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Research article

The conjugation diameters of finite dihedral groups

Fawaz Aseeri*

Mathematics Department, Faculty of Sciences, Umm Al-Qura University, Makkah 21955, Saudi Arabia

* **Correspondence:** Email: fiaseeri@uqu.edu.sa; Tel: +966542963559.

Abstract: Let G be a group. A subset S of G is said to normally generate G if the normal closure of S in G is equal to G itself. This means that every element of G can be represented as a product of conjugates of elements of S and their inverses. Given an element g of G and a normally generating set S, we define the length of g with respect to S as the smallest number of conjugates of elements of S or their inverses needed to express g as a product. Then, for each such S, the diameter of G with respect to S is defined as the supremum of the lengths of elements of G with respect to S. The conjugacy diameter of G is the supremum of all diameters of G over all finite normally generating subsets. It measures how efficiently G is normally generated by its finite normally generating subsets.
In this paper, we found the conjugacy diameters of finite dihedral groups. It is worth noting that the conjugacy diameters of other families, such as semidihedral 2-groups, generalized quaternion groups, and modular p-groups, have already been investigated.

Keywords: dihedral group; normally generating subsets; word norm; conjugacy diameter
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1. Introduction

1.1. Conjugation invariant n orms

Let G be a group. A *norm* on G is a function $\nu : G \longrightarrow [0, \infty)$ that satisfies the following axioms:

(i) $\nu(g) = 0$ if and only if $g = 1$;

(ii) $\nu(g) = \nu(g^{-1})$ for all $g \in G$;

(iii) $\nu(gh) \leq \nu(g) + \nu(h)$ for all $g, h \in G$.

A norm ν is said to be *conjugation-invariant* if, in addition, it satisfies

$$\nu(g^{-1}hg) = \nu(h) \quad \text{for all } g, h \in G.$$

Given such a norm, the *diameter* of G with respect to ν is defined by

$$\mathrm{diam}_\nu(G) := \sup\{\nu(g) \mid g \in G\}.$$

A group G is called *bounded* if $\mathrm{diam}_\nu(G) < \infty$ for every conjugation-invariant norm ν on G.

The paper by Burago et al. [8] formally introduced the concept of conjugation-invariant norms on groups and the associated notion of bounded groups. They systematically developed the foundational properties of these norms and examined their behavior in several classes of groups that arise naturally in geometry. Their work was the starting point of the modern study of conjugation-invariant norms, connecting ideas from geometric group theory, topology, and dynamical systems. Building on this foundation, several subsequent works have explored structural and geometric properties of groups via these norms.

For instance, Bardakov et al. [5] investigated the generation of groups by conjugation-invariant sets, demonstrating how such sets naturally give rise to conjugation-invariant norms. More recently, Kędra [11] studied groups with conjugation-invariant norms, highlighting their links to quasimorphisms and asymptotic geometric structures.

In a related direction, Brandenbursky et al. [6] studied bi-invariant word metrics on groups, which naturally define conjugation-invariant norms arising from generating sets that are invariant under conjugation. Brandenbursky and Kędra [7], inspired by Burago et al. [8], later constructed explicit conjugation-invariant norms known as fragmentation norms in transformation groups.

Similarly, Kawasaki [10], motivated by the same framework, constructed an explicit conjugation-invariant norm on a group of geometric transformations, providing concrete examples of such norms in geometric contexts. Muranov [15] further contributed to this area by presenting examples of finitely generated infinite simple groups that are unbounded with respect to conjugation-invariant norms in the sense of Burago et al. [8].

Kędra et al. [12] further developed the notion of boundedness introduced by Burago et al. [8], introducing stronger concepts called *strong boundedness* and *uniform boundedness*. To ensure clarity for the reader, we will introduce and define these notions in the next section, accompanied by illustrative examples. We will then employ these notions to first present our main theorem in Subsection 1.3 and then provide its proof in Section 2.

