Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 10(11): 27073–27102.
DOI:10.3934/math.20251190
Received: 07 July 2025
Revised: 26 September 2025
Accepted: 23 October 2025
Published: 21 November 2025

Research article

Incorporating complex bipolar fuzzy set with subrings and application in decision making

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Abstract: A complex bipolar fuzzy set ($\mathbb{CBFS}$) is an extension of a complex fuzzy set and a $\mathbb{BF}$ set with a wide range of values. A $\mathbb{CBFS}$ is differentiated from a $\mathbb{BF}$ set by the incorporation of negative and positive membership functions to the unit circle in the complex plane, which empowers one to handle the vagueness more effectively. We aim to generalize the notions of $\mathbb{CBFS}s$ by proposing a general algebraic structure to dealing with complex bipolar ($\mathbb{CB}$) fuzzy data by integrating the idea of $\mathbb{CBFS}s$ and subrings. The structure of $\mathbb{CB}$ fuzzy subrings, such as the $\mathbb{CB}$ fuzzy ring isomorphism, $\mathbb{CB}$ fuzzy quotient ring, and $\mathbb{CB}$ fuzzy ring homomorphism, are examined in this paper. We develop the $(\delta,\alpha;\sigma,\beta)$-cut of a $\mathbb{CBFS}$ and explore its algebraic interpretations. Additionally, we describe the $\mathbb{CB}$ fuzzy support set and demonstrate some significant characteristics associated with this idea. Furthermore, we use the concept of a naturally occurring complex ring homomorphism to describe a $\mathbb{CB}$ fuzzy homomorphism. Additionally, we prove a $\mathbb{CB}$ fuzzy homomorphism between the $\mathbb{CB}$ fuzzy subrings of the ring and the $\mathbb{CB}$ fuzzy subring of the complex quotient ring. We demonstrate a strong connection among two $\mathbb{CB}$ fuzzy subrings of complex quotient rings under a specific $\mathbb{CB}$ fuzzy surjective homomorphism. We construct a complex fuzzy isomorphism between both associated $\mathbb{CB}$ fuzzy subrings. Furthermore, we introduce three basic results of the $\mathbb{CB}$ fuzzy isomorphism to explain the relationship between two $\mathbb{CB}$ fuzzy subrings. Finally, we use a complex bipolar fuzzy subring in decision making.

Keywords: $\mathbb{CB}$ fuzzy subring; $\mathbb{CB}$ fuzzy ideal; $\mathbb{CB}$ fuzzy isomorphism
Mathematics Subject Classification: 03E72I, 08A72, 13E15

1. Introduction

In mathematics, ring theory is a crucial aspect of computational algebra. Ring structure is an exploration of ring mathematical frameworks that define two binary operations, addition and multiplication, in algebra. Ring theory provides an effective structure to analyzing an item that demonstrates symmetry. It is essential to classifying the symmetry of particles, molecules, regular polygons, and crystalline structures. Using ring theory helped us figure out important physical laws, such as how relativity and symmetry were discovered in quantum mechanics. To explore rings, mathematicians have generated a number of ideas that divide them into smaller, easier-to-handle components, such as ideals, quotient rings, and fundamental rings. Since rings have an importance in numerous fields, they are incredibly important to number theory and the theories it draws from commutative arithmetic and the geometry of algebra. In our daily conversations, we are unable to constantly say "yes" or "no". Sometimes, we lack sufficient information to make a decision. To overcome amigiousness and uncertainty in 1965, Zadeh [1] introduced the concept of fuzzy sets ($\mathbb{FS}s$) and outlined their preliminary outcomes. Fuzzy theory is utilized for the representation of real numbers between zero and one. The notion of $\mathbb{FS}s$ has become more prevalent in numerous disciplines. The fields of artificial intelligence, information technology, medical procedures, automation, control design, the theory of decisions, computational logic and the science of management and operational research are a few fields in which this theory recognizes applications.

Rosenfeld [2] initially presented the notion of fuzzy subgroups ($\mathbb{FSG}$) in 1971. The notion of Engel FSG was presented by Ameri et al. [3], which examined the basic results associated with the left and right fuzzy Engel components. In 1983, Liu [4] introduced the idea of fuzzy ideals ($\mathbb{FI}$s) and fuzzy sub-rings ($\mathbb{FSR}$s). Addis et al. [5] illustrated fuzzy homomorphism theorems for fuzzy rings. A fuzzy subset η of ring $\mathfrak{H}$ is a $\mathbb{FSR}$ if $\eta(d-h) \geq \{\eta(d) \wedge \eta(h)\}$ and $\eta(dh) \geq \{\eta(d) \wedge \eta(h)\}$. A $\mathbb{FS}$ of ring $\mathfrak{H}$ is $\mathbb{FI}$ if $\eta(d-h) \geq \{\eta(d) \wedge \eta(h)\}$ and $\eta(dh) \geq \{\eta(d) \vee \eta(h)\}$. Deniz [6] examined several approximations for a fuzzy ring homomorphism ($\mathbb{FRH}$). Emniyet and Sahin [7] proposed an algebraic framework among normed rings and $\mathbb{FS}s$. In 2009, Fotea and Davvaz [8] explored the novel concept of fuzzy hyper-rings. Motameni et al. [9] investigated particular types of fuzzy hyper-ideals and expanded this theory to include fuzzy hyper-ring homomorphisms for both prim and maximum fuzzy hyper-ideals. Malik and Mordeson [10] examined the characteristics of L-$\mathbb{FSR}$ for direct sum operations. Atanassov [11] presented the enlarged form of the $\mathbb{FS}$, and included a new element known as an intuitionistic fuzzy set ($\mathbb{IFS}$).

