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Research article

Eigenvalue ratios for the conformable fractional vibrating string equations with single-well densities

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Abstract: In this paper, we investigated in a class of Sturm-Liouville vibrating string problems involving conformable fractional derivatives and studied the behavior of their eigenvalue ratios. Under the assumptions that the potential function is of single-barrier type or the density function is of single-well type, we rigorously established the optimal upper bound for the eigenvalue ratios λ_n/λ_m, namely

$$\frac{\lambda_n}{\lambda_m} \le \left(\frac{n}{m}\right)^2 \quad (n > m).$$

Moreover, we showed that equality holds if and only if the density or the potential is constant. Our novelty was in extending the classical Sturm–Liouville eigenvalue inequalities to the conformable fractional setting.

Keywords: eigenvalue ratio; conformable fractional; Sturm-Liouville; Prüfer transformation; single-well
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1. Introduction

Let $\alpha \in (0, 1]$, and consider the conformable fractional Sturm–Liouville problem

$$-D_x^\alpha D_x^\alpha u + q(x)\, u = \lambda V(x)\, u, \quad x \in (0, 1), \tag{1.1}$$

subject to the Dirichlet boundary condition

$$u(0) = u(1) = 0, \tag{1.2}$$

where V and q are continuous on $[0, 1]$ and satisfy $q(x) \ge 0$ and $V(x) > 0$. Specially, if $q \equiv 0$, we say

$$-D_x^\alpha D_x^\alpha u = \lambda V(x) u, \quad x \in (0, 1), \tag{1.3}$$

is a conformable fractional vibrating string problem.

By referring to Lemma 2.4 in [5, 12], we know the eigenvalues in (1.1)–(1.2) can be arranged as:

$$0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots \to \infty,$$

and the corresponding eigenfunction u_n has exactly $(n-1)$ zeros in $(0,1)$.

For classical Sturm-Liouville problems, the issue of obtaining optimal estimates for the eigenvalue ratios $\frac{\lambda_n}{\lambda_m}$ has garnered significant attention [2, 3, 6–8]. Ashbaugh [3] demonstrated that

$$\frac{\lambda_n}{\lambda_1} \le \frac{Kn^2}{k},$$

under the conditions $q \ge 0$ and $0 < k \le pq(x) \le K$. In addition, Huang [7] studied the eigenvalue problem for the vibrating string equation

$$-u'' = \lambda V(x)u$$

with Dirichlet boundary conditions satisfying $\frac{\lambda_2}{\lambda_1} \le 4$ for symmetric single-well density V and $\frac{\lambda_2}{\lambda_1} \ge 4$ for symmetric single-barrier density V. In 2006, Kiss in [11] further showed that $\frac{\lambda_n}{\lambda_1} \le n^2$ for symmetric single-well densities and $\frac{\lambda_n}{\lambda_1} \ge n^2$ for symmetric single-barrier densities.

Khalil was the first to study the conformable fractional derivative D_x^α. For further exploration and key findings, referring to works like [1] and [4] is beneficial. Mortazaasl and Akbarfam, in [12], have focused on scrutinizing the trace formula of eigenvalues. They investigated in the inverse nodal problem of the conformable fractional Sturm-Liouville equation, accompanied by conformable fractional boundary conditions of a specific order denoted by α, where $0 < \alpha \le 1$.

In this work, we prove that, under the single-well assumption on the density V specified in (1.3)–(1.2), the corresponding eigenvalues satisfy the sharp ratio estimate $\frac{\lambda_n}{\lambda_m} \le \frac{n^2}{m^2}$. The key step in proving this bound involves the inequality $\frac{\lambda_n(V)}{\lambda_m(V)} \le \frac{\lambda_n(S)}{\lambda_m(S)}$, where S is a step function. Additionally, we extend our analysis to the conformable fractional Dirichlet Sturm-Liouville problems described in equations (1.1)–(1.2). More precisely, under the assumptions that q is single-barrier and V is single-well with transition point $x_0 = \frac{1}{2}$, and that $0 < \min\{\hat{\nu}_1, \tilde{\nu}_1\}$, where $\hat{\nu}_1$ and $\tilde{\nu}_1$ denote the first eigenvalues of the Neumann problems on $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, respectively, we obtain the sharp inequality $\frac{\lambda_n}{\lambda_m} \le \frac{n^2}{m^2}$. The proof proceeds by applying an inverse Liouville transformation, which converts (1.1) into the string equation (1.3) with a single-well density. We then prove the eigenvalue ratio estimates for string equations with single-well densities established in Section 2 to conclude the argument.

