F = A$
(2) $(m)^F = F$*, where* $m \in \mathcal{M}_{Max.elts}$
(3) $s \leq t$ *implies* $(t)^F \subseteq (s)^F$
(4) $(s \wedge t)^F = (s)^F \vee (t)^F$
(5) $(s \vee t)^F = (s)^F \cap (t)^F$
(6) $(s)^F = F$ *if and only if* $s \in F$.

Proof. (1) Now, $(0)^F = [0) \vee F = A \vee F = A$.
(2) Now, $(m)^F = [m) \vee F = \{m\} \vee F \subseteq F$. Clearly, we have $F \subseteq (m)^F$. Therefore, $F = (m)^F$.
(3) Let $s \leq t$. Then, $[t) \subseteq [s)$. Now, $(t)^F = [t) \vee F \subseteq [s) \vee F = (s)^F$. Therefore, $(t)^F \subseteq (s)^F$.
(4) Clearly, we have that $[s\wedge t) = [s)\vee[t)$. Now, $(s\wedge t)^F = [s\wedge t)\vee F = [s)\vee[t)\vee F = ([s)\vee F)\vee([t)\vee F)) = (s)^F \vee (t)^F$. Therefore, $(s \wedge t)^F = (s)^F \vee (t)^F$.
(5) Since $s \leq s \vee t$ and $t \leq t \vee s$, then $[s \vee t) \subseteq [s)$ and $[t \vee s) \subseteq [t)$. Since $[s \vee t) = [t \vee s)$, then $[s\vee t) \subseteq [s)\cap[t)$. Let $b \in [s)\cap[t)$. Then, $b \in [s)$, $b \in [t)$. This gives $b\vee s = b$, $b\vee t = b$. Now, $b\wedge(s\vee t) = (b \wedge s) \vee (b \wedge t) = s \vee t$. This implies $b \vee (s \vee t) = b$, and hence $b \in [s \vee t)$. Therefore, $[s) \cap [t) \subseteq [s \vee t)$. Thus, $[s \vee t) = [s) \cap [t)$. Now, $(s \vee t)^F = [s \vee t) \vee F = ([s) \cap [t)) \vee F = ([s) \vee F) \cap ([t) \vee F) = (s)^F \cap (t)^F$. Hence, $(s \vee t)^F = (s)^F \cap (t)^F$.
(6) Assume that $(s)^F = F$. Then, $[s) \vee F = F$. This implies $[s) \subseteq F$, and hence $s \in F$. Conversely, assume that $s \in F$. Then, $[s) \subseteq F$. This implies that $[s) \vee F \subseteq F$. As $F \subseteq [s) \vee F$, it follows that $F = [s) \vee F$. Therefore $(s)^F = F$. □

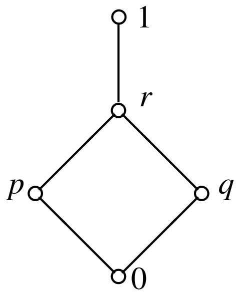
We represent $\mathfrak{F}(A)$, $\mathfrak{F}^F(A)$, and $\mathfrak{F}^{\mathcal{P}F}(A)$ as the collections of all filters, F−filters, and principal F−filters of A, respectively.

Theorem 3.5. *The collection $\mathfrak{F}^F(A)$ constitutes a distributive lattice that is contained in $\mathfrak{F}(A)$, while $\mathfrak{F}^{\mathcal{P}F}(A)$ serves as a sublattice of $\mathfrak{F}^F(A)$.*

Definition 3.6. *An $F-$filter (C) is called proper if $C \subsetneq A$. A proper $F-$filter C is said to be maximal if there exists no other proper $F-$filter of A that strictly contains it. Furthermore, a proper $F-$filter C in A is called prime if it satisfies the standard definition of primeness in A.*

We represent $Max_F(A)$ and $\mathrm{Spec}_F(A)$ as the collections of all maximal $F-$filters and prime $F-$filters of A, respectively.

Example 3.7. *Let $Y = \{0, p, q, r, 1\}$ represent a distributive lattice, and let $D = \{0', p'\}$ denote a discrete ADL.*



It is evident that
$A = D \times Y = \{(0', 0), (0', p), (0', q), (0', r), (0', 1), (p', 0), (p', p), (p', q), (p', r), (p', 1)\}$ *is an ADL with zero element* $(0, 0')$.
Consider the filters
$F_1 = \{(0', p), (0', r), (0', 1), (p', p), (p', 1), (p', r)\}$
$F_2 = \{(0', 1), (p', 1)\}$
$F_3 = \{(0', q), (0', r), (0', 1), (p', 1), (p', q), (p', r)\}$
$F_4 = \{(p', p), (p', r), (p', 1)\}$
$F_5 = \{(p', q), (p', r), (p', 1)\}$
$F_6 = \{(p', r), (p', 1)\}$
$F_7 = \{(0', r), (p', r), (p', 1), (0', 1)\}$

Here, F_1, F_3, and F_7 are F_2-filters. Clearly, F_1 and F_3 are prime F_2-filters. But, F_7 is not a prime F_2-filter, because $(p', q) \vee (p', p) = (p', r) \in F_7$, while $(p', q), (p', p) \notin F_7$. Here, F_4 and F_5 are F_6-filters. F_5 is a prime F_6-filter. But, F_4 is not a prime F_6-filter, because $(0', p) \vee (p', q) = (p', r) \in F_4$, while $(0', p) \notin F_4$ and $(p', q) \notin F_4$.

Theorem 3.8. *For each $C \in \mathfrak{F}^F(A)$ of A, the subsequent conditions hold true equivalently:*
(1) $C \in Spec_F(A)$
(2) for every $J, Y \in \mathfrak{F}^F(A)$, $J \cap Y \subseteq C \Rightarrow J \subseteq C$ or $Y \subseteq C$
(3) for every $s, t \in A, (s)^F \cap (t)^F \subseteq C \Rightarrow s \in C$ or $t \in C$.

Proof. (1) $\Rightarrow$ (2) : Assume (1). Let J, $Y \in \mathfrak{F}^F(A)$ with the condition that $J \cap Y \subseteq C$. We want to show that either $J \subseteq C$ or $Y \subseteq C$. Assume, for contradiction, that neither J nor Y is a subset of C. Therefore, we can select elements s and t such that $s \in J \setminus C$ and $t \in Y \setminus C$. Given that s is in J and t is in Y, it follows that the join $s \vee t$ should belong to the intersection $J \cap Y$, which is contained in C.