In 1989, Biswas [12] explored the idea of intuitionistic $\mathbb{FSG}$ and their features. Hur et al. [13] proposed the notions of an intuitionistic fuzzy subgroup ($\mathbb{IFSG}$) and $\mathbb{FSR}$ and an evaluated certain characteristics. In 2003, Banerjee [14] conducted further research on intuitionistic fuzzy ideals ($\mathbb{IFI}$) and $\mathbb{IFSR}$. Alhaleem and Ahmad [15] introduced the concept of on intuitionistic fuzzy ($\mathbb{IF}$) normed ring. Marashdeh and Salleh [16] presented a novel analysis of the intuitionistic fuzzy ring ($\mathbb{IFR}$) that explored the concept of the $\mathbb{IF}$ space. Nakkasen [17] examined the characteristics of Artnians and Noetherian tertiary near-rings on $\mathbb{IFI}s$. Gulzar et al. [18] analyzed the new level of t-$\mathbb{IFSG}s$. Yamin and Sharma [19] investigated the $\mathbb{IFR}$ with operators, as well as the $\mathbb{IFI}$ and quotient ring ($\mathbb{QR}$).

An extension of $\mathbb{FS}$ is the a bipolar $\mathbb{FS}$, since the $\mathbb{FS}$ membership grade belongs to the range of [0,1]. Positive membership levels for a bipolar fuzzy set ($\mathbb{BFS}$) fall between [0,1], whereas negative membership levels fall between [-1,0]. It is normal for humans to hold numerous opinions at a time.

Zhang [20] presented a novel idea of $\mathbb{BFS}$ to tackle double-sided opinions. The extension of real numbers with positive values to negative values is comparable to the expansion of the $\mathbb{FS}$ to the $\mathbb{BFS}$. In decision-making, $\mathbb{BFS}s$ are recognized as a novel technique to handling uncertainty. Some other researchers found more advancements about $\mathbb{BFS}s$ in [21–23]. The $\mathbb{BFS}s$, in contrast to Zadeh's $\mathbb{FS}$, are recognized to be a more useful tool to study cases of ambiguity since they handle degrees of membership, both positive and negative. Although both $\mathbb{BFS}s$ and $\mathbb{IFS}s$ have the same appearance, Lee [24] claimed that both are essentially different ideas. Both degrees of membership and non-membership in the $\mathbb{IFS}$ belong to $[0,1]$, and their sum is less than or equal to one; however, in a $\mathbb{BFS}$, a positive membership degree is associated with $[0,1]$, whereas a negative membership degree is associated with $[-1,0]$. There are several applications of $\mathbb{BFS}s$ in real-world problems [25]. The majority of scholars [26–29] possessed significant achievements in extending the notion of $\mathbb{BFS}s$ to contemporary mathematics and decision-making processes. Akram et al. [30] developed a collaborative method for making decisions with numerous criteria over bipolar fuzzy PROMETHEE operations in order to choose sustainable manufacturers. The opinions of the self-focused bipolar fuzzy graph, as well as its magnitude, diameter, eccentricity, and spacing, were examined in [31].

Mahmood and Munir [32] explored the idea of a $\mathbb{BF}$ subgroup. Additionally, they derived some basic results utilizing the notion of a bipolar fuzzy ($\mathbb{BFS}$) in the theory of groups. The generalization of the $\mathbb{FSG}$ translational of a sub-semigroup and $\mathbb{FS}$ comparability association with $\mathbb{BFS}$ was demonstrated by Sardar et al. [33]. The perspectives of the $\mathbb{BF}$ subalgebra and k-fold bipolar fuzzy ideals ($\mathbb{BFI}s$) were demonstrated by Jun et al. [34]. Baik [35] established a connection throughout near ring theory ideals and $\mathbb{BFS}s$. This connection is undoubtedly a fundamental component of classical $\mathbb{FSR}$, as it offers new solutions to a number of problems in near ring theory. Maheswari et al. [36] proposed the concept of a $\mathbb{BF}$ subring. A more extended form of the $\mathbb{FSR}$ is the $\mathbb{BFSR}$. A bipolar fuzzy subset $\eta = \{d, (\eta^+(d), \eta^-(d)), \forall\, d \in \mathfrak{H}\}$ of ring $\mathfrak{H}$ is a bipolar fuzzy sub-ring ($\mathbb{BFSR}$) if it satisfies: both of the requirements for a positive membership level and the two requirements for a negative one: $\eta^+(d-h) \geq \{\theta^+(d) \wedge \eta^+(h)\}$, $\eta^+(dh) \geq \{\eta^+(d) \wedge \eta^+(h)\}$; and $\eta^-(d-h) \leq \{\eta^-(d) \vee \eta^-(h)\}$, $\eta^-(dh) \leq \{\theta^-(d) \vee \eta^-(h)\}$. The advanced structure of a $\mathbb{BFSR}$ was proposed in 2021 by Alolaiyan et al. [37]. This structure included the bipolar fuzzy ring homomorphism ($\mathbb{BFRH}$), isomorphism, and $\mathbb{QR}s$. Furthermore, the (α, β)-cut of the $\mathbb{BFS}$ was described. Moreover, the $\mathbb{BFS}$ supporting set was explained, and some of its key characteristics were illustrated. Additionally, three fundamental bipolar fuzzy isomorphism ($\mathbb{BFIS}$) theorems were given to examine the connection between two $\mathbb{BFSR}$. Subbian and Kamaraj [38] presented the idea of $\mathbb{BFI}s$ and examined their extension. A $\mathbb{BFS}$ $\eta = \{d, (\eta^+(d), \eta^-(d)), \forall\, d \in \mathfrak{H}\}$ of ring $\mathfrak{H}$ is a $\mathbb{BSI}$ if it fulfills the following conditions: $\eta^+(d-h) \geq \{\eta^+(d) \wedge \eta^+(h)\}$, $\eta^+(dh) \geq \{\eta^+(d) \vee \eta^+(h)\}$; and $\eta^-(d-h) \leq \{\theta^-(d) \vee \eta^-(h)\}$, $\eta^-(dh) \leq \{\eta^-(d) \wedge \eta^-(h)\}$. The recent advancement in $\mathbb{BFS}s$ in the semi-group and BCK/BCI algebras notions might be perceived in [39, 40]. Many physical problems may be solved in a significant manner by frequently using the $\mathbb{BFS}s$'. Furthermore, the algebraic characteristics of $\mathbb{BFSR}s$ make their study important.

