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Research article

An SMC strategy for neutral-type systems with time-varying delays using improved integral inequality and practical applications

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Abstract: In this paper, we investigates the sliding mode control (SMC) strategy for neutral-type systems with distributed time-varying delay using novel improved integral inequality, which has significant applications in fields such as control systems, communication networks, and biological systems. A standard Lyapunov-Krasovskii functional is introduced, complemented by improved integral inequality techniques and two types of time-delay methods (neutral type and distributed). These methodologies enabled the derivation of sufficient conditions, formulated as linear matrix inequalities, that ensured the asymptotic stability of the system utilizing the SMC technique. The proposed approach reduced conservatism in stability criteria by investigating improved integral inequalities and delay-dependent techniques, offering more accurate and efficient stability conditions. Numerical examples are presented to validate the theoretical findings with the practical application of partial element equivalent circuit (PEEC), showcasing the effectiveness and superiority of the proposed methodology over existing results.

Keywords: time-varying delay; linear matrix inequality; Lyapunov-Krasovskii functional; sliding-mode control; improved integral inequality
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1. Introduction

As communication network technology advances, closed-loop control systems are widely implemented in networked contexts. Nonetheless, inevitable delays in communication channels can substantially impair system performance and, in key instances, result in instability [10, 33]. Mitigating these delays is crucial for ensuring instability and attaining optimal performance in contemporary control systems. In practical systems, delays frequently occur due to sensing, computing, and transmission operations, with transmission delays usually being the most significant. Neutral-type systems, distinguished by delays in both the state and its derivative, commonly manifest in uncertain neural networks, multi-agent coordination, and switching systems with mixed delays [23–25]. These delays induce memory effects that hamper stability analysis, frequently increasing oscillations and degrading transient behavior. For example, He et al. [9] investigated the stability of uncertain neutral systems with mixed delays. Research has enhanced the stability and control of neutral-type systems characterized by delays. Aghayan et al. [1] proposed a robust delay-dependent PD controller for uncertain variable fractional-order neutral systems characterized by time-varying delays. Li et al. [11] proposed the stability of stochastic neutral systems characterized by multiple delays. Sharma et al. [22] investigated controllability in impulsive fractional stochastic integro-differential systems. Li et al. [13] examined local stability in sampled-data neutral systems considering actuator saturation constraints.

Distributed delays, commonly represented as finite integrals across historical intervals, inherently occur in several engineering and physical systems, including process control, population dynamics, and electromagnetic field modeling [4]. In these applications, delays arise from the transmission and propagation of signals or materials, as observed in communication networks, smart grids, and partial element equivalent circuit (PEEC) models utilized in electromagnetic simulations. The presence of dispersed and time-varying delays presents significant analytical difficulties, sometimes leading to excessively cautious stability requirements. As a result, several researchers have introduced advanced delay-dependent methodologies employing segmentation and decomposition techniques to increase precision in stability evaluation [8, 17, 36]. These advancements underscore the increasing need for sophisticated control techniques that guarantee resilient stability and superior performance in systems influenced by intricate and variable delays. The study of neutral-type systems with delays has progressed recently. With applications to water pollution control, Shanmugam et al. [17] examined the finite-time boundedness of switching delay systems under actuator saturation. Moreover, to accommodate time-varying delays in real-world scenarios, Shanmugam et al. [18] highlighted resilient H_∞ performance for neutral-type neural networks. To improve finite-time passivity analysis, Saravanan et al. [16] used integral inequalities based on auxiliary functions. Neutral-type neural networks with random time-varying delays were studied for robust finite-time stability in stochastic contexts [2]. Stabilization techniques for uncertain switched neutral systems with interval time-varying delays were proposed by the researchers in [7]. Together, these studies highlight the need for advanced approaches to enhance neutral-type systems' performance, stability, and robustness in dynamic and unpredictable circumstances.

Sliding mode control (SMC) is an effective method for addressing these difficulties because of its inherent resilience to parameter uncertainty and external disruptions [26, 28]. In recent decades, SMC has been effectively utilized across categories of systems characterized by delays and uncertainties, encompassing Markovian jump systems, adaptive and observer-based frameworks, and event-triggered schemes that minimize computational expenses [6, 20]. Nevertheless these advancements, the proper integration of SMC design with less conservative delay-dependent stability requirements, continues to be an unresolved issue, especially for neutral-type systems exhibiting scattered delays. In practical applications, like networked control, robotics, and electromagnetic circuit modeling, attaining rapid convergence and dependable performance amidst various delay effects remains a significant research impetus [37, 39].

An essential analytical difficulty is to precisely estimate the impacts of delay terms without imposing unwarranted conservatism. Traditional inequalities, like Jensen's and Wirtinger's methods, frequently yield imprecise limits and constrain optimal performance [19,34]. Advancements in integral inequalities, including free-matrix-based, Bessel-Legendre, and generalized reciprocally convex inequalities [15, 21], provide more tighter estimates and facilitate enhanced delay-dependent stability analysis [35]. Incorporating these approaches into Lyapunov-Krasovskii functional (LKF) results in enhanced linear matrix inequality (LMI) conditions characterized by diminished conservatism and better analytical precision. Recent improvements have enhanced the examination of stability in systems characterized by delays. Wang et al. [29] proposed relaxed stability criteria utilizing delay-derivative-dependent slack matrices, thus broadening the stability region while maintaining the computational effectiveness. Wang et al. [27] examined stability in systems characterized by periodically varying delays, utilizing advanced non-continuous piecewise Lyapunov functionals to achieve improved analytical accuracy. The delay-derivative-dependent switched system model method reported in [32] provides an effective way to reduce conservatism for time-delay systems. The work by the researchers in [40] enhances the stability assessment of sampled-data systems through the formulation of novel delay-dependent criteria that consider the effects of sampling and associations with time delays.

As far as we know, few researchers consider the proposed model, including all types of delays (distributed, neutral, and etc.), with sliding mode control. Based on the discussions, we establish a sliding mode control framework for neutral-type distributed delay systems characterized by time-varying delays. By including an enhanced integral inequality into the Lyapunov-Krasovskii stability analysis, fewer conservative delay-dependent criteria are formulated. The suggested design offers a methodical approach to ascertain the sliding gain K, guaranteeing finite-time attainment of the sliding surface and strong asymptotic stability. The proposed model is further validated through a PEEC-based example, showcasing enhanced performance and robustness relative to previous methodologies. This paper's main contributions are summarized as follows:

- We develop a category of neutral-type distributed systems and devise a sliding mode control approach to tackle time-varying delays.
- A designed LKF is utilized to thoroughly encapsulate discrete, neutral, and distributed delay influences.
- Enhanced integral inequalities (Lemmas 1.1 and 1.2) are employed to provide more stringent delay-dependent LMI-based stability criteria with less conservatism.
- A systematic SMC design is formulated to calculate the control gain K, guaranteeing the asymptotic stability.
- The practical usability and superiority of the suggested method are validated using the PEEC model, demonstrating improved stability and performance amid time-varying delays.

Let us consider the following neutral system with a distributed time-delay:

$$\begin{cases} \dot{\varpi}(t) & = A\varpi(t) + B\varpi(t-\tau(t)) + C\dot{\varpi}(t-d) + D\int_{t-h}^{t}\varpi(s)ds + Eu(t), \quad \forall t \geq 0, \\ \varpi(t) & = \phi(t), \quad t \in [\tau, d, h, 0], \end{cases} \tag{1.1}$$

where $\varpi(t) \in \mathbb{R}^n$ denotes the state vector, $u(t) \in \mathbb{R}^m$ denotes the control input, and the initial condition $\phi(t)$ is a continuously differentiable vector-valued function; $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{n\times n}$, $D \in \mathbb{R}^{n\times n}$, and $E \in \mathbb{R}^{n\times m}$ are known matrices; $\tau(t)$ is a time-varying delay with the condition $0 \leq \tau(t) \leq \tau$, $\dot{\tau}(t) \leq \mu$; and d and h are a neutral delay and distributed delay, respectively, with a positive constant.

Our goal is to develop an SMC law that ensures the state trajectories converge to the desired sliding surface and stay on it thereafter. From this result, we choose the sliding surface $s(t)$ as

$$s(t) = H\Big(\varpi(t) - C\varpi(t-d)\Big) - H\int_0^t \Big[(A+EK)\varpi(\theta) - B\varpi(\theta-\tau(\theta)) - D\int_{\theta-h}^{\theta}\varpi(s)ds\Big]d\theta, \tag{1.2}$$

where K and H are real matrices to be determined later. H is designed such that HE is a non-singular.