Therefore, $s \vee t \in C$. However, since $s, t \notin C$, we have $s \vee t \notin C$ as well, leading to a contradiction. This contradiction implies that our assumption must be false. Hence, it follows that either $J \subseteq C$ or $Y \subseteq C$.
(2) $\Rightarrow$ (3) : Assume (2). Let $s, t \in A$ such that $(s)^F \cap (t)^F \subseteq P$. Since both $(s)^F$ and $(t)^F$ belong to $\mathfrak{F}^F(A)$, our assumption implies that either $(s)^F \subseteq P$ or $(t)^F \subseteq P$. Consequently, this leads to the conclusion that $s \in P$ or $t \in P$.
(3) $\Rightarrow$ (1) : Assume (3). Let $s, t \in A$ be such that $s \vee t \in C$. As $C \in \mathfrak{F}^F(A)$, it follows that $(s)^F \cap (t)^F = (s \vee t)^F \subseteq C$. Based on this assumption, we conclude that either $s \in C$ or $t \in C$. Therefore, we can deduce that C is prime. □

Theorem 3.9. *In an ADL A, each maximal $F-$filter is prime.*

Proof. Let $W \in Max_F(A)$. Consider two elements $p, q \in A$ such that both p and q are not in W. This means that we have $W \vee (p)^F = A$ and $W \vee (q)^F = A$. From these relationships, we can deduce that

$$A = W \vee ((p)^F \cap (q)^F) = W \vee (p \vee q)^F$$

If both $p \vee q \in W$, then we would conclude that W is equal to A, resulting in a contradiction. Hence, we must conclude that $p \vee q \notin W$. This establishes that W is a prime filter.

□

Corollary 3.10. *Consider the maximal $F-$filters $W_1, W_2, \ldots, W_n$ and W in A. If it is given that $\bigcap_{i=1}^{n} W_i \subseteq W$, then it can be concluded that there is some $j \in \{1, 2, \ldots, n\}$ which satisfies $W_j \subseteq W$.*

Theorem 3.11. *A proper $F-$filter C in A is prime if and only if the complement $A \setminus C \in Spec_F(A)$ and $(A \setminus C) \cap F = \emptyset$.*

Proof. Assume that C is a prime $F-$filter of A. It follows that $A \setminus C$ forms a prime ideal in A. We will demonstrate that $(A \setminus C) \cap F = \emptyset$. Suppose, for contradiction, that $(A \setminus C) \cap F$ is not empty. This implies that there exists some element $s \in (A \setminus C) \cap F$. Consequently, s belongs to F, and since $F \subseteq C$, it must also be true that $s \in C$. This results in a contradiction, as s cannot belong to both C and $A \setminus C$. Thus, we conclude that $(A \setminus C) \cap F = \emptyset$. Conversely, let's assume that $A \setminus C$ is a prime ideal of A and that $(A \setminus C) \cap F = \emptyset$. Under these conditions, it can be shown that C is indeed a prime filter of A. Furthermore, since $F \subseteq A \setminus (A \setminus C) = C$, we conclude that $C \in \mathrm{Spec}_F(A)$. □

Theorem 3.12. *Let $J \in \mathfrak{F}^F(A)$, and Q be a non-void subset of A that is closed under $\vee$, which satisfies $J \cap Q = \emptyset$. Then, there is $F-$filter $C \in Spec_F(A)$ that contains J and satisfies $C \cap Q$ is empty.*

Proof. Consider the collection $\mathfrak{F} = \{Y \mid Y \in \mathfrak{F}^F(A), J \subseteq Y, Y \cap Q = \emptyset\}$. By applying Zorn's lemma, it follows that there is at least one maximal element within $\mathfrak{F}$, which we denote as C. Therefore, C qualifies as an $F-$filter of A with the properties that $J \subseteq C$ and $C \cap Q = \emptyset$. Now, consider elements $s, t \in A$ such that $s \vee t \in C$. We will show that at least one of s or t must belong to C. Assume, for contradiction, that neither s nor t is in C.

If this is true, then both $C \vee (s)^F$ and $C \vee (t)^F$ are $F-$filters of A, and it holds that $C \subsetneq C \vee (s)^F$ and $C \subsetneq C \vee (t)^F$.

Since C is maximal in $\mathfrak{F}$, we conclude that both $(C \vee (s)^F) \cap Q \neq \emptyset$ and $(C \vee (t)^F) \cap Q \neq \emptyset$.

Next, let a be an element from $(C \vee (s)^F) \cap Q$ and b be an element from $(C \vee (t)^F) \cap Q$. Thus, we have $a \in (C \vee (s)^F)$, $b \in (C \vee (t)^F)$, and both a and b are members of Q. Since Q is closed under the operation $\vee$, we can assert that $a \vee b \in Q$. We can express $a \vee b$ as follows:

$$a \vee b = \{C \vee (s)^F\} \cap \{C \vee (t)^F\} = C \vee \{(s)^F \cap (t)^F\} = C \vee (s \vee t)^F.$$

Since $s \vee t \in C$, then $a \vee b \in C$. However, since $a \vee b$ is also in Q, we reach a contradiction with the assertion $a \vee b \in C \cap Q$, violating the condition that $C \cap Q = \emptyset$. As a result, we conclude that at least one of s or t must be in C. Thus, we confirm that $C \in \mathrm{Spec}_F(A)$. □

Corollary 3.13. *If $J \in \mathfrak{F}^F(A)$ with $s \notin J$, then there is $C \in Spec_F(A)$ in A such that J is contained in C and s is not included in C.*

Corollary 3.14. *For any $F-$filter J of an ADL A, $J = \bigcap\{C|C$ is a prime $F-$filter of A and $J \subseteq C\}$.*

Corollary 3.15. *F can be represented as the intersection of every prime $F-$filter in A.*

Proof. Let C be any prime $F-$filter of A. It is evident that $F \subseteq \bigcap C$. Now, let C be a prime $F-$filter of A, and suppose $s \in \bigcap C$. If we assume $s \notin F$, then there exists a prime ideal W such that $s \in W$ and $W \cap F = \emptyset$. This leads to $s \notin A \setminus W$ and implies that $F \subseteq A \setminus W$. Thus, $A \setminus W \in \mathrm{Spec}_F(A)$, and since $s \notin A \setminus W$, we reach a contradiction. Thus, we conclude that $s \in F$, which means $\bigcap C \subseteq F$. Therefore, we establish that $F = \bigcap C$. □

Theorem 3.16. *The equivalence of the following statements holds in an ADL:*
(1) Any proper $F-$filter is prime;
(2) $\mathfrak{F}^F(A)$ is a totally ordered set;
(3) $\mathfrak{F}^{\mathcal{P}F}(A)$ is a totally ordered set.