2. Eigenvalue ratios for the vibrating string problems

In this section, we recall the fundamental notions and basic properties [1, 5] of conformable fractional calculus and state several auxiliary lemmas that will be used in the proofs of our major theorems.

Definition 2.1. [5] Let $f : [0,\infty) \to \mathbb{R}$ be a given function and let $\alpha \in (0,1]$.

(1) The conformable fractional derivative of f of order α is defined by

$$D_x^\alpha f(x) \equiv \lim_{h\to 0} \frac{f(x + hx^{1-\alpha}) - f(x)}{h}, \quad D_x^\alpha f(0) = \lim_{x\to 0^+} D_x^\alpha f(x)$$

for all $x > 0$.

(2) The conformable fractional integral of f of order α is defined by

$$I_\alpha f(x) \equiv \int_0^x f(t) d_\alpha t = \int_0^x t^{\alpha-1} f(t) dt$$

for all $x > 0$. Here, the last integral is the usual Riemann improper integral.

Theorem 2.2. *[5] Let f, g be α-differentiable in $x \in (0, \infty)$.*

(1) $D_x^\alpha(af + bg) = aD_x^\alpha f + bD_x^\alpha g$ for all a, $b \in \mathbb{R}$.

(2) $D_x^\alpha(x^p) = px^{p-\alpha}$ for $p \in \mathbb{R}$.

(3) $D_x^\alpha(c) = 0$ for $c \in \mathbb{R}$.

(4) $D_x^\alpha(fg) = gD_x^\alpha f + fD_x^\alpha g$.

(5) $D_x^\alpha\left(\frac{f}{g}\right) = \frac{gD_x^\alpha f - fD_x^\alpha g}{g^2}$.

(6) If f is differentiable, then $D_x^\alpha f(x) = x^{1-\alpha} f'(x)$.

In this paper, we use $\{\lambda_n, u_n\}_{n\in\mathbb{N}}$ to denote the normalized eigenpairs, i.e., $\int_0^1 V(x)u_n^2(x)d_\alpha x = 1$. From [5], it follows that u_n has exactly $(n-1)$ zeros in the open interval $(0, 1)$ and the zeros of the $n-th$ and $(n-1)-th$ eigenfunctions interlace. We denote by (a_i) the zeros of u_n and (b_i) the zeros of u_{n-1}, and then we have $a_i < b_i$. For convenience, we may assume that $u_n > 0$, $u_{n-1} > 0$ in $(0, a_1)$, so we have that $\frac{u_n}{u_{n-1}}$ is strictly decreasing in $(0, 1)$. Let $\phi(x) = \frac{u_n(x)}{u_{n-1}(x)}$, so that,

$$D_x^\alpha \phi(x) = \frac{1}{u_{n-1}^2(x)} \left[u_{n-1}(x)D_x^\alpha u_n(x) - u_n(x)D_x^\alpha u_{n-1}(x)\right].$$

Let,

$$w(x) = u_{n-1}(x)D_x^\alpha u_n(x) - u_n(x)D_x^\alpha u_{n-1}(x),$$

then,

$$\begin{aligned} D_x^\alpha w(x) =& D_x^\alpha u_{n-1}(x)D_x^\alpha u_n(x) + u_{n-1}(x)D_x^\alpha D_x^\alpha u_n(x) \\ &- D_x^\alpha u_n(x)D_x^\alpha u_{n-1}(x) - u_n(x)D_x^\alpha D_x^\alpha u_{n-1}(x) \\ =& u_{n-1}(x)(-\lambda_n V(x)u_n(x)) - u_n(x)(-\lambda_{n-1}(x)V(x)u_{n-1}(x)) \\ =& (\lambda_{n-1} - \lambda_n)V(x)u_n(x)u_{n-1}(x). \end{aligned}$$

Since $D_x^\alpha w(x) = x^{1-\alpha}w'(x)$, this implies that $w(x) < 0$ in $(0, 1)$. Moreover, $D_x^\alpha \phi(x) = x^{1-\alpha}\phi'(x)$, i.e., $\phi'(x) < 0$. Hence, $\frac{u_n(x)}{u_{n-1}(x)}$ is strictly decreasing in $(0, 1)$.