To design SMC: The derivative of the $s(t)$ from (1.2) is

$$\begin{aligned} \dot{s}(t) &= H\Big(\dot{\varpi}(t) - C\dot{\varpi}(t-d)\Big) - H(A+EK)\varpi(t) - HB\varpi(t-\tau(t)) - HD\int_{t-h}^{t}\varpi(s)ds, \\ &= -HEK\varpi(t) + HEu(t). \end{aligned} \tag{1.3}$$

As the state trajectories approach the sliding mode, both $s(t) = 0$ and $\dot{s}(t) = 0$ hold. Hence, the system is governed by the equivalent controller:

$$u_{eq}(t) = K\varpi(t). \tag{1.4}$$

When we enforce $\dot{s}(t) = 0$, the control $u(t)$ that satisfies this condition is called the equivalent control, denoted by $u_{eq}(t)$, and we substitute $u_{eq}(t)$ (1.4) into system (1.1)

$$\dot{\varpi}(t) = (A+EK)\varpi(t) + B\varpi(t-\tau(t)) + C\dot{\varpi}(t-d) + D\int_{t-h}^{t}\varpi(s)ds, \quad \forall t \geq 0. \tag{1.5}$$

Next, we present key lemmas that establish essential conditions for analyzing the system's behavior. These results are foundational for proving the main theorems of the paper.

Lemma 1.1. *[38] Let $\varpi(t)$ be any continuously differentiable function, and let Q_4 be a positive definite symmetric matrix that satisfies the following inequality:*

$$-\int_{t-\tau}^{t}\dot{\varpi}^T(s)Q_4\dot{\varpi}(s)\,ds \leq \eta^T(t)\Xi\eta(t),$$

where

$$\Xi = \begin{bmatrix} \frac{-18Q_4}{\tau} & \frac{6Q_4}{\tau} & 0 & 0 & \frac{-96Q_4}{\tau^2} & 0 & \frac{480Q_4}{\tau^3} \\ * & \frac{-36Q_4}{\tau} & \frac{6Q_4}{\tau} & \frac{-96Q_4}{\tau^2} & \frac{144Q_4}{\tau^2} & \frac{480Q_4}{\tau^3} & \frac{-480Q_4}{\tau^3} \\ * & * & \frac{-18Q_4}{\tau} & \frac{144Q_4}{\tau^2} & 0 & \frac{-480Q_4}{\tau^3} & 0 \\ * & * & * & \frac{-1536Q_4}{\tau^3} & 0 & \frac{5760Q_4}{\tau^4} & 0 \\ * & * & * & * & \frac{-1536Q_4}{\tau^3} & 0 & \frac{5760Q_4}{\tau^4} \\ * & * & * & * & * & \frac{-23040Q_4}{\tau^5} & 0 \\ * & * & * & * & * & * & \frac{-23040Q_4}{\tau^5} \end{bmatrix},$$

$$\eta(t) = \Big[\varpi^T(t)\ \varpi^T(t-\frac{\tau}{2})\ \varpi^T(t-\tau)\ \int_{t-\tau}^{t-\frac{\tau}{2}} \varpi^T(s)\,ds\ \int_{t-\frac{\tau}{2}}^{t} \varpi^T(s)\,ds$$
$$\int_{t-\tau}^{t}\int_{u}^{t-\frac{\tau}{2}} \varpi^T(s)\,ds\,du\ \int_{t-\frac{\tau}{2}}^{t}\int_{u}^{t} \varpi^T(s)\,ds\,du\Big]^T.$$

Lemma 1.2. *[19] For a symmetric matrix R, scalars a and b, where $a > 0$, $b > 0$, and vector $\varpi \in [a, b]$, the corresponding integral is defined by the following inequality:*

$$(b-a)\int_a^b \dot{\varpi}^T(s)R\dot{\varpi}(s)ds \geq \chi_1^T R\chi_1 + 3\chi_2^T R\chi_2, \tag{1.6}$$

where

$$\chi_1 = \varpi(b) - \varpi(a),\ \chi_2 = \varpi(b) - \varpi(a) - \frac{2}{b-a}\int_a^b \varpi(s)ds.$$

2. Main results

In this section, we establish asymptotically stable criteria by constructing the corresponding positive definite LKF. We proceed by designing a SMC and deriving a control law that guarantees the trajectories of the time-delay system in (1.1) to converge to the predefined sliding surface $s(t) = 0$ within a finite time.

Theorem 2.1. *Consider K as the known matrix, and let τ, h, d, and μ be given scalars. System* (1.5) *is asymptotically stable if there exist positive definite matrices P, $Q_1, \ldots, Q_6$, and Z_2 that satisfy the following matrix inequalities:*

$$[\omega] = [\omega_{n\times n}],\ n = 1, 2, ..., 12 < 0, \tag{2.1}$$

where

$$\omega_{11} = Q_1 + Q_3 - \frac{18Q_4}{\tau} + 2Q_5 + Q_6 + F_1A + A^TF_1^T + F_1EK + K^TE^TF_1^T - \frac{1}{d}Z_2,$$
$$\omega_{13} = F_1C + \frac{1}{d}Z_2,\ \omega_{14} = \frac{6Q_4}{\tau},\ \omega_{16} = -\frac{96Q_4}{\tau},\ \omega_{18} = \frac{480Q_4}{\tau^3},\ \omega_{19} = 2P - F_1 + A^TF_1^T + K^TE^TF_1^T,$$
$$\omega_{1,10} = -2Q_5,\ \omega_{1,11} = \frac{6Q_5}{h} + F_1D,\ \omega_{1,12} = F_1B,\ \omega_{22} = -Q_1 - \frac{18Q_4}{\tau},\ \omega_{25} = \frac{144Q_4}{\tau^2},$$
$$\omega_{27} = -\frac{480Q_4}{\tau^3},\ \omega_{33} = -Q_2 - \frac{1}{d}Z_2,\ \omega_{39} = C^TF_1^T,\ \omega_{44} = -\frac{36Q_4}{\tau},\ \omega_{45} = -\frac{96Q_4}{\tau^2},\ \omega_{46} = \frac{144Q_4}{\tau^2},$$
$$\omega_{47} = \frac{480Q_4}{\tau^3},\ \omega_{48} = -\frac{480Q_4}{\tau^3},\ \omega_{55} = -\frac{1536Q_4}{\tau^3},\ \omega_{57} = \frac{5760Q_4}{\tau^4},\ \omega_{66} = -\frac{1536Q_4}{\tau^3},\ \omega_{68} = \frac{5760Q_4}{\tau^4},$$
$$\omega_{77} = -\frac{23040Q_4}{\tau^5},\ \omega_{88} = -\frac{23040Q_4}{\tau^5},\ \omega_{99} = Q_2 + \tau Q_4 + h^2Q_5 - F_1 - F_1^T + dZ_2,\ \omega_{9,11} = F_1D,$$
$$\omega_{9,12} = F_1B,\ \omega_{10,10} = -Q_3 - 4Q_5,\ \omega_{10,11} = \frac{6Q_5}{h},\ \omega_{11,11} = -\frac{12Q_5}{h^2},\ \omega_{12,12} = -(1-\mu)Q_6.$$

Proof. Consider the following LKF:

$$V(t) = \sum_{j=1}^{7} V_j(t), \tag{2.2}$$

where

$$\begin{aligned}
V_1(t) &= \varpi^T(t)P\varpi(t),\\
V_2(t) &= \int_{t-\tau}^{t} \varpi^T(s)Q_1\varpi(s)ds + \int_{t-\tau(t)}^{t} \varpi^T(s)Q_6\varpi(s)ds,\\
V_3(t) &= \int_{t-d}^{t} \dot{\varpi}^T(s)Q_2\dot{\varpi}(s)ds,\\
V_4(t) &= \int_{t-h}^{t} \varpi^T(s)Q_3\varpi(s)ds,\\
V_5(t) &= \int_{\tau}^{0}\int_{t+\theta}^{t} \dot{\varpi}^T(s)Q_4\dot{\varpi}(s)dsd\theta,\\
V_6(t) &= h\int_{-h}^{0}\int_{t+\theta}^{t} \dot{\varpi}^T(s)Q_5\dot{\varpi}(s)dsd\theta,\\
V_7(t) &= \int_{-d}^{0}\int_{t+s}^{t} \dot{\varpi}^T(\theta)Z_2\dot{\varpi}(\theta)d\theta ds.
\end{aligned}$$

