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Research article

Analysis of Type-II censored data from the Kies distribution with classical and generalized approaches

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Abstract: This paper investigated statistical inference for the Kies distribution under the Type-II censoring scheme; specifically, classical and alternative generalized inferential approaches were proposed for parameter estimation. From the classical likelihood perspective, maximum likelihood estimators for unknown parameters were derived, and the existence and uniqueness of these estimators was also established. The corresponding asymptotic confidence intervals were constructed based on the observed Fisher information matrix. For comparison, an alternative generalized estimation method was conducted based on the constructed pivotal quantities. Finally, the performance of different estimation methods was evaluated via extensive simulation studies , and meanwhile, two real-world data examples were presented to illustrate the applications of the proposed methods.

Keywords: Kies distribution; Type-II censoring; parameter estimation; maximum likelihood method; generalized method
Mathematics Subject Classification: 62F01, 62N02

1. Introduction

Lifetime distributions play a crucial role in statistical inference and data analysis, and various distributions (e.g., Weibull, gamma, lognormal and exponential distributions) are widely applied in practice. It is noted that such distributions exhibit infinite support by definition; however, the range of lifetimes may be limited, and in turn, distributions with bounded support may be more appropriate in data analysis (e.g., Jodrá [16], Jodra and Jimenez-Gamero [17], Korkmaz [18], Krishna et al. [21]). Specially, unit bounded distributions with support $(0, 1)$ have attracted much attention due to the popularity of unit data in practice, and such data frequently appears in various fields, including biological studies, finance, mortality, actuarial science, and measurement science, among others. Various unit-bounded distributions have been proposed in practice, including the Kumaraswamy distribution (Al-Babtain et al. [3]), Topp-Leone distribution (Atchadé et al. [6]), unit Burr-XII distribution (Korkmaz and Chesneau [19]), unit log-log distribution (Korkmaz and Korkmaz [20]), and unit-Gompertz distribution (Mazucheli et al. [22]), among others. In this study, another distribution with unit support-referred to as the Kies distribution-is discussed. Its cumulative distribution function (CDF) and probability density function (PDF) can be expressed as follows:

$$F(x;\alpha,\beta) = 1 - e^{-\alpha\left(\frac{x}{1-x}\right)^{\beta}}, \quad 0 < x < 1, \tag{1.1}$$

and

$$f(x;\alpha,\beta) = \frac{\alpha\beta x^{(\beta-1)}}{(1-x)^{(\beta+1)}} e^{-\alpha\left(\frac{x}{1-x}\right)^{\beta}}, \quad 0 < x < 1, \tag{1.2}$$

where $\alpha > 0$ and $\beta > 0$ denote the scale and shape paramers of the distribution, respectively. For simplicity, the Kies distribution with parameters α and β is hereafter denoted as $\mathrm{K}(\alpha,\beta)$. Compared to modern bounded models such as the aforementioned unit distributions, although the Kies distribution was proposed earlier, it still offers advantages for modeling bounded lifetime data. Specifically, its CDF employs a logit transformation that elegantly maps the bounded interval to $(0, 1)$ and incorporates an exponential-power structure that enables flexible hazard rate modeling. This formulation also provides superior parameter interpretability and mathematical tractability. Correspondingly, the survival function (SF) and hazard rate function (HRF) of the Kies distribution at mission time t are presented as follows:

$$S(t;\alpha,\beta) = \exp\left\{-\alpha\left(\frac{t}{1-t}\right)^{\beta}\right\} \quad \text{and} \quad H(t;\alpha,\beta) = \alpha\left(\frac{t}{1-t}\right)^{\beta}. \tag{1.3}$$

For illustration, plots of PDF and HRF of the Kies distribution are presented in Figure 1 for different parameter values. It is noted that the PDF of the Kies distribution exhibits diverse characteristics, which enables it to effectively fit data with different features. Meanwhile, the HRF of the Kies distribution exhibits two typical shapes, namely bathtub-shaped and monotone increasing, which are consistent with practical reliability phenomena-including aging tests, service life characteristics, and wear-out periods. Therefore, the Kies distribution offers greater flexibility in data analysis.

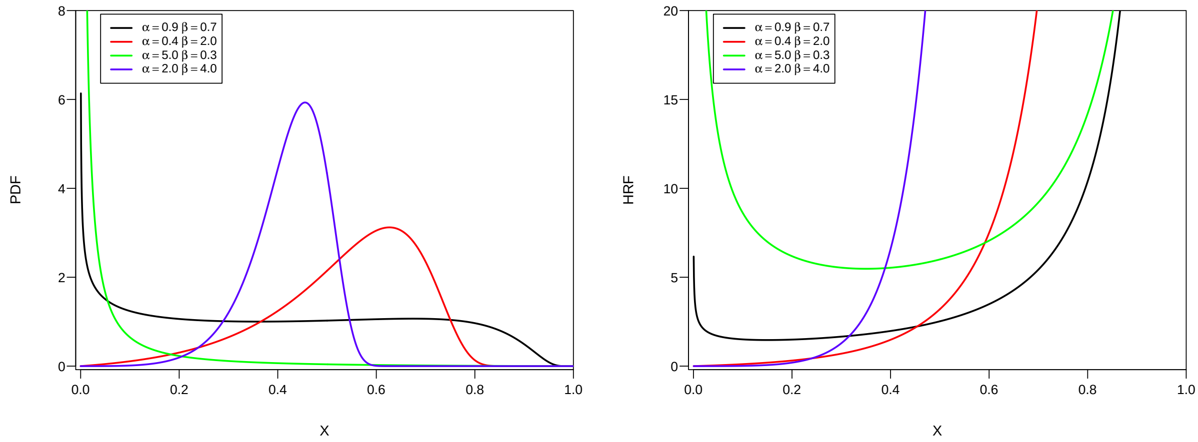

Figure 1. Plots of PDF and HRF for the Kies distribution.

With the development of technology, modern products often exhibit high reliability and long life cycles; conducting full life tests is often impractical due to practical time and cost constraints. Therefore, censoring schemes are frequently employed in lifetime experiments to enhance experimental efficiency. Although various censoring schemes are used in practice—including Type-I censoring (e.g., Almetwally et al. [5]), Type-II censoring (e.g., Almetwally et al. [4]), progressive censoring (e.g., Chandra et al. [8], Dey et al. [9], ElGazar et al. [10]), hybrid censoring (e.g., Elshahhat and Abu El Azm [11]), and generalized censoring (e.g., Aboul-Fotouh Salem et al. [2])-Type-II censoring, as one of the traditional schemes, still attracts much attention in lifetime experiments. In this scheme, n identical and independent units are tested, and the test terminates when the first r failure times are observed. In this paper, Type-II censoring is adopted for statistical inference, and this choice is justified by its strong theoretical foundations and practical advantages in reliability analysis. Theoretically, Type-II censoring has a well-established probabilistic framework, which provides analytical tractability that simplifies statistical inference compared to more complex schemes (e.g., progressive or hybrid censoring). For instance, the likelihood function under Type-II censoring retains a straightforward structure, which may enable closed-form maximum likelihood estimation for many common reliability models (e.g., exponential, Rayleigh, and Weibull distributions). This reduces computational burden-a critical consideration for small sample sizes or non-regular datasets commonly encountered in data analysis. Practically, Type-II censoring also ensures experimental efficiency: it only requires pre-specifying the number of failures before test termination, thus avoiding the logistical complexity of time-dependent decisions or progressive unit removal-issues inherent in modern alternative schemes. This simplicity minimizes experimental bias and operational costs, particularly in scenarios with constrained resources (e.g., equipment or human supervision). Furthermore, Type-II censoring also aligns closely with field applications in reliability engineering, where halting testing after observing a fixed number of failures is standard practice for balancing data informativeness and resource constraints. Given these advantages, this paper focuses on statistical inference for the Kies distribution under the Type-II censoring scheme.

In statistical inference, classical likelihood-based estimation—particularly maximum likelihood estimation (MLE) - remains a cornerstone of parametric estimation. Its compelling theoretical advantages include consistency, asymptotic normality, and asymptotic efficiency; additionally, it fully exploits data information by leveraging the joint distribution, retains invariance under parameter transformations, and relies on a mature theoretical framework adaptable to diverse models. However, these properties may be heavily affected by sample size, especially when the number of observations is limited due to practical time and cost constraints. Motivated by these limitations of MLE, this paper explores parameter estimators for the Kies distribution under the Type-II censoring scheme. Classical likelihood methods and generalized approaches are proposed for comparison. For completeness and clarity, some potential contributions of this paper are provided as follows: First, maximum likelihood estimators for unknown parameters are derived, and the corresponding existence and uniqueness of these estimators are established for the Kies distribution under the Type-II censoring scheme. Second, two types of pivotal quantities are constructed, and in turn, alternative generalized inferential approaches are proposed for parameter estimation. From simulation studies and real-world examples, it is noted that the proposed generalized estimation methods perform better in most cases than classical likelihood-based results-even for small sample sizes. However, the generalized estimators based on the proposed pivotal quantities are relatively complex, which increases the computational burden.

The remainder of this paper is organized as follows. Section 2 discusses the likelihood estimation for the Kies parameters under the Type-II censoring scheme. Two generalized estimation methods are proposed in Section 3 based on the constructed pivotal quantities. Section 4 conducts extensive simulation studies to illustrate the performance of various methods, and two real-life examples are presented in Section 5 to demonstrate the applications. Finally, some concluding remarks are given in Section 6.

2. Classical inference

In this section, we derive the MLE for the Kies distribution under Type-II censoring, obtain the MLE for the unknown parameters, and accordingly construct the asymptotic confidence intervals (ACI) for these parameters.

2.1. Maximum likelihood estimation

Suppose there are n units tested in the experiment, and $X_1 < X_2 < \cdots < X_r$ are the Type-II censored data of size r from the Kies distribution $\mathrm{K}(\alpha,\beta)$. The likelihood function of parameters α and β can be constructed as follows:

$$\begin{aligned} L(\alpha,\beta|X) &= \prod_{i=1}^{r} f(x_i;\alpha,\beta)\,(S(x_r;\alpha,\beta))^{(n-r)}, \\ &= (\alpha\beta)^r \prod_{i=1}^{r} \frac{x_i^{(\beta-1)}}{(1-x_i)^{(\beta+1)}} e^{-\alpha\left(\frac{x_i}{1-x_i}\right)^{\beta}-\alpha(n-r)\left(\frac{x_r}{1-x_r}\right)^{\beta}}, \end{aligned} \tag{2.1}$$

and the the corresponding log-likelihood function is given by

$$\ell(\alpha,\beta) = r\ln\alpha + r\ln\beta + \sum_{i=1}^{r} \ln\frac{x_i^{(\beta-1)}}{(1-x_i)^{(\beta+1)}} - \alpha\sum_{i=1}^{r}\left(\frac{x_i}{1-x_i}\right)^{\beta}$$

$$- \alpha(n-r)\left(\frac{x_r}{1-x_r}\right)^{\beta}. \tag{2.2}$$

Based on Eq (2.2), the MLE of parameters α and β, denoted as $(\hat{\alpha}, \hat{\beta})$, could be obtained by solving the following equations:

$$\frac{\partial \ell(\alpha, \beta)}{\partial \alpha} = 0 \quad \text{and} \quad \frac{\partial \ell(\alpha, \beta)}{\partial \beta} = 0, \tag{2.3}$$

where the first-order partial derivatives of the log-likelihood function $\ell(\alpha, \beta)$ can be directly computed; details are omitted here for concision. To solve for the MLE, numerical methods (e.g., Newton-Raphson and quasi-Newton methods) can be used to find the parameter estimates. However, these methods can be computationally time-consuming and resource-intensive. Alternatively, this paper uses the profile likelihood method to obtain the MLE of the unknown parameters by maximizing the profile likelihood function. For the specific implementation process, details of this method could be found in Barndor-Nielsen and Cox [7].