1.2. Strong and uniform boundedness of groups

Let G be a group and $S \subseteq G$. Recall that S *normally generates* G if the normal closure of S in G equals G :

$$G = \langle h^{-1} s^{\pm 1} h \mid h \in G, s \in S \rangle.$$

In other words, every element of G can be expressed as a product of elements of

$$\mathrm{Conj}_G(S^{\pm 1}) := \{h^{-1} s h \mid h \in G,\ s \in S \text{ or } s^{-1} \in S\}. \tag{1.1}$$

We define the *diameter* of $g \in G$ with respect to a normally generating subset S of G by

$$\|g\|_S := \min\left\{n \in \mathbb{N} \,\middle|\, g = h_1^{-1} s_1 h_1 \cdots h_n^{-1} s_n h_n, \text{for some } s_i \in S \cup S^{-1},\ h_i \in G\right\}.$$

It is essential to emphasize that the diameter of the identity element $1 \in G$ with respect to S, denoted $\|1\|_S$, equals zero. The *word norm* $\| \,.\, \|_S : G \to [0, \infty)$, defined by $g \mapsto \|g\|_S$, is a conjugation-invariant norm. The *diameter* of G with respect to this word norm is

$$\|G\|_S := \sup\{\|g\|_S \mid g \in G\}.$$

Example 1.1. *Let $G := S_3$, the symmetric group of degree 3. Then*

$$G = \{\mathrm{id}_{\{1,2,3\}}, (1\,2), (1\,3), (2\,3), (1\,2\,3), (1\,3\,2)\}.$$

Let

$$S := \{(1\,2)\},$$

and note that

$$\mathrm{Conj}_G(S^{\pm 1}) = \{(1\,2), (1\,3), (2\,3)\}.$$

We have

$$(1\,2\,3) = (1\,2) \circ (2\,3) \quad \textit{and} \quad (1\,3\,2) = (1\,2) \circ (1\,3).$$

It follows that S normally generates G. For $g \in G$, we have

$$\|g\|_S = \begin{cases} 1, & \textit{if } g \textit{ is a transposition}, \\ 2, & \textit{if } g \textit{ is a 3-cycle}, \\ 0, & \textit{if } g = \mathrm{id}_{\{1,2,3\}}. \end{cases}$$

Hence,

$$\begin{aligned} \|G\|_S &= \sup\{\|g\|_S \mid g \in G\} \\ &= \sup\{0, 1, 2\} \\ &= 2. \end{aligned}$$

Arguing similarly, we can see that $\|G\|_S$ is also 2 when $S = \{(1\,3)\}$ or $S = \{(2\,3)\}$.

For every group G and natural number $n \geq 1$, let

$$\Gamma_n(G) := \{S \subseteq G \mid |S| \leq n \text{ and } S \text{ normally generates } G\},$$

$$\Gamma(G) := \{S \subseteq G \mid |S| < \infty \text{ and } S \text{ normally generates } G\}.$$

Kędra et al. [12] showed that in finitely normally generated groups, boundedness is completely determined by the behavior of the word norms $\|G\|_S$, where $S \in \Gamma(G)$, and introduced the following important result:

Lemma 1.2. *([12, Corollary 2.5]) Let G be a finitely normally generated group. The following are equivalent:*

1) G is bounded;

2) $\|G\|_S < \infty$ for every $S \in \Gamma(G)$;

3) $\|G\|_S < \infty$ for some $S \in \Gamma(G)$.

Lemma 1.2 emphasizes the central role of word norms and their diameters, defined with respect to finite normally generating subsets, in studying the boundedness of groups.

Word norms are also applied in studying refined versions of the notion of bounded groups. These refinements, introduced by Kędra et al. [12], strengthen the classical concept of boundedness by considering the behavior of word norms over finite normally generating subsets. To define them, we introduce the following notation: for a group G and a natural number $n \geq 1$, set

$$\Delta_n(G) := \sup\{\|G\|_S \mid S \in \Gamma_n(G)\},$$
$$\Delta(G) := \sup\{\|G\|_S \mid S \in \Gamma(G)\}.$$

Now, we have the following definition:

Definition 1.3. *([12, Definition 1.1]) A finitely normally generated group G is said to be strongly bounded if $\Delta_n(G) < \infty$ for every $n \in \mathbb{N}$. It is called uniformly bounded if $\Delta(G) < \infty$.*

Example 1.4. *We consider Example 1.1, and adopt the notation from there. Note that* $\mathrm{id}_{\{1,2,3\}}$, $\langle(1\,2\,3)\rangle$, *and G are the only normal subgroups of G. This implies that a subset of G normally generates G if and only if it contains a transposition. In particular, we have*

$$\Gamma_1(G) = \{\{(12)\}, \{(13)\}, \{(23)\}\}.$$

Applying Example 1.1, we conclude that

$$\begin{aligned}\Delta_1(G) &= \sup\{\|G\|_S \mid S \in \Gamma_1(G)\}\\ &= \sup\{2\}\\ &= 2.\end{aligned}$$