We calculate the derivatives $\dot{V}_j(t)$, $j = 1, 2, ..., 7$ along the trajectories of system (1.1), which gives

$$\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t) + \dot{V}_4(t) + \dot{V}_5(t) + \dot{V}_6(t) + \dot{V}_7(t),$$

where

$$\dot{V}_1(t) = 2\varpi^T(t)P\dot{\varpi}(t), \tag{2.3}$$

$$\begin{aligned}
\dot{V}_2(t) \le{}& \varpi^T(t)Q_1\varpi(t) - \varpi^T(t-\tau)Q_1\varpi(t-\tau) + \varpi^T(t)Q_6\varpi(t)\\
&- (1-\mu)\varpi^T(t-\tau(t))Q_6\varpi(t-\tau(t)),
\end{aligned} \tag{2.4}$$

$$\dot{V}_3(t) = \dot{\varpi}^T(t)Q_2\dot{\varpi}(t) - \dot{\varpi}^T(t-d)Q_2\dot{\varpi}(t-d), \tag{2.5}$$

$$\dot{V}_4(t) = \varpi^T(t)Q_3\varpi(t) - \varpi^T(t-h)Q_3\varpi(t-h), \tag{2.6}$$

$$\begin{aligned}
\dot{V}_5(t) &= \int_{-\tau}^{0} \dot{\varpi}^T(t)Q_4\dot{\varpi}(t)d\theta - \int_{-\tau}^{0} \dot{\varpi}^T(t+\theta)Q_4\dot{\varpi}(t+\theta)d\theta\\
&= \dot{\varpi}^T(t)Q_4\dot{\varpi}(t)\int_{-\tau}^{0} d\theta - \int_{t-\tau}^{t} \dot{\varpi}^T(s)Q_4\dot{\varpi}(s)ds,\\
&= \tau\dot{\varpi}^T(t)Q_4\dot{\varpi}(t) - \int_{t-\tau}^{t} \dot{\varpi}^T(s)Q_4\dot{\varpi}(s)ds.
\end{aligned} \tag{2.7}$$

Next, consider $-\int_{t-\tau}^{t} \dot{\varpi}^T(s)Q_4\dot{\varpi}(s)ds$ as follows:
Using Lemma 1.1, we have

$$-\int_{t-\tau}^{t} \dot{\varpi}^T(s)Q_4\dot{\varpi}(s)ds \le \eta^T(t)\Xi\eta(t), \tag{2.8}$$

where

$$\Xi = \begin{bmatrix} \frac{-18Q_4}{\tau} & \frac{6Q_4}{\tau} & 0 & 0 & \frac{-96Q_4}{\tau^2} & 0 & \frac{480Q_4}{\tau^3} \\ * & \frac{-36Q_4}{\tau} & \frac{6Q_4}{\tau} & \frac{-96Q_4}{\tau^2} & \frac{144Q_4}{\tau^2} & \frac{480Q_4}{\tau^3} & \frac{-480Q_4}{\tau^3} \\ * & * & \frac{-18Q_4}{\tau} & \frac{144Q_4}{\tau^2} & 0 & \frac{-480Q}{\tau^3} & 0 \\ * & * & * & \frac{-1536Q_4}{\tau^3} & 0 & \frac{5760Q_4}{\tau^4} & 0 \\ * & * & * & * & \frac{-1536Q_4}{\tau^3} & 0 & \frac{5760Q_4}{\tau^4} \\ * & * & * & * & * & \frac{-23040Q_4}{\tau^5} & 0 \\ * & * & * & * & * & * & \frac{-23040Q_4}{\tau^5} \end{bmatrix},$$

$$\begin{aligned} \eta(t) = \Big[& \varpi^T(t)\varpi^T(t-\tfrac{\tau}{2})\varpi^T(t-\tau) \textstyle\int_{t-\tau}^{t-\frac{\tau}{2}} \varpi^T(s)ds \\ & \textstyle\int_{t-\frac{\tau}{2}}^{t} \varpi^T(s)ds \int_{t-\tau}^{t}\int_{u}^{t-\frac{\tau}{2}} \varpi^T(s)dsdu \int_{t-\frac{\tau}{2}}^{t}\int_{u}^{t} \varpi^T(s)dsdu \Big], \end{aligned}$$

$$\begin{aligned} \dot{V}_6(t) &= \int_{-h}^{0} \dot{\varpi}^T(t)Q_5\dot{\varpi}(t)d\theta - h\int_{-h}^{0} \dot{\varpi}^T(t+\theta)Q_5\dot{\varpi}(t+\theta)d\theta \\ &= h\dot{\varpi}^T(t)Q_5\dot{\varpi}(t)\int_{-h}^{0} d\theta - h\int_{t-h}^{t} \dot{\varpi}^T(s)Q_5\dot{\varpi}(s)ds \\ &= h^2\dot{\varpi}^T(t)Q_5\dot{\varpi}(t) - h\int_{t-h}^{t} \dot{\varpi}^T(s)Q_5\dot{\varpi}(s)ds. \end{aligned} \tag{2.9}$$

Next, using Lemma 1.2, we get

$$(b-a)\int_{a}^{b} \dot{\varpi}^T(s)R\dot{\varpi}(s)ds \geq \chi_1^T R\chi_1 + 3\chi_2^T R\chi_2, \tag{2.10}$$

where

$$\begin{aligned} \chi_1 &= \varpi(b) - \varpi(a), \\ \chi_2 &= \varpi(b) - \varpi(a) - \frac{2}{b-a}\int_{a}^{b} \varpi(s)ds. \end{aligned}$$

Given $a = t-h,\ b = t$ and $R = Q_5$, we get

$$\begin{aligned} -h\int_{t-h}^{t} \dot{\varpi}^T(s)Q_5\dot{\varpi}(s)ds \leq & -\varpi^T(t)Q_5\varpi(t) + \varpi^T(t)Q_5\varpi(t-h) + \varpi^T(t-h)Q_5\varpi(t) \\ & - \varpi^T(t-h)Q_5\varpi(t-h) - \varpi^T(t)3Q_5\varpi(t) - \varpi^T(t)3Q_5\varpi(t-h) \\ & + \varpi^T(t)3Q_5\frac{2}{h}\int_{t-h}^{t} \varpi(s)ds - \varpi^T(t-h)3Q_5\varpi(t) - \varpi^T(t-h)Q_5\varpi(t-h) \\ & + \varpi^T(t-h)3Q_5\frac{2}{h}\int_{t-h}^{t} \varpi(s)ds + \frac{2}{h}\int_{t-h}^{t} \varpi^T(s)ds3Q_5\varpi(t) \\ & + \frac{2}{h}\int_{t-h}^{t} \varpi^T(s)ds3Q_5\varpi(t-h) - \frac{2}{h}\int_{t-h}^{t} \varpi^T(s)ds3Q_5\frac{2}{h}\int_{t-h}^{t} \varpi^T(s)ds, \end{aligned} \tag{2.11}$$

$$\dot{V}_7(t) = d\dot{\varpi}^T(t)Z_2\dot{\varpi}(t) - \int_{t-d}^{t} \dot{\varpi}^T(\theta)Z_2\dot{\varpi}(\theta)d\theta. \tag{2.12}$$

By applying Jensen's inequality and the Newton-Leibniz formula $\int_{t-d}^{t} \dot{\varpi}(\theta)d\theta = \varpi(t) - \varpi(t-d)$, we get

$$\begin{aligned} -\int_{t-d}^{t} \dot{\varpi}^T(\theta)Z_2\dot{\varpi}(\theta)d\theta &\leq -\frac{1}{d}\left[\int_{t-d}^{t} \dot{\varpi}(\theta)d\theta\right]^T Z_2 \left[\int_{t-d}^{t} \dot{\varpi}(\theta)d\theta\right] \\ &= -\frac{1}{d}[\varpi(t) - \varpi(t-d)]^T Z_2 \left[\varpi(t) - \varpi(t-d)\right]. \end{aligned} \tag{2.13}$$