Therefore, we have points $x_i \in (a_i, b_i)$, where $u_n^2(x_i) = u_{n-1}^2(x_i)$, such that

$$\begin{cases} u_n^2(x) > u_{n-1}^2(x), & x \in (x_{2i}, x_{2i+1}), \\ u_n^2(x) < u_{n-1}^2(x), & x \in (x_{2i+1}, x_{2i+2}). \end{cases}$$

Lemma 2.3. *For Eq* (1.3)*, let* $V(x, s)$ *be a one-parameter family of piecewise continuous functions such that* $\frac{\partial}{\partial s}V(x, s)$ *exist. Denote by* $\{\lambda_n(s), u_n(x, s)\}_{n\in\mathbb{N}}$ *as the normalized eigenpairs. Then we have:*

$$\frac{d}{ds}\lambda_n(s) = -\lambda_n(s)\int_0^1 \frac{\partial V}{\partial s}(x, s)u_n^2(x, s)d_\alpha x. \tag{2.1}$$

Moreover,

$$\frac{d}{ds}\frac{\lambda_n(s)}{\lambda_m(s)} = \frac{\lambda_n(s)}{\lambda_m(s)}\int_0^1 \frac{\partial V}{\partial s}(x, s)\left(u_m^2(x, s) - u_n^2(x, s)\right)d_\alpha x. \tag{2.2}$$

Proof. For Eq (2.1), we refer to [5]. Using (2.1), one can directly calculate

$$\begin{aligned}\frac{d}{ds}\left(\frac{\lambda_n(s)}{\lambda_m(t)}\right) =&\frac{1}{\lambda_m^2(s)}\left(\lambda_m(s)\frac{d}{ds}\lambda_n(s) - \lambda_n(s)\frac{d}{ds}\lambda_m(s)\right)\\ =&\frac{1}{\lambda_m^2(s)}\Big(- \lambda_n(s)\lambda_m(s)\int_0^1 \frac{\partial V}{\partial s}(x, s)u_n^2(x, s)d_\alpha x\\ &+ \lambda_m(s)\lambda_n(s)\int_0^1 \frac{\partial V}{\partial s}(x, s)u_m^2(x, s)d_\alpha x\Big)\\ =&\frac{\lambda_n(s)}{\lambda_m(s)}\int_0^1 \frac{\partial V}{\partial t}(x, s)\left(u_m^2(x, s) - u_n^2(x, s)\right)d_\alpha x.\end{aligned}$$

□

Let $u(x, z)$ denote the unique solution to the initial value problem given by

$$\begin{cases} - D_x^\alpha D_x^\alpha u = z^2V(x)u, \quad x \in (0, 1), \quad z > 0,\\ u(0) = 0, \quad D_x^\alpha u(0) = V^{\frac{1}{4}}(0), \end{cases} \tag{2.3}$$

We apply system (2.3), the Prüfer transformation similar to [9, 11],

$$u(x, z) = \frac{w(x, z)}{z}V^{-\frac{1}{4}}sin\psi(x, z), \tag{2.4}$$

$$D_x^\alpha u(x, z) = w(x, z)V^{\frac{1}{4}}cos\psi(x, z), \tag{2.5}$$

$$\psi(0, z) = 0, \tag{2.6}$$

where $w(x, z) > 0$, and then let $\theta(x, z) = \frac{\psi(x,z)}{z}$. Here and after, we denote by prime (resp. dot) the derivative with respect to x (resp. z).

From (2.4) and (2.5), we can directly calculate:

$$D_x^\alpha u(x, z) = x^{1-\alpha}\left(\frac{w'}{z}V^{-\frac{1}{4}}sin\psi - \frac{1}{4}\frac{w}{z}V^{-\frac{5}{4}}V'sin\psi + \frac{w}{z}V^{-\frac{1}{4}}cos\psi\cdot\psi'\right) = wV^{\frac{1}{4}}cos\psi \tag{2.7}$$

⇒

$$\frac{w'}{w} = \frac{1}{x^{1-\alpha}}zV^{\frac{1}{2}}cot\psi - \psi'cot\psi + \frac{1}{4}V^{-1}V' \tag{2.8}$$

By (2.4), (2.5), and (2.3),

$$
\begin{aligned}
-D_x^\alpha D_x^\alpha u(x,z) = -x^{1-\alpha}\left(w'V^{\frac{1}{4}}cos\psi + \frac{1}{4}wV^{-\frac{3}{4}}V'cos\psi - wV^{\frac{1}{4}}sin\psi\cdot\psi'\right) &= wV^{\frac{3}{4}}wsin\psi \\
-x^{1-\alpha}\left(\frac{w'}{w}V^{\frac{1}{4}}cos\psi + \frac{1}{4}V^{-\frac{3}{4}}V'cos\psi - V^{\frac{1}{4}}sin\psi\cdot\psi'\right) &= zV^{\frac{3}{4}}sin\psi
\end{aligned}
\tag{2.9}
$$

