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Research article

Solving the 1D neutron diffusion kinetic equation under mixed boundary conditions: Explicit solution

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Abstract: Obtaining accurate solutions for mathematical models of neutron diffusion systems may lead to a deeper understanding of processes in reactor physics. The present paper applies the Laplace transform to the time-dependent neutron diffusion equation (together with the delayed neutron precursor equation) under a reflective boundary condition at one edge. The residue theorem is employed to obtain the inverse transform, leading to a series solution structured as a modal expansion associated with the eigenvalues of a transcendental equation. Moreover, the obtained series solution is theoretically proven to converge. The numerical results show acceptable accuracy based on residual errors. Physically, the neutron flux exhibits oscillatory behavior within the spatial domain, resulting in a wave-like alternating surface. Additionally, the delayed neutron precursor concentration stabilizes over time, gradually approaching a stationary profile, which is consistent with the physical expectations. The results also support the effectiveness of the Laplace transform technique in capturing the early-time behavior of the system. Differences between the present results and those reported in the relevant literature are explained.

Keywords: neutron diffusion; reactor physics; partial differential equation; analytic solution; Laplace transform
Mathematics Subject Classification: 83C15, 44A10, 35Q70

1. Introduction

This paper focuses on the following system in reactor physics [1,2]:

$$\frac{1}{V}\frac{\partial\phi(x,t)}{\partial t} = D\frac{\partial^2\phi(x,t)}{\partial x^2} + \left(-\Sigma_a + (1-\beta)\nu\Sigma_f\right)\phi(x,t) + \lambda C(x,t), \tag{1}$$

$$\frac{\partial C(x,t)}{\partial t} = \beta\nu\Sigma_f\phi(x,t) - \lambda C(x,t), \tag{2}$$

where $\phi(x,t)$ and $C(x,t)$ represent the neutron flux and the delayed neutron precursor concentration, respectively. The parameters D, V, β, ν, λ, Σ_a, and Σ_f are defined as the diffusion coefficient, the neutron speed, the delayed neutron fraction, the average number of neutrons per fission, the decay constant of delayed neutron precursors, the macroscopic absorption cross-section, and the macroscopic fission cross-section for the fuel, respectively; see Ref. [3] for details. The boundary conditions (BCs) are taken as

$$\phi(0,t) = 0, \qquad \frac{\partial\phi}{\partial x}(L,t) = 0, \qquad t > 0, \tag{3}$$

while initial conditions (ICs) are

$$\phi(x,0) = \phi_0, \quad C(x,0) = \frac{\beta\nu\Sigma_f}{\lambda}\phi_0, \quad 0 < x < L. \tag{4}$$

The second boundary condition in Eq (3) corresponds to a reflective boundary condition at one edge of the reactor, mainly at $x = L$. In the literature, several authors applied various analytical and numerical approaches to solve the coupled partial differential equations (PDEs) (1-2) under distinct BCs:

$$\phi(0,t) = 0, \qquad \phi(L,t) = 0, \qquad t > 0. \tag{5}$$

For example, Ceolin et al. [1] applied the General Integral Transform Technique (GITT) to solve the coupled PDEs (1-2) subject to the ICs/BCs (4-5). Their approach was based on adding the fictitious diffusion term $\epsilon\frac{\partial^2 C}{\partial x^2}$ to the right-hand side of Eq (2), where ϵ is an artificial auxiliary parameter. However, Khaled [2] ignored the addition of this term and systematically employed the Laplace transform (LT) technique to obtain closed-form series solutions for $\phi(x,t)$ and $C(x,t)$. Although the LT method used by Khaled [2] was found to be effective, it encountered computational difficulties, particularly when producing the curves describing the behavior of $\phi(x,t)$ and $C(x,t)$ at the initial time $t = 0$ using the Computer Algebra System (CAS) Wolfram Mathematica. Moreover, several authors have resorted to other analytical techniques [4–6] in addition to numerical methods [7–9] to analyze different neutron diffusion systems under various scenarios. On the other hand, the solution reported by Khaled [2] was later reproduced by Al-Sharif et al. [10], who employed an ansatz method, and by Al-Jeaid [11] through a simpler approach. Although the authors in Refs. [10,11] showed some advantages of their analysis over the LT technique, they ignored the computational problems encountered in Ref. [2], described above. Very recently, the results obtained in Refs. [10,11] are generalized by the authors [12] through extending the ansatz method [10] to deal with the system (1-2) under the BCs (5) and general ICs with arbitrary functions. In the literature, the LT was effectively implemented to solve numerous mathematical models with applications in different fields governed by ordinary differential equations (ODEs) [13–16] and PDEs [17–20]. Although other methods have been proposed to solve ODEs and PDEs, such as the DTM [21], the HAM [22,23], the HPM [24–26], and the ADM [27–29], the LT has its own advantage over such methods because of its capability of determining the solution in exact/closed form. Presenting a solution for the problem (1)–(4) via LT is of interest in studying the neutron diffusion equation under new different ICs/BCs. Regarding this, it was recently reported by Cruz-López [30] and Espinosa-Paredes [31] that the solutions of classical (or integer models) that were obtained with the LT can be straightforwardly extended to the fractional case. The objective of this work is to explore the applicability of the LT to solve the system (1)–(4), which differs from the previous neutron diffusion system [1,2]. Our problem includes the second BC (3) instead of the second BC (5), and accordingly, different behavior for the physical system is expected in contrast to Refs. [1,2]. So, the LT is suggested in this paper to extend the work of Khaled [2], where the boundary condition at $x = L$ is change, and a new analysis related to the confirmation of the initial conditions is employed. On this occasion, it will be stated later that the present analysis leads to closed-form solutions without similar computational problems at $t = 0$ as arose in Ref. [2]. The paper is organized as follows. Section 2 applies the LT to the problem, while the residues method is employed in Section 3. Derivation of the analytic solution and its convergence are presented in Sections 4 and 5, respectively. Section 6 presents some theoretical and numerical results for the solution and its physical explanation.

2. Application of the LT

Let us first denote

$$\mu = V\left(-\Sigma_a + (1-\beta)\nu\Sigma_f\right), \quad \sigma = \beta\nu\Sigma_f, \quad \rho = \frac{\beta\nu\Sigma_f}{\lambda} = \frac{\sigma}{\lambda}, \tag{6}$$

then using a similar procedure to the one followed by Khaled [2], we obtain from the system given in Eqs (1) and (2) that

$$\overline{\phi}(x,s) = A_1(s)\cosh(\omega(s)x) + A_2(s)\sinh(\omega(s)x) + \frac{g(s)}{\omega^2(s)}, \tag{7}$$

where $\overline{\phi}(x,s) := \mathcal{L}\{\phi(x,t), s\}$ and

$$\omega^2(s) = \frac{1}{VD}\left[\frac{s^2 - (\mu-\lambda)s - \lambda(\mu + V\sigma)}{s+\lambda}\right], \quad g(s) = \frac{\phi_0}{VD}\left[\frac{s + \lambda(1+V\rho)}{s+\lambda}\right]. \tag{8}$$