Theorem 2.1. *Suppose $X_1 < X_2 < \cdots < X_r$ are the Type-II censored data of size r from the Kies distribution $K(\alpha, \beta)$. For $r > 0$ and given β, the MLE $\tilde{\alpha}$ of parameter α can be expressed as*

$$\tilde{\alpha} = \frac{r}{\sum_{i=1}^{r} \left(\frac{x_i}{1-x_i}\right)^{\beta} + (n-r)\left(\frac{x_r}{1-x_r}\right)^{\beta}}. \tag{2.4}$$

Proof. See Appendix A. □

Replacing α in Eq (2.2) with $\tilde{\alpha}$, the log-likelihood function of the profile of β can be obtained as

$$\begin{aligned} \ell(\beta) \propto r \ln \frac{r\beta}{\sum_{i=1}^{r} \left(\frac{x_i}{1-x_i}\right)^{\beta} + (n-r)\left(\frac{x_r}{1-x_r}\right)^{\beta}} + (\beta - 1) \sum_{i=1}^{r} \ln x_i \\ - (\beta + 1) \sum_{i=1}^{r} \ln(1 - x_i). \end{aligned} \tag{2.5}$$

Correspondingly, the MLE $\hat{\beta}$ of the parameter β is shown in the following.

Theorem 2.2. *Suppose $X_1 < X_2 < \cdots < X_r$ are the Type-II censored data of size r from the Kies distribution $K(\alpha, \beta)$. The MLE $\hat{\beta}$ of β exists uniquely being the solution to the equation $\Omega(\beta) = 0$ with*

$$\begin{aligned} \Omega(\beta) = \frac{r}{\beta} - r \frac{\sum_{i=1}^{r} \left(\frac{x_i}{1-x_i}\right)^{\beta} \ln\left(\frac{x_i}{1-x_i}\right) + (n-r)\left(\frac{x_r}{1-x_r}\right)^{\beta} \ln\left(\frac{x_r}{1-x_r}\right)}{\sum_{i=1}^{r} \left(\frac{x_i}{1-x_i}\right)^{\beta} + (n-r)\left(\frac{x_r}{1-x_r}\right)^{\beta}} \\ + \sum_{i=1}^{r} \ln\left(\frac{x_i}{1-x_i}\right). \end{aligned} \tag{2.6}$$

Proof. See Appendix B. □

It is noted that there is no closed-form of $\hat{\beta}$, and an algorithm called Algorithm 1 is proposed to compute $\hat{\beta}$. Once the $\hat{\beta}$ is obtained, $\hat{\alpha}$ could be obtained via Theorem 2.1 as

$$\hat{\alpha} = \frac{r}{\sum_{i=1}^{r}\left(\frac{x_i}{1-x_i}\right)^{\hat{\beta}} + (n-1)\left(\frac{x_r}{1-x_r}\right)^{\hat{\beta}}}. \tag{2.7}$$

Algorithm 1: Iterative method for obtaining $\hat{\beta}$.

Step 1 Set an initial guess $\beta^{(0)}$ of β with $q = 0$.

Step 2 Calculate $\beta^{(q+1)} = \Omega_0(\beta^{(q)})$ with function

$$\Omega_0(\beta) = \frac{r}{r\frac{\sum_{i=1}^{r}\left(\frac{x_i}{1-x_i}\right)^{\beta}\ln\left(\frac{x_i}{1-x_i}\right)+(n-r)\left(\frac{x_r}{1-x_r}\right)^{\beta}\ln\left(\frac{x_r}{1-x_r}\right)}{\sum_{i=1}^{r}\left(\frac{x_i}{1-x_i}\right)^{\beta}+(n-r)\left(\frac{x_r}{1-x_r}\right)^{\beta}}-\sum_{i=1}^{r}\ln\left(\frac{x_i}{1-x_i}\right)}.$$

Step 3 Set $q = q + 1$ with $q = 1, 2, \cdots$.

Step 4 Given a predetermined precision level ε, if $|\beta^{(q+1)} - \beta^{(q)}| < \varepsilon$, terminate the iterative process; otherwise, repeat Steps 2 and 3 until convergence.

Furthermore, the MLE of SF and HRF denoted as $\hat{S}(t)$ and $\hat{H}(t)$, can be obtained based on the invariance principle as follows

$$\hat{S}(t) = S(t; \hat{\alpha}, \hat{\beta}) = \exp\left\{-\hat{\alpha}\left(\frac{t}{1-t}\right)^{\hat{\beta}}\right\} \tag{2.8}$$

and

$$\hat{H}(t) = H(t; \hat{\alpha}, \hat{\beta}) = \hat{\alpha}\left(\frac{t}{1-t}\right)^{\hat{\beta}}. \tag{2.9}$$

2.2. Asymptotic confidence intervals

Since the precise distribution of the MLE is not readily available, we construct the ACI in this section based on asymptotic theory and the delta method. From Eq (2.2), the observed Fisher information matrix for $\hat{\alpha}$ and $\hat{\beta}$ can be expressed as

$$I(\hat{\alpha}, \hat{\beta}) = \begin{pmatrix} -\frac{\partial^2 \ell(\alpha,\beta)}{\partial \alpha^2} & -\frac{\partial^2 \ell(\alpha,\beta)}{\partial \alpha \partial \beta} \\ -\frac{\partial^2 \ell(\alpha,\beta)}{\partial \alpha \partial \beta} & -\frac{\partial^2 \ell(\alpha,\beta)}{\partial \beta^2} \end{pmatrix}_{(\alpha,\beta)=(\hat{\alpha},\hat{\beta})},$$

where

$$\frac{\partial^2 \ell(\alpha,\beta)}{\partial \alpha^2} = -\frac{r}{\alpha^2},$$
$$\frac{\partial^2 \ell(\alpha,\beta)}{\partial \alpha \partial \beta} = \frac{\partial^2 \ell(\alpha,\beta)}{\partial \beta \partial \alpha} = -\sum_{i=1}^{r}\left(\frac{x_i}{1-x_i}\right)^{\beta}\ln\left(\frac{x_i}{1-x_i}\right)$$

$$- (n-r)\left(\frac{x_r}{1-x_r}\right)^{\beta}\ln\left(\frac{x_r}{1-x_r}\right),$$

$$\frac{\partial^2 \ell(\alpha,\beta)}{\partial \beta^2} = -\frac{r}{\beta^2} - \alpha\sum_{i=1}^{r}\left(\frac{x_i}{1-x_i}\right)^{\beta}\ln^2\left(\frac{x_i}{1-x_i}\right)$$

$$- \alpha(n-r)\left(\frac{x_r}{1-x_r}\right)^{\beta}\ln^2\left(\frac{x_r}{1-x_r}\right).$$

Let $g(v)$ be the arbitrary continuous function of parameter v with $v = (\alpha,\beta)$, and $\hat{g}(v)$ be the MLE of $g(v)$ with $\hat{v} = (\hat{\alpha},\hat{\beta})$, then under mild regularity conditions, the asymptotic distribution of $\hat{g}(v)$ can be constructed based on asymptotic theory and the delta method (e.g., Oehlert [25]) as

$$\hat{g}(v) \sim N\left(g(v), \widehat{var(g(v))}\right),$$

where

$$\widehat{var(g(v))} = \nabla(g(\hat{v}))I^{-1}(\hat{\alpha},\hat{\beta})[\nabla(g(\hat{v}))]',$$

with $\nabla(g(\hat{v})) = \left(\frac{\partial g(v)}{\partial \alpha}, \frac{\partial g(v)}{\partial \beta}\right)\Big|_{(\alpha,\beta)=(\hat{\alpha},\hat{\beta})}$.

Therefore, for arbitrary $0 < \gamma < 1$, the $100(1-\gamma)\%$ confidence intervals of $g(v)$ could be constructed by

$$\left(\hat{g}(v) - u_{\gamma/2}\sqrt{\widehat{var(g(v))}}, \hat{g}(v) + u_{\gamma/2}\sqrt{\widehat{var(g(v))}}\right),$$

where u_γ denotes the upper $100\gamma\%$ percentile of the standard normal distribution. Sometimes, the lower confidence bounds may yield negative values-values that are not meaningful for the positive parameters. In such a case, the logarithmic transformation and the delta method could be used to achieve asymptotic normality of the distribution of $\ln(g(\hat{v}))$, as shown below:

$$\ln(g(\hat{v}) - \ln(g(v)) \sim N(0, var(\ln(g(\hat{v}))),$$

where $var(\ln(g(\hat{v}))) = \widehat{var(g(v))}/(g(\hat{v}))^2$. Therefore, the $100(1-\gamma)\%$ modified confidence interval for $g(v)$ could be established as follows:

$$\left(\frac{\hat{g}(v)}{\exp\left\{u_{\gamma/2}\sqrt{\mathrm{Var}(\hat{g}(v))}\right\}}, \hat{g}(v)\exp\left\{u_{\gamma/2}\sqrt{\mathrm{Var}(\hat{g}(v))}\right\}\right).$$

In this manner, the ACIs of parameters α,β, SF $S(t)$, HRF $H(t)$ could be constructed consequently due to above results, and the details are omitted here for concision and saving space.

3. Generalized estimation

In this section, two distinct pivotal quantities are constructed based on Type-II censored data from the Kies distribution, and alternative generalized point and interval estimation methods are introduced for parameters α and β as well as for the reliability indices. Similar studies based on the generalized inferential approaches have been reported by many authors, interested readers may refer to the works of Guo et al. [13], Toulis [28] for a review.

3.1. FC pivotal based generalized estimation

In this subsection, generalized point and interval estimators are proposed based on pivotal quantities, which are constructed using F-distributed and chi-square-distributed statistics (FC). For concision, these generalized estimators are hereafter referred to as FC generalized estimators.

Theorem 3.1. *Denote pivotal quantities*

$$G_1(\beta) = (r-1)\frac{n}{(n-r+1)\left(\frac{x_r(1-x_1)}{x_1(1-x_r)}\right)^{\beta} + \sum_{i=1}^{r-1}\left(\frac{x_i(1-x_1)}{x_1(1-x_i)}\right)^{\beta} - n}, \tag{3.1}$$

and

$$B_1(\alpha,\beta) = 2\alpha(n-r+1)\left(\frac{x_r}{1-x_r}\right)^{\beta} + 2\alpha\sum_{i=1}^{r-1}\left(\frac{x_i}{1-x_i}\right)^{\beta}. \tag{3.2}$$

Then, $G_1(\beta)$ follows the F-distribution with 2 and $2(r-1)$ degrees of freedom, and $B_1(\alpha,\beta)$ follows the chi-square distribution with $2r$ degrees of freedom. Additionally, $G_1(\beta)$ and $B_1(\alpha,\beta)$ are statistically independent.

Proof. See Appendix C □

Lemma 3.1. *For arbitrary numbers a and b with $0 < a < b < 1$, denote function $g_1(t) = \left(\frac{b(1-a)}{a(1-b)}\right)^t$, $t > 0$, then function $g_1(t)$ increases in t with $\lim_{t\to 0} g_1(t) = 1$ and $\lim_{t\to+\infty} g_1(t) = +\infty$.*

Proof. The results could be established by direct computation, and the detailed proof is omitted for concision. □

According to Lemma 3.1, the following result is obtained in consequence, and the detailed proof is omitted for concision.

Corollary 3.1. *According to Lemma 3.1, the pivotal quantity $G_1(\beta)$ is decreasing in β with range $(0,+\infty)$.*

Based on Theorem 3.1 and Corollary 3.1, for given random number $g_1 \sim F_{(2,2r-2)}$, equation $G_1(\beta) = g_1$ possesses a unique solution with respect to parameter β, and the solution is denoted as $\hat{\beta}_{FC} = \beta(g_1|x)$ that could be regarded as the pivotal based generalized estimate from the perspective of inverse moment estimation. Subsequently, it is deduced from Theorem 3.1 that parameter α can be expressed as

$$\alpha = \frac{b_1}{2(n-r+1)\left(\frac{x_r}{1-x_r}\right)^{\beta} + 2\sum_{i=1}^{r-1}\left(\frac{x_i}{1-x_i}\right)^{\beta}} \text{ with } b_1 \sim \chi^2_{2r}.$$

Therefore, using the substitution method of Weerahandi [31], a generalized pivotal quantity for parameter α is constructed by substituting $\hat{\beta}_{FC}$ for β as

$$\hat{\alpha}_{FC} = \frac{b_1}{2(n-r+1)\left(\frac{x_r}{1-x_r}\right)^{\hat{\beta}_{FC}} + 2\sum_{i=1}^{r-1}\left(\frac{x_i}{1-x_i}\right)^{\hat{\beta}_{FC}}}. \tag{3.3}$$

Further, let $\theta = (\alpha,\beta)$ and $R(\theta)$ be an arbitrary function of parameter θ, then based on the substitution method of Weerahandi [31], the associated FC pivotal quantity-based generalized estimator of $R(\theta)$ can be constructed as

$$\hat{R}_{FC}(\theta) = R(\hat{\alpha}_{FC},\hat{\beta}_{FC}). \tag{3.4}$$

Specifically, when the function $R(\theta)$ is chosen as the reliability indices (e.g., $S(t;\alpha,\beta)$ and $H(t;\alpha,\beta)$) and the original model parameters α and β, we can accordingly establish the FC-based generalized estimator using FC pivotal quantities; the detailed expressions are omitted for brevity.