Now, let $S \in \Gamma(G)$. Then, S contains a transposition, and so we have $T \subseteq S$ for some $T \in \Gamma_1(G)$. Notice that

$$\begin{aligned}\|G\|_S &= \sup\{\|g\|_S \mid g \in G\}\\ &\leq \sup\{\|g\|_T \mid g \in G\}\\ &= \|G\|_T\\ &= 2.\end{aligned}$$

Hence, $\Delta(G) = \sup\{\|G\|_S \mid S \in \Gamma(G)\} = 2$.

It follows from Lemma 1.2 and Definition 1.3 that within the class of finitely normally generated groups, we have the following inclusions:

$$\{\text{uniformly bounded groups}\} \subseteq \{\text{strongly bounded groups}\} \subseteq \{\text{bounded groups}\}. \tag{1.2}$$

All inclusions in (1.2) are proper. As shown in [12, Theorem 1.2(b)], every semisimple Lie group with a finite center and a nontrivial compact factor is bounded but not strongly bounded. Moreover, for any $n \geq 3$, the group $SL_n(\mathbb{Z})$ is strongly bounded but not uniformly bounded [12, Theorem 1.4]. These examples demonstrate that each refinement of boundedness strictly strengthens the previous one.

It immediately follows from the above definitions that

$$\Delta_1(G) \leq \Delta_2(G) \leq \Delta_3(G) \leq \cdots,$$

and

$$\Delta_n(G) \leq \Delta(G) \text{ for all positive integers } n.$$

1.3. The conjugacy diameter of groups

For any group G, we call $\Delta(G)$ the *conjugacy diameter* of G. This definition differs from the *absolute diameter* of a group as considered by Klopsch and Lev [13], where they focused only on finite groups. In their context, the *diameter* of a finite group G with respect to a generating set A is the smallest integer n such that every element of G can be expressed as a product of at most n elements of $A \cup A^{-1}$. This notion corresponds exactly to the diameter of the Cayley graph $\mathrm{Cay}(G, A)$ of G with respect to A. The *absolute diameter* of G is the maximum of all diameters of G with respect to a generating subset of G. In contrast, the conjugacy diameter $\Delta(G)$ is, in the finite case, the maximal diameter of G with respect to a normally generating subset $S \subseteq G$.

The key differences between conjugacy diameters and absolute group diameters are as follows: Klopsch and Lev [13] used generating sets, not necessarily normally generating subsets. Moreover, their definition of the diameter of a finite group with respect to a generating subset A does not involve the conjugates of the elements of $A \cup A^{-1}$. By contrast, conjugacy diameters emphasize normal generation, and involve the consideration of conjugates. However, note that the concepts of conjugacy diameters and absolute group diameters coincide for finite abelian groups.

Calculating $\Delta(G)$ is generally a challenging problem. For infinite groups, several authors have studied specific classes where $\Delta(G)$ can be bounded or explicitly determined. For example, [3, Theorems 1.4 and 1.5] determined the conjugacy diameters $\Delta(G)$ for $PSL_2(\mathbb{C})$ and $SL_2(\mathbb{C})$, as well as their direct products $(PSL_2(\mathbb{C}))^n$ and $(SL_2(\mathbb{C}))^n$, providing explicit values of $\Delta(G)$ and showing that all these groups are uniformly bounded. Furthermore, Trost [16, Theorems 1.2, 1.3, and Corollary 5.13] studied the conjugacy diameter $\Delta(G)$ for $SL_2(R)$ defined over rings of S-algebraic integers with infinitely many units, showing that $SL_2(R)$ is strongly bounded and providing explicit upper bounds for $\Delta(SL_2(R))$.

For finite groups, computing $\Delta(G)$ has also attracted considerable attention. Libman and Tarry [14] showed that for the symmetric group S_n, $\Delta(S_n) = n - 1$, for all $n \geq 2$ (see [14, Thereom 1.2]). Aseeri and Kaspczyk [4] further determined $\Delta(G)$ for non-abelian finite p-groups with cyclic maximal subgroups, including semidihedral 2-groups (see [4, Theorem 1.4]), generalized quaternion groups (see [4, Theorem 1.5]), and modular p-groups of order p^n (see ([4, Theorem 1.6]), where p is a prime number and $n \geq 3$. In [13], the absolute diameters of the finite abelian groups (and hence their conjugacy diameters) were found.