The unknowns $A_1(s)$ and $A_2(s)$ are to be determined from the transformed BCs:

$$\overline{\phi}(0,s) = 0, \qquad \frac{\partial\overline{\phi}(L,s)}{\partial x} = 0, \tag{9}$$

and thus

$$A_1(s) = -\frac{g(s)}{\omega^2(s)}, \qquad A_2(s) = \frac{g(s)\sinh(\omega(s)L)}{\omega^2(s)\cosh(\omega(s)L)}. \tag{10}$$

From (7) and (10), we obtain

$$\overline{\phi}(x,s) = \frac{g(s)}{\omega^2(s)}\left[1 - \frac{\cosh(\omega(s)(x-L))}{\cosh(\omega(s)L)}\right], \tag{11}$$

which can be simplified to

$$\overline{\phi}(x, s) = 2g(s) \cdot \frac{\sinh\left(\omega(s)\frac{x}{2}\right)}{\omega(s)} \cdot \frac{\sinh\left(\omega(s)\frac{2L-x}{2}\right)}{\omega(s)} \cdot \frac{1}{\cosh\left(\omega(s)L\right)}, \tag{12}$$

or in the form:

$$\overline{\phi}(x, s) = 2\phi_0 \left[\frac{(s + \lambda(1 + V\rho))\sinh\left(\omega(s)\frac{x}{2}\right)\sinh\left(\omega(s)\frac{2L-x}{2}\right)}{(s^2 - (\mu - \lambda)s - \lambda(\mu + V\sigma))\cosh\left(\omega(s)L\right)}\right]. \tag{13}$$

3. The residues method

The flux $\phi(x, t)$ can be obtained by computing the inverse LT of Eq (12) using the well-known residues method. It is possible to observe from this expression that any singularity associated with $\omega(s) = 0$ is removable. Indeed, using that:

$$\lim_{\omega(s)\to 0} \frac{\sinh\left(\omega(s)\alpha\right)}{\omega(s)} = \alpha, \quad \alpha > 0, \quad \lim_{\omega(s)\to 0} \cosh\left(\omega(s)\beta\right) = 1, \tag{14}$$

and considering that $\lim_{s\to s_0} \omega(s) = 0$, by the limit composition it follows that:

$$\lim_{s\to s_0} \overline{\phi}(x, s) = 2g(s_0) \cdot \lim_{s\to s_0} \frac{\sinh\left(\omega(s)\frac{x}{2}\right)}{\omega(s)} \cdot \frac{\sinh\left(\omega(s)\frac{2L-x}{2}\right)}{\omega(s)} \cdot \frac{1}{\cosh\left(\omega(s)L\right)} = \frac{x(2L - x)}{2} g(s_0), \tag{15}$$

which is finite (note that $g(s)$ in Eq (8) is regular at s_0 since $s_0 \neq -\lambda$). Thus, the zeros of $\omega^2(s)$ do not generate poles; they are removable singularities. The only genuine poles relevant for inversion are from the transcendental condition:

$$\cosh\left(\omega(s)L\right) = 0, \tag{16}$$

which requires

$$\omega^2(s) = -(2n + 1)^2 \frac{\pi^2}{4L^2}, \quad n \in \mathbb{Z}. \tag{17}$$

In Eq (16), the condition $\cosh\left(\omega(s)L\right) = 0$ leads to $\omega(s)L = \pm i(2n + 1)\frac{\pi}{2}$, with $n = 0, 1, 2, \ldots$ Consequently, for each value of n one obtains two distinct solutions of the quadratic equation (17), namely:

$$s_{n,\pm} = \frac{1}{2}\left[E_n \pm \sqrt{E_n^2 + 4F_n}\right], \tag{18}$$

with E_n and F_n given by

$$E_n = \mu - \lambda - (2n + 1)^2 VD \frac{\pi^2}{4L^2}, \quad F_n = \lambda\left[\mu + V\sigma - (2n + 1)^2 VD \frac{\pi^2}{4L^2}\right]. \tag{19}$$

Therefore, the poles arising from Eq (16) form an infinite sequence of simple poles $\{s_{n,\pm}\}_{n\geq 0}$.

By this, one can rewrite Eq (13) in the form:

$$\overline{\phi}(x, s) = h(x, s)\left[\frac{s + \lambda(1 + V\rho)}{(s^2 - (\mu - \lambda)s - \lambda(\mu + V\sigma))\cosh\left(\omega(s)L\right)}\right], \tag{20}$$

where $h(x,s)$ is defined by

$$h(x,s) = 2\phi_0 \sinh\left(\omega(s)\frac{x}{2}\right)\sinh\left(\omega(s)\frac{2L-x}{2}\right), \tag{21}$$

and observing that:

$$h(x,s_{n,\pm}) = -2\phi_0 \sin\left[\frac{(2n+1)\pi x}{4L}\right]\sin\left[\frac{(2n+1)\pi}{4L}(2L-x)\right]. \tag{22}$$

In order to calculate the residues at the poles $s_{n,\pm}$, we have from Eq (12) that

$$\begin{aligned}\text{Res}\,\{e^{st}\overline{\phi}(x,s)\}_{|s=s_{n,\pm}} &= e^{s_{n,\pm}t}\cdot\frac{2g(s_{n,\pm})}{\omega^2(s_{n,\pm})}\sinh\left(\omega(s_{n,\pm})\frac{x}{2}\right)\sinh\left(\omega(s_{n,\pm})\frac{2L-x}{2}\right)\\ &\quad\times\lim_{s\to s_{n,\pm}}\frac{s-s_{n,\pm}}{\cosh(\omega(s)L)},\\ &= e^{s_{n,\pm}t}\cdot\frac{g(s_{n,\pm})}{\phi_0\omega^2(s_{n,\pm})}h(x,s_{n,\pm})\lim_{s\to s_{n,\pm}}\frac{s-s_{n,\pm}}{\cosh(\omega(s)L)},\end{aligned} \tag{23}$$

where $s_{n,\pm}$ are the roots of the equation:

$$\omega^2(s) = \frac{1}{VD}\frac{s^2-(\mu-\lambda)s-\lambda(\mu+V\sigma)}{s+\lambda} = -\frac{(2n+1)^2\pi^2}{4L^2}. \tag{24}$$

The limit in Eq (23) is undetermined and accordingly, the L'Hôpital's rule gives

$$\lim_{s\to s_{n,\pm}}\left(\frac{s-s_{n,\pm}}{\cosh(\omega(s)L)}\right) = \frac{1}{L\,\omega'(s_{n,\pm})\sinh(\omega(s_{n,\pm})L)}. \tag{25}$$

Using the following relationships:

$$\omega'(s_{n,\pm}) = \frac{(s_{n,\pm}+\lambda)^2+\lambda V\sigma}{2VD(s_{n,\pm}+\lambda)^2\omega(s_{n,\pm})},\quad \sinh(\omega(s_{n,\pm})L) = \sinh\left(\pm i(2n+1)\frac{\pi}{2}\right) = \pm i(-1)^n, \tag{26}$$

it then follows

$$\lim_{s\to s_{n,\pm}}\left(\frac{s-s_{n,\pm}}{\cosh(\omega(s)L)}\right) = \frac{2VD(s_{n,\pm}+\lambda)^2\omega(s_{n,\pm})}{L\,\left((s_{n,\pm}+\lambda)^2+\lambda V\sigma\right)\sinh(\omega(s_{n,\pm})L)} = \frac{(2n+1)(-1)^nVD\frac{\pi}{L^2}(s_{n,\pm}+\lambda)^2}{((s_{n,\pm}+\lambda)^2+\lambda V\sigma)}. \tag{27}$$