It is noted from the above results that the proposed generalized pivotal quantities for parameters α,β and $R(\theta)$ could not be estimated directly. Consequently, an FC-based Monte-Carlo sampling approach-namely Algorithm 2-is introduced to compute the associated point and interval estimates; the associated FC-based results are referred as FC-based point estimates (FCPE) and the FC-based confidence intervals (FCCI), respectively.

Algorithm 2: FC-based Generalized Estimation pivotal quantities.

Step 1 Generate a random data $g_1 \sim F_{(2,2r-2)}$, and a generalized observation of $\hat{\beta}_{FC}$ is obtained from equation $G_1(\beta) = g_1$.

Step 2 Generate a random data b_1 from χ^2_{2r}, and a generalized observation for β is obtained from Eq (3.3).

Step 3 Repeat Steps 1 and 2 N times, and N estimators of α and β are obtained as $\hat{\alpha}^{(1)}_{FC}, \hat{\alpha}^{(2)}_{FC}, \ldots, \hat{\alpha}^{(N)}_{FC}$ and $\hat{\beta}^{(1)}_{FC}, \hat{\beta}^{(2)}_{FC}, \ldots, \hat{\beta}^{(N)}_{FC}$.

Step 4 Using substitution method, N generalized estimates of $R(\theta)$ are further constructed as $\hat{R}^{(i)}_{FC} = R(\hat{\alpha}^{(i)}_{FC},\hat{\beta}^{(i)}_{FC}),\ i = 1, 2, \ldots, N$.

Step 5 Let Θ be parameters, α,β or $R(\theta)$, respectively, and based on the estimates from Steps 3 and 4, a natural FCPE of Θ is constructed as $\hat{\Theta}_{FC} = \frac{1}{N}\sum_{s=1}^{N} \hat{\Theta}^{(s)}_{FC}$.

Step 6 To construct the FCCI of Θ, arrange estimates $\hat{\Theta}^{(1)}_{FC}, \hat{\Theta}^{(2)}_{FC}, \ldots, \hat{\Theta}^{(N)}_{FC}$ in ascending order as $\hat{\Theta}^{[1]}_{FC}, \hat{\Theta}^{[2]}_{FC}, \ldots, \hat{\Theta}^{[N]}_{FC}$. For arbitrary $0 < \gamma < 1$, a series of $100(1-\gamma)\%$ FCCI of Θ can be expressed as $(\hat{\Theta}^{[l]}_{FC}, \hat{\Theta}^{[l+N-\lfloor N\gamma\rfloor]}_{FC})$, where $l = 1, 2, \cdots, \lfloor N\gamma\rfloor$ and $\lfloor . \rfloor$ refers to the ceiling function. Therefore, the $100(1-\gamma)\%$ FCCI of Θ is selected as the l^*th interval estimate satisfying

$$\hat{\Theta}^{[l^*+N-\lfloor N\gamma+1\rfloor]}_{FC} - \hat{\Theta}^{[l^*]}_{FC} = \min_{l=1}^{\lfloor N\gamma\rfloor}\left(\hat{\Theta}^{[l+N-\lfloor N\gamma+1\rfloor]}_{FC} - \hat{\Theta}^{[l]}_{FC}\right).$$

3.2. *CC pivotal based generalized estimation*

In this subsection, generalized point and interval estimators are proposed based on pivotal quantities, which are constructed using two proposed types of chi-square distributions (CC). For concision, these generalized results are hereafter referred to as CC based generalized estimators.

Theorem 3.2. *Denote pivotal quantities*

$$G_2(\beta) = -2\sum_{i=1}^{r-1} \ln \frac{\sum_{j=1}^{i-1}\left(\frac{x_j(1-x_i)}{x_i(1-x_j)}\right)^{\beta} + (n-i+1)}{\sum_{k=1}^{r-1}\left(\frac{x_k(1-x_i)}{x_i(1-x_k)}\right)^{\beta} + (n-r+1)\left(\frac{x_r(1-x_i)}{x_i(1-x_r)}\right)^{\beta}}, \tag{3.5}$$

and

$$B_2(\alpha,\beta) = 2\alpha(n-r+1)\left(\frac{x_r}{1-x_r}\right)^{\beta} + 2\alpha\sum_{i=1}^{r-1}\left(\frac{x_i}{1-x_i}\right)^{\beta}. \tag{3.6}$$

Quantity $G_2(\beta)$ and $B_2(\alpha,\beta)$ follow chi-square distribution with $2(r-1)$ and $2r$ degrees of freedom, respectively. Additionally, $G_2(\beta)$ and $B_2(\alpha,\beta)$ are statistically independent.

Proof. See Appendix D □

Using Lemma 3.1, the following result holds consequently.

Corollary 3.2. *The pivotal quantity $G_2(\beta)$ increasing in β with range $(0,+\infty)$.*

Based on Theorem 3.2 and Corollary 3.2, for given random number $g_2 \sim \chi^2_{(2,2r-2)}$, equation $G_2(\beta) = g_2$ possesses a unique solution with respect to parameter β, and the solution is denoted as $\hat{\beta}_{CC} = \beta(g_2|x)$, which could be regarded as the pivotal based generalized estimate from the perspective of inverse moment estimation. Subsequently, it is deduced from Theorem 3.2 that parameter α can be expressed as

$$\alpha = \frac{b_2}{2(n-r+1)\left(\frac{x_r}{1-x_r}\right)^{\beta} + 2\sum_{i=1}^{r-1}\left(\frac{x_i}{1-x_i}\right)^{\beta}} \text{ with } b_2 \sim \chi^2_{2r}.$$

Therefore, using the substitution method of Weerahandi [31], a generalized pivotal quantity for parameter α is constructed by substituting $\hat{\beta}_{CC}$ for β as

$$\hat{\alpha}_{CC} = \frac{b_2}{2(n-r+1)\left(\frac{x_r}{1-x_r}\right)^{\hat{\beta}_{CC}} + 2\sum_{i=1}^{r-1}\left(\frac{x_i}{1-x_i}\right)^{\hat{\beta}_{CC}}}. \tag{3.7}$$

Similarly to Subsection 3.1, the generalized estimator based on the CC pivotal quantities, $\hat{R}_{CC}(\theta)$ for parameter function $R(\theta)$ could be constructed based on the substitution method of Weerahandi [31].

Consequently, the reliability indices SF and HRF could be obtained when the function $R(\theta)$ takes its special expressions, and the details are omitted to save space and concision. Similarly, as the CC based generalized estimators could not be used directly, another Monte-Carlo sampling approach, namely, Algorithm 3, is also introduced for computing the associated CC-based point (CCPE) and confidence interval estimates (CCCI), which are called CCPE and CCCI, respectively.

Algorithm 3: CC-based Generalized estimation pivotal quantities.

Step 1 Generate a random data $g_2 \sim \chi^2_{(2,2r-2)}$, and a generalized observation of $\hat{\beta}_{CC}$ can be obtained from equation $G_2(\beta) = g_2$.

Step 2 Generate a random data b_2 from χ^2_{2r}, and a generalized observation for β is obtained from Eq (3.7).

Step 3 Repeat Steps 1 and 2 N times, and N estimators of α and β are obtained as $\hat{\alpha}^{(1)}_{CC}, \hat{\alpha}^{(2)}_{CC}, \ldots, \hat{\alpha}^{(N)}_{CC}$ and $\hat{\beta}^{(1)}_{CC}, \hat{\beta}^{(2)}_{CC}, \ldots, \hat{\beta}^{(N)}_{CC}$.

Step 4 Using the substitution method, N generalized estimates of $R(\theta)$ are further constructed as $\hat{R}^{(i)}_{CC} = R(\hat{\alpha}^{(i)}_{CC}, \hat{\beta}^{(i)}_{CC}),\ i = 1, 2, \ldots, N$.

Step 5 Let Θ be parameters, α, β or $R(\theta)$, respectively, and based on the estimates from Steps 3 and 4, a natural CCPE of Θ is constructed as $\hat{\Theta}_{CC} = \frac{1}{N}\sum_{s=1}^{N} \hat{\Theta}^{(s)}_{CC}$.

Step 6 To construct the CCCI of Θ, arrange estimates $\hat{\Theta}^{(1)}_{CC}, \hat{\Theta}^{(2)}_{CC}, \ldots, \hat{\Theta}^{(N)}_{CC}$ in ascending order as $\hat{\Theta}^{[1]}_{CC}, \hat{\Theta}^{[2]}_{CC}, \ldots, \hat{\Theta}^{[N]}_{CC}$. For arbitrary $0 < \gamma < 1$, a series of $100(1-\gamma)\%$ CCCI of Θ can be expressed as $(\hat{\Theta}^{[l]}_{CC}, \hat{\Theta}^{[l+N-\lfloor N\gamma \rfloor]}_{CC})$, where $l = 1, 2, \cdots, \lfloor N\gamma \rfloor$ and $\lfloor . \rfloor$ refers to the ceiling function. Therefore, the $100(1-\gamma)\%$ CCCI of Θ is selected as the l^*th interval estimate satisfying

$$\hat{\Theta}^{[l^*+N-\lfloor N\gamma+1 \rfloor]}_{CC} - \hat{\Theta}^{[l^*]}_{CC} = \min_{l=1}^{\lfloor N\gamma \rfloor} \left(\hat{\Theta}^{[l+N-\lfloor N\gamma+1 \rfloor]}_{CC} - \hat{\Theta}^{[l]}_{CC} \right).$$

4. Simulation study

In this section, extensive simulation studies are conducted to evaluate the performance of the proposed methodology. Absolute bias (AB) and mean square error (MSE) are used to assess the performance of point estimates, while average lengths (AL) and average lower/upper bounds are used to evaluate the performance of intervals at the 95% confidence level.

In numerical experiments, sample size n, number of failures, r and parameters (α, β) are randomly selected. Criteria quantities for model parameters and reliability indices (i.e., SF, HRF) are calculated over 10,000 replications, with results in Tables 1–8. In addition, visualizations of α and β are provided in Figures 2–4 for illustration. Similar visualizations for SF and HRF could also be constructed. The simulation studies were conducted using R software, and the associated codes are provided in the supplement.