Additionally, according to the system $\varpi(t)$, for any appropriately sized matrix F_1, the following conditions must be satisfied:

$$2\left[\varpi^T(t)F_1 + \dot{\varpi}^T(t)F_1\right]\left[(A+EK)\varpi(t) + B\varpi(t-\tau(t)) + C\dot{\varpi}(t-d) + D\int_{t-h}^{t} \varpi(s)ds - \dot{\varpi}(t)\right] = 0. \tag{2.14}$$

We obtain the following zero equation:

$$\begin{aligned} 0 =& 2\varpi^T(t)F_1(A+EK)\varpi(t) + 2\varpi^T(t)F_1B\varpi(t-\tau(t)) + 2\varpi^T(t)F_1C\dot{\varpi}(t-d) + 2\varpi^T(t)F_1D\int_{t-h}^{t} \varpi(s)ds \\ &- 2\varpi^T(t)F_1\dot{\varpi}(t) + 2\dot{\varpi}^T(t)F_1(A+EK)\varpi(t) + 2\dot{\varpi}^T(t)F_1B\varpi(t-\tau(t)) + 2\dot{\varpi}^T(t)F_1C\dot{\varpi}(t-d) \\ &+ 2\dot{\varpi}^T(t)F_1D\int_{t-h}^{t} \varpi(s)ds - 2\dot{\varpi}^T(t)F_1\dot{\varpi}(t). \end{aligned} \tag{2.15}$$

Combining the results of the proofs from (2.3) to (2.15), we obtain

$$\dot{V}(t) \leq \zeta^T(t)\omega\zeta(t), \tag{2.16}$$

where

$$\begin{aligned} \zeta^T(t) =& \Big[\varpi^T(t)\ \varpi^T(t-\tau)\dot{\varpi}^T(t-d)\varpi^T(t-\tfrac{\tau}{2}) \int_{t-\tau}^{t-\frac{\tau}{2}} \varpi^T(s)ds \int_{t-\frac{\tau}{2}}^{t} \varpi^T(s)ds \\ & \int_{t-\tau}^{t}\int_{u}^{t-\frac{\tau}{2}} \varpi^T(s)dsdu \int_{t-\frac{\tau}{2}}^{t}\int_{u}^{t} \varpi^T(s)dsdu\dot{\varpi}^T(t)\varpi^T(t-h) \int_{t-h}^{t} \varpi^T(s)ds\ \varpi^T(t-\tau(t))\Big]. \end{aligned}$$

If this is equivalent to the linear matrix inequality, it follows from expression (2.16) that $\dot{V}(t) < 0$, which ensures the asymptotic stability of the system (2.5), thus concluding the proof.

Theorem 2.2. *Suppose that there exist matrices $P > 0$, $Q_i > 0$ $(i = 1, 2, ...6)$, $Z_2 > 0$; and let τ, h, d, and μ be given scalars, and the sliding surface is given by* (1.2). *Then, the neutral delay system* (1.5) *is asymptotically stable if the following inequalities holds:*

$$[\widehat{\omega}] = [\widehat{\omega}_{n\times n}],\ n = 1, 2, ..., 12 < 0, \tag{2.17}$$

where

$$\widehat{\omega}_{11} = Q_1 + Q_3 - \frac{18Q_4}{\tau} + 2Q_5 + Q_6 + F_1A + A^TF_1^T - \frac{1}{d}Z_2 + EY + Y^TE^T,\ \widehat{\omega}_{13} = F_1C + \frac{1}{d}Z_2,$$

$\widehat{\omega}_{14} = \dfrac{6Q_4}{\tau}$, $\widehat{\omega}_{16} = -\dfrac{96Q_4}{\tau}$, $\widehat{\omega}_{18} = \dfrac{480Q_4}{\tau^3}$, $\widehat{\omega}_{19} = 2P - F_1 + A^T F_1^T + Y^T E^T$, $\widehat{\omega}_{1,10} = -2Q_5$,
$\widehat{\omega}_{1,11} = \dfrac{6Q_5}{h} + F_1 D$, $\widehat{\omega}_{1,12} = F_1 B$, $\widehat{\omega}_{22} = -Q_1 - \dfrac{18Q_4}{\tau}$, $\widehat{\omega}_{25} = \dfrac{144Q_4}{\tau^2}$,
$\widehat{\omega}_{27} = -\dfrac{480Q_4}{\tau^3}$, $\widehat{\omega}_{33} = -Q_2 - \dfrac{1}{d}Z_2$, $\widehat{\omega}_{39} = C^T F_1^T$, $\widehat{\omega}_{44} = -\dfrac{36Q_4}{\tau}$, $\widehat{\omega}_{45} = -\dfrac{96Q_4}{\tau^2}$, $\widehat{\omega}_{46} = \dfrac{144Q_4}{\tau^2}$,
$\widehat{\omega}_{47} = \dfrac{480Q_4}{\tau^3}$, $\widehat{\omega}_{48} = -\dfrac{480Q_4}{\tau^3}$, $\widehat{\omega}_{55} = -\dfrac{1536Q_4}{\tau^3}$, $\widehat{\omega}_{57} = \dfrac{5760Q_4}{\tau^4}$, $\widehat{\omega}_{66} = -\dfrac{1536Q_4}{\tau^3}$, $\widehat{\omega}_{68} = \dfrac{5760Q_4}{\tau^4}$,
$\widehat{\omega}_{77} = -\dfrac{23040Q_4}{\tau^5}$, $\widehat{\omega}_{88} = -\dfrac{23040Q_4}{\tau^5}$, $\widehat{\omega}_{99} = Q_2 + \tau Q_4 + h^2 Q_5 - F_1 - F_1^T + dZ_2$, $\widehat{\omega}_{9,11} = F_1 D$,
$\widehat{\omega}_{9,12} = F_1 B$, $\widehat{\omega}_{10,10} = -Q_3 - 4Q_5$, $\widehat{\omega}_{10,11} = \dfrac{6Q_5}{h}$, $\widehat{\omega}_{11,11} = -\dfrac{12Q_5}{h^2}$, $\widehat{\omega}_{12,12} = -(1-\mu)Q_6$.

Proof. We begin with the term $F_1 K = Y$. To isolate K, we multiply both sides by F_1^{-1}, obtaining

$$K = F_1^{-1} Y. \tag{2.18}$$

Substituting this expression for K into LMI (2.1), we can conclude the result in LMI (2.17). Thereby, the proof of Theorem 2.2 is completed.

Remark 2.1. *Consider systems of the following type without being affected by distributed time delays:*

$$\begin{cases} \dot{\varpi}(t) = A\varpi(t) + B\varpi(t-\tau(t)) + C\dot{\varpi}(t-d), \\ \varpi(t) = \phi(t). \end{cases} \tag{2.19}$$

According to Theorem 2.1, we have the following proof for stability depending on the system's inherent delay.

Corollary 2.1. *Given a scalar τ, we assume the existence of positive definite matrices P, Q_1, Q_2, Q_3, Q_4, Q_6, and F_1 in $\mathbb{R}^{n\times n}$. These matrices must satisfy the following LMIs, which ensure the stability and proper behavior of the system under consideration:*

$$[\omega] = [\omega_{n\times n}],\ n = 1, 2, ..., 10 < 0, \tag{2.20}$$

and the other elements are defined in Theorem 2.1.

Proof. Consider the LKF with the same constraints as in Theorem 2.1, $V_1(t)$, $V_2(t)$, $V_3(t)$, $V_4(t)$, $V_5(t)$, $V_7(t)$. The proof of this statement is similar to Theorem 2.1, so it is omitted.

$$\begin{aligned} \zeta^T(t) = [&\varpi^T(t), \varpi^T(t-\tau), \dot{\varpi}^T(t-d), \varpi^T(t-\frac{\tau}{2}), \int_{t-\tau}^{t-\frac{\tau}{2}} \varpi^T(s)ds, \\ &\int_{t-\frac{\tau}{2}}^{t} \varpi^T(s)ds, \int_{t-\tau}^{t}\int_{u}^{t-\frac{\tau}{2}} \varpi^T(s)dsdu, \int_{t-\frac{\tau}{2}}^{t}\int_{u}^{t} \varpi^T(s)dsdu, \dot{\varpi}^T(t)\varpi^T(t-\tau(t))]. \end{aligned} \tag{2.21}$$

At the next step of the sliding-mode control design, in the following, we investigate the reachability of the sliding surface.