Thus

$$\text{Res}\,\{e^{st}\overline{\phi}(x,s)\}_{|s=s_{n,\pm}} = -\frac{4(-1)^nh(x,s_{n,\pm})}{(2n+1)\pi}e^{s_{n,\pm}t}\left[\frac{(s_{n,\pm}+\lambda)(s_{n,\pm}+\lambda(1+V\rho))}{(s_{n,\pm}+\lambda)^2+\lambda V\sigma}\right], \tag{28}$$

which is

$$\text{Res}\,\{e^{st}\overline{\phi}(x,s)\}_{|s=s_{n,\pm}} = k(x,n)e^{s_{n,\pm}t}\left[\frac{(s_{n,\pm}+\lambda)(s_{n,\pm}+\lambda(1+V\rho))}{(s_{n,\pm}+\lambda)^2+\lambda V\sigma}\right], \tag{29}$$

where

$$\begin{aligned}k(x,n) &= -\frac{4(-1)^nh(x,s_{n,\pm})}{(2n+1)\pi}\\ &= \frac{8(-1)^n\phi_0}{(2n+1)\pi}\sin\left[\frac{(2n+1)\pi x}{4L}\right]\sin\left[\frac{(2n+1)\pi}{4L}(2L-x)\right],\ n=0,1,2\ldots.\end{aligned} \tag{30}$$

Using trigonometric identities, one can show that

$$\sin\left[\frac{(2n+1)\pi x}{4L}\right]\sin\left[\frac{(2n+1)\pi}{4L}(2L-x)\right]=\frac{1}{2}\cos\left[\frac{(2n+1)\pi}{2L}(L-x)\right]. \quad (31)$$

Therefore

$$k(x,n)=\frac{4(-1)^n\phi_0}{(2n+1)\pi}\cos\left[\frac{(2n+1)\pi}{2L}(L-x)\right],\ n=0,1,2\ldots. \quad (32)$$

4. The analytic solution

Given

$$f(t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{st}F(s)ds,$$

with $\gamma > R(s_k)$ for all poles of $F(s)$, the Cauchy residue theorem ensures that, provided the contribution from the large contour vanishes, one can write

$$f(t)=\sum_k \mathrm{Res}\left(e^{st}F(s),\, s=s_k\right).$$

This representation holds even when $F(s)$ has infinitely many poles, as discussed in Schiff [32, pages 160–161]. Based on the residue method, the solution $\phi(x,t)$ is given as

$$\phi(x,t)=\left(\sum_{n=0}^{\infty}\mathrm{Res}\,\{e^{st}\overline{\phi}(x,s)\}_{|s=s_{n,+}}+\sum_{n=0}^{\infty}\mathrm{Res}\,\{e^{st}\overline{\phi}(x,s)\}_{|s=s_{n,-}}\right). \quad (33)$$

In view of the results of the previous section, we obtain

$$\phi(x,t)=\sum_{n=0}^{\infty}k(x,n)\left[\left(\frac{(s_{n,+}+\lambda)(s_{n,+}+\lambda(1+V\rho))}{(s_{n,+}+\lambda)^2+\lambda V\sigma}\right)e^{s_{n,+}t}+\left(\frac{(s_{n,-}+\lambda)(s_{n,-}+\lambda(1+V\rho))}{(s_{n,-}+\lambda)^2+\lambda V\sigma}\right)e^{s_{n,-}t}\right], \quad (34)$$

which can be expressed as

$$\phi(x,t)=\sum_{n=0}^{\infty}k(x,n)\left[A_ne^{s_{n,+}t}+B_ne^{s_{n,-}t}\right], \quad (35)$$

where

$$A_n=\frac{(s_{n,+}+\lambda)(s_{n,+}+\lambda(1+V\rho))}{(s_{n,+}+\lambda)^2+\lambda V\sigma},\quad B_n=\frac{(s_{n,-}+\lambda)(s_{n,-}+\lambda(1+V\rho))}{(s_{n,-}+\lambda)^2+\lambda V\sigma}. \quad (36)$$

One can obtain $C(x,t)$ as

$$C(x,t)=\rho\phi_0e^{-\lambda t}+\sigma\int_0^t e^{-\lambda(t-\tau)}\phi(x,\tau)d\tau, \quad (37)$$

or

$$C(x,t)=\rho\phi_0e^{-\lambda t}+\sigma e^{-\lambda t}\int_0^t e^{\lambda\tau}\phi(x,\tau)d\tau. \quad (38)$$

Performing the integration in (38) yields

$$C(x,t)=\rho\phi_0e^{-\lambda t}+\sigma\sum_{n=0}^{\infty}k(x,n)\left[\frac{A_n}{s_{n,+}+\lambda}e^{s_{n,+}t}+\frac{B_n}{s_{n,-}+\lambda}e^{s_{n,-}t}-\left(\frac{A_n}{s_{n,+}+\lambda}+\frac{B_n}{s_{n,-}+\lambda}\right)e^{-\lambda t}\right], \quad (39)$$

where $k(x,n)$ is defined by Eq (32).

5. Convergence analysis

This section concerns the theoretical analysis of the convergence of the series given in Eq (35). A bound for the function $|k(x, n)|$ can be stated as

$$|k(x, n)| \leq \frac{4\phi_0}{(2n + 1)\pi}. \tag{40}$$

Define a rational function (related to the coefficients A_n and B_n) in the form:

$$\frac{(s + \lambda)(s + \lambda(1 + V\rho))}{(s + \lambda)^2 + \lambda V\sigma} = \frac{y(y + \lambda V\rho)}{y^2 + \lambda V\sigma}, \quad y = s + \lambda. \tag{41}$$

Using this representation, together with the relation (17), it follows that:

$$y^2 + (a_n - (\mu + \lambda))y - \lambda V\sigma = 0, \quad a_n = VD(2n + 1)^2\pi^2/4L^2. \tag{42}$$

The roots of Eq (42) can be written as:

$$y_n^{(\pm)} = \frac{-(a_n - (\mu + \lambda)) \pm \sqrt{(a_n - (\mu + \lambda))^2 + 4\lambda V\sigma}}{2}, \tag{43}$$

and they satisfy the relationship $s_{n,\pm} = y_n^{(\pm)} - \lambda$. Now, it is possible to establish the following bound:

$$0 < y_n^{(+)} \leq \frac{2\lambda V\sigma}{a_n}, \quad y_n^{(-)} \leq -\frac{a_n}{2}. \tag{44}$$