Table 1. Results of point estimates with $(\alpha, \beta) = (0.5, 0.3)$ and mission time $t = 0.5$.

n	r	para.	MLE		FCPE		CCPE	
			AB	MSE	AB	MSE	AB	MSE
14	7	α	0.2759	0.0963	0.1799	0.0526	0.1732	0.0490
		β	0.2130	0.0864	0.0823	0.0100	0.0704	0.0081
		$S(t)$	0.0951	0.0133	0.0728	0.0078	0.0481	0.0035
		$H(t)$	0.2796	0.1540	0.2001	0.0781	0.1239	0.0306
	10	α	0.2096	0.0852	0.1387	0.0287	0.1263	0.0189
		β	0.1697	0.0514	0.0619	0.0069	0.0422	0.0025
		$S(t)$	0.0651	0.0069	0.0409	0.0023	0.0333	0.0013
		$H(t)$	0.1549	0.0687	0.0817	0.0131	0.0601	0.0057
24	16	α	0.2041	0.0551	0.1000	0.0179	0.0953	0.0165
		β	0.1634	0.0473	0.0506	0.0049	0.0411	0.0023
		$S(t)$	0.0607	0.0053	0.0361	0.0020	0.0306	0.0010
		$H(t)$	0.1285	0.0397	0.0746	0.0108	0.0534	0.0037
	19	α	0.1455	0.0293	0.0892	0.0144	0.0721	0.0081
		β	0.1493	0.0418	0.0445	0.0029	0.0340	0.0017
		$S(t)$	0.0429	0.0022	0.0194	0.0005	0.0157	0.0003
		$H(t)$	0.0603	0.0045	0.0256	0.0008	0.0226	0.0006

Table 2. Results of interval estimates with $(\alpha, \beta) = (0.5, 0.3)$ and mission time $t = 0.5$.

n	r	para.	ACI			FCCI			CCCI		
			Lower	Upper	AL	Lower	Upper	AL	Lower	Upper	AL
14	7	α	0.0728	1.2401	1.1673	0.2195	1.0945	0.8749	0.2233	0.9287	0.7054
		β	0.1376	1.0272	0.8896	0.2345	0.9924	0.7579	0.2655	0.8888	0.6233
		$S(t)$	0.5015	0.8655	0.3639	0.4808	0.8095	0.3286	0.4603	0.7736	0.3133
		$H(t)$	0.1281	0.8036	0.6756	0.2003	0.7256	0.5252	0.2554	0.7804	0.5250
	10	α	0.0196	0.7823	0.7627	0.1309	0.6653	0.5344	0.1443	0.5823	0.4380
		β	0.1537	0.7410	0.5873	0.1323	0.5064	0.3741	0.1457	0.4361	0.2904
		$S(t)$	0.5008	0.8508	0.3500	0.4726	0.7953	0.3226	0.4741	0.7767	0.3026
		$H(t)$	0.1720	0.7902	0.6182	0.2140	0.7341	0.5201	0.2511	0.7531	0.5020
24	16	α	0.2226	0.8419	0.6193	0.3148	0.8680	0.5532	0.3484	0.8116	0.4632
		β	0.2123	0.6950	0.4827	0.1577	0.5001	0.3424	0.1928	0.3827	0.1898
		$S(t)$	0.6494	0.9201	0.2707	0.6344	0.8783	0.2439	0.6315	0.8581	0.2266
		$H(t)$	0.0622	0.4519	0.3897	0.1306	0.4569	0.3263	0.1535	0.4615	0.3080
	19	α	0.2208	0.6745	0.4537	0.3325	0.7454	0.4129	0.3230	0.7196	0.3966
		β	0.2441	0.6154	0.3713	0.1471	0.3911	0.2440	0.1907	0.3728	0.1821
		$S(t)$	0.6484	0.9117	0.2633	0.6304	0.8581	0.2277	0.6351	0.8574	0.2223
		$H(t)$	0.0783	0.4478	0.3695	0.1526	0.4612	0.3085	0.1466	0.4447	0.2981

Table 3. Results of point estimates with $(\alpha, \beta) = (0.4, 2.0)$ and mission time $t = 0.5$.

n	r	para.	MLE		FCPE		CCPE	
			AB	MSE	AB	MSE	AB	MSE
16	12	α	0.1664	0.0513	0.1415	0.0322	0.1243	0.0268
		β	0.6578	0.5127	0.4445	0.2600	0.3503	0.1926
		$S(t)$	0.1105	0.0145	0.0808	0.0066	0.0630	0.0040
		$H(t)$	0.3094	0.1691	0.1700	0.0330	0.1267	0.0176
	14	α	0.1239	0.0243	0.1162	0.0224	0.0987	0.0218
		β	0.6555	0.4946	0.3775	0.1773	0.3405	0.1474
		$S(t)$	0.0806	0.0097	0.0536	0.0032	0.0414	0.0019
		$H(t)$	0.2282	0.0994	0.1117	0.0152	0.0843	0.0085
26	20	α	0.1220	0.0191	0.1087	0.0152	0.0688	0.0070
		β	0.5211	0.3305	0.2186	0.0709	0.2018	0.0638
		$S(t)$	0.0643	0.0056	0.0449	0.0029	0.0382	0.0018
		$H(t)$	0.1073	0.0192	0.0749	0.0088	0.0612	0.0052
	23	α	0.0932	0.0140	0.0728	0.0105	0.0426	0.0028
		β	0.4984	0.3218	0.1978	0.0536	0.1870	0.0535
		$S(t)$	0.0345	0.0017	0.0269	0.0016	0.0112	0.0008
		$H(t)$	0.0426	0.0024	0.0366	0.0026	0.0116	0.0016

Table 4. Results of interval estimates with $(\alpha, \beta) = (0.4, 2.0)$ and mission time $t = 0.5$.

n	r	para.	ACI			FCCI			CCCI		
			Lower	Upper	AL	Lower	Upper	AL	Lower	Upper	AL
16	12	α	0.2180	0.8678	0.6498	0.3243	0.7604	0.4362	0.3212	0.6893	0.3681
		β	1.7007	3.1072	1.4065	1.2377	2.4363	1.1986	1.2850	2.3861	1.1012
		$S(t)$	0.5378	0.8022	0.2644	0.5611	0.7835	0.2224	0.5769	0.7730	0.1961
		$H(t)$	0.2677	0.7278	0.4601	0.2374	0.5755	0.3381	0.2495	0.5468	0.2973
	14	α	0.2154	0.7525	0.5371	0.2899	0.6751	0.3851	0.2969	0.6434	0.3465
		β	1.8342	3.1478	1.3136	1.2838	2.4504	1.1666	1.3621	2.4002	1.0381
		$S(t)$	0.5602	0.7993	0.2391	0.5720	0.7913	0.2193	0.5899	0.7799	0.1901
		$H(t)$	0.3024	0.6809	0.3785	0.2243	0.5537	0.3294	0.2460	0.5270	0.2810
26	20	α	0.2598	0.6859	0.4261	0.4126	0.6797	0.2671	0.3660	0.5647	0.1987
		β	3.0963	4.0195	0.9232	1.3701	1.9476	0.5775	1.3299	1.7885	0.4586
		$S(t)$	0.5678	0.7875	0.2197	0.5943	0.8054	0.2111	0.6043	0.7876	0.1833
		$H(t)$	0.3382	0.6744	0.3363	0.2186	0.5230	0.3044	0.2390	0.5056	0.2666
	23	α	0.2290	0.6179	0.3890	0.3995	0.6551	0.2556	0.4421	0.6196	0.1775
		β	3.4663	4.3697	0.9033	1.3767	1.9329	0.5562	1.3939	1.7204	0.3265
		$S(t)$	0.6938	0.8855	0.1918	0.7140	0.8702	0.1562	0.7326	0.8780	0.1454
		$H(t)$	0.1317	0.3804	0.2488	0.1396	0.3381	0.1984	0.1305	0.3122	0.1817

Table 5. Results of point estimates with $(\alpha, \beta) = (2.0, 0.8)$ and mission time $t = 0.5$.

n	r	para.	MLE		FCPE		CCPE	
			AB	MSE	AB	MSE	AB	MSE
18	13	α	0.6549	0.4354	0.4409	0.2720	0.3278	0.1788
		β	0.3210	0.1099	0.2480	0.0684	0.1870	0.0417
		$S(t)$	0.0807	0.0076	0.0743	0.0065	0.0645	0.0048
		$H(t)$	0.1224	0.0177	0.1060	0.0122	0.0930	0.0097
	17	α	0.4435	0.2198	0.3783	0.2040	0.3136	0.1728
		β	0.2603	0.0801	0.2135	0.0522	0.1646	0.0331
		$S(t)$	0.0472	0.0030	0.0434	0.0025	0.0398	0.0022
		$H(t)$	0.0931	0.0132	0.0785	0.0073	0.0699	0.0067
28	22	α	0.3144	0.1194	0.2841	0.1091	0.2631	0.0909
		β	0.2235	0.0632	0.1921	0.0473	0.0905	0.0119
		$S(t)$	0.0471	0.0027	0.0281	0.0009	0.0197	0.0004
		$H(t)$	0.0777	0.0068	0.0540	0.0037	0.0392	0.0021
	25	α	0.3008	0.1031	0.2622	0.1029	0.1804	0.0525
		β	0.2050	0.0484	0.1132	0.0178	0.0842	0.0102
		$S(t)$	0.0380	0.0020	0.0190	0.0005	0.0040	0.0001
		$H(t)$	0.0701	0.0061	0.0330	0.0013	0.0134	0.0004

Table 6. Results of interval estimates with $(\alpha, \beta) = (2.0, 0.8)$ and mission time $t = 0.5$.

n	r	para.	ACI			FCCI			CCCI		
			Lower	Upper	AL	Lower	Upper	AL	Lower	Upper	AL
18	13	α	0.2772	2.4130	2.1358	0.9420	2.3084	1.3664	1.1381	2.4594	1.3213
		β	0.0520	1.1140	1.0619	0.3234	0.8015	0.4781	0.4328	0.7867	0.3539
		$S(t)$	0.4009	0.8315	0.4306	0.6737	0.8150	0.1413	0.6035	0.7197	0.1162
		$H(t)$	0.2227	0.8688	0.6461	0.2071	0.4012	0.1941	0.3217	0.4992	0.1775
	17	α	0.5573	2.5557	1.9984	1.1059	2.2694	1.1635	1.2788	2.4136	1.1348
		β	0.0619	1.0231	0.9612	0.3864	0.8045	0.4181	0.4769	0.7888	0.3120
		$S(t)$	0.4259	0.8268	0.4009	0.6475	0.7801	0.1326	0.5988	0.7133	0.1145
		$H(t)$	0.2089	0.8412	0.6324	0.2480	0.4353	0.1873	0.3130	0.4884	0.1754
28	22	α	0.8545	2.5167	1.6622	1.4129	1.8957	0.4828	1.6066	2.0375	0.4309
		β	0.1219	1.0311	0.9092	0.6633	0.9342	0.2709	0.6693	0.8919	0.2227
		$S(t)$	0.4244	0.8206	0.3963	0.6482	0.7629	0.1148	0.6254	0.7142	0.0889
		$H(t)$	0.2209	0.8385	0.6176	0.2723	0.4368	0.1645	0.3288	0.4599	0.1311
	25	α	0.7494	2.2035	1.4541	1.2341	1.6722	0.4381	1.5324	1.9016	0.3692
		β	0.1796	1.0104	0.8308	0.6232	0.8800	0.2568	0.6522	0.8570	0.2048
		$S(t)$	0.4379	0.8297	0.3917	0.6284	0.7387	0.1103	0.6200	0.6969	0.0769
		$H(t)$	0.2158	0.8194	0.6036	0.2936	0.4525	0.1589	0.3637	0.4796	0.1159

Table 7. Results of point estimates with $(\alpha, \beta) = (1.8, 2.0)$ and mission time $t = 0.5$.

n	r	para.	MLE		FCPE		CCPE	
			AB	MSE	AB	MSE	AB	MSE
20	15	α	0.6923	0.4941	0.5562	0.3410	0.5351	0.3338
		β	0.5906	0.4950	0.3829	0.2333	0.3048	0.1608
		$S(t)$	0.1041	0.0133	0.0971	0.0099	0.0803	0.0071
		$H(t)$	0.2800	0.1434	0.2292	0.0714	0.2104	0.0706
	18	α	0.6123	0.4000	0.5315	0.3258	0.4717	0.2990
		β	0.4969	0.3336	0.2571	0.0844	0.1992	0.0700
		$S(t)$	0.0717	0.0075	0.0534	0.0038	0.0525	0.0035
		$H(t)$	0.1692	0.0546	0.1265	0.0282	0.1220	0.0248
30	24	α	0.3951	0.2199	0.3302	0.1349	0.2712	0.0902
		β	0.3211	0.1414	0.2075	0.0827	0.1425	0.0234
		$S(t)$	0.0453	0.0027	0.0417	0.0023	0.0275	0.0013
		$H(t)$	0.1268	0.0383	0.1177	0.0264	0.0931	0.0211
	27	α	0.3352	0.2086	0.2041	0.0494	0.1923	0.0459
		β	0.2555	0.1097	0.1917	0.0620	0.1080	0.0218
		$S(t)$	0.0428	0.0021	0.0309	0.0012	0.0165	0.0004
		$H(t)$	0.1016	0.0165	0.0827	0.0110	0.0537	0.0067

Table 8. Results of interval estimates with $(\alpha, \beta) = (1.8, 2.0)$ and mission time $t = 0.5$.