Proof. (1) ⇒ (2) : Assume (1). It is evident that $(\mathfrak{F}^F(A), \subseteq)$ forms a poset. Let H and X be two proper $F-$filters of A. According to (1), the intersection $H \cap X$ is prime. Since $H \cap X \subseteq H$ and $H \cap X \subseteq X$, it follows that either $H \subseteq H \cap X \subseteq X$ or $X \subseteq H \cap X \subseteq H$. Thus, we conclude that $\mathfrak{F}^F(A)$ forms a totally ordered set.
(2) ⇒ (3) : It is straightforward.
(3) ⇒ (1) : Assume (3). Suppose J is a proper $F-$filter of A. We will demonstrate that $J \in \mathrm{Spec}_F(A)$. Consider the elements $s, t \in A$ with $(s)^F \cap (t)^F \subseteq J$. By our assumption, it follows that either $(s)^F \subseteq (t)^F$ or $(t)^F \subseteq (s)^F$. This leads us to conclude that $s \in (s)^F = (s)^F \cap (t)^F \subseteq J$ or $t \in (t)^F = (s)^F \cap (t)^F \subseteq J$. Thus, we establish that $J \in \mathrm{Spec}_F(A)$. □

The concept of a relative annihilator is introduced below.

Definition 3.17. *For any nonempty subset H of A, define $(H, F) = \{p \in A \mid a \vee p \in F$ for all $a \in H\}$. This set is referred as a relative annihilator of H with respect to F.*

When $H = \{a\}$, we write $(\{a\}, F)$ as (a, F).

Lemma 3.18. *Consider nonempty subsets H and X in A. The statements below are true:*
(1) $(A, F) = F = (\{0\}, F)$
(2) $(F, F) = A$
(3) $F \subseteq (H, F)$

(4) $(H, F) \in \mathfrak{F}^F(A)$
(5) If $H \subseteq F$ *iff* $(H, F) = A$
(6) $H \subseteq X$ *implies* $(X, F) \subseteq (H, F)$ *and* $((H, F), F) \subseteq ((X, F), F)$
(7) $H \subseteq ((H, F), F)$
(8) $(((H, F), F), F) = (H, F)$
(9) $(H, X) = ([H), F)$
(10) $\bigcap_{i\in\triangle}(H_i, F) = \left(\bigcup_{i\in\triangle} H_i, F\right)$
(11) $(H, F) \subseteq (H \cap X, (X, F))$
(12) If $H \subseteq X$ *then* $(H, (X, F)) = (H, F)$
(13) $(H \cup X, F) \subseteq (H, (X, F)) \subseteq (H \cap X, F)$
(14) $(H, (H, F)) = (H, F)$.

Proof. (1). Assume $s \in (A, F)$. Then, for any $p \in A$, we have $p \vee s \in F$. Thus, $s \vee s \in F$, which leads to the conclusion that $s \in F$. Therefore, we can assert that $(A, F) \subseteq F$. Next, consider $s \in F$. In this case, for every $p \in A$, it follows that $p \vee s \in F$. Consequently, we conclude that $s \in (A, F)$, resulting in $F \subseteq (A, F)$. Thus, we establish that $(A, F) = F$. It is evident that $(\{0\}, F) = F$.
(2). Let $s \in F$. Then, for every $p \in A$, we have $s \vee p \in F$. Since this holds for all $s \in F$, we deduce that $p \in (F, F)$ for all $p \in A$. Consequently, we can conclude that $A \subseteq (F, F)$, leading to the result that $A = (F, F)$.
(3). Let $s \in F$. Thus, for all $t \in A$, it holds that $t \vee s \in F$. Thus, for all $p \in H \subseteq A$, we have $p \vee s \in F$ as well. This means $s \in (H, F)$. Therefore, we can conclude that $F \subseteq (H, F)$.
(4). Let $p, q \in (H, F)$. For any $a \in H$, we have $a \vee p$ and $a \vee q$ in F. This leads to $(a \vee p) \wedge (a \vee q) \in F$, which implies that $a \vee (p \wedge q) \in F$. Therefore, $p \wedge q$ is an element of (H, F). Now, assume $p \in (H, F)$ and $q \in A$ such that $p \leq q$. For every $a \in H$, we get $a \vee p \in F$ and $a \vee p \leq a \vee q$. Since F is a filter and $a \vee p \in F$, we can conclude that $a \vee q \in F$. Hence, q lies in (H, F) for all $a \in H$, showing that (H, F) forms a filter on A. Since $F \subseteq (H, F)$, we get that (H, F) is an F-filter of A.
(5). Assume that $(H, F) = A$. Then, 0 belongs to (H, F). This means that for any $p \in H$, we have $p = p \vee 0 \in F$. Hence, $p \in F$ for all $p \in H$, which implies F containing H. Conversely, suppose $H \subseteq F$. Let $s \in A$. From a filter F, it gives $p \vee s \in F$ for any $p \in H \subseteq F$. Thus, $s \in (H, F)$, which leads to $(H, F) = A$.
(6). Let us consider the case where $H \subseteq X$. If $p \in (X, F)$, then for every $b \in X$, it holds that $b \vee p \in F$. Given that $H \subseteq X$, it follows that for all $a \in H$, $a \vee p \in F$. This indicates that p belongs to (H, F). Therefore, we have $(X, F) \subseteq (H, F)$, which leads us to conclude that $((H, F), F) \subseteq ((X, F), F)$.
(7). Assume $s \in (H, F)$. For every $a \in H$, it holds that $a \vee s \in F$. This implies that $s \vee a \in F$ for all $s \in (H, F)$. Thus, we can conclude that $a \in ((H, F), F)$ for each $a \in H$. Therefore, we establish that $H \subseteq ((H, F), F)$.
(8). From (7), we have $(((H, F), F), F) \subseteq (H, F)$. Now, suppose $s \notin (((H, F), F), F)$. This implies there is $p \notin ((H, F), F)$ such that $p \vee s \notin F$. Given that $H \subseteq ((H, F), F)$, we conclude that $p \notin H$. Consequently, we find that $p \vee s \notin F$ and $a \notin H$. Therefore, $s \notin (H, F)$. This leads to the conclusion that $(H, F) \subseteq (((H, F), F), F)$. Thus, we arrive at the equality $(((H, F), F), F) = (H, F)$.
(9). From $H \subseteq [H)$, we can conclude that $([H), F) \subseteq (H, F)$. Let s be an element of (H, F). For every $p \in H \subseteq [H)$, it follows that $p \vee s \in F$. This leads us to conclude that $s \in ([H), F)$. Thus, we have $(H, F) \subseteq ([H), F)$. Hence, we arrive at the equality $(H, F) = ([H), F)$.