Therefore, it can be shown that (using elementary inequalities):

$$\left|\frac{(s_{n,+} + \lambda)(s_{n,+} + \lambda(1 + V\rho))}{(s_{n,+} + \lambda)^2 + \lambda V\sigma}\right| = \frac{y_n^{(+)}(y_n^{(+)} + \lambda V\rho)}{\left(y_n^{(+)}\right)^2 + \lambda V\sigma} \leq \frac{2\lambda V\rho}{a_n} + \frac{4\lambda V\sigma}{a_n^2}, \tag{45}$$

and

$$\left|\frac{(s_{n,-} + \lambda)(s_{n,-} + \lambda(1 + V\rho))}{(s_{n,-} + \lambda)^2 + \lambda V\sigma}\right| = \frac{y_n^{(-)}(y_n^{(-)} + \lambda V\rho)}{\left(y_n^{(-)}\right)^2 + \lambda V\sigma} \leq c_1, \tag{46}$$

where $c_1 = 1 + 2\lambda V\rho/a_0$, with $a_0 = VD\pi^2/4L^2$. One of the crucial steps is showing that the exponential terms are bounded. For this task it can be shown that

$$s_{n,+} = y_n^{(+)} - \lambda \leq -\frac{\lambda}{2}, \quad (n \geq N), \tag{47}$$

for some sufficiently large N, and:

$$s_{n,-} = y_n^{(-)} - \lambda \leq -\frac{a_n}{2} - \frac{\lambda}{2} \leq c_2(2n + 1)^2, \tag{48}$$

where $c_2 = VD\pi^2/16L^2 > 0$. These inequalities show that, for $t \geq 0$, the exponential factors can be uniformly bounded as:

$$e^{s_{n,+}t} \leq e^{-\frac{\lambda}{2}t}, \quad e^{s_{n,-}t} \leq e^{-c_2(2n+1)^2 t}. \tag{49}$$

6. Verification and results

This section is divided mainly into two parts. The first part verifies the given ICs&BCs through the obtained solution as a vital/mandatory task before launching to the second part, which focuses on the behavior of the system with some numerical results.

6.1. Verification of the ICs&BCs

This subsection shows theoretically that our solutions for $\phi(x,t)$ and $C(x,t)$ given in (35) and (39), respectively (along with $k(x,n)$ defined in (32) and A_n and B_n given in (36)), satisfy the given ICs&BCs (3,4). To make such a verification as clear as possible, we rewrite the solutions (35) and (39) in the forms:

$$\begin{cases} \phi(x,t) = \sum_{n=0}^{\infty} k(x,n) T_n(t), \\ C(x,t) = \rho\phi_0 e^{-\lambda t} + \sigma \sum_{n=0}^{\infty} k(x,n)\tau_n(t), \end{cases} \tag{50}$$

such that

$$\begin{cases} k(x,n) = \frac{4(-1)^n\phi_0}{(2n+1)\pi} \cos\left[\frac{(2n+1)\pi}{2L}(L-x)\right], \\ T_n(t) = A_n e^{s_{n,+}t} + B_n e^{s_{n,-}t}, \\ \tau_n(t) = \frac{A_n}{s_{n,+}+\lambda} e^{s_{n,+}t} + \frac{B_n}{s_{n,-}+\lambda} e^{s_{n,-}t} - \left(\frac{A_n}{s_{n,+}+\lambda} + \frac{B_n}{s_{n,-}+\lambda}\right) e^{-\lambda t}, \\ s_{n,\pm} = \frac{1}{2}\left[E_n \pm \sqrt{E_n^2 + 4F_n}\right], \\ E_n = \mu - \lambda - (2n+1)^2 VD\frac{\pi^2}{4L^2}, \quad F_n = \lambda\left[\mu + V\sigma - (2n+1)^2 VD\frac{\pi^2}{4L^2}\right]. \end{cases} \tag{51}$$

It can be directly seen from Eqs (50) and (51) that the conditions $\phi(x,t)\mid_{x=0} = 0$ and $\frac{\partial\phi(x,t)}{\partial x}\mid_{x=L} = 0$, where $k(x,n)\mid_{x=0} = 0$ and $\frac{\partial k(x,n)}{\partial x}\mid_{x=L} = 0\ \forall\ n = 0,1,2,\ldots$. Although the conditions $\phi(x,t)\mid_{x=0} = 0$ and $\frac{\partial\phi(x,t)}{\partial x}\mid_{x=L} = 0$ can be verified in a straightforward way; the verification of the IC $\phi(x,0) = \phi_0$ is not an easy task due to the difficulty of calculating the infinite sum of spatial functions in x over the interval $0 < x \leq L$. We have from Eq (50) that

$$\phi(x,t)\mid_{t=0} = \phi(x,0) = \sum_{n=0}^{\infty} k(x,n) T_n(0). \tag{52}$$

From Eq (51), we have

$$T_n(0) = A_n + B_n, \tag{53}$$

where A_n and B_n are already given by Eq (36). By algebraic manipulation, one can prove that

$$T_n(0) = A_n + B_n = 1. \tag{54}$$

Accordingly, Eq (52) becomes

$$\phi(x,0) = \sum_{n=0}^{\infty} k(x,n). \tag{55}$$

In the following steps, it is indicated that the infinite sum of the spatial functions $k(x,n)$ approaches the constant value ϕ_0 over the interval $0 < x \leq L$. Employing $k(x,n)$ in (32), then

$$\phi(x,0) = \frac{4\phi_0}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} \cos\left[\frac{(2n+1)\pi}{2L}(L-x)\right], \tag{56}$$

or

$$\phi(x,0) = \phi_0 f(x), \tag{57}$$

where

$$f(x) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}\cos\left[\frac{(2n+1)\pi}{2L}(L-x)\right]. \tag{58}$$

Now, our target is to show that $f(x) \to 1$ over the domain $0 < x \leq L$. One can theoretically find that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}\cos\left[\frac{(2n+1)\pi}{2L}(L-x)\right] &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}\cos\left[(2n+1)\left(\frac{\pi}{2}-a\right)\right], \quad a = \frac{\pi x}{2L}, \\ &= \sum_{n=0}^{\infty}\frac{\sin((2n+1)a)}{(2n+1)}.\end{aligned} \tag{59}$$

Introducing the Abel's parameter $(0 < r < 1)$, it follows that

$$\sum_{n=0}^{\infty}\frac{\sin((2n+1)a)}{(2n+1)} = \lim_{r\to1^-}\text{Im}\left(\sum_{n=0}^{\infty}\frac{r^{2n+1}e^{i(2n+1)a}}{2n+1}\right) = \lim_{r\to1^-}\text{Im}\left(\frac{1}{2}\ln\left(\frac{1+re^{ia}}{1-re^{ia}}\right)\right), \tag{60}$$

which by continuity gives

$$\sum_{n=0}^{\infty}\frac{\sin((2n+1)a)}{(2n+1)} = \text{Im}\left(\frac{1}{2}\ln\left(\frac{1+e^{ia}}{1-e^{ia}}\right)\right). \tag{61}$$

Applying the imaginary part and considering the sign of $\sin(a) = +$ (where $\sin(a)$ is always positive over the problem's domain $0 < x \leq L$), then

$$\sum_{n=0}^{\infty}\frac{\sin((2n+1)a)}{(2n+1)} = \frac{1}{2}\arg\left(\frac{e^{-\frac{ia}{2}}+e^{\frac{ia}{2}}}{e^{-\frac{ia}{2}}-e^{\frac{ia}{2}}}\right) = \frac{1}{2}\arg\left(i\cot\left(\frac{a}{2}\right)\right) = \frac{\pi}{4}\text{sgn}(\sin(a)) = \frac{\pi}{4}. \tag{62}$$

In this case, Abel's theorem guarantees the convergence (see Ref. [33, page 9]). Thus, $f(x) = 1$ which implies that $\phi(x,0) = \phi_0$. Also, it is easy to verify that the IC $C(x,0) = \rho\phi_0$ is satisfied. As a final note, it can be verified that the present closed-form series solutions for $\phi(x,t)$ and $C(x,t)$ satisfy the system of PDEs (1-2) if the infinity is replaced by any finite number of terms.