n	r	para.	ACI			FCCI			CCCI		
			Lower	Upper	AL	Lower	Upper	AL	Lower	Upper	AL
20	15	α	0.4348	1.7806	1.3458	0.8730	1.8138	0.9408	0.9871	1.7512	0.7641
		β	0.0524	3.5039	3.4515	1.3375	2.9516	1.6141	1.4406	2.3984	0.9578
		$S(t)$	0.2616	0.6930	0.4313	0.4326	0.6688	0.2362	0.4232	0.6201	0.1969
		$H(t)$	0.7445	1.2555	0.5111	0.4146	0.8487	0.4341	0.4708	0.8510	0.3802
	18	α	0.5687	1.8068	1.2381	0.9836	1.8694	0.8857	1.0708	1.8242	0.7534
		β	0.5658	2.8607	2.2949	1.3648	2.4703	1.1055	1.4359	2.3109	0.8750
		$S(t)$	0.4247	0.8122	0.3875	0.5719	0.7854	0.2135	0.5299	0.7259	0.1960
		$H(t)$	0.2990	0.8055	0.5065	0.2244	0.5421	0.3177	0.3051	0.6223	0.3172
30	24	α	0.8720	2.0450	1.1729	1.0634	1.8734	0.8100	1.2687	1.9752	0.7065
		β	0.9068	2.7010	1.7941	1.4883	2.1219	0.6336	1.7297	2.2062	0.4766
		$S(t)$	0.4273	0.7849	0.3576	0.5608	0.7505	0.1898	0.5856	0.7719	0.1862
		$H(t)$	0.3621	0.7851	0.4230	0.2548	0.5499	0.2950	0.2604	0.5374	0.2770
	27	α	0.8641	2.0360	1.1719	1.1564	1.9647	0.8083	1.1921	1.7588	0.5666
		β	1.0692	2.8230	1.7538	1.6547	2.2234	0.5687	1.8117	2.2563	0.4445
		$S(t)$	0.5605	0.8656	0.3051	0.6744	0.8589	0.1845	0.7134	0.8534	0.1400
		$H(t)$	0.1803	0.5354	0.3550	0.1363	0.3754	0.2392	0.1596	0.3380	0.1784

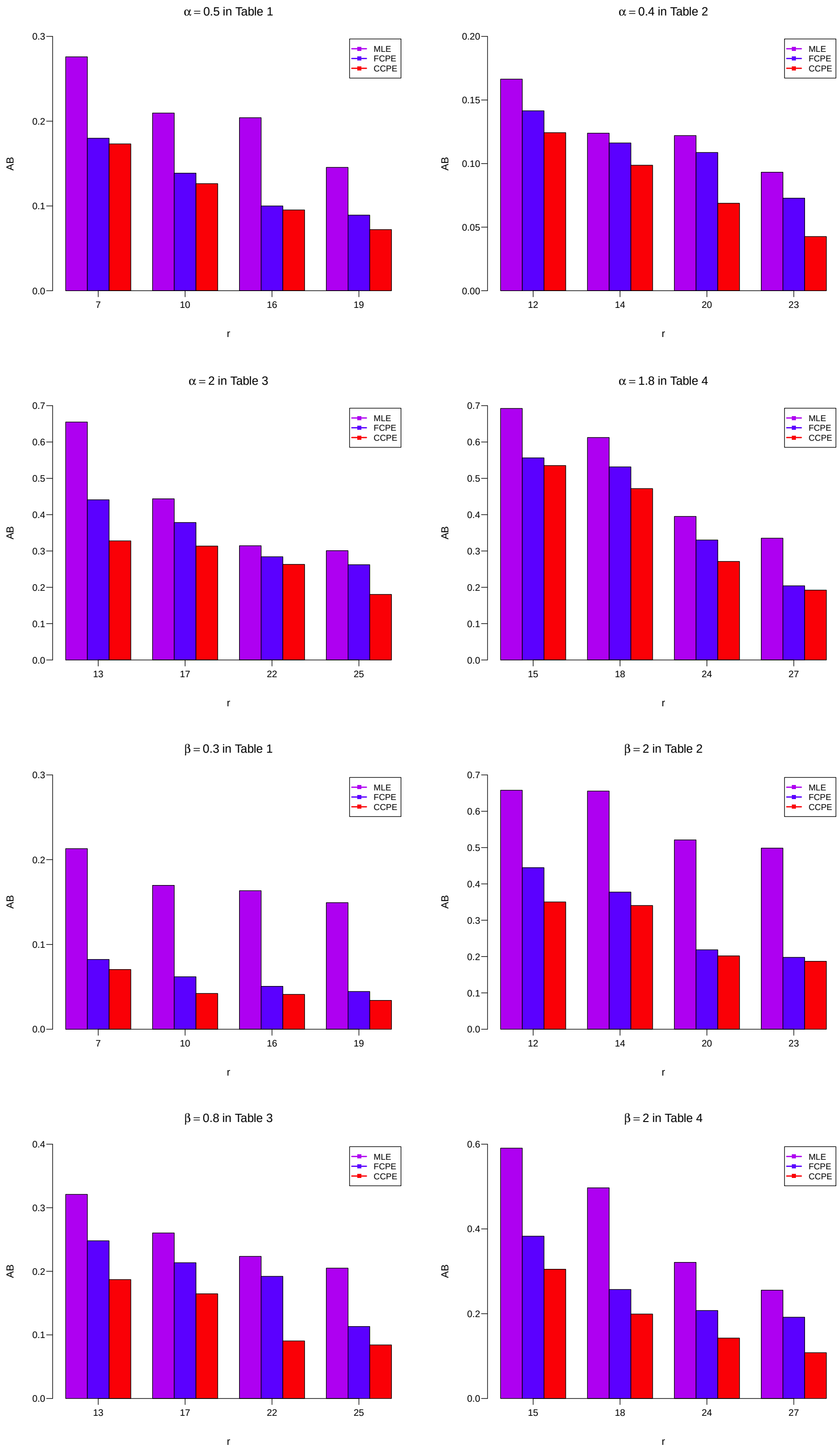

Figure 2. Criteria ABs of point estimates for parameters α and β.

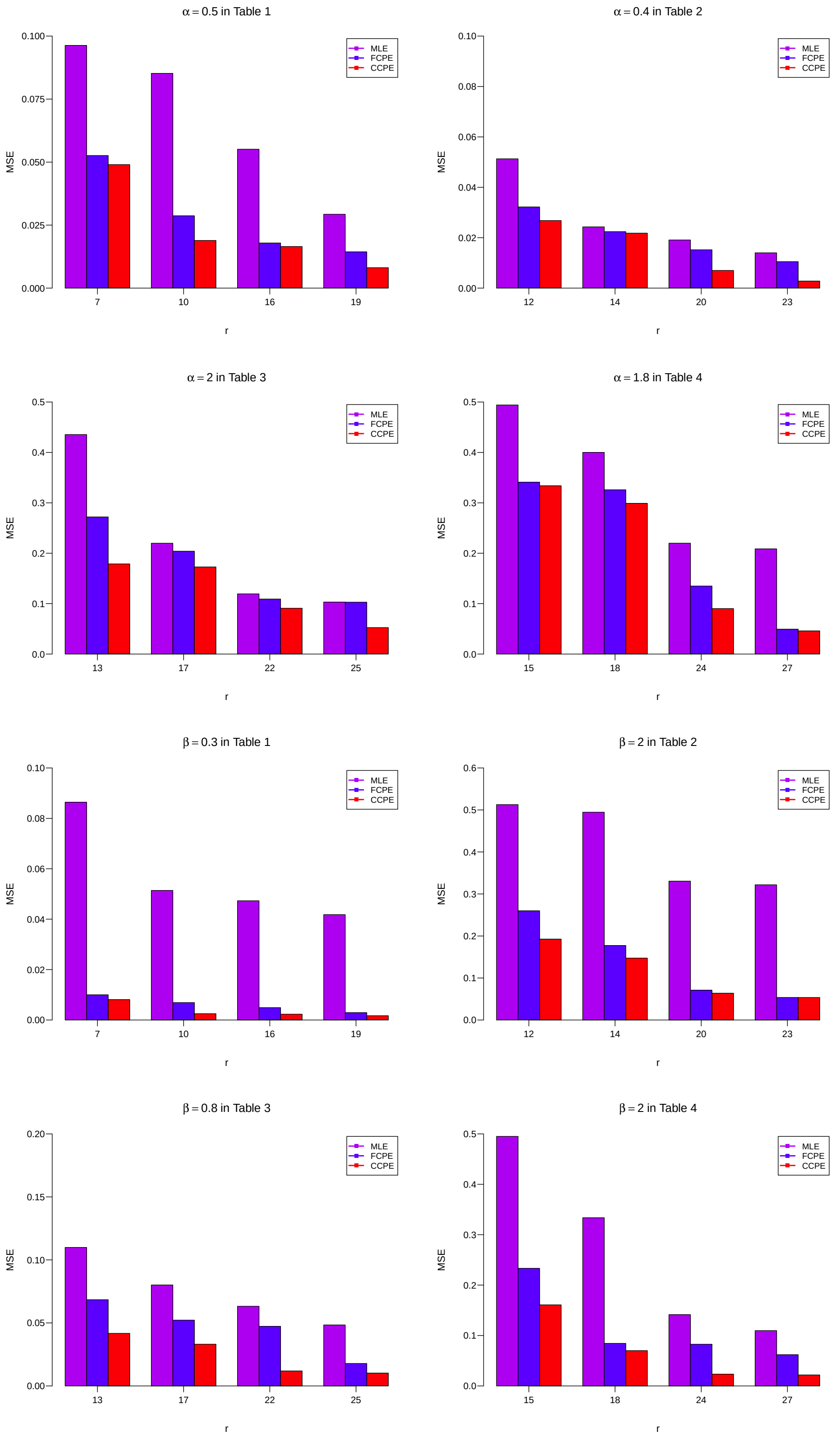

Figure 3. Criteria MSEs of point estimates for parameters α and β.

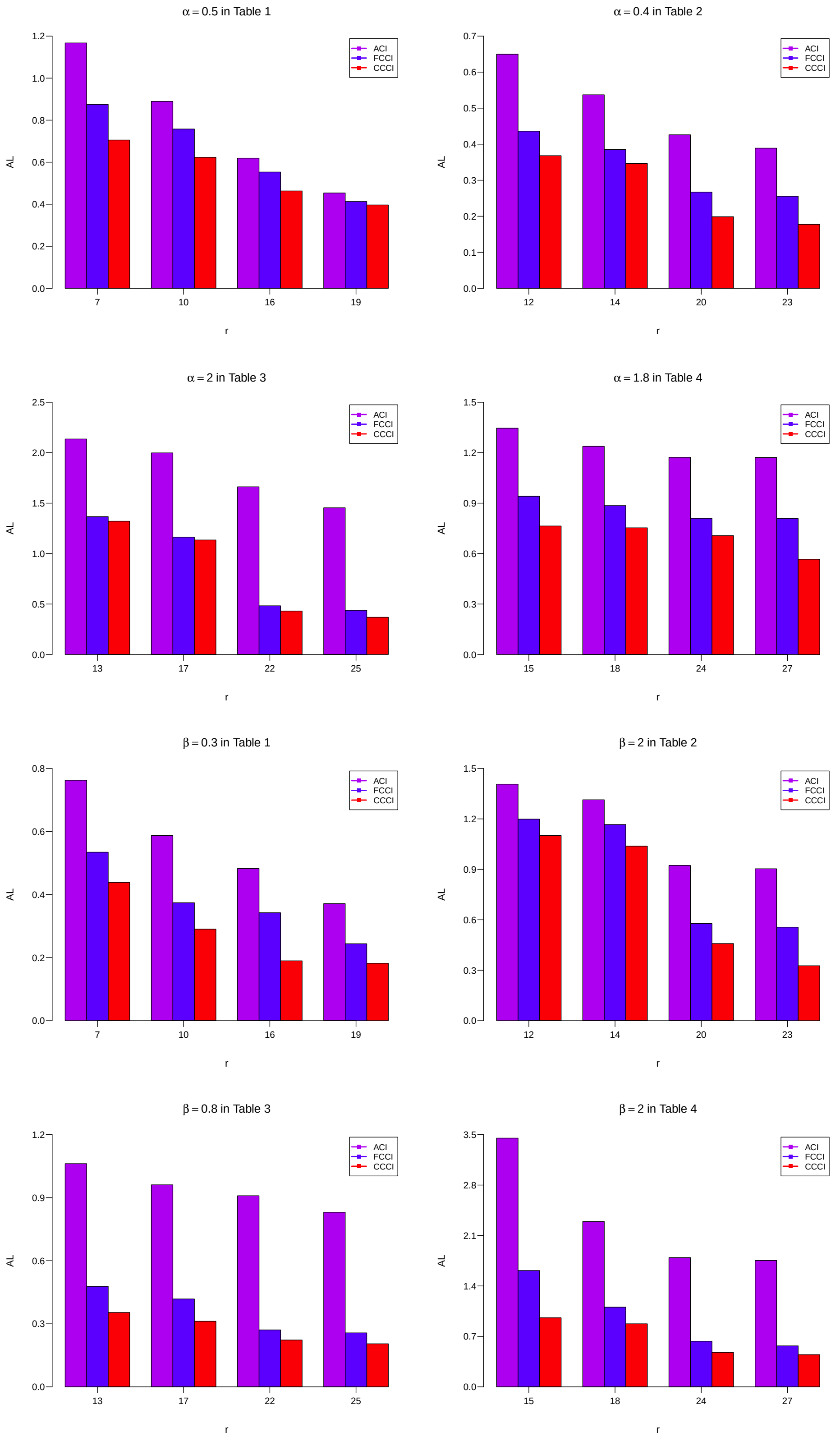

Figure 4. Criteria ALs of interval estimates for parameters α and β.