(10). Since $S_i \subseteq \bigcup_{i\in\triangle} H_i$ for all $i \in \triangle$, it gives $\left(\bigcup_{i\in\triangle} H_i, F\right) \subseteq (H_i, F)$ for every $i \in \triangle$. This implies that $\left(\bigcup_{i\in\triangle} H_i, F\right) \subseteq \bigcap_{i\in\triangle}(H_i, F)$. Let $s \in \bigcap_{i\in\triangle}(H_i, F)$. Then, s belongs to (H_i, F) for all $i \in \triangle$. This means that $p \vee s \in F$ for all $p \in H_i \subseteq \bigcup H_i$. Thus, we conclude that $\bigcap_{i\in\triangle}(H_i, F) \subseteq \left(\bigcup_{i\in\triangle} H_i, F\right)$. Therefore, we arrive at the equality $\bigcap_{i\in\triangle}(H_i, F) = \left(\bigcup_{i\in\triangle} H_i, F\right)$.
(11). As F is a filter in A, it follows that $F \subseteq (X, F)$. Consequently, we can deduce that $(H, F) \subseteq (H, (X, F))$. Since $H \cap X \subseteq H$, we get $(H, (X, F)) \subseteq (H \cap X, (X, F))$. Thus, we conclude that $(H, F) \subseteq (H \cap X, (X, F))$.
(12). Let H and X be two non-void subsets of A with $H \subseteq X$. As $F \subseteq (X, F)$, it concludes $(H, F) \subseteq (H, (X, F))$. Now, take $s \in (H, (X, F))$. For every $p \in H$, it follows that $p \vee s \in (X, F)$. Consequently, this means $p \vee s \in (H, F)$ for all $p \in H$. Since $p \vee s$ belongs to (H, F), we also have $a \vee (p \vee s) \in F$ for all $a \in H$, which indicates that $p \vee s \in F$ for every $p \in H$. As a result, we conclude that $s \in (H, F)$. This leads $(H, (X, F)) \subseteq (H, F)$. Therefore, we arrive at the final equality $(H, (X, F)) = (H, F)$.
(13). It is evident that $(H \cup X, F) \subseteq (H, F)$ and $F \subseteq (X, F)$. Consequently, we can deduce that $(H, F) \subseteq (H, (X, F))$. Moreover, since $H \cap X \subseteq H$, we find that $(H, (X, F)) \subseteq (H \cap X, F)$. Thus, it follows that $(H \cup X, F) \subseteq (H, (X, F)) \subseteq (H \cap X, F)$.
(14). It is evident from (12). □

Proposition 3.19. *The statements below are true for all $H,\ X \in \mathfrak{F}(A)$:*
(1) $(H, F) \cap ((H, F), F) = F$
(2) $(H \vee X, F) = (H, F) \cap (X, F)$
(3) $((H \cap X, F), F) \subseteq ((H, F), F) \cap ((X, F), F)$.

Proof. (1). Clearly $F \subseteq (H, F) \cap ((H, F), F)$. Let $s \in (H, F) \cap ((H, F), F)$. Then, $s \in (H, F)$ and $s \in ((H, F), F)$. Since $s \in ((H, F), F))$, we have that $p \vee s \in F$, for all $p \in (H, F)$. Since $s \in (H, F)$, we get that $s \in F$. Therefore, $(H, F) \cap ((H, F), F) \subseteq F$. Hence, $(H, F) \cap ((H, F), F) = F$.
(2). As $H \subseteq H \vee X$ and $X \subseteq H \vee X$, we get $((H \vee X), F) \subseteq (H, F)$ and $((H \vee X), F) \subseteq (X, F)$, which gives $((H \vee X), F) \subseteq (H, F) \cap (X, F)$. Let $s \in (H, F) \cap (X, F)$. Then, $s \in (H, F)$ and $s \in (X, F)$. This leads to that $a \vee s \in F$ for every $a \in H$, and $b \vee s \in F$ for every $b \in X$. This implies $(a \vee s) \wedge (b \vee s) \in F$, and hence $(a \wedge b) \vee s \in F$. Since $a \in H$ and $b \in X$, $a \wedge b \in H \vee X$. It follows that $(a \wedge b) \vee s \in F$, for every $a \wedge b \in H \vee X$. This leads to $s \in (H \vee X, F)$. It follows that $(H, F) \cap (X, F) \subseteq (H \vee X, F)$. This concludes that $(H, F) \cap (X, F) = (H \vee X, F)$.
(3). As $H \cap X \subseteq H$ and $H \cap X \subseteq X$, it follows that $(H, F) \subseteq (H \cap X, F)$ and $(X, F) \subseteq (H \cap X, F)$. This leads to $((H \cap X, F), F) \subseteq ((H, F), F)$ and $((H \cap X, F), F) \subseteq ((X, F), F)$. Therefore, $((H \cap X, F), F) \subseteq ((H, F), F) \cap ((X, F), F)$.

□

Theorem 3.20. *Assume $H \subseteq A$ is non-empty. Then $(H, F) = \bigcap_{a\in H}([a), F)$.*

Proof. Let $s \in \bigcap_{a\in H}([a), F)$. Then, $s \in ([a), F)$ for all $a \in H$. We get $b \vee s \in F$ for every $b \in [a)$ and $a \in H$. This gives $a \vee s \in F$ for every $a \in H$. It follows that $s \in (H, F)$. Thus, $s \in \bigcap_{a\in H}([a), F) \subseteq (H, F)$.

Let $a \in H$ with $b \in [a)$. Then, we obtain $b \vee a = b$. Now, $s \in (H, F)$. This gives $a \vee s \in F$ for every $a \in H$. This implies $b \vee a \vee s \in F$ for every $b \in [a) \subseteq H$ and for every $a \in H$. From this, we obtain $b \vee s \in F$ for every $b \in [a)$ and $a \in H$. It follows that $[a) \vee s \subseteq F$ for every $a \in H$, which implies $s \in ([a), F)$ for every $a \in H$. Therefore, $s \in \bigcap_{a\in H}([a), F)$ and hence $(H, F) \subseteq \bigcap_{a\in H}([a), F)$. Thus, $(H, F) = \bigcap_{a\in H}([a), F)$. □

Corollary 3.21. *Consider an element* $s \in A$ *and let* H *be any subset of* A. *It follows that* $(H, [s))$ *can be expressed as* $\bigcap_{p\in H}(p, s)$.