Theorem 2.3. *Consider the system in* (1.1)*, where the initial condition* ϕ *is a continuously differentiable vector-valued function. Assume that the matrices A, B, C, D, and E are constant, and* $\tau(t)$*, d, and h are time-delay parameters that satisfy* $0 \leq \tau(t) \leq \tau$ *and* $d, h > 0$*. Define the sliding surface* $s(t)$ *as from* (1.3)*:*

$$s(t) = H\Big(\varpi(t) - C\varpi(t-d)\Big) - H\Big(\int_0^t \left[(A+EK)\varpi(\theta) - B\varpi(\theta - \tau(\theta)) - D\int_{\theta-h}^{\theta} \varpi(s)\,ds\right] d\theta\Big), \quad (2.22)$$

where H is a real matrix and K is a positive definite matrix to be determined. Suppose that a control law $u(t)$ *is designed as*

$$u(t) = K\varpi(t) - \eta\,\mathrm{sign}(s(t)), \quad (2.23)$$

where η *is a positive scalar gain, and sign(s(t)) denotes the component-wise sign function. Then, under the assumption that HE is nonsingular, all system trajectories reach the sliding surface* $s(t) = 0$ *in finite time. Furthermore, the finite reaching time T satisfies*

$$T \leq \frac{\lambda_{\max}((HE)^{-1})\,\|s(0)\|}{2\eta}. \quad (2.24)$$

Proof. Consider the Lyapunov candidate function:

$$V(s(t)) = \frac{1}{2}s^T(t)(HE)^{-1}s(t), \quad (2.25)$$

which is nonnegative and vanishes only when $s(t) = 0$, implying that $V(s(t))$ serves as a measure of the distance from the sliding surface.

Differentiating $V(s(t))$ with respect to time along the trajectory of $s(t)$, we get

$$\dot{V}(s(t)) = s^T(t)(HE)^{-1}\dot{s}(t). \quad (2.26)$$

Substitute the expression for $\dot{s}(t)$:

$$(HE)^{-1}\dot{s}(t) = -HEK\varpi(t) + HEu(t). \quad (2.27)$$

Therefore,

$$\dot{V}(s(t)) = s^T(t)(HE)^{-1}\left(-HEK\varpi(t) + HEu(t)\right). \quad (2.28)$$

Using the sliding-mode control law $u(t) = K\varpi(t) - \eta\,\mathrm{sign}(s(t))$, we substitute into the derivative of $V(s(t))$:

$$\dot{V}(s(t)) = s^T(t)\left(K\varpi(t) - K\varpi(t) - \eta\,\mathrm{sign}(s(t))\right). \quad (2.29)$$

Simplifying, we have

$$\dot{V}(s(t)) = -\eta\|s(t)\|, \quad (2.30)$$

where $\|s(t)\|$ denotes the Euclidean norm of $s(t)$. This expression implies that $\dot{V}(s(t)) \leq 0$, showing that $V(s(t))$ is a nonincreasing function.

To prove finite-time reachability, we observe that $\dot{V}(s(t)) = -\eta\|s(t)\|$ ensures that $V(s(t))$ decreases as long as $s(t) \neq 0$.

$$\dot{V}(s(t)) \leq -\eta\|s(t)\|. \tag{2.31}$$

Since $V(s) \leq \frac{1}{2}\lambda_{\max}((HE)^{-1})\|s(t)\|^2$, we can write

$$\|s(t)\| \geq \sqrt{\frac{2V(s(t))}{\lambda_{\max}((HE)^{-1})}}. \tag{2.32}$$

Substituting this relation into the inequality (2.31) gives

$$\dot{V}(s(t)) \leq -\eta\sqrt{\frac{2V(s(t))}{\lambda_{\max}((HE)^{-1})}} = -\frac{\sqrt{2}\eta}{\sqrt{\lambda_{\max}((HE)^{-1})}}\sqrt{V(s(t))}. \tag{2.33}$$

Integrating both sides from $t = 0$ to $t = T$, we obtain

$$\int_0^T \frac{\dot{V}(s(t))}{\sqrt{V(s(t))}} \leq -\frac{\sqrt{2}\,\eta}{\sqrt{\lambda_{\max}((HE)^{-1})}}\int_0^T dt, \tag{2.34}$$

which simplifies to

$$2(\sqrt{V(T)} - \sqrt{V(0)}) \leq -\frac{\sqrt{2}\,\eta}{\sqrt{\lambda_{\max}((HE)^{-1})}}T. \tag{2.35}$$

Since $\sqrt{V(T)} \geq 0$, the finite reaching time satisfies

$$T \leq \frac{\sqrt{2\,\lambda_{\max}((HE)^{-1})}\,\sqrt{V(0)}}{\eta}. \tag{2.36}$$

Substituting $V(0) = \frac{1}{2}s^{\mathrm{T}}(0)((HE)^{-1})s(0) \leq \frac{1}{2}\lambda_{\max}((HE)^{-1})\|s(0)\|^2$, we obtain

$$T \leq \frac{\lambda_{\max}((HE)^{-1})\,\|s(0)\|}{2\eta}. \tag{2.37}$$

Thus, the trajectories of the system reach the sliding surface $s(t) = 0$ in finite time, and the sliding mode control law maintains the state trajectories on the surface thereafter. This completes the proof.

Remark 2.2. *Traditional Lyapunov-Krasovskii functionals and classical inequalities (e.g., Jensen's, reciprocal convex) usually lead to conservative results. The proposed improved integral inequalities (Lemmas 1.1 and 1.2) provide tighter estimations of delay-dependent terms, thereby reducing conservatism. Moreover, unlike SMC designs that treat neutral-type and distributed delays separately, the proposed method simultaneously considers both types, ensuring stability under combined delay effects. In addition, the proposed framework accommodates time-varying delays with less restrictive bounds compared to conventional constant or fixed-delay assumptions and guarantees the asymptotic stability of system* (1.1).

Remark 2.3. *The use of improved integral inequalities, such as the extended reciprocally convex matrix inequality, helps make the stability analysis less conservative and more precise. These inequalities provide better estimates for integral terms, leading to more reliable results. While the use of extra free weighting matrices adds to the computational effort, the benefits are clear. The method shows better performance by achieving stability conditions and more accurate control gains, as proven through Examples 3.2 and 3.3 and comparisons with existing methods.*

3. Numerical examples

In this section, we have given numerical examples to demonstrate the validity of the proposed design method. We also compare our findings with those in [9] and [12] to illustrate the reduced conservativeness of our findings.

Example 3.1. Consider the following neutral-type system:

$$\begin{cases} \dot{\varpi}(t) & = A\varpi(t) + B\varpi(t-\tau(t)) + C\dot{\varpi}(t-d) + D\int_{t-h}^{t}\varpi(s)ds + Eu(t), \quad \forall t \geq 0, \\ \varpi(t) & = \phi(t), \ \forall t \in [\tau, d, h, 0], \end{cases} \tag{3.1}$$

where $\tau(t) = 0.5 + 0.1 sin(t), \ h = 0.2, \ d = 0.1$, and $\mu = 0.2$.

The corresponding coefficient matrices are selected as follows:

$$A = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} -0.5 & 0 \\ -0.4 & -0.5 \end{bmatrix}, \quad C = \begin{bmatrix} 0.3 & 0 \\ 0 & -0.3 \end{bmatrix}, \quad D = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad E = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}.$$

Utilizing the value of the above parameter and MATLAB to solve the LMIs in Theorem 2.2, we then establish the asymptotic stability conditions for system (3.1) and the possibility of the calculation P, Q_i $(i = 1, 2, ..., 6)$, as given as follows:

$$P = \begin{bmatrix} 16.3900 & 0.1423 \\ 0.1423 & 15.8629 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 4.4174 & -0.5816 \\ -0.5816 & 4.0314 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 3.1408 & 0.0102 \\ 0.0102 & 2.1430 \end{bmatrix},$$

$$Q_3 = \begin{bmatrix} 3.6090 & -0.5196 \\ -0.5196 & 3.3001 \end{bmatrix}, \quad Q_4 = \begin{bmatrix} 0.8516 & 0.0001 \\ 0.0001 & 0.8517 \end{bmatrix}, \quad Q_5 = \begin{bmatrix} 0.8004 & -0.0582 \\ -0.0582 & 0.7804 \end{bmatrix},$$

$$Q_6 = \begin{bmatrix} 6.6344 & 0.2636 \\ 0.2636 & 5.0700 \end{bmatrix}, \quad Z_2 = \begin{bmatrix} 5.1141 & 0.1494 \\ 0.1494 & 3.7658 \end{bmatrix}.$$

According to Theorem 2.2 LMIs with the designed SMC, we obtain the following controller gain K matrix:

$$K = \begin{bmatrix} 1.9577 & -0.0777 \\ -0.0777 & 2.6585 \end{bmatrix}.$$

The simulation results are presented in Figures 1–6. Under the parameters given above, Figure 1 illustrates the evolution of the state variables $\varpi_1(t)$ and $\varpi_2(t)$ for a closed-loop system with different initial states $\varpi(0) = [0.6 \;\; -0.6]^T$ and $\varpi(0) = [0.1 \;\; -0.1]^T$. In contrast, Figure 2 shows that the open-loop system, initialized with $\varpi(0) = [0.6 \;\; -0.6]^T$, is unstable. However, under the designed SMC law, the proposed neutral-type system with distributed delay achieves asymptotic stability. This highlights the effectiveness of the SMC in stabilizing the system.