Building on the study of conjugacy diameters in both finite and infinite groups, a particularly interesting case arises with dihedral groups. For the infinite dihedral group $D_\infty = \langle a, b \mid b^2 = 1,\ bab = a^{-1} \rangle$, it is known that D_∞ is uniformly bounded with $\Delta(D_\infty) \leq 4$ (see [12, Example 2.8]). This provides a natural motivation to investigate the corresponding finite case. Let

$$D_n = \langle a, b \mid a^n = b^2 = 1,\ bab = a^{-1} \rangle, \quad n \geq 3.$$

Our main result can be stated as follows:

Theorem 1.5. *Let $n \geq 3$ be a natural number and $G := D_n$. Then,*

$$\Delta(G) = \begin{cases} 2 & \text{if } n \geq 3 \text{ and } n \text{ odd}, \\ 2 & \text{if } n = 4, \\ 3 & \text{if } n \geq 6 \text{ and } n \text{ even}. \end{cases}$$

2. Proof of Theorem 1.5

In this section, we present some results and the notation needed for the proof of Theorem 1.5.

Definition 2.1. *([12, Section 2]) Let X be a subset of a group G. For any $n \geq 0$, we define $B_X(n)$ to be the set of all elements of G, which can be written as a product of at most n conjugates of elements of X and their inverses.*

By Definition 2.1, we have

$$\{1\} = B_X(0) \subseteq B_X(1) \subseteq B_X(2) \subseteq \ldots$$

The next result follows from the above definition.

Lemma 2.2. *([12, Lemma 2.3 (iii)]) Let G be a group, $X \subseteq G$, and $n, m \in \mathbb{N}$. Then, $B_X(n)B_X(m) = B_X(n+m)$.*

Recall that a finite group is dihedral if and only if it is generated by two involutions, and we have the following theorem.

Theorem 2.3. *([2, Theorem 45.2]) Let x and y be involutions in a finite group G such that $n = |xy|$ and $G = \langle x, y \rangle$. Then, the following holds:*

1) Every element in $G \backslash \langle xy \rangle$ is an involution;

2) If n is odd, then all involutions in G are conjugate;

3) if n is even, then each involution in G is conjugate to exactly one of x, y, or z, where z is the only involution in $\langle xy \rangle$. Moreover, z lies in the center of G.

From now on, set $G := D_n = \langle a, b | a^n = 1 = b^2, bab = a^{-1} \rangle$, where $n \geq 3$ is a natural number. In fact, G is generated by b and $a^m b$ such that the greatest common divisor of m and n , denoted by $\gcd(n, m)$, is 1 (see [1, Lemma 6]). We have the following results for G.

The following well-known result describes the proper normal subgroups of G.

Proposition 2.4. *([9, Proposition 2.1]) In G, we have the following:*

1) For $n \geq 3$ and n odd, the only proper normal subgroups of G are the subgroups of $\langle a \rangle$;

2) For $n \geq 4$ and n even, the proper normal subgroups of G are $\langle a^2, b \rangle, \langle a^2, ab \rangle$, and the subgroups of $\langle a \rangle$.

Lemma 2.5. *Assume that $n \geq 3$ is odd. Let $S \in \Gamma(G)$. Then, S contains an involution $u \in G$.*

Proof. Let $S \in \Gamma(G)$. Suppose that $S \subseteq \langle a\rangle$. Then, $\langle\langle S\rangle\rangle \leq \langle a\rangle$ since $\langle a\rangle \trianglelefteq G$ (see Proposition 2.4 (1)), contradicting the assumption that S normally generates G. Therefore, $S \nsubseteq \langle a\rangle$. So there is some $0 \leq i \leq n-1$ with $u := a^i b \in S$. □