Corollary 3.22. *Given any elements* s *and* t *from* A, *the following statements hold:*
(1) $([s), F) = (s, F)$
(2) $s \leq t \Rightarrow (s, F) \subseteq (t, F)$
(3) $(s \wedge t, F) = (s, F) \cap (t, F)$
(4) $((s \vee t, F), F) = ((s, F), F) \cap ((t, F), F)$
(5) $(s, F) = A \Leftrightarrow s \in F$.

Proof. (1). Let $a \in ([s), F)$. Then, $b \vee a \in F$, for all $b \in [s)$. Since $s \in [s)$, we have $s \vee a \in F$, and hence $a \in (s, F)$. From this we can conclude that $([s), F) \subseteq (s, F)$. Let $a \in (s, F)$. Then, $s \vee a \in F$. Let $b \in [s)$. Then, $b \vee s = b$. Since $s \vee a \in F$, we get $b \vee a = b \vee s \vee a \in F$, and hence $b \vee a \in F$ for all $b \in [s)$. Therefore, $a \in ([s), F)$. Thus, $(s, F) \subseteq ([s), F)$. Therefore, $([s), F) = (s, F)$.
(2). Assume that $s \leq t$. Let $a \in (s, F)$. Then, $s \vee a \in F$ and hence, $s \vee t \vee a \in F$. This implies $t \vee a \in F$. Therefore, $a \in (t, F)$.
(3). Clearly, we have that $(s \wedge t, F) = (t \wedge s, F)$, and hence $(s \wedge t, F) \subseteq (s, F) \cap (t, F)$. Let $a \in (s, F) \cap (t, F)$. Then, $s \vee a \in F$ and $t \vee a \in F$. This implies $a \vee s, a \vee t \in F$, and hence $(a \vee s) \wedge (a \vee t) \in F$. Therefore, $s \vee (s \wedge t) \in F$. This gives that $(s \wedge t) \vee a \in F$. Hence, $s \in (s \wedge t, F)$. Thus, $(s \wedge t, F) = (s, F) \cap (t, F)$.
(4). As $(s \vee t, F) = (t \vee s, F)$, it is verified easily.
(5). Assume that $(s, F) = A$. Then, $0 \in (s, F)$ and hence $s \vee 0 \in F$. Therefore $s \in F$. Conversely, assume that $s \in F$. Then, $s \vee a \in F$ for all $a \in A$. Therefore, $a \in (s, F)$ for all $a \in A$. Hence, $(s, F) = A$. □

Proposition 3.23. *For every prime* $F-$*filter* C *of* A, $p \notin C \Rightarrow (p, F) \subseteq C$ *for any* $p \in A$.

Proof. Let C be any prime $F-$filter A with $p \notin C$. Suppose $(p, F) \nsubseteq C$. Then, there exists an element $a \in (p, F)$ such that $a \notin C$. Then, $p \vee a \in F \subseteq C$. Since C is prime and $a \notin C$, we get $p \in C$, which is a contradiction to $p \notin C$. Hence, $(p, F) \subseteq C$. □

4. On minimal prime $F-$filters

In this section, we prove that $O^F(M)$ can be expressed as the intersection of all minimal prime $F-$filters contained in M, where M is a prime $F-$filter. Finally, the notation of $F-$normal ADLs is introduced and characterized in terms of relative annihilators with respect to a filter F.

The definition is stated as follows.

Definition 4.1. *A prime* $F-$*filter* U *of an ADL* A *that contains an* $F-$*filter* J *is referred to as minimal belonging to* J *if there does not exist any prime* $F-$*filter* W *for which* $J \subseteq W \subseteq U$.

We represent $Min_F(A)$ as the collections of all minimal prime $F-$filters of A. It is important to observe that if we set $F = J$ in the definition above, then we refer to U as a minimal prime $F-$filter.

Example 4.2. *From Example 3.7, we see that F_3 is a prime F_6-filter, while F_5 is an F_6-filter of A. It is evident that $F_5 \subseteq F_3$. Furthermore, there does not exist a F_6-filter W of A such that $F_5 \subseteq W \subseteq F_3$. Therefore, we conclude that F_3 is a minimal prime F_6-filter belonging to F_5.*

Proposition 4.3. *Let $J \in \mathfrak{F}^F(A)$ and $U \in Spec_F(A)$ such that $J \subseteq U$. Then, U is minimal belonging to J if and only if $A \setminus U \in Max_F(A)$ and $(A \setminus U) \cap J = \emptyset$.*

Proof. Note that $A \setminus U$ is a proper ideal, and it follows that $(A \setminus U) \cap J = \emptyset$. Let W be a proper ideal of A such that $W \cap J = \emptyset$ and $A \setminus U \subseteq W$. This implies that $J \subseteq A \setminus W$, and thus $A \setminus W \subseteq U$. As $U \in Min_F(A,)$ we conclude that $A \setminus W = U$. Hence, we establish that $A \setminus U$ is maximal with $(A \setminus U) \cap J = \emptyset$. Conversely, assume that $A \setminus U$ is maximal with $(A \setminus U) \cap J = \emptyset$. We now show that U is minimal. Suppose W is any prime $F-$filter of A such that $F \subseteq J \subseteq W \subseteq U$. This gives that $A \setminus W$ is an ideal for which $A \setminus U \subseteq A \setminus W$ and $(A \setminus W) \cap J = \emptyset$, leading to a contradiction. Thus, $U \in Min_F(A)$ and $U \subseteq J$. □

Theorem 4.4. *Let $J \in \mathfrak{F}^F(A)$ and $U \in Spec_F(A)$ with $J \subseteq U$. Then, U is minimal prime $F-$filter contained in J iff for every $p \in U$, there is $q \notin U$ satisfies $p \vee q \in J$.*

Proof. Let $U \in Min_F(A)$ with $U \subseteq J$. It follows that $A \setminus U$ is a maximal ideal, satisfying $(A \setminus U) \cap J = \emptyset$. Take any $p \in U$. Since $p \notin A \setminus U$, it gives $A \setminus U \subsetneq (A \setminus U) \vee [p)$. By the maximality of $A \setminus U$, we must have $((A \setminus U) \vee [p)) \cap J \neq \emptyset$. Now, let $a \in ((A \setminus U) \vee [p)) \cap J$. Then, there are $q \in A \setminus U$ and $a \in J$ such that $a = q \vee p$, where $q \in A \setminus U$ and $a \in J$. Therefore, $q \vee p \in J$. Conversely, assume that for each $p \in U$ there is some $q \notin U$ such that $p \vee q \in J$. Suppose, for contradiction, that U is not minimal prime $F-$filter contained in J. Then, there must exist a prime $F-$filter W such that $F \subseteq J \subseteq W \subsetneq U$. Choose some $p \in U \setminus W$. By assumption, there is some $q \notin U$ such that $p \vee q \in J \subseteq W$. Since $p \notin W$, it follows that $q \in W \subseteq U$, which gives a contradiction. Thus, U must be minimal prime $F-$filter contained in J. □