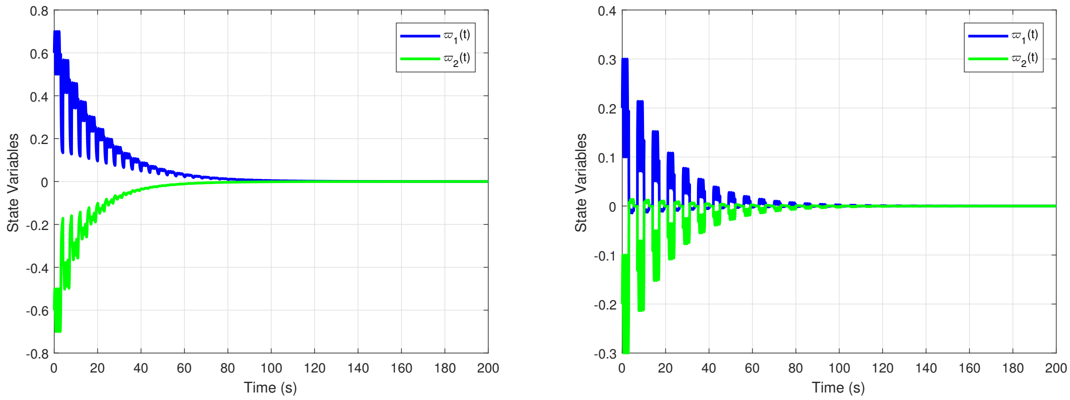

Figure 1. State responses of $\varpi_1(t)$ and $\varpi_2(t)$ of Example 3.1.

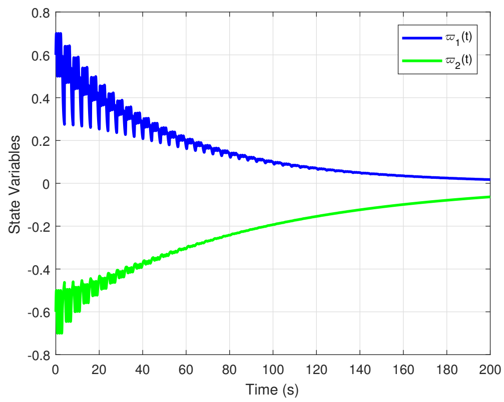

Figure 2. State responses of $\varpi_1(t)$ and $\varpi_2(t)$ of Example 3.1 without control.

Figures 3 and 4 compare the state variable trajectories with and without the SMC, demonstrating the system's improved tracking performance under the control law. Figure 5 displays the sliding variable $s(t)$, which converges to zero, confirming the system's ability to reach the sliding surface. The control input $u(t)$ is shown in Figure 6. Overall, the simulation results in Figures 1 and 2 indicate that the state $\varpi(t)$ achieves stability, validating the proposed stability conditions and the effectiveness of the designed controller presented in Theorem 2.2.

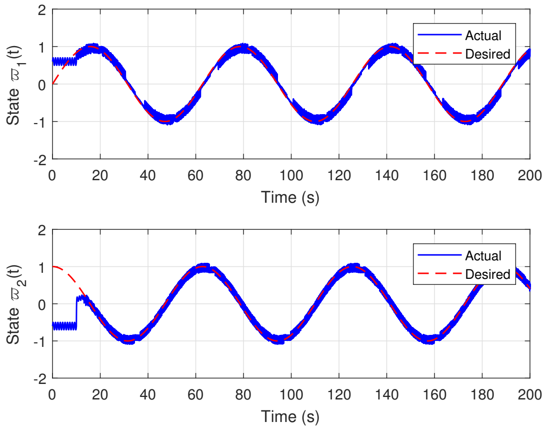

Figure 3. State of $\varpi_1(t)$ and $\varpi_2(t)$ tracking with SMC under control in Example 3.1.

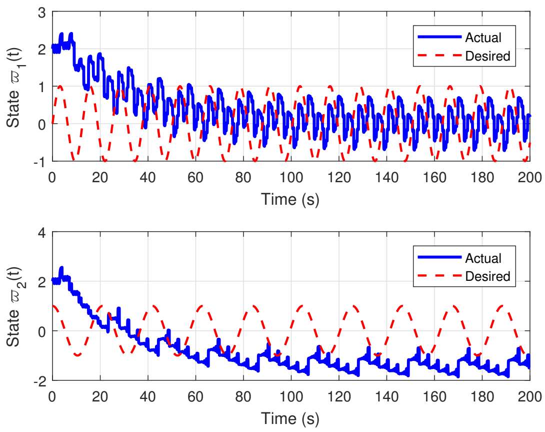

Figure 4. State of $\varpi_1(t)$ and $\varpi_2(t)$ tracking with SMC not under control in Example 3.1.

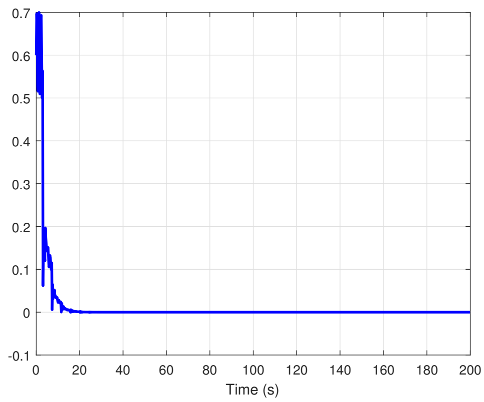

Figure 5. Sliding variable $s(t)$ in Example 3.1.

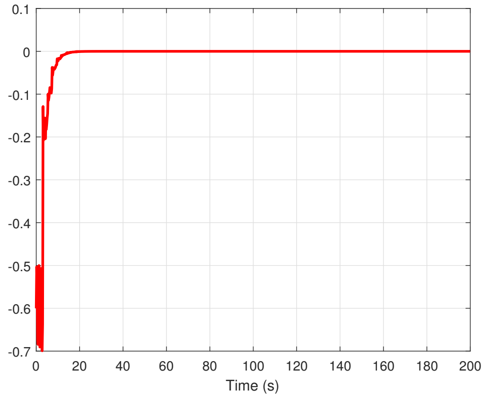

Figure 6. Control input $u(t)$ in Example 3.1.

Example 3.2. Consider the following delayed neutral type system with the following parameters:

$$\dot{\varpi}(t) - \begin{bmatrix} -0.2 & 0 \\ 0.2 & -0.1 \end{bmatrix} \dot{\varpi}(t-d) = \begin{bmatrix} -0.9 & 0.2 \\ 0.1 & -0.9 \end{bmatrix} \varpi(t) + \begin{bmatrix} -1.1 & -0.2 \\ -0.1 & -1.1 \end{bmatrix} \varpi(t-\tau(t)). \tag{3.2}$$

By solving the LMIs presented in Corollary 2.1, Table 1 lists the maximum upper bounds for the state delay τ that guarantee the stability of system (3.2) for varying values of parameter d, with $\mu = 0.01$. The findings demonstrate that the results obtained in this study exhibit considerably lower error rates compared to those reported in other works, such as [9] and [12].

Table 1. Maximum upper bounds for different values of τ (Example 3.2).

d	0.1	0.5	1.0	1.5	1.6527
[9]	1.7100	1.6718	1.6543	1.6527	1.6527
[12]	1.7844	1.7495	1.7201	1.7191	1.7191
Corollary 2.1	1.7932	1.7592	1.7230	1.7200	1.7195

Table 1 provides the maximum allowable state delays that ensure system stability as d ranges from 0.1 to 1.6527, based on the stability criterion derived in Corollary 2.1. In contrast, our system has a time-varying state delay $\tau(t)$, making the proposed criterion more general and useful for a wider range of systems. This innovation enhances the results in [9] and [12]. For example, with $d = 1.6527$, the maximum permissible state delay τ is 1.7195, highlighting how the new stability criterion extends the range of permissible time-varying delays.