Lemma 2.6. *Assume that $n \geq 3$ is odd. Then, $\Delta_1(G) = \Delta(G)$.*

Proof. Since $\Gamma_1(G) \subseteq \Gamma(G)$, we have $\Delta_1(G) \leq \Delta(G)$. Now, we want to show that $\Delta(G) \leq \Delta_1(G)$. Let $S \in \Gamma(G)$. Hence, S is a subset of G that normally generates G. Lemma 2.5 shows that S must contain an involution, say $u := a^i b$ for some $0 \leq i \leq n-1$. Set $X := \langle\langle\{u\}\rangle\rangle$. Then, X is a normal subgroup of G that is not contained in $\langle a\rangle$. So $X = G$ by Proposition 2.4 (1), whence $\{u\}$ normally generates G. Since $\{u\} \subseteq S$, we have $\|G\|_S \leq \|G\|_{\{u\}}$. Set $T := \{u\}$. We have $T \in \Gamma_1(G)$ and $\|G\|_S \leq \|G\|_T$. Thus,

$$\begin{aligned}\Delta(G) &= \sup\{\|G\|_S \mid S \in \Gamma(G)\}\\ &\leq \sup\{\|G\|_T \mid T \in \Gamma_1(G)\}\\ &= \Delta_1(G).\end{aligned}$$

□

The next result is useful to find the possible normally generating sets of G, where $n \geq 4$ and n even.

Lemma 2.7. *Assume that $n \geq 4$ is even. Let $S \in \Gamma(G)$. Then, there exist $u, v \in S$ such that $u \in G\backslash Z(G)$ is an involution and is not conjugate to v, $\langle\langle\{u, v\}\rangle\rangle = G$, and $v \neq a^l$, where $0 \leq l \leq n-1$ is even.*

Proof. Let $S \in \Gamma(G)$. Suppose that $S \subseteq \langle a\rangle$. Then, $\langle\langle S\rangle\rangle \leq \langle a\rangle$ since $\langle a\rangle \trianglelefteq G$ (see Proposition 2.4 (2)), contradicting the assumption that S normally generates G. Therefore, $S \nsubseteq \langle a\rangle$. So there is some $0 \leq i \leq n-1$ with $u := a^i b \in S$. By Theorem 2.3 (1), u is an involution of G, and it is easy to see that u is not in the center of G.

Assume that every element of S has the form a^l, where $0 \leq l \leq n-1$ is even, or $a^m b$, where $0 \leq m \leq n-1$ and $m \equiv i \bmod 2$. Then $\mathrm{Conj}_G(S) \subseteq \langle a^2\rangle\langle b\rangle$ if i is even or $\mathrm{Conj}_G(S) \subseteq \langle a^2\rangle\langle ab\rangle$ if i is odd. In both cases, we have $\langle\langle S\rangle\rangle = \langle \mathrm{Conj}_G(S)\rangle \neq G$ since $\langle a^2, b\rangle$ and $\langle a^2, ab\rangle$ are proper normal subgroups of G (see Proposition 2.4 (2)). This is a contradiction.

Assume that there is some $0 \leq k \leq n-1$ such that $v := a^k b \in S$ and such that k is even if i is odd and odd if i is even. We see from Proposition 2.4 (2) that u and v are not conjugate and that $\langle\langle\{u, v\}\rangle\rangle = G$, as required.

Now assume that there is some odd $1 \leq l \leq n-1$ such that $v := a^l \in S$. Clearly, u and v are not conjugate. From Proposition 2.4 (2) we see that $\langle\langle\{u, v\}\rangle\rangle = G$, as needed.

□

Corollary 2.8. *Assume that $n \geq 4$ is even. Then, $\Gamma_1(G) = \emptyset$.*

Proof. This follows from Lemma 2.7. □

Lemma 2.9. *Assume that $n \geq 4$ is even. Then, $\Delta_2(G) = \Delta(G)$.*

Proof. Since $\Gamma_2(G) \subseteq \Gamma(G)$, we have $\Delta_2(G) \leq \Delta(G)$. Now, we want to show that $\Delta(G) \leq \Delta_2(G)$. Let $S \in \Gamma(G)$. Hence, S is a subset of G that normally generates G. Lemma 2.7 shows that S must contain an involution $u \in G$ and some $v \in G$ such that $u \neq v$ and $\{u, v\}$ normally generates G. Since $\{u, v\} \subseteq S$, we have $\|G\|_S \leq \|G\|_{\{u,v\}}$. Set $T := \{u, v\}$. We have $T \in \Gamma_2(G)$ and $\|G\|_S \leq \|G\|_T$. Thus,

$$\begin{aligned}\Delta(G) &= \sup\{\|G\|_S \mid S \in \Gamma(G)\}\\ &\leq \sup\{\|G\|_T \mid T \in \Gamma_2(G)\}\\ &= \Delta_2(G).\end{aligned}$$

□

The following result is easy to check.