Substituting (2.8) into (2.9), we have:

$$
\begin{aligned}
-x^{1-\alpha}\left(\frac{1}{x^{1-\alpha}}zV^{\frac{1}{2}}cot\psi - \psi'cot\psi + \frac{1}{4}V^{-1}V'\right)V^{\frac{1}{4}}cos\psi & \\
-\frac{x^{1-\alpha}}{4}V^{-\frac{3}{4}}V'cos\psi + x^{1-\alpha}V^{\frac{1}{4}}sin\psi\cdot\psi' &= zV^{\frac{3}{4}}sin\psi \\
\left(-zV^{\frac{1}{2}}cot\psi + x^{1-\alpha}\psi'cot\psi - \frac{x^{1-\alpha}}{4}V^{-1}V'\right)V^{\frac{1}{4}}cos\psi & \\
-\frac{x^{1-\alpha}}{4}V^{-\frac{3}{4}}V'cos\psi + x^{1-\alpha}V^{\frac{1}{4}}sin\psi\cdot\psi' &= zV^{\frac{3}{4}}sin\psi \\
-zV^{\frac{3}{4}}cot\psi cos\psi + x^{1-\alpha}\psi'V^{\frac{1}{4}}cot\psi cos\psi - \frac{x^{1-\alpha}}{4}V^{-\frac{3}{4}}V'cos\psi & \\
-\frac{x^{1-\alpha}}{4}V^{-\frac{3}{4}}V'cos\psi + x^{1-\alpha}V^{\frac{1}{4}}sin\psi\cdot\psi' &= zV^{\frac{3}{4}}sin\psi \\
-zV^{\frac{3}{4}}cos^2\psi + x^{1-\alpha}\psi'V^{\frac{1}{4}}cos^2\psi - \frac{x^{1-\alpha}}{2}V^{-\frac{3}{4}}V'cos\psi sin\psi + x^{1-\alpha}V^{\frac{1}{4}}sin^2\psi\cdot\psi' &= zV^{\frac{3}{4}}sin^2\psi \\
x^{1-\alpha}\psi'V^{\frac{1}{4}}cos^2\psi - \frac{x^{1-\alpha}}{2}V^{-\frac{3}{4}}V'cos\psi sin\psi + x^{1-\alpha}V^{\frac{1}{4}}sin^2\psi\cdot\psi' &= zV^{\frac{3}{4}} \\
x^{1-\alpha}V^{\frac{1}{4}}\cdot\psi' - \frac{x^{1-\alpha}}{4}V^{-\frac{3}{4}}sin2\psi &= zV^{\frac{3}{4}}
\end{aligned}
$$

$\Rightarrow$

$$
\psi' = \frac{1}{x^{1-\alpha}}zV^{\frac{1}{2}} + \frac{1}{4}V^{-1}V'sin2\psi \tag{2.10}
$$

Moreover, we can rewrite (2.8) as:

$$
\frac{w'}{w} = -\frac{1}{4}V^{-1}V'cos2\psi \tag{2.11}
$$

Lemma 2.4.

$$
\dot{\psi} = \int_0^x \frac{V^{\frac{1}{2}}(s)}{s^{1-\alpha}}\cdot\frac{w^2(s,z)}{w^2(x,z)}ds. \tag{2.12}
$$

Proof. Differential equation (2.10) with respect to z:

$$
\begin{aligned}
\dot{\psi}' &= \frac{1}{x^{1-\alpha}}V^{\frac{1}{2}} + \frac{1}{4}V^{-1}V'cos2\psi\cdot 2\dot{\psi} \\
&= \frac{1}{x^{1-\alpha}}V^{\frac{1}{2}} - \frac{w'}{w}\cdot 2\dot{\psi},
\end{aligned}
$$

Multiplying both sides by $e^{\int_0^x \frac{w'}{w}dt}$ results in:

$$
\dot{\psi} = \int_0^x \frac{V^{\frac{1}{2}}(s)}{s^{1-\alpha}}\cdot\frac{w^2(s,z)}{w^2(x,z)}ds.
$$

□

We now present a key computation that is essential to the proof of Proposition 2.7.

Lemma 2.5.

$$\dot{\theta}(x,z) = \frac{1}{z^2 w^2(x,z)} \int_0^x w^2(s)\left[2z\frac{V^{\frac{1}{2}(s)}}{s^{1-\alpha}} + \frac{1}{4}V^{-1}(s)V'(s)(sin2\psi + 2\psi(s)cos2\psi)\right]ds$$

Proof.