From the results presented in Tables 1 to 8 and Figures 2 to 4, the following conclusions can be drawn:

- With the increase of sample size n and number of failures r, the AB and MSE of MLEs and generalized estimates generally decrease. This indicates that the performance of different methods is satisfactory, exhibiting consistency as the effective sample size increases.
- For fixed n and r, the generalized point estimates (e.g., FCPEs and CCPEs) have smaller ABs and MSEs than those of MLEs, indicating that generalized estimates are superior to classical likelihood-based MLEs.
- For fixed sample size n and r, CCPEs generally outperform FCPEs in terms of both ABs and MSEs.
- For interval estimates, AL generally decreases as n and r increase. Additionally, the ALs of different interval estimates perform well even when the sample size is relatively small.
- For different interval estimates, the CCCIs and FCCIs have relatively shorter interval lengths than ACIs for both model parameters and reliability indices. This phenomenon persists when the sample size is small (less than 30), indicating the better performance of generalized intervals.
- For fixed n and r, CCCIs outperform FCCIs in terms of ALs in most cases, revealing that CC-based generalized interval estimates are superior to FC-based intervals.

In summary, the simulation results indicate that generalized methods exhibit better performance than the classical likelihood based methods in terms of the evaluation criteria. Furthermore, CC-based estimates outperform FC-based ones consequently. For clarity, the overall performance ranking for both point and interval estimation is generally as MLE(ACI) < FCPE(FCCI) < CCPE(CCCI) in general.

5. Real data illustration

In this section, two real-life examples illustrate the practical implications of the proposed methods.

5.1. Example one: system failure data

In this subsection, a real-life dataset introduced by Meeker et al. [23] is used for illustration, which consists of 30 failure times of a device from the field-tracking study of a larger system. To ensure the data falls within the $(0, 1)$ range for fitting, all observations were normalized by dividing by 120; the normalized data is presented in Table 9.

Table 9. Dataset of system failure.

0.0008	0.0017	0.0083	0.0167	0.0250	0.0500	0.0583	0.0917	0.1000	0.1500
0.1750	0.2667	0.3000	0.3333	0.3750	0.3833	0.3917	0.4167	0.4583	0.5000
0.5250	0.5583	0.6000	0.6250	0.6583	0.6833	0.6917	0.7000	0.7083	0.7167

Before proceeding, the Kies distribution and several candidate distributions (e.g., Topp-Leone, Beta, exponential, Weibull and gamma distributions) were used to fit the data. The Kolmogorov-Smirnov (K-S) distances and the associated p-values are presented in Table 10. It is noted that the Kies distribution provides the best goodness-of-fit in this example compared with other distributions. Additionally, plots of the empirical cumulative distribution function (ECDF) versus the fitted Kies distribution, along with probability-probability (P-P) and quantile-quantile (Q-Q) plots for the Kies distribution, are presented in Figure 5 for visual inspection. From the empirical distribution plots, the comparison between the empirical data distribution (histogram or density plot) and the fitted Kies distribution curve shows good concordance. Meanwhile, points in the P-P plot lie tightly along the diagonal line, validating that the cumulative probabilities align closely with their theoretical counterparts (from the Kies distribution). The Q-Q plot further shows that observed quantiles align closely with theoretical quantiles along the reference line. These plots thus indicate that the Kies distribution is an appropriate model for fitting this data.

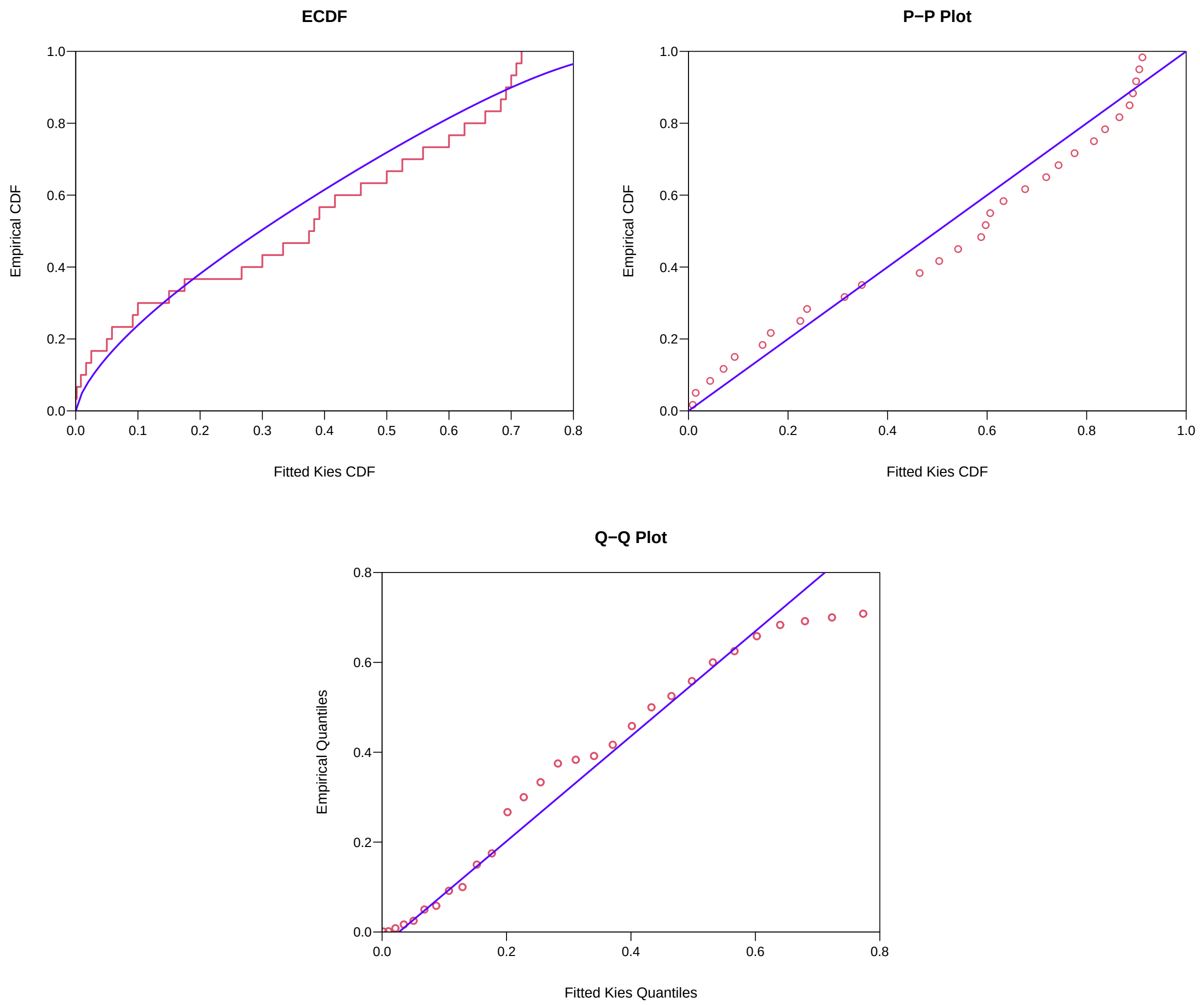

Figure 5. ECDF, P-P and Q-Q plots of the Kies distribution for system failure dataset.

Table 10. K-S test of system failure dataset under different distribution.

distribution	K-S distance	p-value
Kies distibution	0.1214	0.7239
Topp-Leone distribution	0.2586	0.0294
Beta distribution	0.1607	0.3799
exponential distribution	0.1884	0.2091
Weibull distribution	0.1928	0.1882
Gamma distribution	0.2004	0.1564

Furthermore, two sets of Type-II censored data were obtained with $r = 17$ and 21, respectively. The corresponding point and interval estimates were calculated, and the results are presented in Table 10 at the 95% significance level for interval estimates, with interval lengths provided in square brackets.

Table 11. Point and interval estimates for Kies model for failure data with mission time $t = 0.5$.

r	para.	MLE	ACI	FCPE	FCCI	CCPE	CCCI
17	α	1.0308	(0.4556,1.4539) [0.9982]	1.0280	(0.6994,1.3390) [0.6396]	1.0375	(0.7335,1.3204) [0.5869]
	β	0.5552	(0.1950,0.7228) [0.5278]	0.5475	(0.3418,0.7464) [0.4045]	0.5672	(0.4464,0.6863) [0.2398]
	$S(t)$	0.5263	(0.3475,0.7051) [0.3576]	0.4799	(0.3798,0.5758) [0.1960]	0.4838	(0.3861,0.5770) [0.1909]
	$H(t)$	0.7424	(0.2833,1.2015) [0.9181]	0.7500	(0.5415,0.9586) [0.4171]	0.7417	(0.5410,0.9433) [0.4023]
21	α	1.0786	(0.6024,1.5439) [0.9415]	1.0734	(0.8040,1.3341) [0.5301]	1.0731	(0.8176,1.3386) [0.5210]
	β	0.5776	(0.3007,0.8276) [0.5268]	0.5761	(0.3798,0.7736) [0.3937]	0.5932	(0.4851,0.6921) [0.2070]
	$S(t)$	0.5086	(0.3622,0.6550) [0.2928]	0.4803	(0.3789,0.5681) [0.1892]	0.4796	(0.3991,0.5706) [0.1716]
	$H(t)$	0.8334	(0.4027,1.2641) [0.8614]	0.7487	(0.5507,0.9551) [0.4043]	0.7478	(0.5555,0.9108) [0.3553]

From Table 11, with a fixed sample size ($n = 30$) and two different censoring levels ($r = 17, 21$), the generalized point estimates and MLEs are close in these two scenarios. Additionally, the generalized interval estimates(i.e., FCCIs and CCCIs) are superior to the ACIs in terms of interval lengths. Furthermore, CCCIs for model parameters and reliability indices perform better than FCCIs based on their interval lengths. It is also noted that despite the finite sample size ($n = 30$) in this example,both classical likelihood-based ACIs and generalized interval estimates are still acceptable. For completeness, plots of the profile log-likelihood function for parameter β and associated contour plots for the log-likelihood function $\ell(\alpha, \beta)$ are presented in Figures 6 and 7, respectively, which are consistent with Theorem 2.2.