Lemma 2.10. *If m_1, m_2, m_3, and $m_4 \in \mathbb{Z}$, then the following holds:*

(i) $a^{m_1}b \cdot a^{m_2}b = a^{m_1-m_2}$,

(ii) $a^{m_3} \cdot a^{m_4} = a^{m_3+m_4}$,

(iii) $a^{m_1}b \cdot a^{m_3} = a^{m_1-m_3}b$,

(iv) $a^{m_3} \cdot a^{m_1}b = a^{m_3+m_1}b$.

Remark 2.11. *Let n be a natural number.*

1) Assume that $n \geq 4$ is even and that $1 \leq m < n$ is odd. Set $x := a^m b, y := b$ and $z := a^{n/2}$. Theorem 2.3 (3) shows that G has precisely three conjugacy classes of involutions with representatives x, y and z. We know that $z = a^{\frac{n}{2}} \in Z(G)$ and thus has only one element in its conjugacy class. A direct calculation shows that

$$\{a^v b \mid 0 \leq v \leq n-1 \text{ is even}\}$$

is the conjugacy class of y. As G has only three conjugacy classes of involutions, it follows that

$$\{a^{\hat{o}} b \mid 0 < \hat{o} \leq n-1 \text{ is odd}\}$$

is the conjugacy class of x.

2) Let $x = a^r$, where $1 \leq r \leq n-1$ is fixed. For any $g \in G$, we have $g^{-1} \cdot x \cdot g$ is either a^r or a^{-r}.

Proof of Theorem 1.5. Let $G := D_n = \langle a, b | a^n = 1 = b^2, bab = a^{-1}\rangle$, where $n \geq 3$ is a natural number. Let $0 < \hat{o_1}, \hat{o_2} \leq n-1$ be fixed and odd, $0 \leq v_1 \leq n-1$ be fixed and even, and $0 \leq \ell \leq n-1$ be fixed. If n is odd, we set:

$$S_1 := \{a^\ell b\}.$$

If n is even, we set the following:

$$S_2 := \{a^{\hat{o_1}} b, a^{v_1} b\},$$

$$S_3 := \{a^{\hat{o_1}} b, a^{\hat{o_2}}\},$$

$$S_4 := \{a^{v_1} b, a^{\hat{o_2}}\}.$$

If n is odd, then $\Delta_1(G) = \Delta(G)$ by Lemma 2.6. By Lemma 2.5, any normally generating subset in $\Gamma_1(G)$ consists of an involution; hence, we may assume without loss of generality that such a subset equals S_1. Thus, we need to consider $||G||_{S_1}$ in order to find $\Delta(G)$. If n is even, then Lemmas 2.7 and 2.9 show that we need to consider $||G||_{S_i}$, where $2 \leq i \leq 4$.

In the sense of (1.1), and in view of Theorem 2.3 and Remark 2.11, if n is odd, we have

$$C_1 := \mathrm{Conj}_G(S_1^{\pm 1}) = \{a^k b \big| 0 \leq k \leq n-1\}.$$

If n is even, we have

$$C_2 := \mathrm{Conj}_G(S_2^{\pm 1}) = \{a^k b \big| 0 \leq k \leq n-1\},$$

$$C_3 := \mathrm{Conj}_G(S_3^{\pm 1}) = \{a^{\hat{o}} b \big| 0 < \hat{o} \leq n-1 \text{ is odd}\} \cup \{a^{\pm \hat{o}_2}\},$$

$$C_4 := \mathrm{Conj}_G(S_4^{\pm 1}) = \{a^{v} b \big| 0 \leq v < n-1 \text{ is even}\} \cup \{a^{\pm \hat{o}_2}\}.$$

For the reader's convenience, and to make the proof easy to follow, we set the following:

$$\hat{C}_1 := G \setminus C_1 = \langle a \rangle,$$

$$\hat{C}_2 := G \setminus C_2 = \langle a \rangle,$$

$$\hat{C}_3 := G \setminus C_3 = \{a^{v} b \big| 0 \leq v < n-1 \text{ is even}\} \cup \langle a \rangle \setminus \{a^{\pm \hat{o}_2}\},$$

$$\hat{C}_4 := G \setminus C_4 = \{a^{\hat{o}} b \big| 0 < \hat{o} \leq n-1 \text{ is odd}\} \cup \langle a \rangle \setminus \{a^{\pm \hat{o}_2}\}.$$

Next, we study $||G||_{S_i}$, where $1 \leq i \leq 4$.