Figures 7–9 present the state trajectories for Example 3.2, where the initial conditions are set as $\varpi(0) = [0.6, -0.6]^T$, $\varpi(0) = [0.2, -0.2]^T$, and $\varpi(0) = [0.1, -0.1]^T$, while the time-varying delay is defined as $\tau(t) = 0.5|\sin(t)|$. The results shown in these figures indicate that both state variables asymptotically approach zero, confirming the effectiveness of the proposed stability criterion. This outcome is particularly noteworthy as it demonstrates that the system remains stable despite the presence of time-varying delays, extending the applicability of the stability analysis to a broader class of systems than those typically considered in other studies. This further emphasizes the robustness and generality of the proposed approach in handling time-varying delay systems.

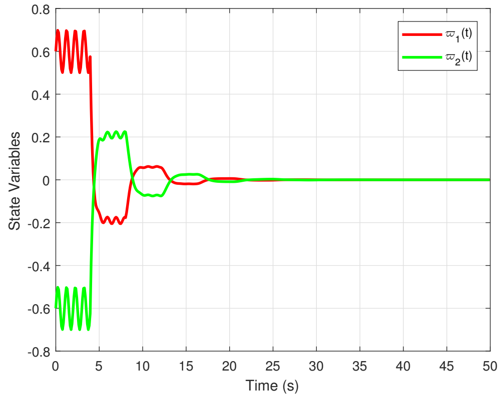

Figure 7. State responses of variables $\varpi_i(t)$, $i = 1, 2$ with the initial condition $\varpi(0) = [0.6, -0.6]^T$ in Example 3.2.

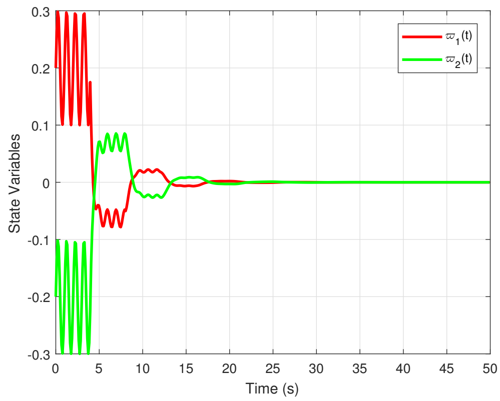

Figure 8. Variation of $\varpi_1(t)$ and $\varpi_2(t)$ with the different condition $\varpi(0) = [0.2, -0.2]^T$ in Example 3.2.

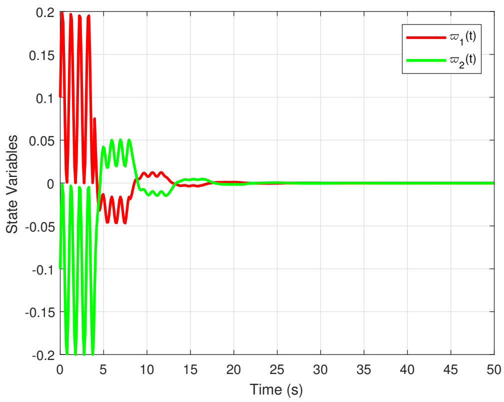

Figure 9. Responses of $\varpi_i(t)$, $i = 1, 2$ with the initial condition $\varpi(0) = [0.1, -0.1]^T$ in Example 3.2.

Remark 3.1. *In [5, 41], we investigated stability criteria for neutral systems with delays using the free weighting matrix method. However, this technique significantly increases the number of decision variables, resulting in higher computational complexity and greater time requirements, particularly when implemented with MATLAB's LMI Toolbox. In comparison, we have implemented novel integral inequality techniques, yielding less conservative results and offering a more computationally efficient solution compared to the methods in Table 2 from the literature [5, 41].*

Table 2. Maximum upper bounds of τ for different values of h (Example 3.3).

h	0.1	0.2	0.5	1.0	1.5	Decision variables
[41]	1.6953	1.6720	1.6546	1.4995	1.4022	$10.5n^2 + 2.5n + 2$
[5]	3.5250	3.4208	3.1362	2.7400	2.4174	$21n^2 + 7n + 3$
Theorem 2.1	4.1250	4.0241	3.5251	3.2210	2.9895	$6n^2 + 4n + 4$

Example 3.3. Consider a neutral-type system described by the following differential equation:

$$\dot{\varpi}(t) = A\varpi(t) + B\varpi(t - \tau + C\dot{\varpi}(t-d) + D\int_{t-h}^{t} \varpi(s)ds, \quad \forall t \geq 0,$$

$$A = \begin{bmatrix} -0.9 & 0.2 \\ 0.1 & -0.9 \end{bmatrix}, \quad B = \begin{bmatrix} -1.1 & -0.2 \\ -0.1 & -1.1 \end{bmatrix}, \quad C = \begin{bmatrix} -0.2 & 0 \\ 0 & -0.1 \end{bmatrix}, \quad D = \begin{bmatrix} -0.2 & 0 \\ 0.2 & -0.1 \end{bmatrix}.$$

By solving the same linear matrix inequalities presented in Theorem 2.1, using the same LKF in Eq (2.2) and setting $Q_6 = 0$, Table 2 shows the maximum allowable upper bound (MAUB) of the state delay τ for various values of h, ensuring the system's stability. Additionally, Table 3 provides the MAUB of h for the system's stability across different values of τ. The results demonstrate that the stability criterion proposed in this paper is much less conservative compared to those in [41] and [5], indicating an improved and more practical approach to handling time delays in the system.

Table 3. Maximum upper bounds of h for different values of τ (Example 3.3).

τ	0.1	0.2	0.5	1.0	1.5
[41]	17.3740	17.1379	12.3343	4.2393	0.9975
[5]	17.3732	17.1055	12.6229	6.3284	3.7774
Theorem 2.1	18.2121	17.5210	13.1245	8.1241	4.0013

Figures 10–12 illustrate the state trajectories for Example 3.3, with initial conditions $\varpi(0) = [0.6, -0.6]^T$, $\varpi(0) = [0.2, -0.2]^T$, and $\varpi(0) = [0.1, -0.1]^T$, and a constant time delay $\tau = 0.5$. These figures confirm that, regardless of the initial conditions, both state variables converge to zero over time. This demonstrates the effectiveness of the stability criterion in ensuring the system's stability, even in the presence of time delays. The results show that the proposed method guarantees stable system behavior, even when subjected to constant delays, which is critical for real-world applications where such delays are often unavoidable. This further emphasizes the robustness of the stability criterion in handling time delays effectively.

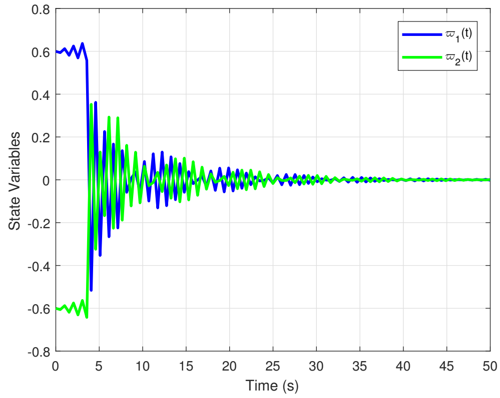

Figure 10. Trajectories of the state variable $\varpi_1(t)$ and $\varpi_2(t)$ along the initial condition $\varpi(0) = [0.6, -0.6]^T$ in Example 3.3.

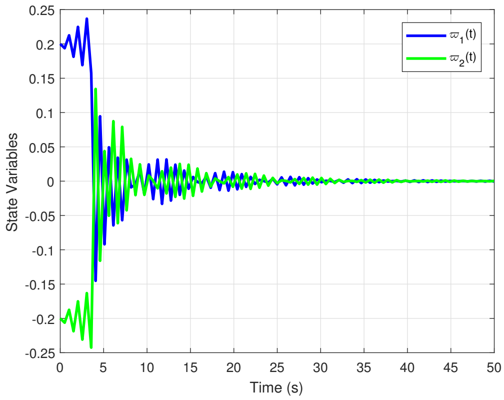

Figure 11. The state curves of $\varpi_1(t)$ and $\varpi_2(t)$ with $\varpi(0) = [0.2, -0.2]^T$ in Example 3.3.