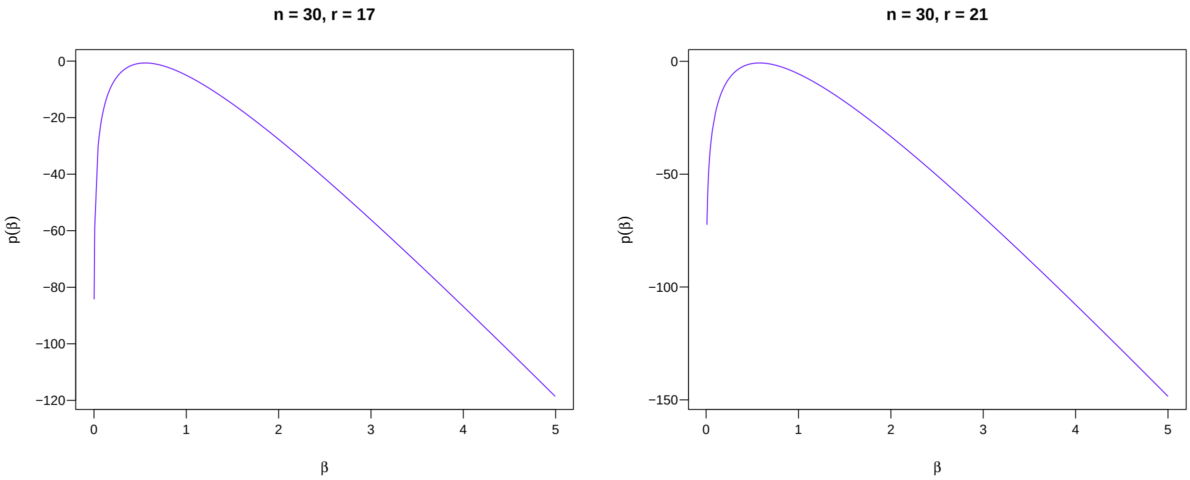

Figure 6. Profile plots of log-likelihood function for β under practical failure times.

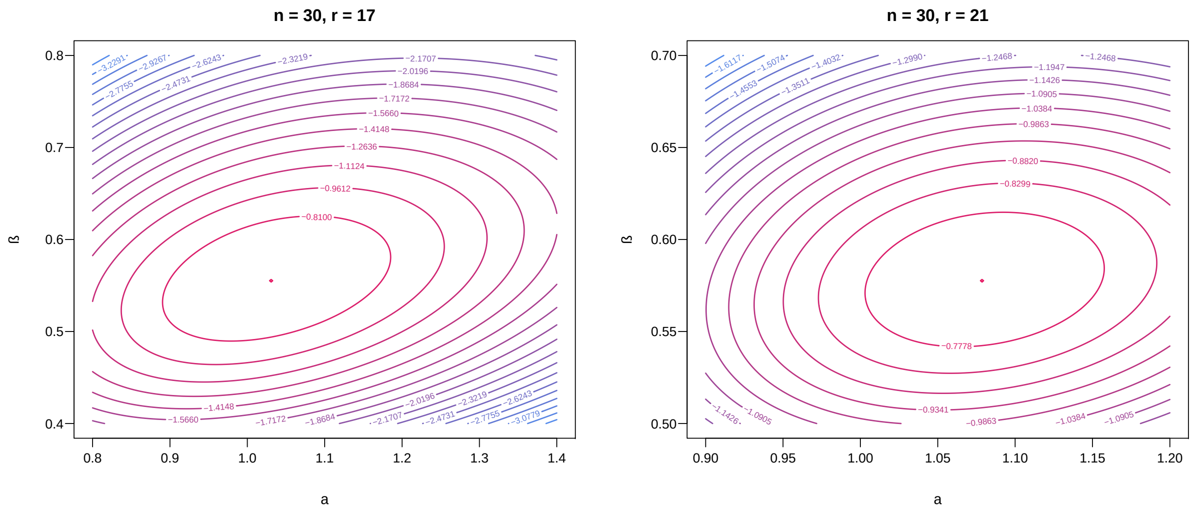

Figure 7. Contour plots of log-likelihood function $\ell(\alpha, \beta)$ under practical failure data.

5.2. *Example two: remission times data*

Another real-life dataset is discussed, consisting of 22 survival times(in weeks) of patients suffering from acute Myelogeneous Leukaemia. The original data was introduced by Feigl and Zelen [12] and has also been analyzed by Nassar et al. [24], Abouelmagd et al. [1], and Sen et al. [26]. For illustration, the original data was divided by 200, and the transformed data is presented in Table 12.

Table 12. Dataset of remission times.

0.0050	0.0100	0.0150	0.0200	0.0250	0.0350	0.0400	0.0800	0.0850	0.1100
0.1300	0.1500	0.1950	0.2150	0.2800	0.3250	0.5000	0.5400	0.6050	0.6700
0.7150	0.7800								

Before proceeding, goodness-of-fit tests are conducted for some candidate models, with results presented in Table 13. In addition, empirical distribution plots, P-P plots, and Q-Q plots for the Kies distribution are presented in Figure 8. Similar to Example 1, it is visually observed that the Kies distribution is a proper model for this real-world dataset.

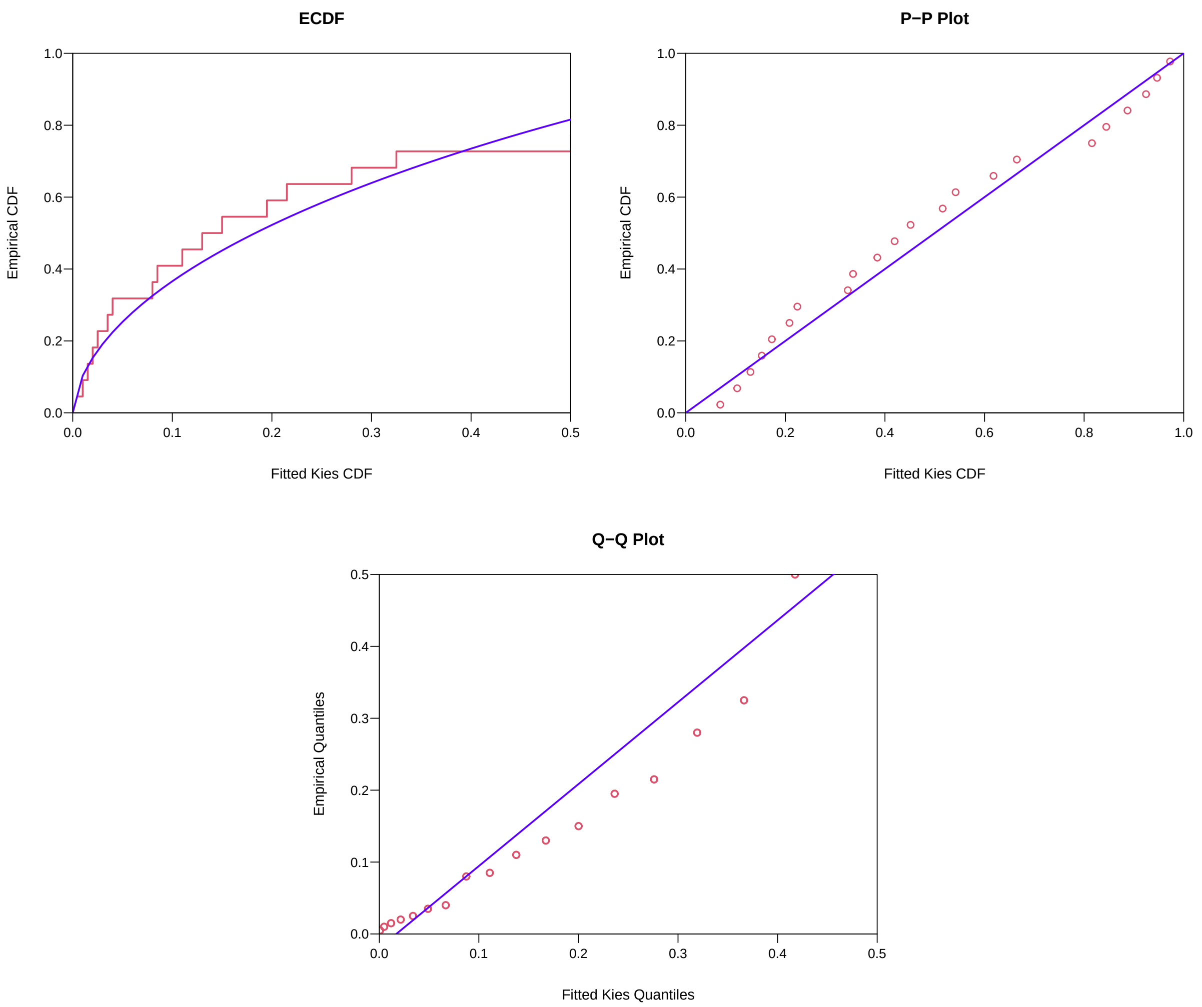

Figure 8. ECDF, P-P and Q-Q plots of the Kies distribution for remission times dataset.

Table 13. K-S test of remission times dataset under different distribution

distribution	K-S distance	p-value
Kies distibution	0.0945	0.9788
Topp-Leone distribution	0.4185	0.0005
Beta distribution	0.1083	0.9342
exponential distribution	0.1711	0.4877
Weibull distribution	0.8379	0.2300
Gamma distribution	0.1241	0.8466

Furthermore, for Type-II censored data with $r = 15$ and $r = 20$, different point and interval estimates are presented in Table 14 at the 95% confidence level. A similar phenomenon to Example 1 is observed: the generalized methods outperform classical likelihood-based methods. For illustration, the profile log-likelihood function of β and contour plots for the log-likelihood function $\ell(\alpha,\beta)$ are provided in Figures 9 and 10, respectively.

Table 14. Point and interval estimates for Kies model for remission data with mission time $t = 0.5$.

r	para.	MLE	ACI	FCPE	FCCI	CCPE	CCCI
15	α	2.2913	(0.3238,2.5865) [2.2627]	3.0909	(2.1260,3.2957) [1.1697]	2.3525	(1.9301,2.6794) [0.7493]
	β	0.6690	(0.0383,0.7564) [0.7181]	0.8764	(0.6940,0.9826) [0.2886]	0.7268	(0.6441,0.7754) [0.1312]
	$S(t)$	0.4978	(0.3319,0.6636) [0.3318]	0.5106	(0.4563,0.5502) [0.0939]	0.4301	(0.4040,0.4674) [0.0634]
	$H(t)$	0.9314	(0.4556,1.4071) [0.9514]	0.7579	(0.6169,0.8543) [0.2374]	0.8248	(0.7301,0.9136) [0.1835]
20	α	1.6271	(0.8671,2.3038) [1.4367]	1.6597	(1.4262,1.8486) [0.4223]	1.6683	(1.5035,1.8601) [0.3566]
	β	0.5749	(0.2638,0.7575) [0.4937]	0.7901	(0.6267,0.8499) [0.2232]	0.6102	(0.5618,0.6379) [0.0761]
	$S(t)$	0.4999	(0.3387,0.6611) [0.3223]	0.4907	(0.4330,0.5216) [0.0886]	0.4086	(0.3853,0.4435) [0.0582]
	$H(t)$	0.9443	(0.4928,1.3957) [0.9029]	0.7461	(0.6140,0.8088) [0.1949]	0.8732	(0.7803,0.9201) [0.1399]

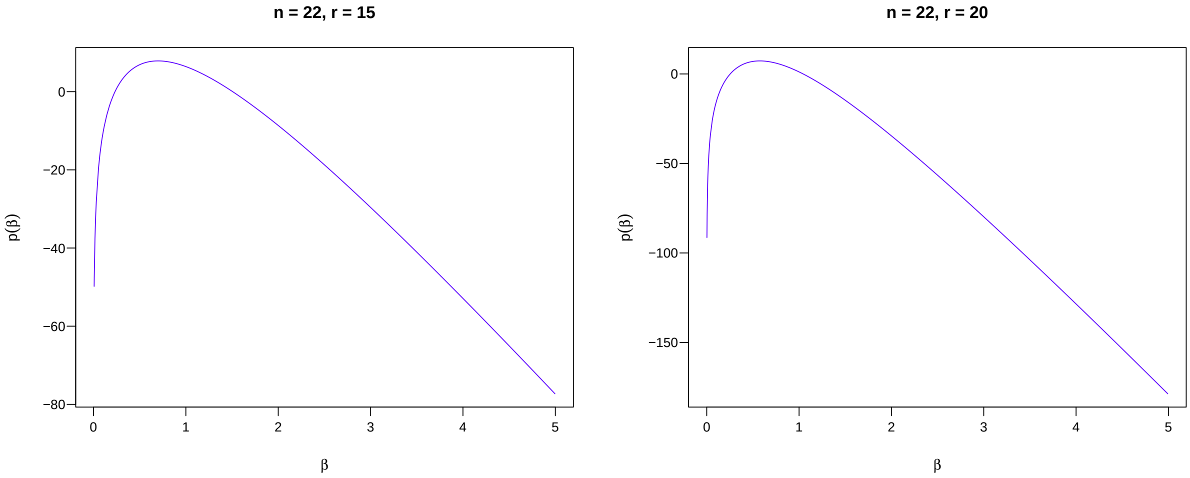

Figure 9. Profile plots of log-likelihood function for β under different censoring schemes for remission times dataset.