The complex $\mathbb{FS}$ theory was introduced by Ramot et al. [41]. An extension of the idea of a $\mathbb{FS}$ to the complex plane is known as a complex fuzzy set ($\mathbb{CFS}$), which is defined as a collection of complex numbers ($\mathbb{CN}$s) that have a modulus that is less than or equal to 1. Thus, compared to being limited to the range [0, 1], the range was extended to the unit disc's complex plane. Tamir et al. [42] reformulated the concepts of complex fuzzy logic ($\mathbb{CFL}$) and $\mathbb{CFS}$. Moreover, the current condition of $\mathbb{CFL}$, $\mathbb{CFS}$ theory, and related applications were studied. Many researchers have studied on $\mathbb{FS}s$ and

$\mathbb{CN}$; Buckley [43], Nguyen et al. [44], Zhang et al. [45] combined the concepts of fuzzy and $\mathbb{CN}$s to create fuzzy real as well as imaginary elements. The idea of a complex $\mathbb{FSG}$ was presented and few of its features were examined by Alsarahead and Ahmad [46]. Alsarahead and Ahmad [47] created the complex fuzzy ideal ($\mathbb{CFI}$) and complex fuzzy subring ($\mathbb{CFSR}$) and utilizing several of its properties.

The notion of complex institutions fuzzy set ($\mathbb{CIFS}$) was derived from the terminology of $\mathbb{CFS}$ that Alkouri and Salleh [48] created in 2012 by defining $\mathbb{CFS}$s and included the non-membership grade. Gulzar et al. [49] proposed an entirely novel notion of complex $\mathbb{IFSG}$ and demonstrated that any complex $\mathbb{IFSG}$ gives rise to two $\mathbb{IFSG}s$. Furthermore, they constructed a few novel findings about the direct product of complex $\mathbb{IFSG}s$. In 2017, the idea that a complex (FSG) was presented and few of its features were examined by Alsarahead and Ahmad [50]. Gulzar et al. [51] established the notion of a direct product between two complex $\mathbb{IFSR}s$. A complex $\mathbb{IFSR}s$ is also shown to be the homomorphic representation (pre-image) of its direct product $\mathbb{IFSR}s$ using the idea of classical homomorphism. Alolaiyan et al. [52] proposed a new algebraic framework of (α,β)-$\mathbb{CFSG}s$. Gulzar [53] proposed the novel framework of Q- $\mathbb{CFSR}s$. Alkouri et al. [54] established the idea of a complex $\mathbb{BFS}$. It has been changed so that the ranges of the elements in the complex $\mathbb{BFS}$ go from [0, 1] to $[0,1]e^{2\pi i[0,1]}$ for positive levels and from [-1, 0] to $[-1,0]e^{2\pi i[-1,0]}$ for negative levels.

Motivation

1) Bipolar, conflicting, and ambiguous information are common in real-world decision-making situations, where both positive and negative features must be modeled at the same time.
2) Partial solutions are offered by classical $\mathbb{FS}$s and even $\mathbb{BFS}$s, but they are unable to represent the complex-valued character of uncertainty or the algebraic structure needed for scientific evaluations.
3) Ring and subring structures are ideal to model structured decision environments because they provide a robust algebraic foundation that guarantees closure and consistency in operations.
4) By combining complex numbers, bipolarity, and subring theory, the complex bipolar fuzzy subring ($\mathbb{CBFSR}$) is developed to overcome these difficulties and enable more expressive and rigorous mathematical modeling of uncertain systems.

Contributions

1) We present a novel theoretical framework that combines $\mathbb{CBFS}$s and algebraic subrings: the $\mathbb{CBFSR}$.
2) $\mathbb{CBFSR}$ improves the ability to model uncertainty in several dimensions by offering a dual-support representation (positive and negative components) using both real and imaginary portions.
3) By adding complex-valued membership functions into algebraic subrings, the suggested structure broadens the application of fuzzy algebra and strengthens the mathematical underpinnings of fuzzy theory.
4) The complex bipolar fuzzy set ($\mathbb{CBFS}$) theory extends the $\mathbb{BFS}$ and complex fuzzy theory to effectively handle ambiguous and uncertain information in real-life applications. Unlike the traditional $\mathbb{BFS}$, the $\mathbb{CBFS}$ incorporates both positive and negative membership degrees using complex values, thus ensuring that no critical information is lost by neglecting imaginary parts. This makes the $\mathbb{CBFS}$ a more reliable framework. Motivated by its significance, we introduce the $\mathbb{CBFSR}$ as a generalization of the $\mathbb{BFS}$. The $\mathbb{CBFS}$ covers both negative and positive aspects of complex values membership degree. The structure of $\mathbb{CBFSR}$, such as the fuzzy quotient ring, and

the $\mathbb{CB}$ $\mathbb{FRH}$, $(\delta, \alpha; \sigma, \beta)$-cut of a $\mathbb{CBFS}$ are examined. Additionally, the well known fundamental theorems of isomorphisms under the influence of $\mathbb{CBF}$ environment are investigates. The idea of complex bipolar fuzzy subrings are used in decision making. Figure 1 illustrated the relationship between the purposed and existing models.