6.2. Numerical results and behavior of the system

Numerically, one can show that $f(x) \to 1$ over the domain $0 < x \leq L$. For this purpose, let us approximate the infinity in (58) by a finite number N of terms to give the finite sum:

$$S_N(x) = \frac{4}{\pi}\sum_{n=0}^{N}\frac{(-1)^n}{(2n+1)}\cos\left[\frac{(2n+1)\pi}{2L}(L-x)\right]. \tag{63}$$

Hence,

$$f(x) = \lim_{N\to\infty} S_N(x). \tag{64}$$

Our target is now moved to prove that $S_N(x) \to 1$ as $N \to \infty$; this yields $f(x) \to 1$. Consider $L = 22.9$ cm as taken in Ref. [2]; the curves of $S_N(x)$ are displayed in Figures 1–3 at different values of N. It is observed from Figures 1 and 2 that $S_N(x)$ oscillates about 1 at relatively low numbers of terms. As N is increased, one can observe in Figure 3 that $S_N(x) \to 1$, i.e., $\lim_{N\to\infty} S_N(x)$ over the domain $0 < x \leq L$. This confirms that $f(x) \to 1$ as $N \to \infty$ and consequently, the satisfaction of the IC $\phi(x, 0) = \phi_0$. This conclusion can also be confirmed through performing calculations at other values for L, such as $L = 160$ cm [34, page 62]; see Figure 4.

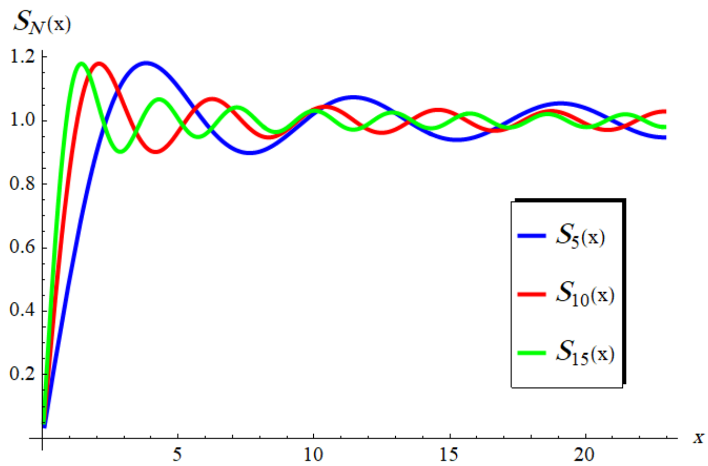

Figure 1. Plots of approximations $S_N(x)$ ($N = 5, 10, 15$) over the domain $0 < x \leq L$, $L = 22.9$ cm.

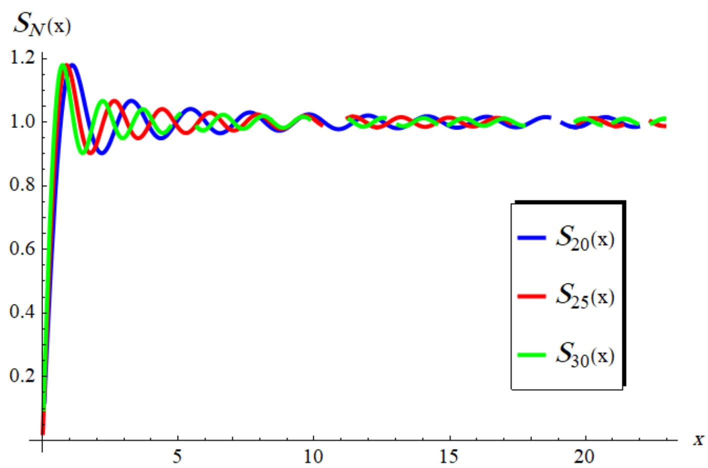

Figure 2. Plots of approximations $S_N(x)$ ($N = 20, 25, 30$) over the domain $0 < x \leq L$, $L = 22.9$ cm.

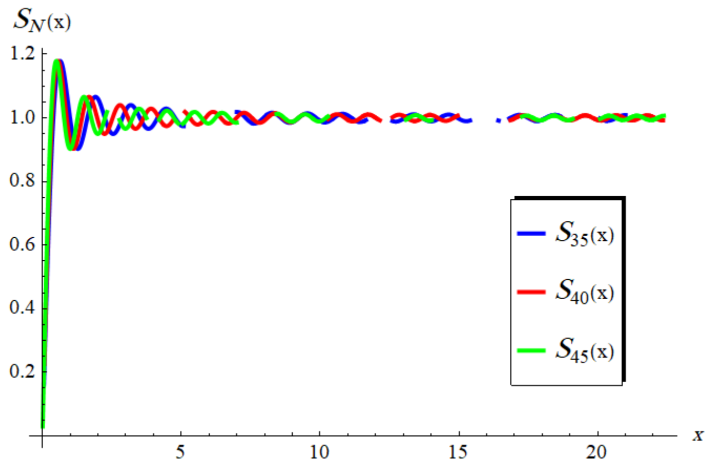

Figure 3. Plots of approximations $S_N(x)$ ($N = 35, 40, 45$) over the domain $0 < x \leq L$, $L = 22.9$ cm.

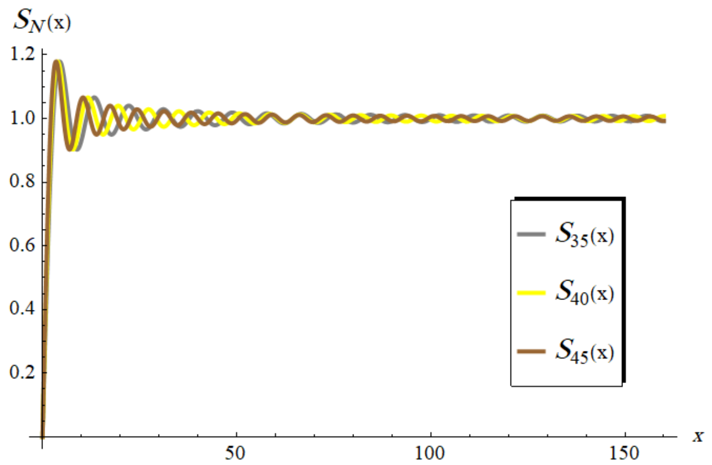

Figure 4. Plots of approximations $S_N(x)$ ($N = 35, 40, 45$) over the domain $0 < x \leq L$, $L = 160$ cm [34].