Corollary 4.5. *A prime $F-$filter U of A is minimal iff for any $p \in U$, there is $q \notin U$ such that $p \vee q \in F$.*

Definition 4.6. *For any $U \in Spec_F(A)$, consider the set $\mathcal{O}^F(U)$ as follows:*

$$\mathcal{O}^F(U) = \{s \in A \mid s \in (t, F),\ \textit{for some } t \notin U\}$$

It is evident that $\mathcal{O}^F(U) = \bigcup_{t \notin P}(t, F)$.

Lemma 4.7. *Let U be a prime $F-$filter of A. Then, $\mathcal{O}^F(U) \in \mathfrak{F}^F(A)$ and $\mathcal{O}^F(U) \subseteq U$.*

Proof. Let $p, q \in \mathcal{O}^F(U)$. There are elements $a \notin U$ snd $b \notin U$ such that $p \in (a, F)$ and $q \in (b, F)$. This implies that $((a, F), F) \subseteq (p, F)$ and $((b, F), F) \subseteq (q, F)$. Consequently, we get

$$((a \vee b, F), F) = ((a, F), F) \cap ((b, F), F) \subseteq (p, F) \cap (q, F) = (p \wedge q, F)$$

Thus, $p \wedge q$ belongs to $((p \wedge q, F), F) \subseteq (((a \vee b, F), F), F) = (a \vee b, F)$. Since $a \vee b \notin U$, it follows that $p \wedge q \in \mathcal{O}^F(U)$. Now, let $p \in \mathcal{O}^F(U)$ and suppose $p \leq q$. There is $a \notin U$ such that $p \in (a, F)$. As

$(a, F) \in \mathfrak{F}(A)$, it follows that $q \in (a, F)$. Therefore, $q \in O^F(U)$, confirming that $O^F(U)$ is a filter in A. Furthermore, it is evident that $F \subseteq O^F(U)$. Thus, $O^F(U)$ is an $F-$filter in A. Now, let $p \in O^F(U)$. Then, there exists $a \notin U$ such that $p \in (a, F)$. This implies that $p \vee a \in F \subseteq U$. Since U is a prime filter, we conclude that $p \in U$. Therefore, we have $O^F(U) \subseteq U$.

□

Corollary 4.8. *For any $U \in Spec_F(A)$, $O^F(U) = U$ if and only if $U \in Min_F(A)$.*

Theorem 4.9. *Every member of $Min_F(A)$ is a member of $O^F(U)$ and contained in U.*

Proof. Let $W \in Min_F(A)$ and belong to $O^F(U)$. Assume, for the sake of contradiction, that $W \nsubseteq U$. Choose $p \in W \setminus U$. Then, there is $q \notin W$ such that $p \vee q \in O^F(U)$. Thus, we have $p \vee q \in (a, F)$ for some $a \notin U$, which leads to $q \vee (p \vee a) \in F \subseteq U$. Since $p \notin U$ and $a \notin U$, and U is a prime filter, it gives $p \vee a \notin U$. Consequently, we find that $q \in O^F(U) \subseteq W$, leading to a contradiction. Therefore, we conclude that $W \subseteq U$. □

Theorem 4.10. *For any $(C \in Spec_F(A))$, the set $O^F(C)$ is expressed as the intersection of all members of $Min_F(A)$ contained in C.*

Proof. Consider $\{X_i | i \in \triangle\}$ as the class of members of $Min_F(A)$ with $X_i \subseteq C$ for all $i \in \triangle$. Let $p \in O^F(C)$. Then, $p \in (q, F)$ for some $q \notin C$. This implies $p \vee q \in F \subseteq X_i$ for all $i \in \triangle$. Since $X_i \subseteq C$, we have that $q \notin X_i$ for every $i \in S$. Since $q \notin X_i$ for every $i \in \triangle$, and each X_i is prime, we get that $p \in X_i$ for all $i \in S$. Therefore, $p \in \cap X_i$, and hence $O^F(C) \subseteq \cap X_i$. Let $p \notin O^F(C)$. Take $X = (A \setminus C) \vee [p)$. We prove that $F \cap X = \emptyset$. Suppose $F \cap X \neq \emptyset$. Then, we can choose $q \in F \cap X$. Hence, $q \in X$ and $q \in F$, and there exists $s \in A \setminus C$ such that $q = a \vee p$ and $a \vee p \in F$. This implies $p \in (a, F)$. Since $a \notin C$, we get that $p \in O^F(C)$, which leads a contradiction. Therefore, $X \cap F = \emptyset$. It follows that there is a maximal ideal W of A satisfying $X \subseteq W$, $W \cap F = \emptyset$. Hence, $A \setminus W$ is a minimal prime F filter, and $(A \setminus W) \subseteq C$, $p \notin A \setminus W$, which gives $p \notin \cap_{i \in \triangle} X_i$. Hence, $\cap X_i \subseteq O^F(C)$. Therefore, $O^F(C) = \cap_{i \in S} X_i$. □

Proposition 4.11. *Consider two prime $F-$filters, U_1 and U_2, in A, with $U_1 \subseteq U_2$. It then follows that $O^F(U_2)$ is a subset of $O^F(U_1)$.*

Proof. Assume $s \in O^F(U_2)$. Then, there is some $p \notin U_2$ with $s \in (p, F)$. Since $p \notin U_1$ as well, it leads to that $s \in O^F(U_1)$. Consequently, $O^F(U_2) \subseteq O^F(U_1)$. □

Proposition 4.12. *Given any element $p \in A$ that is not maximal and satisfies $p \notin F$, there is a member of $Min_F(A)$ not containing p.*

Proof. Let p be a non-maximal element of A such that p is not an element of F. By the assertion in Corollary-3.13, there is a prime $F-$filter P in A that does not contain p. Now, consider the collection $\mathfrak{K}$ consisting of all prime $F-$filters C in A that satisfy $p \notin C$ and are contained within P. This collection satisfies the conditions of Zorn's lemma, which guarantees the existence of a minimal element within it. We will denote this minimal element as U. Hence, U is minimal and does not include p. □