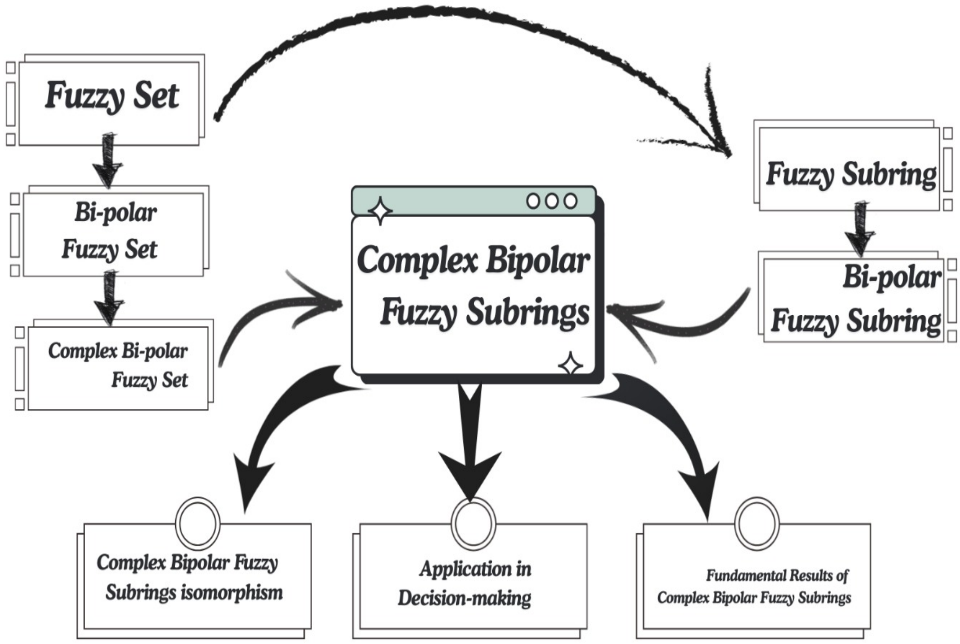

Figure 1. The relationship between proposed and existing models.

The rest of this paper is organized as follows: in Section 2, the notions of $\mathbb{CBFS}s$, $\mathbb{CBFSR}s$, and associated findings are described; We cover several significant algebraic properties of $\mathbb{CBFSR}s$ and examined the notions of the $(\delta, \alpha; \sigma, \beta)$-cut of $\mathbb{CBFS}s$ in Section 3; using the concept of $(\delta, \alpha; \sigma, \beta)$-cut of $\mathbb{CBFS}s$, we express the support of $\mathbb{CBFS}$ and demonstrate how the support of $\mathbb{CBFI}s$ of the ring constitutes a natural ideal of the ring; a natural ring homomorphism of a $\mathbb{CBFSR}$ under a $\mathbb{CBF}$ homomorphism is described in Section 4, and we demonstrate how the addition and multiplication operation specified in the $\mathbb{CBFSR}$ is maintained by the $\mathbb{BF}$ homomorphism. Additionally, we establish an intense relationship over a specified surjective homomorphism between two $\mathbb{CBFSR}s$ of the $\mathbb{QR}s$ and demonstrate fundamental $\mathbb{CBF}$ homomorphism theorems for this specific the $\mathbb{FSR}$. We explore the three basic results of $\mathbb{CBF}$ isomorphism of $\mathbb{CBFSR}s$; lastly, we discuss c$\mathbb{CBFSR}s$ in decision making.

2. Preliminaries

In this section, we review the fundamental concepts of a $\mathbb{CBFS}$, $\mathbb{CBFSR}$, and $\mathbb{CBFI}$ that are related to the examination of this paper.

Definition 2.1. *[20] A $\mathbb{BFS}$ η of the universe Q is presented with elements of the following structure: $\eta = \{(d, \eta^+(d), \eta^-(d)) : d \in Q\}$, where $\eta^+(d) : Q \to [0,1]$ is the positive membership degree ($\mathbb{PMD}$),*

$\eta^{-}(d) : Q \rightarrow [-1, 0]$, *is the negative membership degree* ($\mathbb{NMD}$), $\forall d \in Q$, *and satisfies the norm,* $-1 \leq \eta^{+}(d) + \eta^{-}(d) \leq +1$.

Definition 2.2. *[36] A $\mathbb{BFS}$ η of a ring $\mathfrak{H}$ is known as $\mathbb{BFSR}$ of a $\mathfrak{H}$ if each of the following norms are satisfied:*

1) $\eta^{+}(d - h) \geq \{\eta^{+}(d) \wedge \eta^{+}(h)\}$,
2) $\eta^{+}(dh) \geq \{\eta^{+}(d) \wedge \eta^{+}(h)\}$,
3) $\eta^{-}(d - h) \leq \{\eta^{-}(d) \vee \eta^{-}(h)\}$,
4) $\eta^{-}(dh) \leq \{\eta^{-}(d) \vee \eta^{-}(h)\}, \forall d, h \in \mathfrak{H}$.

Definition 2.3. *[36] Suppose that $\eta = \{d, \eta^{+}(d), \eta^{-}(d) : d \in X\}$ and $\mu = \{h, \mu^{+}(h), \mu^{-}(h) : h \in Y\}$ are both $\mathbb{BFS}$s, where X and Y are non-empty sets. The cartesian product of η and μ is indicated by $\eta \times \mu$, and is described as follows:*

$\eta \times \mu = \{< (d, h), (\eta^{+} \times \mu^{+})(d, h), (\eta^{-} \times \mu^{-})(d, h) >: d \in X, h \in Y\}$,
where $(\eta^{+} \times \mu^{+})(d, h) = \{\eta^{+}(d) \wedge \mu^{+}(h)\}$ and $(\eta^{-} \times \mu^{-})(d, h) = \{\eta^{-}(d) \vee \mu^{-}(h)\}$.

Definition 2.4. *[38] A $\mathbb{BFS}$ η of a ring $\mathfrak{H}$ is known as a $\mathbb{BF}$ left ideal of $\mathfrak{H}$, if each of the subsequent norms are satisfied:*

1) $\eta^{+}(d - h) \geq \{\eta^{+}(d) \wedge \eta^{+}(h)\}$,
2) $\eta^{+}(dh) \geq \eta^{+}(h)$,
3) $\eta^{-}(d - h) \leq \{\eta^{-}(d) \vee \eta^{-}(h)\}$,
4) $\eta^{-}(dh) \leq \eta^{-}(h)$, for all $d, h \in \mathfrak{H}$.

Definition 2.5. *[38] A $\mathbb{BFS}$ η of a ring $\mathfrak{H}$ is known as a $\mathbb{BF}$ right ideal of $\mathfrak{H}$, if each of the following norms are satisfied:*

1) $\eta^{+}(d - h) \geq \{\eta^{+}(d) \wedge \eta^{+}(h)\}$,
2) $\eta^{+}(dh) \geq \eta^{+}(d)$,
3) $\eta^{-}(d - h) \leq \{\eta^{-}(d) \vee \eta^{-}(h)\}$,
4) $\eta^{-}(dh) \leq \eta^{-}(d)$, for all $d, h \in \mathfrak{H}$.