$$\begin{aligned}
\dot{\theta}(x,z) =& \frac{\dot{\psi}}{z} - \frac{\psi}{z^2} \\
=& \frac{1}{z}\int_0^x \frac{V^{\frac{1}{2}}(s)}{s^{1-\alpha}} \cdot \frac{w^2(s,z)}{w^2(x,z)}ds - \frac{\psi}{z^2} \\
=& \frac{1}{z^2 w^2(x,z)}\left[\int_0^x z\frac{V^{\frac{1}{2}}(s)}{s^{1-\alpha}} w^2(s,z)ds - w^2(x,z)\psi(x,z)\right] \\
=& \frac{1}{z^2 w^2(x,z)}\left[\int_0^x z\frac{V^{\frac{1}{2}}(s)}{s^{1-\alpha}} w^2(s,z)ds - 2\int_0^x w(s)w'(s)\psi(s,z)ds + \int_0^x w^2(s)\psi'(s,z)ds\right] \\
=& \frac{1}{z^2 w^2(x,z)}\left[\int_0^x w^2(s,z)\left(z\frac{V^{\frac{1}{2}}(s)}{s^{1-\alpha}} + \psi'(s,z)\right)ds - 2\int_0^x w(s)w'(s)\psi(s,z)ds\right] \\
=& \frac{1}{z^2 w^2(x,z)}\left[\int_0^x w^2(s,z)\left(z\frac{V^{\frac{1}{2}}(s)}{s^{1-\alpha}} + \psi'(s,z)\right)ds - 2\int_0^x w^2(s,z)\frac{w'(s,z)}{w(s,z)}\psi(s,z)ds\right] \\
=& \frac{1}{z^2 w^2(x,z)}\Bigg[\int_0^x w^2(s,z)\left(z\frac{V^{\frac{1}{2}}(s)}{s^{1-\alpha}} + \psi'(s,z)\right)ds \\
& + \frac{2}{4}\int_0^x w^2(s,z)V^{-1}(s,z)V'(s,z)\psi(s,z)cos2\psi ds\Bigg] \\
=& \frac{1}{z^2 w^2(x,z)}\int_0^x w^2(s)\left[2z\frac{V^{\frac{1}{2}(s)}}{s^{1-\alpha}} + \frac{1}{4}V^{-1}(s)V(s)(sin2\psi + 2\psi(s)cos2\psi)\right]ds
\end{aligned}$$

The step from the first to the second equality is justified by Lemma 2.4. □

Now, we carry out our major results.

Proposition 2.6. *For Eq* (1.3)*, assume that the V is positive and monotone decreasing on* $[0,1]$*. Let $\{x_i\}$ denote the points at which $u_n^2(x_i) = u_{n-1}^2(x_i)$, and define*

$$S(x) := V(x_{2i+1}).$$

Then,

$$\frac{\lambda_n(V)}{\lambda_m(V)} \le \frac{\lambda_n(S)}{\lambda_m(S)},$$

and equality holds if and only if $V \equiv S$.

Proof. Define $V(x,s) = sV(x) + (1-s)S(x)$. Using (2.2), we have

$$\frac{d}{ds}\left(\frac{\lambda_n(s)}{\lambda_{n-1}(s)}\right) = \frac{\lambda_n(s)}{\lambda_{n-1}(s)} \int_0^1 \frac{\partial V}{\partial s}(x,s)\left(u_{n-1}^2(x,s) - u_n^2(x,s)\right) d_\alpha x.$$
$$= \frac{\lambda_n(s)}{\lambda_{n-1}(s)} \sum_{i=0}^{n} \int_{x_{2i}}^{x_{2i+2}} (V(x) - S(x))\left(u_{n-1}^2(x,s) - u_n^2(x,s)\right) d_\alpha x.$$

We notice that

$$\int_{x_{2i}}^{x_{2i+2}} (V(x) - S(x))\left(u_{n-1}^2(x,s) - u_n^2(x,s)\right) d_\alpha x < 0.$$

It then follows that $\frac{d}{ds}\left(\frac{\lambda_n(s)}{\lambda_{n-1}(s)}\right) \leq 0$. Thus, using the continuity of the eigenvalues with respect to the parameter, we have

$$\frac{\lambda_n(V)}{\lambda_{n-1}(V)} = \frac{\lambda_n(1)}{\lambda_{n-1}(1)} \leq \frac{\lambda_n(0)}{\lambda_{n-1}(0)} = \frac{\lambda_n(S)}{\lambda_{n-1}(S)}.$$

$\Rightarrow$

$$\frac{\lambda_n(V)}{\lambda_m(V)} \leq \frac{\lambda_n(S)}{\lambda_m(S)}$$

Equality holds if and only if $V \equiv S$. □

Proposition 2.7. *For the problem* (1.3)–(1.2)*, assume that the density V is positive and nonincreasing on* [0, 1]*. Then the corresponding eigenvalues* $\{\lambda_k\}$ *satisfy, for every pair of indices* $1 \leq m < n$*,*

$$\frac{\lambda_n}{\lambda_m} \leq \frac{n^2}{m^2},$$

and equality holds if and only if V is constant on [0, 1].