This subsection also extracts some numerical results about the behavior of the neutron flux and the delayed neutron precursor concentration. In order to do that, the infinite sums appearing in Eq (50) should be replaced with a finite number of terms. Regarding, the m-term approximations of $\phi(x, t)$ and $C(x, t)$ are defined as

$$\phi(x, t) \approx \sum_{n=0}^{m} k(x, n) T_n(t), \quad C(x, t) \approx \rho \phi_0 e^{-\lambda t} + \sigma \sum_{n=0}^{m} k(x, n) \tau_n(t). \tag{65}$$

The accuracy of the above approximations can be estimated by calculating the residuals from the governing PDEs (1-2) as

$$RE_\phi = \left| \frac{1}{V} \frac{\partial \phi}{\partial t} - D \frac{\partial^2 \phi}{\partial x^2} - \left(- \sum\nolimits_a + (1 - \beta) \nu \sum\nolimits_f \right) \phi(x, t) - \lambda C(x, t) \right|, \tag{66}$$

$$RE_C = \left| \frac{\partial C}{\partial t} - \beta \nu \sum\nolimits_f \phi(x,t) + \lambda C(x,t) \right|. \quad (67)$$

The following parameter values are implemented, as considered in Ref. [2], to produce the numerical results: $D = 0.96343$ [cm], $V = 1.103497 \times 10^7$ [cm/s], $\sum_a = 1.58430 \times 10^{-2}$ [1/cm], $\nu \sum_f = 3.33029 \times 10^{-2}$ [1/cm], $L = 22.9$ [cm], $\beta = 0.0045$, and $\lambda = 0.08$ [1/s].

In Figures 5 and 6, the residuals RE_ϕ and RE_C are depicted using $m = 700$ to ensure accuracy of the approximations. It is obvious from these figures that the obtained residuals reflect acceptable accuracy. Hence, the present approach may be viewed as an effective tool to deal with the system (1)–(4) with acceptable accuracy. Here, it should also be noted that the number of terms m can be increased to achieve the desired accuracy. This point can be seen from the plots in Figures 7 and 8 for the residuals RE_ϕ and RE_C at different values of m when $t = 10$ and $L = 22.9$ cm. Figures 9 and 10 illustrate the surface plots of the neutron flux and the delayed neutron precursor concentration, respectively, over the domain $0 < x \leq 22.9$ and $0 < t \leq 100$. Figure 9 shows that the neutron flux exhibits oscillatory behavior within the spatial domain, resulting in a wave-like, alternating surface. This differs from the findings in Ref. [2], where a purely asymptotic decay was observed under the boundary condition $\phi(L,t) = 0$. The current oscillatory pattern arises due to the boundary condition $\frac{\partial \phi}{\partial x}(L,t) = 0$, which represents a reflective boundary at one edge of the reactor. Physically, this condition implies that no neutron current exists in the domain at the edge $x = L$, leading to partial reflection of neutron flux and constructive/destructive interference of neutron modes. This reflective effect gives rise to non-uniform oscillations that are fully consistent with the reactor physics under such boundary conditions.

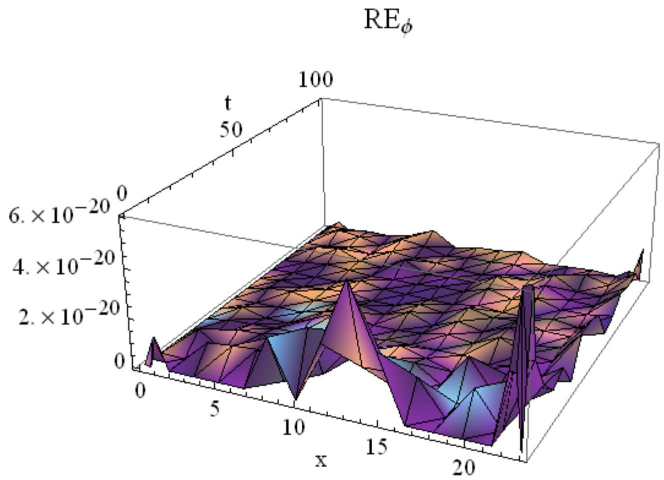

Figure 5. Plots of the residual RE_ϕ (Eq (66)) at $m = 700$.

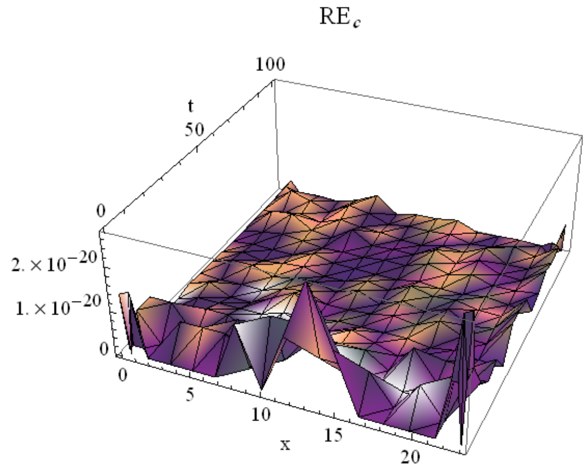

Figure 6. Plots of the residual RE_C (Eq (67)) at $m = 700$.

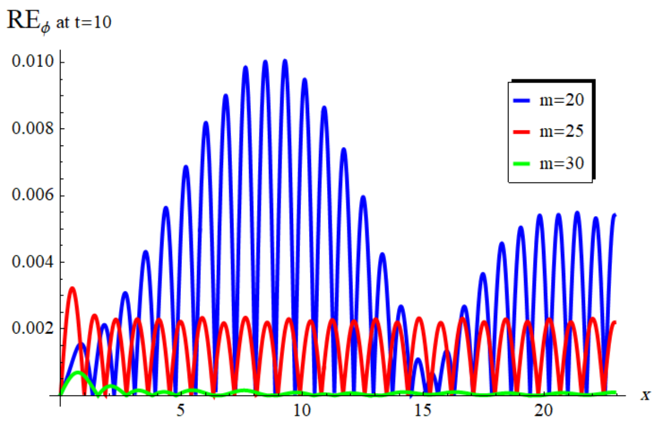

Figure 7. Plots of the residual RE_ϕ (Eq (66)) at different values of m when $t = 10$, $L = 22.9$ cm.

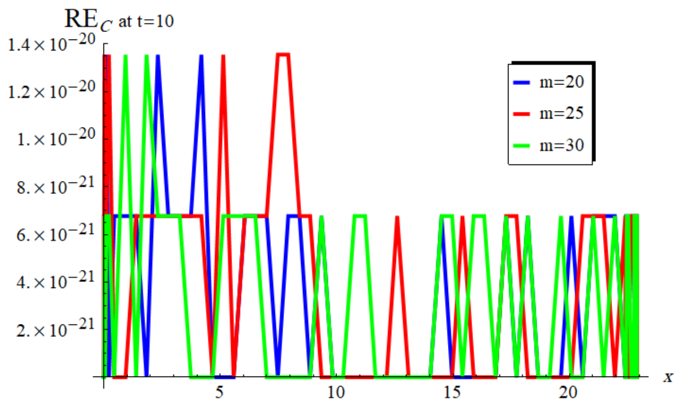

Figure 8. Plots of the residual RE_C (Eq (67)) at different values of m when $t = 10$, $L = 22.9$ cm.