Definition 2.6. *[38] A $\mathbb{CBFS}$ η of a ring $\mathfrak{H}$ is known as a $\mathbb{CBFI}$ of $\mathfrak{H}$, if each of the following norms are satisfied:*

1) $\eta^{+}(d - h) \geq \{\gamma^{+}(d)e^{i\varpi^{+}(d)} \wedge \gamma^{+}(h)e^{i\varpi^{+}(h)}\}$,
2) $\eta^{+}(dh) \geq \{\gamma^{+}(d)e^{i\varpi^{+}(d)} \wedge \gamma^{+}(h)e^{i\varpi^{+}(h)}\}$,
3) $\eta^{-}(d - h) \leq \{\gamma^{-}(d)e^{i\varpi^{-}(d)} \vee \gamma^{-}(h)e^{i\varpi^{-}(h)}\}$,
4) $\eta^{-}(dh) \leq \{\gamma^{-}(d)e^{i\varpi^{-}(d)} \vee \gamma^{-}(h)e^{i\varpi^{-}(h)}\} \; \forall d, h \in \mathfrak{H}$.

Definition 2.7. *[41] Suppose that η is $\mathbb{CFS}$ over the universal collection Q and the elements are expressed as follows: $\eta = \{(d, \gamma(d)e^{i\varpi(d)} : d \in Q\}$, where the amplitude term is $\gamma(d) \in [0, 1]$ and the phase term $\varpi(d) \in [0, 2\pi]$.*

Definition 2.8. *Suppose that η is a $\mathbb{CBFS}$ on a ring $\mathfrak{H}$. A $\mathbb{CBFS}$ η of a ring $\mathfrak{H}$ is defined as $\mathbb{CBFSR}$ of a $\mathfrak{H}$ where Q is not an empty set if each of the following axioms are satisfied:*

$\eta = \{q, \gamma^{+}(q)e^{i\varpi^{+}(q)}, \gamma^{-}(q)e^{i\varpi^{-}(q)} : q \in Q\}$, where a $\mathbb{PMD} : \gamma^{+}(q)e^{i\varpi^{+}(q)} : \gamma^{+}(q) \in [0,1], \varpi^{+}(q) \in [0, 2\pi]$ and $\mathbb{NMD} : \gamma^{-}(q)e^{i\varpi^{-}(q)} : \gamma^{-}(q) \in [-1, 0], \varpi^{-}(q) \in [-2\pi, 0]$;

1) $\gamma^{+}(d-h)e^{i\varpi^{+}(d-h)} \geq \{\gamma^{+}(d) \wedge \gamma^{+}(h)\}e^{\iota\{\varpi^{+}(d)\wedge\varpi^{+}(h)\}}$,
2) $\gamma^{+}(dh)e^{i\varpi^{+}(d-h)} \geq \{\gamma^{+}(d) \wedge \gamma^{+}(h)\}e^{\iota\{\varpi^{+}(d)\wedge\varpi^{+}(h)\}}$,
3) $\gamma^{-}(d-h)e^{i\varpi^{-}(d-h)} \leq \{\gamma^{-}(d) \vee \gamma^{-}(h)\}e^{\iota\{\varpi^{-}(d)\vee\varpi^{-}(h)\}}$,
4) $\gamma^{-}(dh)e^{i\varpi^{-}(d-h)} \leq \{\gamma^{-}(d) \vee \gamma^{-}(h)\}e^{\iota\{\varpi^{-}(d)\vee\varpi^{-}(h)\}}$, $\forall d, h \in \mathfrak{H}$.

Definition 2.9. *A $\mathbb{CBFS}$ η of a ring $\mathfrak{H}$ is known as a $\mathbb{CBF}$ left ideal of $\mathfrak{H}$, if each of the subsequent norms are satisfied:*

1) $\eta^{+}(d-h) \geq \{\gamma^{+}(d)e^{i\varpi^{+}(d)} \wedge \gamma^{+}(h)e^{i\varpi^{+}(h)}\}$,
2) $\eta^{+}(dh) \geq \gamma^{+}(h)e^{i\varpi^{+}(h)}$,
3) $\eta^{-}(d-h) \leq \{\gamma^{-}(d)e^{i\varpi^{-}(d)} \vee \gamma^{-}(h)e^{i\varpi^{-}(h)}\}$,
4) $\eta^{-}(dh) \leq \gamma^{-}(h)e^{i\varpi^{-}(h)}$, for all $d, h \in \mathfrak{H}$.

Definition 2.10. *A $\mathbb{CBFS}$ η of a ring $\mathfrak{H}$ is known as a $\mathbb{CBF}$ right ideal a $\mathbb{CBFRI}$ of $\mathfrak{H}$, if each of the subsequent norms are satisfied:*

1) $\eta^{+}(d-h) \geq \{\gamma^{+}(d)e^{i\varpi^{+}(d)} \wedge \gamma^{+}(h)e^{i\varpi^{+}(h)}\}$,
2) $\eta^{+}(dh) \geq \gamma^{+}(d)e^{i\varpi^{+}(d)}$,
3) $\eta^{-}(d-h) \leq \{\gamma^{-}(d)e^{i\varpi^{-}(d)} \vee \gamma^{-}(h)e^{i\varpi^{-}(h)}\}$,
4) $\eta^{-}(dh) \leq \gamma^{-}(d)e^{i\varpi^{-}(h)}$, for all $d, h \in \mathfrak{H}$.