Theorem 4.13. *The statements given below are equivalent for any prime $F-$filter U in A :*
(1) U is minimal
(2) $U = O^F(U)$
(3) the filter U contains exactly one of the elements s or (s, F), for every $s \in A$,

Proof. (1) ⇒ (2) : Assuming (1), let $s \in U$. There is $t \notin U$ such that $s \vee t \in F$. Consequently, this means $s \in \mathcal{O}^F(U)$. Thus, we have $U \subseteq \mathcal{O}^F(U)$. Given that $\mathcal{O}^F(U) \subseteq U$, we obtain $U = \mathcal{O}^F(U)$.
(2) ⇒ (3) : Assume (2). Let s be an element of A with $s \notin U$. Consider an element p in (s, F). Since $p \vee s$ is in F, it follows that $p \vee s$ is also an element of U. This leads to the conclusion that p must be in U. Since s is not in U, we can deduce that the entire interval (s, F) is contained in U.
(3) ⇒ (1) : Consider any prime $F-$filter C in A such that $C \subsetneq U$. Choose an element s from U that is not in C. This situation implies that (s, F) is contained in C while also not being contained in U. Consequently, this leads to the conclusion that (s, F) cannot be a subset of U, resulting in a contradiction. □

Definition 4.14. *A is referred to as an $F-$semi-complemented if, for any $0 \neq s \in A$, there exists a $t \notin \mathcal{M}_{Max.elts}$ which is not in F such that $s \vee t \in F$.*

Example 4.15. *Let $D_1 = \{0, p\}$ and $D_2 = \{0, q_1, q_2\}$ be two discrete ADLs. Then, $A = D_1 \times D_2 = \{(0,0), (0,q_1), (0,q_2), (p,0), (p,q_1), (p,q_2)\}$. Then, $(A, \wedge, \vee, 0)$ is an ADL, but not a lattice, because $(p,q_1) \wedge (p,q_2) = (p,q_2) \neq (p,q_1) = (p,q_2) \wedge (p,q_1)$. Clearly, $F = \{(0,q_1), (0,q_2), (p,0), (p,q_1), (p,q_2)\}$ is an $F-$filter of A. It is evident that for any non-zero element s in A, there exists a $t \notin \mathcal{M}_{Max.elts}$ which is not in F for which $s \vee p$ lies in F. This demonstrates that A is an $F-$semi-complemented ADL.*

Theorem 4.16. *An ADL A is $F-$semi-complemented if and only if the intersection of all maximal ideals that are not intersecting with F is equal to $\{0\}$.*

Proof. Assume that A is $F-$semi-complemented. Let us define

$$Q = \bigcap \{U \mid U \text{ is a maximal ideal of } A \text{ such that } U \cap F = \emptyset\}.$$

We aim to show that $Q = \{0\}$. Suppose $s \in Q$ and $s \neq 0$. This implies that s is an element of every maximal ideal U that does not intersect F, hence $s \notin F$. Given that s is non-zero and A is $F-$semi complemented, there exists a non-maximal element $t \notin F$ such that $s \vee t \in F$. This leads us to conclude that $s \vee t \notin U$, which further implies that $U \vee (s \vee t) = A$. Since t is non-maximal in F, we can find a minimal prime $F-$filter W in A such that $t \notin W$. Consequently, t belongs to $A \setminus W$, and we also have $(A \setminus W) \cap F = \emptyset$, indicating that $A \setminus W$ is maximal of A. Thus, both s and t are in $A \setminus W$. Therefore, $s \vee t \in A \setminus W$. This results in $(A \setminus W) \cap F \neq \emptyset$, leading to a contradiction. Hence, we conclude that $s = 0$, establishing that $Q = \{0\}$. Next, we consider the converse. Suppose

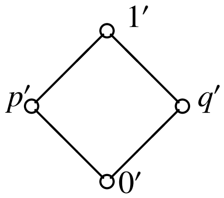
$$\bigcap \{U \mid U \text{ is a maximal ideal of } A \text{ and } U \cap F = \emptyset\} = \{0\}.$$

Let s be any non-zero element in A. This implies that there exists at least one maximal ideal U such that $s \notin U$ and $U \cap F = \emptyset$. Consequently, we have $U \vee (s) = A$. For some element $p \in U$, the expression $p \vee s$ is also maximal. Since p is part of U and $U \cap F = \emptyset$, it follows that $p \notin F$. Moreover, we get $p \vee s \in F$. This represents that for every non-zero element s in A, there is a non-maximal element $p \notin F$ such that $p \vee s \in F$. Thus, we conclude that A is $F-$semi-complemented.

□

Definition 4.17. *We say A as $F-$normal if, for every pair of elements $p, q \in A$ with the property that $p \vee q \in F$, there exist elements $s \in (p, F)$ and $t \in (q, F)$ such that $s \wedge t = 0$.*

Example 4.18. *Consider $G = \{0, p\}$, a discrete ADL, and $K = \{0', p', q', 1'\}$, a distributive lattice. The Hasse diagram for K is shown below:*



Consider $A = G \times K = \{(0,0'), (0,p'), (0,q'), (0,1'), (p,0'), (p,p'), (p,q'), (p,1')\}$. Clearly, A is an ADL with zero element $(0,0')$.
Consider a filter $F = \{(p,0'), (p,p'), (p,q'), (p,1')\}$. Clearly, A is $F-$normal.

The result presented below follows directly from the definition mentioned earlier.

Theorem 4.19. *The condition for A to be $F-$normal is equivalent to the assertion that for any $p, q \in A$ satisfying $p \vee q \in F$, the equation $(p, F) \vee (q, F) = A$ holds true.*

Definition 4.20. *Two $F-$filters J_1 and J_2 of A are said to be co-maximal if $J_1 \vee J_2 = A$.*

Example 4.21. *Based on Example 3.7, it can be observed that F_3 and F_4 are F_6-filters of A. Clearly, $F_3 \vee F_4 = A$. Therefore, F_3 and F_4 are co-maximal. Also, we have that F_5 and F_7 are F_6-filters of A, but not co-maximal.*

Theorem 4.22. *The following statements are equivalent in the context of an ADL A :*
(1) $p \vee q \in F \Rightarrow (p, F) \vee (q, F) = A$, for every $p, q \in A$
(2) $(p, F) \vee (q, F) = (p \vee q, F)$, for every $p, q \in A$
(3) Every two distinct members of $Min_F(A)$ are co-maximal
(4) Every prime $F-$filter contains a unique minimal prime $F-$filter
(5) For any $P \in Spec_F(A)$, $O^F(P) \in Spec_F(A)$.