Proof. Given that $V(x) = V(x_{2i+1})$ for all $x \in (x_{2i}, x_{2i+2})$, we have $V' \equiv 0$ for all $x \in (x_{2i}, x_{2i+2})$. By utilizing Lemma 2.5, we can deduce that:

$$\dot{\theta}(x,z) = \frac{2}{zw^2(x,z)} \int_0^x \frac{w^2(s)}{s^{1-\alpha}} V^{\frac{1}{2}(s)} ds \geq 0.$$

Hence, $\dot{\theta}(x,z) \geq 0$. Suppose m is less than n, then $\theta(z_m) = \frac{m\pi}{z_m} \leq \frac{n\pi}{z_n} = \theta(z_n)$, where z_m and z_n denote the spectral parameters corresponding to λ_m and λ_n (2.3). This implies $\frac{z_n}{z_m} \leq \frac{n}{m}$ and $\frac{\lambda_n(V)}{\lambda_m(V)} \leq \frac{n^2}{m^2}$. Consequently, using Proposition 2.6, we can derive:

$$\frac{\lambda_n}{\lambda_m} \leq \frac{n^2}{m^2}.$$

The equality holds if and only if $\frac{\lambda_n(s)}{\lambda_m(s)}$ is a constant. From (2.11), we obtain $\dot{\theta}(x,z) = 0$, and hence S must be constant. This completes the proof. □

Theorem 2.8. *For the Dirichlet problem* (1.3)–(1.2)*, suppose that V is a single-well density on* [0, 1]*. Then, for any m < n, the associated eigenvalues satisfy*

$$\frac{\lambda_n}{\lambda_m} \leq \frac{n^2}{m^2},$$

and this estimate is sharp in the sense that equality occurs if and only if V is constant on [0, 1].

Proof. Let us define $\tilde{S}(x) = S(1-x)$. It follows that $\tilde{S}(x)$ is decreasing on $[0, 1-x_0]$ and increasing on $[1-x_0, 1]$. Referring to Proposition 2.6 and Proposition 2.7, we derive:

$$\frac{\lambda_n}{\lambda_m} = \frac{\lambda_n(V)}{\lambda_m(V)} \leq \frac{\lambda_n(S)}{\lambda_m(S)} \leq \frac{n^2}{m^2}.$$

The equality holds if and only if V is a constant. □

3. Eigenvalue ratios for Sturm-Liouville problems

To apply the eigenvalue ratio estimate λ_n/λ_m for vibrating string equations with single-well densities established in Section 2, we first record some auxiliary material. For Eq (1.1), denote by $l(x,\lambda)$ the unique solution, satisfying

$$l\left(\frac{1}{2}\right) = 1, \quad D_x^\alpha l\left(\frac{1}{2}\right) = 0. \tag{3.1}$$

Define

$$L(x,\lambda) := \frac{D_x^\alpha l(x,\lambda)}{l(x,\lambda)}. \tag{3.2}$$

We denote by $\hat{\nu}_1$ and $\tilde{\nu}_1$ the first eigenvalues of (1.1) corresponding to the mixed boundary conditions

$$u(0) = D_x^\alpha u\left(\tfrac{1}{2}\right) = 0, \tag{3.3}$$

and

$$D_x^\alpha u\left(\tfrac{1}{2}\right) = u(1) = 0, \tag{3.4}$$

respectively.

Lemma 3.1.

(1) The function $L(1,\lambda)$ *is decreasing along on* $(-\infty, \tilde{\nu}_1)$.

(2) The function $L(0,\lambda)$ *is increasing along on* $(-\infty, \hat{\nu}_1)$.

Proof. Let $\lambda, \lambda' \in (-\infty, \tilde{\nu}_1)$ with $\lambda \neq \lambda'$, and denote by $l(x,\lambda)$ and $l(x,\lambda')$ the solutions of (1.1)–(3.1) corresponding to the parameters λ and λ', respectively. Multiplying the equation for $l(\cdot,\lambda)$ by $l(\cdot,\lambda')$, integrating over $[\frac{1}{2}, 1]$ with respect to $d_\alpha s$, and applying integration by parts, we obtain

$$\int_{\frac{1}{2}}^{1} \Big[(q-\lambda)(t)\, l(t,\lambda)\, l(t,\lambda') + D_x^\alpha l(t,\lambda)\, D_x^\alpha l(t,\lambda')\Big]\, d_\alpha t = D_x^\alpha l(1,\lambda)\, l(1,\lambda'). \tag{3.5}$$

which also gives,

$$\int_{\frac{1}{2}}^{1} \Big[(q-\lambda')(t)\, l(t,\lambda')\, l(t,\lambda) + D_x^\alpha l(t,\lambda')\, D_x^\alpha l(t,\lambda)\Big]\, d_\alpha t = D_x^\alpha l(1,\lambda')\, l(1,\lambda). \tag{3.6}$$