Case 1: If n is odd and $n \geq 3$, we show that

$$||G||_{S_1} = 2.$$

For every $g \in C_1$, we have

$$||g||_{S_1} = 1.$$

Now, suppose that $g \in \hat{C}_1 \setminus \{1\}$. Then, g can be written as a product of the following form

$$a^k = a^k b \cdot b \in B_{S_1}(2)\,, \text{where } 0 < k \leq n-1.$$

Now, let $g = 1$; then, by convention, $\{1\} = B_{S_1}(0)$ and, thus, $||g||_{S_1} = 0$. Hence, $||G||_{S_1} = 2$ when n is odd and $n \geq 3$. In view of this and Lemmas 2.5 and 2.6, we have $\Delta(G) = \Delta_1(G) = 2$.

Case 2: If n is even and $n \geq 4$:

(i) We show that

$$||G||_{S_2} = 2.$$

For every $g \in C_2$, we have

$$||g||_{S_2} = 1.$$

Now, suppose that $g \in \hat{C}_2 \setminus \{1\}$. Then, g can be written as a product of the following form

$$a^k = a^k b \cdot b \in B_{S_2}(2)\,, \text{where } 0 < k \leq n-1.$$

Now, let $g = 1$; then, by convention, $\{1\} = B_{S_2}(0)$ and, thus, $||g||_{S_2} = 0$. Hence, $||G||_{S_2} = 2$ when n is even and $n \geq 4$.

(ii) We show that

$$\|G\|_{S_3} = \begin{cases} 2 & \text{if } n = 4, \\ 3 & \text{otherwise.} \end{cases}$$

For every $g \in C_3$, we have

$$\|g\|_{S_3} = 1.$$

Now, suppose that $g \in \hat{C}_3 \setminus \{1\}$. Then, g is either

$$a^v b = a^{\hat{o}_2} \cdot a^{\hat{o}} b \in B_{S_3}(2),$$

where $0 \leq v \leq n-1$ is even and $\hat{o} = v - \hat{o}_2$,
or

$$g \in \langle a \rangle \setminus \{a^{\pm \hat{o}_2}\}.$$

If the former holds, then $\|g\|_{S_3} = 2$. Assume now $g \in \langle a \rangle \setminus \{a^{\pm \hat{o}_2}\}$.

First, if $g = a^v$, where $0 < v \leq n-1$ is even, g can be written as

$$g = a^{\hat{o}} b \cdot ab = a^{\hat{o}-1} \in B_{S_3}(2),$$

where $0 < \hat{o} \leq n-1$ is odd. Thus, $\|g\|_{S_3} = 2$.

Second, if $g = a^{\hat{o}}$, where $0 < \hat{o} \leq n-1$ is odd such that $g \notin \{a^{\pm \hat{o}_2}\}$, g can be written as

$$g = a^v b \cdot ab = a^{v-1} \in B_{S_3}(2) \cdot B_{S_3}(1) = B_{S_3}(3),$$

where $0 < v \leq n-1$ is even. We see from Lemma 2.10 that g cannot be written as a product of exactly two elements of C_3. Thus, $\|g\|_{S_3} = 3$.

To summarize, if $g \in \langle a \rangle \setminus \{a^{\pm \hat{o}_2}\}$, then

$$\|g\|_{S_3} = \begin{cases} 3 & \text{if } g = a^{\hat{o}} \in G \setminus \{a^{\pm \hat{o}_2}\}, \text{ where } 0 < \hat{o} \leq n-1 \text{ is odd,} \\ 2 & \text{otherwise.} \end{cases}$$

Notice that if $n = 4$, then $\|g\|_{S_3} = 2$ since $\{a^1, a^3\} = \{a^{\pm \hat{o}_2}\}$.

Now, let $g = 1$; then, by convention, $\{1\} = B_{S_3}(0)$ and, thus, $\|g\|_{S_3} = 0$. Hence, $\|G\|_{S_3} = 2$ when $n = 4$ and $\|G\|_{S_3} = 3$ when n is even and $n > 4$.