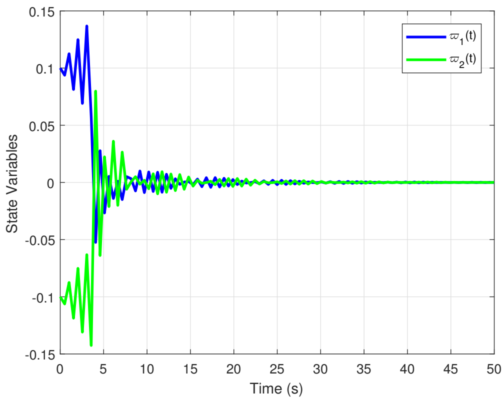

Figure 12. State responses of $\varpi_1(t)$ and $\varpi_2(t)$ with differential initial values $\varpi(0) = [0.1, -0.1]^T$ in Example 3.3.

Example 3.4. As discussed in [3, 14], the PEEC model shown in Figure 13 incorporates additional circuit elements, particularly those that account for retarded mutual coupling between partial inductances of the form $Lp_{ij}i_j(t-\tau)$. This model also includes retarded-dependent current sources, expressed as $p_{ij}/p_{ii}i_{cj}(t-\tau)$, where Lp_{ij} represents partial inductances and p_{ij} denotes partial coefficients related to potential. In this framework, the state vector captures the partial inductance branch currents, while the input corresponds to the unknown nodal voltages. The PEEC model thus offers a generalized approach for modeling complex circuits by integrating the effects of time delays in mutual couplings and current sources, which are essential to accurately analyze the dynamic behavior of such systems in practical applications. In the general form of modeling, PEEC can be modeled as

$$C_0\dot{\varpi}(t)+G_0\varpi(t)+C_1\dot{\varpi}(x-\tau)+G_1\varpi(x-\tau)=Bu(t,t-\tau),\ \varpi(t)=\phi(t),\ t\le t_0. \tag{3.3}$$

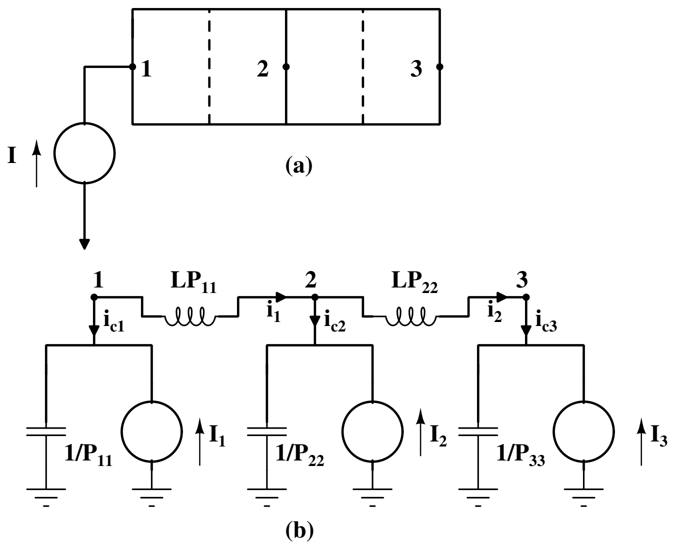

Figure 13. (a) Metal strip with two Lp cells (three capacitive cells dashed) and (b) a small PEEC model for the metal strip.

Choosing the known parameters and utilizing the MATLAB LMI toolbox in Corollary 2.1 LMIs, we conclude that there is a feasible solution of inequalities (2.20) when the delay upper bound is 0.5. Figure 14 illustrates the state responses of the system for the initial condition $\varpi(0)=[0.9\ -0.9\ 0.5]^T$. The responses are shown for two different values of the neutral delay. In both cases, the state variables exhibit clear convergence to zero over time, demonstrating the stability and effectiveness of the proposed approach. The results highlight the ability of the system to handle varying neutral delay values while maintaining stability. This convergence emphasizes the accuracy of the derived stability conditions, confirming their capability to mitigate the effects of neutral delays on system dynamics. Such performance is significant for systems with discrete and neutral delays, as the results validate the theoretical findings and showcase the practical applicability of the proposed method.

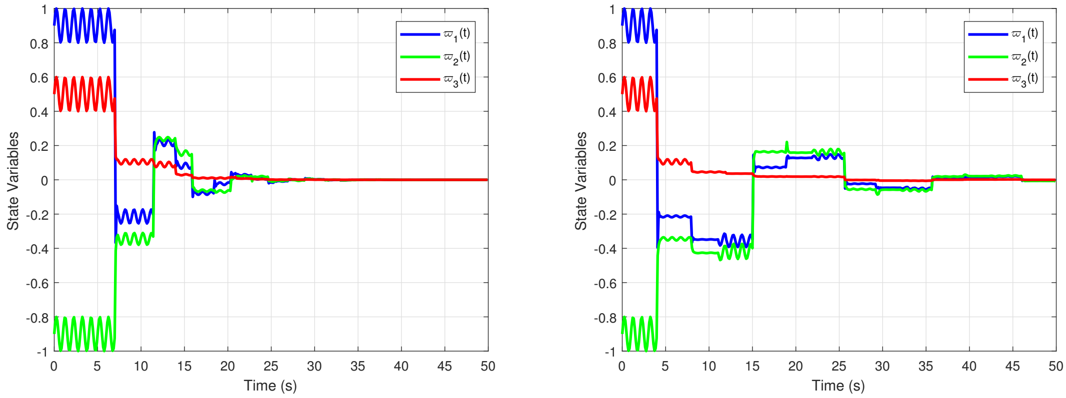

Figure 14. State responses of $\varpi_1(t)$, $\varpi_2(t)$, and $\varpi_3(t)$ in Example 3.4.

If we take the different time-varying delays commonly existing in the modeling of a real circuit into account, a more general form of PEEC (3.3) can be described by the following system [14]:

$$\begin{cases} \dot{\varpi}(t) = A\varpi(t) + B\varpi(t-\tau(t)) + C\dot{\varpi}(t-d), \ \forall t \geq t_0, \\ \varpi(t) = \phi(t), \ \forall t \leq t_0. \end{cases} \tag{3.4}$$

In the model, the coefficient matrices are given as [14]

$$\frac{A}{100} = \begin{bmatrix} -2.105 & 1 & 2 \\ 3 & -9 & 0 \\ 1 & 2 & -6 \end{bmatrix}, \ \frac{B}{100} = \begin{bmatrix} 1 & 0 & -3 \\ -0.5 & -0.5 & -1 \\ -0.5 & -1.5 & 0 \end{bmatrix}, \ C \times 72 = \begin{bmatrix} -1 & 5 & 2 \\ 4 & 0 & 3 \\ -2 & 4 & 1 \end{bmatrix}.$$

4. Conclusions

In this paper, we presented an SMC strategy for stabilizing neutral-type distributed systems with time-varying delays. By employing an LKF and using an improved integral inequality, we provided less conservative stability conditions with tighter bounds, ensuring the system's stability and performance. The proposed method also demonstrated its practical applicability through the PEEC model, showcasing its potential for real-world systems such as electromagnetic field simulations and circuit analysis. The results highlight the effectiveness of the method in handling time delays, offering a more precise solution compared to existing techniques. In future studies, researchers may extend the proposed SMC framework to nonlinear and stochastic systems to enhance its robustness and applicability in uncertain environments [30]. Incorporating event-triggered mechanisms could further improve communication efficiency and system responsiveness [31]. In addition, applying the approach to large-scale interconnected systems, such as power grids and multi-agent networks, while integrating fault detection and adaptive learning strategies, would be a valuable direction. Practical implementation and experimental validation on autonomous platforms like UAVs and UUVs could also provide deeper insights into real-world performance.

Author contributions

Saravanan Shanmugam and Vadivel Rajarathinam: Conceptualization, Writing–original draft; Saravanan Shanmugam and Mohamed Rhaima: Methodology; Nallappan Gunasekaran and K. Chaisena: Formal analysis; Vadivel Rajarathinam: Investigation; Mohamed Rhaima and Nallappan Gunasekaran: Writing–review & editing; K. Chaisena and Mohamed Rhaima: Supervision. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare no potential conflicts of interest.

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