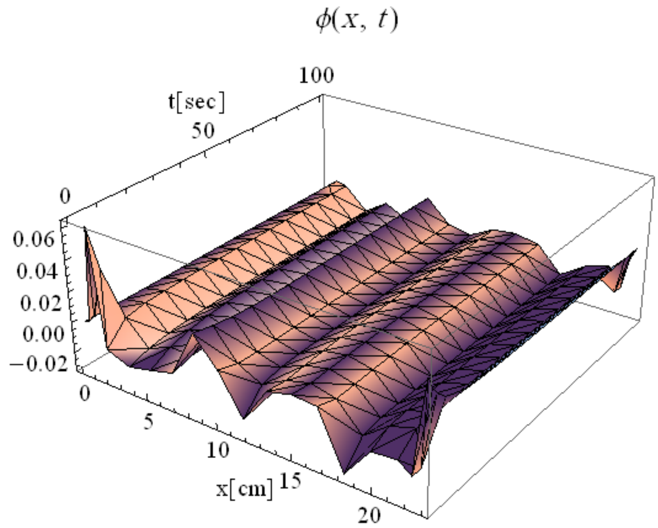

Figure 9. Behavior of the neutron flux $\phi(x, t)$.

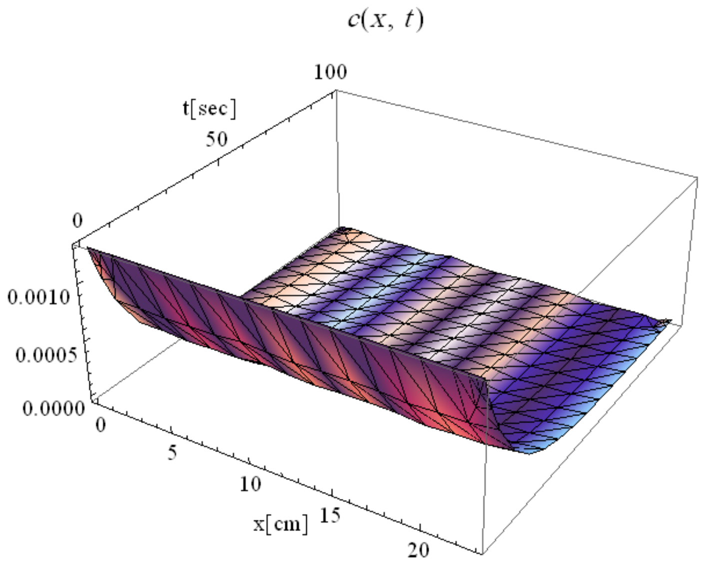

Figure 10. Behavior of the precursor concentration flux $C(x, t)$.

In contrast, Figure 10 shows that the delayed neutron precursor concentration stabilizes over time, gradually approaching a stationary profile. This behavior is consistent with both the physical expectation and the results of Ref. [2], owing to the fact that, regardless of boundary conditions, the delayed neutron precursor concentration tends to a steady state once the balance between its production (driven by the neutron flux) and its radioactive decay is achieved. As a final remark, it is worth noting that while Khaled [2] encountered computational issues with Mathematica at $t = 0$, the present approach successfully overcomes these difficulties. Consequently, no such numerical problems arise in this work, owing to the reflective boundary condition imposed at one edge of the reactor.

7. Conclusions

In this paper, the Laplace Transform (LT) technique was applied to solve the neutron diffusion system under mixed boundary conditions in contrast with the approach of Khaled [2], where the condition $\phi(L,t) = 0$ was replaced by $\partial\phi(L,t) = 0$, which physically corresponds to modelling a reflected reactor rather than a bare one. The transformed system involves an infinite number of poles determined by a transcendental equation. The residue method was then applied to evaluate the residues at these poles, and consequently, the inverse LT was obtained. Accordingly, an explicit analytical solution was derived for the present system. The resulting expressions for the neutron flux and the delayed neutron precursor concentration were established in closed-form series representations. The convergence of the series solution was theoretically proven. Moreover, it was verified that the obtained solutions satisfy the prescribed initial and boundary conditions. Both theoretical and numerical analyses were performed to examine the solution and its physical behavior.

From a physical standpoint, this work revealed oscillatory patterns, distinctive wave-like distributions, in the neutron flux, in contrast to the monotonic decay reported by Khaled [2] under conventional vacuum boundary conditions. These findings provide a novel physical insight relevant to reactor safety analysis. At the same time, the study confirmed that the delayed neutron precursor concentration gradually stabilizes over time. Its equilibrium behavior is aligned with the predictions of nuclear system dynamics, further validating the physical rationality of the current model. Finally, the results reveal that the proposed analysis is straightforward and effective. Moreover, the present analysis may deserve further considerations to include other complex ICs/BCs as future works.

Author contributions

E. R. El-Zahar: Conceptualization, methodology, validation, formal analysis, investigation, writing–review and editing, visualization; A. Ebaid: Conceptualization, methodology, validation, formal analysis, investigation, writing–review and editing; L. F. Seddek: Conceptualization, methodology, software, validation, formal analysis, investigation, data curation, writing–review and editing; S. M. Khaled: Conceptualization, methodology, software, validation, formal analysis, investigation, data curation, writing–review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