Definition 2.11. *Suppose that $\eta = \{d, \gamma^{+}(d)e^{i\varpi^{+}(d)}, \gamma^{-(d)}e^{i\varpi^{-}(d)} : d \in X\}$, and $\mu = \{h, \tau^{+}(h)e^{i\upsilon^{+}(h)}, \tau^{-(h)}e^{i\upsilon^{-}(h)} : h \in Y\}$ are both $\mathbb{BFS}$s, where X and Y are non-empty sets. Then, the cartesian product of η and μ indicated by $\eta \times \mu$ and is described as follows:*

$\eta \times \mu = \{< (d, h), (\gamma^{+}e^{i\varpi^{+}} \times \tau^{+}e^{i\upsilon^{+}}(d, h), (\gamma^{-}e^{i\varpi^{-}} \times \tau^{-}e^{i\upsilon^{-}}(d, h) >: d \in X, h \in Y\}$, where $(\gamma^{+}e^{i\varpi^{+}} \times \tau^{+}e^{i\upsilon^{+}})(d, h) = \{\gamma^{+}(d)e^{i\varpi^{+}(h)} \wedge \tau^{+}(h)e^{i\upsilon^{+}(h)}\}$ and $(\gamma^{-}e^{i\varpi^{-}} \times \tau^{-}e^{i\upsilon^{-}})(d, h) = \{\gamma^{-}(d)e^{i\varpi^{-}(d)} \vee \tau^{-}(h)e^{i\upsilon^{-}(h)}\}$.

3. The basic algebraic characteristics of complex bipolar fuzzy sub-rings

In the following section, we examine the $(\delta, \alpha; \sigma, \beta)$- cut of a $\mathbb{CBFS}$ and explore a few essential aspects of this occurrence. We develop support of a $\mathbb{CBFS}$ describe their compared theoretic features of sets under $\mathbb{CBFSR}$. Additionally, we identify the $\mathbb{CBFSR}$ of the $\mathbb{QR}$.

Definition 3.1. *Suppose that $\eta = \{d, \gamma^{+}(d)e^{i\varpi^{+}(d)}, \gamma^{-}(d)e^{i\varpi^{-}(d)} : d \in X\}$ is a $\mathbb{CBFS}$ of a set X; then, a $(\delta, \alpha; \sigma, \beta)$- cut of η is a crisp set of X and known as $A^{\alpha,\beta}_{\delta,\sigma}(\eta) = \{d \in X \,|\gamma^{+}(d) \geq \delta, \varpi^{+}(d) \geq \alpha, \gamma^{-}(d) \leq \sigma, \varpi^{-}(d) \leq \beta\}$, where $\delta \in [0, 1], \alpha \in [0, 2\pi]$ and $\sigma \in [-1, 0], \beta \in [-2\pi, 0]$.*

Theorem 3.2. *If η is a $\mathbb{CBFS}$ of a ring $\mathfrak{H}$, then $A^{\alpha,\beta}_{\delta,\sigma}(\eta)$ is a subring of $\mathbb{H}$ iff η is a $\mathbb{CBFSR}$ of $\mathfrak{H}$.*

The proof of this theorem is similar to what is illustrated in [50].

Theorem 3.3. *Suppose that η is $\mathbb{CBFSR}$ of $\mathfrak{H}$; then, $A^{\alpha,\beta}_{\delta,\sigma}(\eta) \subseteq A^{\rho',\varrho'}_{\rho,\varrho}(\eta)$ if $\delta \geq \rho$, $\alpha \geq \rho'$ and $\sigma \leq \varrho$, $\beta \leq \varrho'$, where $\delta, \rho \in [0, 1]$, $\alpha, \rho' \in [0, 2\pi]$ and $\sigma, \varrho \in [-1, 0]$, $\beta, \varrho' \in [-2\pi, 0]$.*

Proof. Let $h \in \mathrm{A}^{\alpha,\beta}_{\delta,\sigma}(\eta)$; then, $\eta^{+}(h) \geq \delta, \varpi^{+}(h) \geq \alpha$ and $\eta^{-}(h) \geq \sigma, \varpi^{-}(h) \geq \beta$. Since $\delta \geq \rho$, $\alpha \geq \rho^{`}$ and $\sigma \leq \varrho$, $\beta \leq \varrho^{`}$, we can write, $\gamma^{+}(h) \geq \delta \geq \rho$, $\varpi^{+}(h) \geq \alpha \geq \rho^{`}$ and $\gamma^{-}(h) \leq \sigma \leq \varrho$, $\varpi^{-}(h) \leq \beta \leq \varrho^{`}$. Therefore, $h \in \mathrm{A}^{\rho^{`},\varrho^{`}}_{\rho,\varrho}(\eta)$. □

Theorem 3.4. *Let η and μ be $\mathbb{CBFSR}$s of a ring $\mathfrak{H}$; then, $\mathrm{A}^{\alpha,\beta}_{\delta,\sigma}(\eta \cap \mu) = \mathrm{A}^{\alpha,\beta}_{\delta,\sigma}(\eta) \cap \mathrm{A}^{\alpha,\beta}_{\delta,\sigma}(\mu)$.*

Proof. We have $\mathrm{A}^{\alpha,\beta}_{\delta,\sigma}(\eta \cap \mu) = \{h \in \mathfrak{H} | (\eta^{+} \cap \mu^{+})(h) \geq \delta \geq \alpha, (\eta^{-} \cap \mu^{-})(h) \leq \sigma \leq \beta\}$.
Now, $g \in \mathrm{A}^{\alpha,\beta}_{\delta,\sigma}(\eta \cap \mu)$,

$$
\begin{aligned}
&\Leftrightarrow \quad (\eta^{+} \cap \mu^{+})(h) \geq \delta \geq \alpha, (\eta^{-} \cap \mu^{-})(h) \leq \sigma \leq \beta,\\
&\Leftrightarrow \quad \{\gamma^{+}(h) \wedge \tau^{+}(h)\} \geq \delta, \{\varpi^{+}(h) \wedge \upsilon^{+}(h)\} \geq \alpha, \quad \text{and}\\
&\Leftrightarrow \quad \{\gamma^{-}(h) \vee \tau^{-}(h)\} \leq \sigma, \{\varpi^{-}(h) \vee \upsilon^{-}(h)\} \leq \beta.\\
&\Leftrightarrow \quad \eta^{+}(h), \mu^{+}(h) \geq \delta \geq \alpha \quad \text{and} \quad \eta^{-}(h), \mu^{-}(h) \leq \sigma \leq \beta,\\
&\Leftrightarrow \quad h \in \mathrm{A}^{\alpha,\beta}_{\delta,\sigma}(\eta) \quad \text{and} \quad h \in \mathrm{A}^{\alpha,\beta}_{\delta,\sigma}(\mu)\\
&\Leftrightarrow \quad h \in \mathrm{A}^{\alpha,\beta}_{\delta,\sigma}(\eta) \cap \mathrm{A}^{\alpha,\beta}_{\delta,\sigma}(\mu).
\end{aligned}
$$