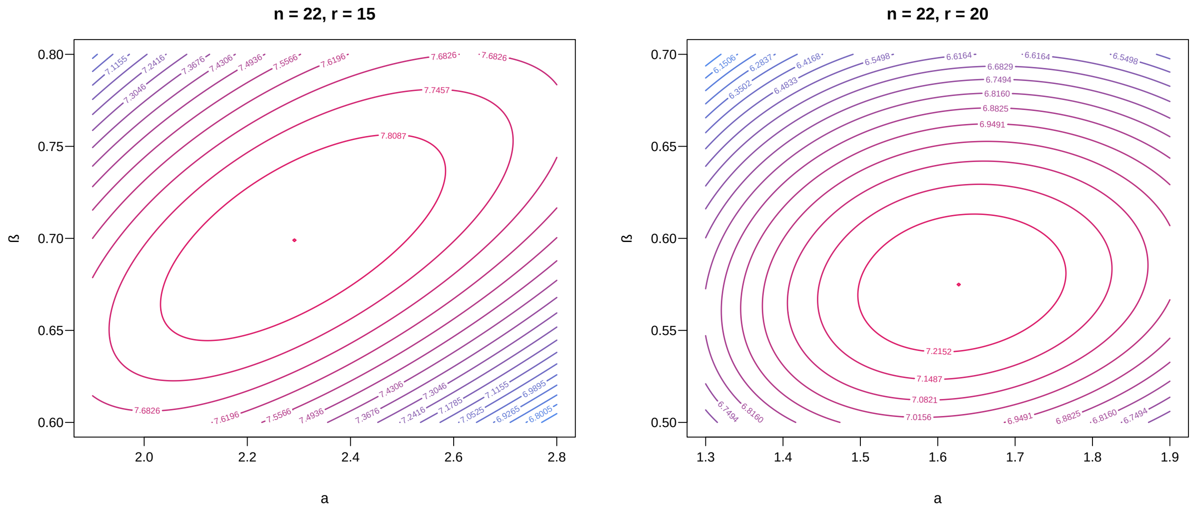

Figure 10. Contour plots of log-likelihood function $\ell(\alpha, \beta)$ under remission times dataset.

From the above two real-world examples, it is noted that our proposed methods perform satisfactorily. For completeness, some practical implications of the results are also summarized. First, due to practical constraints (e.g., cost and time), sample sizes may be limited (e.g., medium or small), and the proposed generalized inferential approaches provide potential ways to investigate product reliability with better performance. Second, the proposed estimates can be applied to various practical scenarios (e.g., lifetime prediction, predictive maintenance, and aging tests), which will help improve the quality of reliability management.

6. Concluding remarks

This paper investigates parameter estimation for the Kies distribution under the Type-II censoring scheme. Classical MLEs and ACIs are derived, and the existence and uniqueness properties of the model parameter MLEs are also established and provided. For comparison, two types of pivotal quantities are constructed, and alternative generalized inferential approaches are accordingly proposed for parameter estimation. Simulation studies and real-world examples demonstrate that the proposed generalized methods outperform the classical likelihood-based method. Additionally, the estimation method based on CC pivotal quantities outperforms that based on FC pivotal quantities. Although the discussion is conducted based on Kies model with Type-II censoring, the results could be extended to other distributions such as Weibull, Kumaraswamy, and Gompertz and other distributions, with minor modifications under general censoring scenarios including adaptive censoring and progressive censoring. For future research, inference with regression models (e.g., Ishag et al. [14] and Ishag et al. [15]), general data types (e.g., large-scale samples and high-dimensional covariates) and other methods (e.g., Bayesian methods, bootstrap resampling approaches) also seem interesting, which will be discussed in the future.

Author contributions

Wei Liu: Methodology, software, writing original draft preparation; Liang Wang: Conceptualization, software, writing original draft preparation; Yuhlong Lio: methodology, supervision; Sanku Dey: visualization, writing review; Min Wu: validation, formal analysis; All authors have read and agreed the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding this research.

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Appendix

A. Proof of Theorem 2.1

By taking derivative of the log-likelihood function (2.2) with respect to α and equating it to zero, the MLE $\tilde{\alpha} = \alpha(\beta)$ can directly derive for a given β. Further, we will show that $\tilde{\alpha}$ achieved the maximum of the log-likelihood function $\ell(\alpha, \beta)$ for given β. Using the inequality $\ln(t) \le t - 1$, $t > 0$ for $t = \frac{\alpha}{\tilde{\alpha}}$, one has that

$$r \ln \alpha = r \ln(\frac{\alpha}{\tilde{\alpha}}) + r \ln \tilde{\alpha} \le r \frac{\alpha}{\tilde{\alpha}} - r + r \ln \tilde{\alpha}.$$

Using the above inequality and ignoring the constant terms, it is noted that

$$\begin{aligned}\ell(\alpha, \beta) \le r \ln \tilde{\alpha} + r \ln \beta + \sum_{i=1}^{r} \ln \frac{x_i^{(\beta-1)}}{(1 - x_i)^{(\beta+1)}} - \tilde{\alpha} \sum_{i=1}^{r} \left(\frac{x_i}{1 - x_i} \right)^{\beta} \\ - \tilde{\alpha}(n - r) \left(\frac{x_r}{1 - x_r} \right)^{\beta} = \ell(\tilde{\alpha}, \beta),\end{aligned}$$

where the equation holds if and only if $\alpha = \tilde{\alpha}(\beta)$. Therefore, the assertion is completed.

B. Proof of Theorem 2.2

By taking the derivative of the profile log-likelihood function, the likelihood equation $\Omega(\beta) = 0$ could be established in consequence.

To show the existence of the MLE $\hat{\beta}$ for parameter β, the limitations of the function $\Omega(\beta)$ are established by taking direct computation at $\beta \to 0$ and $\beta \to \infty$, respectively, as

$$\lim_{\beta \longrightarrow 0} \Omega(\beta) = +\infty, \lim_{\beta \longrightarrow +\infty} \Omega(\beta) = \lim_{\beta \longrightarrow +\infty} \left(\sum_{i=1}^{r} \ln \left(\frac{x_i}{1 - x_i} \right) - r \ln s^* \right) < 0,$$

where $s^* = \max \left\{ \left(\frac{x_1}{1-x_1} \right), \left(\frac{x_2}{1-x_2} \right), \cdots, \left(\frac{x_r}{1-x_r} \right) \right\}$. Further, it is noted that continuous function $\Omega(\beta)$ changes from positive to negative when $\beta \in (0, +\infty)$, then the MLE $\hat{\beta}$ of parameter β exists.

Additionally, by taking derivative of function $\Omega(\beta)$ with respect to β, one has that

$$\begin{aligned}\frac{d\Omega(\beta)}{d\beta} = -\frac{r}{\beta^2} - \frac{r}{\sum_{i=1}^{r} p_i^{\beta} + (n - r) p_r^{\beta}} \\ \times \left\{ \left(\sum_{i=1}^{r} p_i^{\beta} \ln^2(p_i) + (n - r) p_r^{\beta} \ln^2(p_r) \right) * \left(\sum_{i=1}^{r} p_i^{\beta} + (n - r) p_r^{\beta} \right) \right.\end{aligned}$$

$$
-\left(\sum_{i=1}^{r} p_i^{\beta}\ln(p_i)+(n-r)p_r^{\beta}\ln(p_r)\right)^2\Bigg\},
$$

where $p_i=\frac{x_i}{1-x_i},\ i=1,2,\ldots,r$. From the Cauchy-Schwartz inequality, it is noted that the numerator in the second expression of the above derivative is positive, implying that the derivative is $\frac{d\Omega(\beta)}{d\beta}<0$. Therefore, function $\Omega(\beta)$ is monotone and changes from positive to negative within its range $\beta\in(0,\infty)$, the MLE $\hat{\beta}$ uniquely exists, and the assertion is completed.

C. Proof of Theorem 3.2

Suppose $X_1\le X_2\le\cdots\le X_r$ are Type-II censored data of size r from the Kies distribution $\mathrm{K}(\alpha,\beta)$. For $r>0$, and let $L_i=\alpha\left(\frac{x_i}{1-x_i}\right)^{\beta}$ with $i=1,2,\cdots,r$. Then, $L_1,L_2,\cdots,L_r$ are the Type-II censored data from the standard exponential distribution with a sample size of r.

Making transformations

$$
\begin{aligned}
U_1 &= nL_1,\\
U_2 &= (n-1)(L_2-L_1),\\
&\cdots\cdots\\
U_r &= (n-r+1)(L_r-L_{r-1}),
\end{aligned}
$$

it is seen that quantities $U_1,U_2,\cdots,U_r$ are independent random variables from the standard exponential distribution due to the memoryless property of the exponential distribution (e.g., Viveros and Balakrishnan [29], Wang and Ye [30]).

Define $W_1=2U_1$ and $W_2=2\sum_{i=2}^{r}U_i$, and one has $W_1\sim\chi_2^2$ and $W_2\sim\chi^2_{(2r-2)}$. Therefore, quantities

$$
G_1(\beta)=\frac{W_1/2}{W_2/2(r-1)}=(r-1)\frac{n}{(n-r+1)\left(\frac{x_r(1-x_1)}{x_1(1-x_r)}\right)^{\beta}+\sum_{i=1}^{r-1}\left(\frac{x_i(1-x_1)}{x_1(1-x_i)}\right)^{\beta}-n}\sim F_{(2,2r-2)},
$$

and

$$
B_1(\alpha,\beta)=W_1+W_2=2\alpha(n-r+1)\left(\frac{x_r}{1-x_r}\right)^{\beta}+2\alpha\sum_{i=1}^{r-1}\left(\frac{x_i}{1-x_i}\right)^{\beta}\sim\chi^2_{2r},
$$

and that $G_1(\beta)$ and $B_1(\alpha,\beta)$ are statistically independent. Therefore, the assertion is shown.

D. Proof of Theorem 3.2

From the Proof of Theorem 3.1, it is noted that quantities $U_1,U_2,\cdots,U_r$ are independent random variables from exponential distribution. Furthermore, denote quantities

$$
M_1=U_1, M_2=U_1+U_2,\cdots,M_r=U_1+U_2+\cdots+U_r.
$$

It is noted from Stephens [27] that the following quantities

$$
N_1=\frac{M_1}{M_r},\ N_2=\frac{M_2}{M_r},\ \cdots,\ N_r=\frac{M_{r-1}}{M_r}.
$$

are independent random variables from the standard uniform distribution.

Furthermore, pivotal quantities can be constructed as follows

$$G_2(\beta) = -2\sum_{i=1}^{r-1} \ln N_i = -2\sum_{i=1}^{r-1} \ln \frac{\sum_{j=1}^{i-1}\left(\frac{x_j(1-x_i)}{x_i(1-x_j)}\right)^{\beta} + (n-i+1)}{\sum_{k=1}^{r-1}\left(\frac{x_k(1-x_i)}{x_i(1-x_k)}\right)^{\beta} + (n-r+1)\left(\frac{x_r(1-x_i)}{x_i(1-x_r)}\right)^{\beta}} \sim \chi^2_{(2r-2)},$$

and

$$B_2(\alpha,\beta) = 2M_r = 2\alpha(n-r+1)\left(\frac{x_r}{1-x_r}\right)^{\beta} + 2\alpha\sum_{i=1}^{r-1}\left(\frac{x_i}{1-x_i}\right)^{\beta} \sim \chi^2_{2r},$$

and that $G_2(\beta)$ and $B_2(\alpha,\beta)$ are statistically independent. Therefore, the assertion is completed.