Subtracting (3.5) and (3.6), then we have

$$D_x^\alpha l(1,\lambda')\, l(1,\lambda) - D_x^\alpha l(1,\lambda)\, l(1,\lambda') = (\lambda-\lambda') \int_{\frac{1}{2}}^{1} l(t,\lambda)\, l(t,\lambda')\, d_\alpha t. \tag{3.7}$$

Dividing (3.7) by $\lambda' - \lambda$ and rearranging, we obtain

$$l(1,\lambda)\,\frac{D_x^\alpha l(1,\lambda') - D_x^\alpha l(1,\lambda)}{\lambda' - \lambda} - D_x^\alpha l(1,\lambda)\,\frac{l(1,\lambda') - l(1,\lambda)}{\lambda' - \lambda} = -\int_{\frac{1}{2}}^{1} l(t,\lambda)\,l(t,\lambda')\,d_\alpha t. \tag{3.8}$$

Let $\lambda' \to \lambda$ in (3.8), and we obtain

$$l(1,\lambda)\,\frac{\partial}{\partial\lambda}(D_x^\alpha l(1,\lambda)) - D_x^\alpha l(1,\lambda)\,\frac{\partial}{\partial\lambda} l(1,\lambda) = -\int_{\frac{1}{2}}^{1} l^2(t,\lambda)\,d_\alpha t. \tag{3.9}$$

Since $l(1,\lambda) \neq 0$ for all $\lambda \in (-\infty, \tilde{\nu}_1)$, we can divide both sides of (3.9) by $l^2(1,\lambda)$ and obtain

$$\frac{\partial L(1,\lambda)}{\partial\lambda} = -\frac{\int_{\frac{1}{2}}^{1} l^2(t,\lambda)\,d_\alpha t}{l^2(1,\lambda)} < 0, \tag{3.10}$$

which shows that $L(1,\lambda)$ is strictly decreasing on $(-\infty, \tilde{\nu}_1)$. The proof of the corresponding monotonicity of $L(0,\lambda)$ on $(-\infty, \hat{\nu}_1)$ is analogous. □

Theorem 3.2. *For the boundary value problem* (1.1)–(1.2)*, assume that q is a single-barrier potential and that V is a single-well function on* $[0,1]$ *with transition point $x_0 = \frac{1}{2}$, and suppose that*

$$0 < \min\{\hat{\nu}_1, \tilde{\nu}_1\},$$

where $\hat{\nu}_1$ and $\tilde{\nu}_1$ denote the first Neumann eigenvalues on $[0, \frac{1}{2}]$ *and* $[\frac{1}{2}, 1]$*, respectively. Then the Dirichlet eigenvalues satisfy, for any $m < n$,*

$$\frac{\lambda_n}{\lambda_m} \leq \frac{n^2}{m^2}, \tag{3.11}$$

and equality occurs if and only if $q \equiv 0$ and V is constant on $[0,1]$.

Proof. We begin by observing that, in the case $q \equiv 0$, the estimate (3.11) follows directly from Theorem 2.8, so we assume $q \not\equiv 0$. Now we set

$$\hat{\eta}_1 := \min\{\hat{\eta}_1, \tilde{\eta}_1\},$$

and let l denote the unique solution of

$$D_x^\alpha D_x^\alpha u = q(x)\,u \tag{3.12}$$

satisfying the initial conditions (3.1). By the hypothesis $0 < \min(\hat{\nu}_1, \tilde{\nu}_1)$ and the variational characterization of the first eigenvalue, we have

$$0 < \hat{\eta}_1 < \tilde{\nu}_1.$$

From the fact that $l(x, \hat{\nu}_1) > 0$ on $(0, \frac{1}{2}]$, we infer that the corresponding solution l is strictly positive on $[0, \frac{1}{2}]$. Moreover, $L(0, \hat{\eta}_1) = 0$, and Lemma 3.1 yields

$$L(0,\lambda) \leq 0 \qquad \text{for all } \lambda \in (-\infty, \hat{\eta}_1].$$