Proof. (1) $\Rightarrow$ (2) : Suppose $s, t \in A$. Then, we have that $(s, F) \vee (t, F) \subseteq (s \vee t, F)$. Let $b \in (s \vee t, F)$. Then, $b \vee (s \vee t) \in F$, and hence $(b \vee s) \vee (b \vee t) \in F$. By (1), we have that $(b \vee s, F) \vee (b \vee t, F) = A$. So, we have that $b \in (b \vee s, F) \vee (b \vee t, F)$. This implies there exists $s_1 \in (b \vee s, F)$ and $s_2 \in (b \vee t, F)$ such that $s_1 \wedge s_2 = b$. This implies $b \vee s_1 \in (s, F), b \vee s_2 \in (t, F)$, and $b = b \vee b = b \vee (s_1 \wedge s_2) = (b \vee s_1) \wedge (b \vee s_2) \in (s, F) \vee (t, F)$. Therefore, $(s \vee t, F) = (s, F) \vee (t, F)$
(2) $\Rightarrow$ (3) : Assume (2). Suppose U and W are two disjoint minimal prime $F-$filters of A. Thus, there are $p \in U$ and $q \in W$ such that $p \notin W$ and $q \notin U$. Then, $p \vee a \in F$ and $q \vee b \in F \Rightarrow p \vee a \vee q \vee b \in F$, and hence $A = (p \vee a \vee q \vee b)$ for some $a \notin U, b \notin W$). For $q \notin U, a \notin U$, we get $a \vee q \notin U \Rightarrow (a \vee q, F) \subseteq U$. Similarly, we get that $(b \vee p, F) \subseteq W$.
Since $A = (q \vee a, F) \vee (p \vee b, F)$, we get $A \subseteq U \vee W$. Hence, $A = U \vee W$.
(3) $\Rightarrow$ (4): Assume (3). Consider a prime $F-$filter U contained in two distinct minimal prime $F-$filters say Q_1 and Q_2, i.e. $Q_1 \subseteq U, Q_2 \subseteq U$ with $Q_1 \neq Q_2$. By assumption, it follows that $A = Q_1 \vee Q_2 \subseteq U. \Rightarrow A = U$, and we get a contradiction. Hence, (4) holds.
(4) $\Rightarrow$ (5) : It is clear by Corollary 3.29.
(5) $\Rightarrow$ (1) : Assume (5). Let $s, t \in A$ with $s \vee t \in F$. If $(s, F) \vee (t, F) \neq A$, then $(s, F) \vee (t, F) \subseteq U$ for

some maximal $F-$filter U of A. This leads to $(s, F) \subseteq M, (t, F) \subseteq U$. Hence, $s \notin O^F(U)$ and $t \notin O^F(U)$. Since $O^F(U)$ is prime, it gives $s \vee t \notin O^F(U)$. Thus, $F \nsubseteq O^F(U)$. We get a contradiction. Hence, $(s, F) \vee (t, F) = A$. □

Theorem 4.23. *The conditions listed below are equivalent in any ADL:*
(1) A is $F-$normal
(2) For every two disjoint maximal J_1, J_2 in A satisfying $J_1 \cap F = \emptyset$, $J_2 \cap F = \emptyset$, there are $p \notin J_1$, $q \notin J_2$ such that $p \wedge q = 0$
(3) For every maximal ideal J satisfying $J \cap F = \emptyset$, J is the unique maximal containing $A \setminus O^F(P)$.

Proof. (1) ⇒ (2): Assume (i). Let J_1, J_2 be two distinct maximal ideals of A such that $J_1 \cap F = J_2 \cap F = \emptyset$. Consequently, $A \setminus J_1$, $A \setminus J_2$ form distinct minimal prime $F-$filters in A. Given our assumption, it follows that $A \setminus J_1$, $A \setminus J_2$ are co-maximal, which implies $(A \setminus J_1) \vee (A \setminus J_2) = A$. Since $0 \in A$, there exist elements $p \in A \setminus J_1$, $q \in A \setminus J_2$ such that $p \wedge q = 0$.
(2) ⇒ (3): Assume (2). Let J represent any maximal ideal in A such that J does not intersect with F and the complement of $O^F(P)$ in A is contained in J. Now consider another maximal ideal J_1 which also satisfies the condition that $J_1 \cap F = \emptyset$ and $A \setminus O^F(P) \subseteq J_1$. We aim to establish that $J = J_1$. Suppose, for contradiction, that J and J_1 are disjoint. Under this assumption, there exist elements p and q such that $p \notin J_1$ and $q \notin J$, and these elements satisfy $p \wedge q = 0$. Since neither p nor q can belong to $A \setminus O^F(P)$, it follows that both elements must be part of $O^F(P)$. Therefore, we can conclude that $p \wedge q \in O^F(P)$. This implies that $0 \in O^F(P)$, leading us to the conclusion that $O^F(P) = A$. Such a finding contradicts our earlier assumptions. Hence, we deduce that $J = J_1$.
(3) ⇒ (1): Assume (3). Now, consider a prime $F-$filter P in A. Assume that P contains two distinct minimal prime $F-$filters, C_1 and C_2, such that both C_1 and C_2 are subsets of P. This implies that $O^F(P) \subseteq O^F(C_1)$ and $O^F(P) \subseteq O^F(C_2)$. Thus, we can conclude that P is contained within both $O^F(C_1)$ and $O^F(C_2)$. From this, it follows that C_2 is a subset of C_1, and C_1 is a subset of C_2. Consequently, we deduce that $C_1 = C_2$. □

5. Conclusions

In this paper, we introduced the notions of $F-$filters and prime $F-$filters in almost distributive lattices (ADLs). We examined their basic structure and explored several fundamental properties. Equivalent conditions were established for a proper $F-$filter to be prime. It was proved that every maximal $F-$filter in an ADL is necessarily prime. The relationships between maximal and prime $F-$filters were carefully analyzed. We further characterized prime $F-$filters through their minimal counterparts. For any prime $F-$filter $\mathcal{M}$ in an ADL A, the set $O^F(\mathcal{M})$ was shown to be the intersection of all minimal prime $F-$filters contained within $\mathcal{M}$. This result provides an intrinsic representation of prime $F-$filters. Our findings enhance the theoretical understanding of filter systems in ADLs and pave the way for further generalizations in lattice theory.

Author contributions

Every author made an equal contribution to this study. Conceptualization and formal analysis by Ali Yahya Hummdi and N.Rafi; validation and visualization by M. Balaiah; and writing original draft, review and editing by Y. Monikarchana.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest in this article.

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