Therefore, $\mathrm{A}^{\alpha,\beta}_{\delta,\sigma}(\eta \cap \mu) = \mathrm{A}^{\alpha,\beta}_{\delta,\sigma}(\eta) \cap \mathrm{A}^{\alpha,\beta}_{\delta,\sigma}(\mu)$. □

Theorem 3.5. *If $\eta \subseteq \mu$, then $\mathrm{A}^{\alpha,\beta}_{\delta,\sigma}(\eta) \subseteq \mathrm{A}^{\alpha,\beta}_{\delta,\sigma}(\mu)$, where η and μ are $\mathbb{CBFSR}$s of a ring $\mathfrak{H}$.*

Proof. Let $\eta \subseteq \mu$ and $h \in \mathrm{A}^{\alpha,\beta}_{\delta,\sigma}(\eta)$. Then, $\gamma^{+}(h) \geq \delta, \varpi^{+}(h) \geq \alpha$ and $\gamma^{-}(h) \leq \sigma, \varpi^{+}(h) \leq \beta$. Since $\eta \subseteq \mu$, we have $\tau^{+}(h)) \geq \gamma^{+}(h) \geq \delta, \upsilon^{+}(h) \geq \varpi^{+}(h) \geq \alpha$ and $\tau^{-}(h)) \leq \gamma^{-}(h) \leq \sigma, i\upsilon^{-}(h) \leq \varpi^{-}(h) \leq \beta$. Therefore, $h \in \mathrm{A}^{\alpha,\beta}_{\delta,\sigma}(\mu)$. Thus, $\mathrm{A}^{\alpha,\beta}_{\delta,\sigma}(\eta) \subseteq \mathrm{A}^{\alpha,\beta}_{\delta,\sigma}(\mu)$. □

Theorem 3.6. *If η and μ are $\mathbb{CBFSR}$s of a ring $\mathfrak{H}$, then $\mathrm{A}^{\alpha,\beta}_{\delta,\sigma}(\eta \cup \mu) = \mathrm{A}^{\alpha,\beta}_{\delta,\sigma}(\eta) \cup \mathrm{A}^{\alpha,\beta}_{\delta,\sigma}(\mu)$.*

Proof. We have $\mathrm{A}^{\alpha,\beta}_{\delta,\sigma}(\eta \cup \mu) = \{h \in \mathfrak{H} | (\eta^{+} \cup \mu^{+})(g) \geq \delta \geq \alpha, (\eta^{-} \cup \mu^{-})(h) \leq \sigma \leq \beta\}$.
Now, $h \in \mathrm{A}^{\alpha,\beta}_{\delta,\sigma}(\eta \cup \mu)$,

$$
\begin{aligned}
&\Leftrightarrow \quad (\eta^{+} \cup \mu^{+})(h) \geq \delta \geq \alpha, (\eta^{-} \cup \mu^{-})(h) \leq \sigma \leq \beta,\\
&\Leftrightarrow \quad \{\gamma^{+}(h) \wedge \tau^{+}(h)\} \geq \delta, \{\varpi^{+}(h) \wedge \upsilon^{+}(h)\} \geq \alpha, \quad \text{and}\\
&\Leftrightarrow \quad \{\gamma^{-}(h) \vee \tau^{-}(h)\} \leq \sigma, \{\varpi^{-}(h) \vee \upsilon^{-}(h)\} \leq \beta.\\
&\Leftrightarrow \quad \eta^{+}(h), \mu^{+}(h) \geq \delta \geq \alpha \quad \text{and} \quad \eta^{-}(h), \mu^{-}(h) \leq \sigma \leq \beta,\\
&\Leftrightarrow \quad h \in \mathrm{A}^{\alpha,\beta}_{\delta,\sigma}(\eta) \quad \text{and} \quad h \in \mathrm{A}^{\alpha,\beta}_{\delta,\sigma}(\mu)\\
&\Leftrightarrow \quad h \in \mathrm{A}^{\alpha,\beta}_{\delta,\sigma}(\eta) \cup \mathrm{A}^{\alpha,\beta}_{\delta,\sigma}(\mu).
\end{aligned}
$$

Therefore, $\mathrm{A}^{\alpha,\beta}_{\delta,\sigma}(\eta \cup \mu) = \mathrm{A}^{\alpha,\beta}_{\delta,\sigma}(\eta) \cup \mathrm{A}^{\alpha,\beta}_{\delta,\sigma}(\mu)$. □

Proposition 3.7. *If η and μ are two $\mathbb{CBFS}$s of $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively. Then, $\mathrm{A}^{\alpha,\beta}_{\delta,\sigma}(\eta \times \mu) = \mathrm{A}^{\alpha,\beta}_{\delta,\sigma}(\eta) \times \mathrm{A}^{\alpha,\beta}_{\delta,\sigma}(\mu)$, for all $\delta \in [0, 1], \alpha \in [0, 2\pi]$ and $\sigma \in [-1, 0], \beta \in [-2\pi, 0]$.*

Theorem 3.8. *Let η and μ be $\mathbb{CBFSR}$s of $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively. Then, $\eta \times \mu$ is a $\mathbb{CBFSR}$ of ring $\mathfrak{H}_1 \times \mathfrak{H}_2$,*

The proof of this theorem is similar to what is illustrated in [51]

Definition 3.9. *Suppose that $\mathfrak{H}$ is a ring, and η is a $\mathbb{CBFSR}$ of $\mathfrak{H}$. Suppose that $d \in \mathfrak{H}$ is a fixed component. Thus, the set $(d+\eta)(h) = \{< h, d+\gamma^+(h)e^{i\varpi^+(h)}, d+\gamma^-(h)e^{i\varpi^-(h)} >: h \in H\}$, where $d+\eta^+(h) = \gamma^+(h-d)e^{i\varpi^+(h-d)}$ and $f+\eta^-(h) = \gamma^+(h-d)e^{i\varpi^+(h-d)}$, for all $h \in \mathfrak{H}$ is known as a bipolar fuzzy left coset $\mathbb{CBFLC}$ of $\mathfrak{H}$ given by η and d.*