Since $\hat{\eta}_1 > 0$ and $l(0) > 0$, this implies

$$D_x^\alpha l(0) \le 0.$$

Next, use the structural condition on q. For $[0, \frac{1}{2}]$, the potential q is increasing and has at most one zero, say at $b_0 \in [0, \frac{1}{2})$. Combining this with (3.12) gives

$$D_x^\alpha D_x^\alpha l(x) \le 0 \quad \text{for } x \in (0, b_0], \qquad D_x^\alpha D_x^\alpha l(x) \ge 0 \quad \text{for } x \in [b_0, \tfrac{1}{2}],$$

so that $D_x^\alpha l$ is decreasing on $(0, a_0]$ and increasing on $[a_0, \frac{1}{2}]$. Together with $D_x^\alpha l(\frac{1}{2}) = 0$ and $D_x^\alpha l(0) \le 0$, we deduce

$$D_x^\alpha l(x) \le 0 \quad \text{for } x \in (0, \tfrac{1}{2}],$$

hence l is nonincreasing on $(0, \frac{1}{2}]$. A symmetric argument on $[\frac{1}{2}, 1]$ shows that l is nondecreasing there, and therefore l is a single-well function on $[0, 1]$.

We now introduce the inverse Liouville transformation

$$z(x) = \frac{1}{c} \int_0^x \frac{1}{l^2(s)}\, d_\alpha s, \qquad c = \int_0^1 \frac{1}{l^2(s)}\, d_\alpha s,$$

which maps the original problem (1.1)–(1.2) to

$$\begin{cases} -\ddot{u} = \tilde{\lambda}\, \tilde{l}^4(z)\, \rho(z)\, u, \quad z \in (0, 1), \\ u(0) = u(1) = 0, \end{cases} \tag{3.13}$$

where $y = ul$, $\tilde{l}(z) = l(x)$, and $\tilde{\lambda} = c^2 \lambda$. Since l is a single-well for $[0, 1]$, the same property holds for $\tilde{l}$ in the z–variable. Invoking the theorem in [9] for string equations with single-well densities, we obtain

$$\frac{\tilde{\lambda}_n}{\tilde{\lambda}_m} \le \frac{n^2}{m^2},$$

and, because $\tilde{\lambda}_k = c^2 \lambda_k$ for each k, this is equivalent to

$$\frac{\lambda_n}{\lambda_m} \le \frac{n^2}{m^2}.$$

If we suppose the existence of functions $q(x)$ and $V(x)$ such that $\frac{\lambda_n}{\lambda_m} = \frac{n^2}{m^2}$, with $q(x) \ne 0$ or $V(x)$ not being constant in $[0, 1]$, then the density described in (3.13) remains constant if and only if $\tilde{l}^4 = \frac{a}{V}$ for some $a > 0$ in $[0, 1]$. However, the monotonic behavior of $\tilde{l}^4 V$ rules out this possibility, leading to a contradiction with Theorem 1 in [9]. □

Corollary 3.3. *If $q \ge 0$ on $[0, 1]$ and V is a single–well density with transition point $x_0 = \frac{1}{2}$, then for any $1 \le m < n$, the corresponding Dirichlet eigenvalues satisfy*

$$\frac{\lambda_n}{\lambda_m} \le \frac{n^2}{m^2},$$

and equality occurs if and only if $q \equiv 0$ and V is constant on $[0, 1]$.

Proof. Following the proof of Theorem 3.2, let l denote the solution of (3.12)–(3.1). Since, as before, $\min\{\hat{v}_1, \tilde{v}_1\} > 0$, we have $l(x) > 0$ on $[0, 1]$. Moreover, $q(x) > 0$ on $[0, 1]$ and (3.12) together imply that

$$D_x^\alpha l(x) \le 0 \quad \text{for } x \in \left(0, \frac{1}{2}\right], \qquad D_x^\alpha l(x) \ge 0 \quad \text{for } x \in \left[\frac{1}{2}, 1\right],$$

so that l is a single–well function on $[0, 1]$. The remainder of the argument then proceeds exactly as in the proof of Theorem 3.2. □

4. Conclusions

We established a sharp eigenvalue-ratio inequality for conformable fractional Sturm–Liouville problems with Dirichlet boundary conditions. Specifically, for single-well densities or single-barrier potentials, we derived an upper bound for the eigenvalue ratios $\frac{\lambda_n}{\lambda_m}$. By using the conformable fractional calculus, we generalized the classical Sturm-Liouville spectral inequalities.

In future work, we aim to extend these results to Neumann, Robin, and mixed boundary conditions, in the higher dimensional settings. Additionally, considering the stability of sharp constants under perturbations is of crucial independent interest. These directions will be the theoretical and practical uses of optimal inequalities for conformable fractional Sturm–Liouville systems.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that he has no conflict of interest.

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