References

1. C. Ceolin, M. T. Vilhena, S. B. Leite, C. Z. Petersen, *An analytical solution of the one-dimensional neutron diffusion kinetic equation in cartesian geometry*, In: International Nuclear Atlantic Conference – INAC 2009, Rio de Janeiro, RJ, Brazil, September 27 to October 2, 2009.
2. S. M. Khaled, Exact solution of the one-dimensional neutron diffusion kinetic equation with one delayed precursor concentration in Cartesian geometry, *AIMS Math.*, **7** (2022), 12364–12373. https://doi.org/10.3934/math.2022686
3. J. R. Lamarsh, A. J. Baratta, *Introduction to nuclear engineering*, 3 Eds., Prentice Hall, 2001.
4. S. Dulla, P. Ravetto, P. Picca, D. Tomatis, *Analytical benchmarks for the kinetics of accelerator driven systems*, In: Joint International Topical Meeting on Mathematics & Computation and Supercomputing in Nuclear Applications, Monterey-California, on CD-ROM, 2007.
5. A. A. Nahla, M. F. Al-Ghamdi, Generalization of the analytical exponential model for homogeneous reactor kinetics equations, *J. Appl. Math.*, 2012. https://doi.org/10.1155/2012/282367
6. F. Tumelero, C. M. F. Lapa, B. E. J. Bodmann, M. T. Vilhena, Analytical representation of the solution of the space kinetic diffusion equation in a one-dimensional and homogeneous domain, *Braz. J. Radiat. Sci.*, **7** (2019), 1–13. https://doi.org/10.15392/bjrs.v7i2B.389
7. A. A. Nahla, F. Al-Malki, M. Rokaya, Numerical techniques for the neutron diffusion equations in the nuclear reactors, *Adv. Stud. Theor. Phys.*, **6** (2012), 649–664.
8. S. M. Khaled, Power excursion of the training and research reactor of Budapest University, *Int. J. Nucl. Energy Sci. Technol.*, **3** (2007), 42–62. https://doi.org/10.1504/IJNEST.2007.012440
9. S. M. Khaled, F. A. Mutairi, The influence of different hydraulics models in treatment of some physical processes in super critical states of light water reactors, *Int. J. Nucl. Energy Sci. Technol.*, **8** (2014), 290–309. https://doi.org/10.1504/IJNEST.2014.064940
10. M. A. Al-Sharif, A. Ebaid, H. S. Alrashdi, A. H. Alenazy, N. E. Kanaan, A novel ansatz method for solving the neutron diffusion system in Cartesian geometry, *J. Adv. Math. Comput. Sci.*, **37** (2022), 90–99. https://doi.org/10.9734/jamcs/2022/v37i111723
11. H. K. Al-Jeaid, A simple approach for explicit solution of the neutron diffusion kinetic system, *Math. Stat.*, **11** (2023), 107–116, 2023. https://doi.org/10.13189/ms.2023.110112
12. T. M. T. Alsubie, A. Ebaid, A. S. S. Albalawi, S. A. Alghamdi, F. F. M. Alhamdi, O. S. H. Alhamd, et al., Developing ansatz method for solving the neutron diffusion system under general physical conditions, *Adv. Differ. Equ. Contr.*, **31** (2024), 15–26. http://dx.doi.org/10.17654/0974324324002
13. N. Dogan, Solution of the system of ordinary differential equations by combined Laplace transform-Adomian decomposition method, *Math. Comput. Appl.*, **17** (2012), 203–211. https://doi.org/10.3390/mca17030203

14. P. V. Pavani, U. L. Priya, B. A. Reddy, Solving differential equations by using Laplace transforms, *Int. J. Res. Anal. Rev.*, **5** (2018), 1796–1799.

15. A. B. Albidah, A proposed analytical and numerical treatment for the nonlinear SIR Model via a hybrid approach, *Mathematics*, **11** (2023), 2749. https://doi.org/10.3390/math11122749

16. S. M. Khaled, The exact effects of radiation and joule heating on magnetohydrodynamic Marangoni convection over a flat surface, *Therm. Sci.*, **22** (2018), 63–72. https://doi.org/10.2298/TSCI151005050K

17. A. F. Aljohani, A. Ebaid, E. H. Aly, I. Pop, A. O. M. Abubaker, D. J. Alanazi, Explicit solution of a generalized mathematical model for the solar collector/photovoltaic applications using nanoparticles, *Alex. Eng. J.*, **67** (2023), 447–459. https://doi.org/10.1016/j.aej.2022.12.044

18. M. D. Aljoufi, Exact solution of a solar energy model using four different kinds of nanofluids: Advanced application of Laplace transform, *Case Stud. Therm. Eng.*, **50** (2023), 103396. https://doi.org/10.1016/j.csite.2023.103396

19. A. F. Aljohani, A. Ebaid, E. A. Algehyne, Y. M. Mahrous, P. Agarwal, M. Areshi, et al., On solving the chlorine transport model via Laplace transform, *Sci. Rep.*, **12** (2022), 12154. https://doi.org/10.1038/s41598-022-14655-3

20. A. F. Aljohani, A. Ebaid, E. A. Algehyne, Y. M. Mahrous, C. Cattani, H. K. Al-Jeaid, The Mittag-Leffler function for re-evaluating the chlorine transport model: Comparative analysis, *Fractal Fract.*, **6** (2022), 125. https://doi.org/10.3390/fractalfract6030125

21. Y. Liu, K. Sun, Solving power system differential algebraic equations using differential transformation, *IEEE T. Power Syst.*, **35** (2020), 2289–2299. https://doi.org/10.1109/TPWRS.2019.2945512

22. S. Liao, *Beyond perturbation: Introduction to the homotopy analysis method*, CRC Press, Boca Raton, FL, USA, 2003.

23. A. Chauhan, R. Arora, Application of homotopy analysis method (HAM) to the non-linear KdV equations, *Commun. Math.*, **31** (2023), 205–220. https://doi.org/10.46298/cm.10336

24. M. Nadeem, F. Li, He-Laplace method for nonlinear vibration systems and nonlinear wave equations, *J. Low Freq. Noise V. A.*, **38** (2019), 1060–1074. https://doi.org/10.1177/1461348418818973

25. M. Nadeem, S. A. Edalatpanah, I. Mahariq, W. H. F. Aly, Analytical view of nonlinear delay differential equations using Sawi iterative scheme, *Symmetry*, **14** (2022), 2430. https://doi.org/10.3390/sym14112430

26. Z. Ayati, J. Biazar, On the convergence of Homotopy perturbation method, *J. Egypt. Math. Soc.*, **23** (2015), 424–428. https://doi.org/10.1016/j.joems.2014.06.015

27. J. S. Duan, R. Rach, A new modification of the Adomian decomposition method for solving boundary value problems for higher order nonlinear differential equations, *Appl. Math. Comput.*, **218** (2011), 4090–4118. https://doi.org/10.1016/j.amc.2011.09.037

28. S. Bhalekar, J. Patade, An analytical solution of fishers equation using decomposition method, *Am. J. Comput. Appl. Math.*, **6** (2016), 123–127.

29. A. H. S. Alenazy, A. Ebaid, E. A. Algehyne, H. K. Al-Jeaid, Advanced study on the delay differential equation $y'(t) = ay(t) + by(ct)$, *Mathematics*, **10** (2022), 4302. https://doi.org/10.3390/math10224302

30. C. A. Cruz-López, G. Espinosa-Paredes, A new simplified analytical solution to solve the neutron point kinetics equations using the Laplace transform method, *Comput. Phys. Commun.*, **283** (2023), 108564. https://doi.org/10.1016/j.cpc.2022.108564

31. C. A. Cruz-López, G. Espinosa-Paredes, A new solution of the fractional neutron point kinetics equations using symmetry and the Heaviside's expansion formula, *Prog. Nucl. Energ.*, **171** (2024). https://doi.org/10.1016/j.pnucene.2024.105168

32. J. L. Schiff, *The Laplace transform: Theory and applications*, Springer Science & Business Media, 1999.

33. E. C. Titchmarsh, The theory of functions, 2 Eds., Oxford University Press, 1939.

34. C. Ceolin, M. T. Vilhena, B. E. J. Bodmann, *On the analytical solution of the multi-group neutron diffusion kinetic equation in one-dimensional Cartesian geometry by an integral transform technique*, In: Constanda, C., Harris, P. (eds) Integral Methods in Science and Engineering, Birkhäuser Boston, 2011. https://doi.org/10.1007/978-0-8176-8238-5_7