(iii) We show that

$$\|G\|_{S_4} = \begin{cases} 2 & \text{if } n = 4, \\ 3 & \text{otherwise.} \end{cases}$$

For every $g \in C_4$, we have

$$\|g\|_{S_4} = 1.$$

Now, suppose that $g \in \hat{C}_4 \setminus \{1\}$. Then, g is either

$$a^{\hat{o}} b = a^{\hat{o}_2} \cdot a^v b \in B_{S_4}(2),$$

where $0 < \hat{o} \leq n-1$ is odd and $v = \hat{o} - \hat{o}_2$, or

$$g \in \langle a \rangle \setminus \{a^{\pm \hat{o}_2}\}.$$

If the former holds, then $\|g\|_{S_4} = 2$. Assume now that $g \in \langle a \rangle \setminus \{a^{\pm \hat{o}_2}\}$.

First, if $g = a^v$, where $0 < v < n - 1$ is even, g can be written as

$$g = a^v b \cdot b \in B_{S_4}(2),$$

where $0 < v < n - 1$ is even. Thus, $\|g\|_{S_4} = 2$.

Second, if $g = a^{\hat{o}}$, where $0 < \hat{o} \leq n - 1$ is odd such that $g \notin \{a^{\pm \hat{o}_2}\}$, g can be written as

$$g = a^v \cdot a^{\hat{o}_2} \in B_{S_4}(2) \cdot B_{S_4}(1) = B_{S_4}(3),$$

where $v = \hat{o} - \hat{o}_2$. We see from Lemma 2.10 that g cannot be written as a product of exactly two elements of C_4. Thus, $\|g\|_{S_4} = 3$.

To summarize, if $g \in \langle a \rangle \setminus \{a^{\pm \hat{o}_2}\}$, then

$$\|g\|_{S_4} = \begin{cases} 3 & \text{if } g = a^{\hat{o}} \in G \setminus \{a^{\pm \hat{o}_2}\}, \text{where } 0 < \hat{o} \leq n - 1 \text{ is odd,} \\ 2 & \text{otherwise.} \end{cases}$$

Notice that if $n = 4$, then $\|g\|_{S_4} = 2$ since $\{a^1, a^3\} = \{a^{\pm \hat{o}_2}\}$.

Now, let $g = 1$; then, by convention, $\{1\} = B_{S_4}(0)$ and, thus, $\|g\|_{S_4} = 0$. Hence, $\|G\|_{S_4} = 2$ when $n = 4$ and $\|G\|_{S_4} = 3$ when n is even and $n > 4$.

In view of (i)–(iii) and Lemma 2.7, we have $\Delta_2(G) = 2$ if $n = 4$ and $\Delta_2(G) = 3$ if n is even and $n > 4$. So, the result follows from the fact that $\Delta_2(G) = \Delta(G)$ (see Lemma 2.9). □

3. Conclusions

For any natural number $n \geq 3$, we have determined the conjugacy diameters of the finite dihedral groups D_n. Our main result shows that $\Delta(D_n) = 2$ when n is odd or $n = 4$, and $\Delta(D_n) = 3$ for even $n \geq 6$. These findings complement earlier studies on conjugacy diameters for other families of finite groups, such as symmetric, semidihedral, generalized quaternion, and modular p-groups, and contribute to a more complete understanding of how conjugation diameters behave across non-abelian groups.

The methods developed here may also be adapted to the infinite dihedral group. Although it is known that D_∞ is uniformly bounded and that $\Delta(D_\infty) \leq 4$ (see [11, Example 2.8]), our approach may help determine the exact value of $\Delta(D_\infty)$. The techniques used in this paper rely on a fairly complete description of the group structure, in particular of its conjugacy classes. For groups whose internal structure is not as transparent as that of the dihedral groups, our methods may be less effective or require significant modification. However, for families of groups that share structural features with dihedral groups and for which the conjugation structure is well understood, the present approach might remain applicable and could yield comparable results. Thus, further investigations may address classes of groups sharing structural similarities with dihedral groups, for instance dicyclic, metacyclic, or certain Coxeter groups. Because these groups often admit presentations involving involutions and conjugation relations, they form natural candidates for extending the present analysis. It would also be interesting to develop algorithmic or computational tools for estimating or computing conjugacy diameters in larger finite groups, particularly those with complex generating sets. Such investigations could shed light on broader classification problems and on the computational complexity of tasks related to word norms and normal generation.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflict of interest in this paper.

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