Theorem 3.10. *Suppose η is a $\mathbb{BFI}$ of $\mathfrak{H}$ and d is any fixed component of $\mathfrak{H}$. Thus, $d + \mathrm{A}^{\alpha,\beta}_{\delta,\sigma}(\eta) = \mathrm{A}^{\alpha,\beta}_{\delta,\sigma}(\eta)(d+\eta)$.*

Proof. Consider $d + \mathrm{A}^{\alpha,\beta}_{\delta,\sigma}(\eta)$:

$$\begin{aligned} &= d + \{< h \in \mathfrak{H} : \gamma^+(h) \geq \delta, \varpi^+(h) \geq \alpha \text{ and } \gamma^-(h) \leq \sigma, \varpi^-(h) \leq \beta >\} \\ &= \{< f+g \in \mathfrak{H} : \gamma^+(h) \geq \delta, \varpi^+(h) \geq \alpha \text{ and } \gamma^-(h) \leq \sigma, \varpi^-(h) \leq \beta >\}. \end{aligned}$$

Place $d+h = c$ so that $h = c-d$. Then, $f + \mathrm{A}^{\alpha,\beta}_{\delta,\sigma}(\eta)$:

$$\begin{aligned} &= \{< c \in \mathfrak{H} : \gamma^+(c-d) \geq \delta, \varpi^+(c-d) \geq \alpha \geq \alpha \text{ and } \gamma^-(c-d) \leq \sigma, \varpi^-(c-d) \leq \beta >\} \\ &= \{< c \in \mathfrak{H} : d+\gamma^+(c) \geq \delta, \varpi^+(c) \geq \alpha, \text{ and } f+\gamma^-(c) \leq \sigma, \varpi^-(c) \leq \beta >\}. \end{aligned}$$

Thus, $f + \mathrm{A}^{\alpha,\beta}_{\delta,\sigma}(\eta)(\eta) = \mathrm{A}^{\alpha,\beta}_{\delta,\sigma}(\eta)(d+\eta)$, $\forall\, \delta \in [0,1]$, $\alpha \in [0,2\pi]$, and $\sigma \in [-1,0]$, $\beta \in [-2\pi, 0]$. □

Definition 3.11. *Suppose that θ and η are $\mathbb{CBFS}s$ of the universal discourse set Q. Then, the $\mathbb{CBF}$ sum of η and μ is expressed by $\eta+\mu = \{h, (\eta^+ + \mu^+)(h), (\eta^- + \mu^-)(h) : h \in Q\}$, where the following hold:*

$$(\eta^+ + \mu^+)(h) = \begin{cases} \vee\{\gamma^+(c)e^{i\varpi^+(c)} \wedge \tau^+(d)e^{i\upsilon^+(d)}\}, & if\ h = c+d; \\ e^{2\pi}, & \text{otherwise.} \end{cases}$$

$$(\eta^- + \mu^-)(h) = \begin{cases} \wedge\{\gamma^-(c)e^{i\varpi^-(c)} \vee \tau^-(d)e^{i\upsilon^-(d)}\}, & if\ h = c+d; \\ -e^{-2\pi}, & \text{otherwise.} \end{cases}$$

Definition 3.12. *Suppose η is a $\mathbb{CBFS}$ of Q. The support set $\eta_\star$ of η is described as follows:*

$$\eta_\star = \{d \in Q : \gamma^+(d) > 0, \varpi^+(d) > 0, \gamma^-(d) < 0, \varpi^-(d) < 0\}.$$

Remark 3.13. *Let η be a $\mathbb{CBFSR}$ of $\mathfrak{H}$. Then, $\eta_\star$ is a $\mathbb{CBFSR}$ of $\mathfrak{H}$.*

The following theorem demonstrates how a $\mathbb{BFI}$ support set is an ideal of $\mathfrak{H}$.

Theorem 3.14. *Let η be the a $\mathbb{CBF}$ isomorphism of $\mathfrak{H}$; then $\eta_\star$ is a $\mathbb{CBF}$ isomorphism of $\mathfrak{H}$.*

Proof. Note that η is $\mathbb{CBFI}$. Suppose that $f, g \in \eta_\star$. Consider $\eta^+(d-h) \geq \{\gamma^+(d)e^{i\varpi^+(d)} \wedge \gamma^+(h)e^{i\varpi^+(h)}\} > 0$, and $\eta^-(d-h) \leq \{\gamma^-(d)e^{i\varpi^-(d)} \vee \gamma^-(h)e^{i\varpi^-(h)}\} < 0$. This indicates that $d-h \in \eta_\star$. Moreover, suppose that $d \in \eta_\star$ and $h \in \mathfrak{H}$. Then, we have $\eta^+(dh) \geq \{\gamma^+(d)e^{i\varpi^+(h)} \wedge \gamma^+(h)e^{i\varpi^+(h)}\} > 0$, and $\eta^-(d-h) \leq \{\gamma^-(d)e^{i\varpi^-(d)} \vee \gamma^-(h)e^{i\varpi^-(h)}\} < 0$. This indicates that $d-h \in \eta_\star$. Similarly, $\eta^+(hd) > 0$ and $\eta^-(hd) < 0$ indicates that $fh, hd \in \eta_\star$. This indicates that $\eta_\star$ is an ideal of $\mathfrak{H}$. □

The intersection of any two support set $\mathbb{CBFSR}s$ of a ring is demonstrated by the following theorem.

Theorem 3.15. *If η and μ are $\mathbb{CBFSR}s$ of $\mathfrak{H}$; then, $(\eta \cap \mu)_\star = \eta_\star \cap \mu_\star$.*

Proof. For an arbitrary random component, $d \in (\theta \cap \eta)_\star$, then, $(\eta^+ \cap \mu^+)(d) > 0$ and $(\eta^- \cap \eta^-)(d) < 0$. We have $\eta^+(d)$, $\mu{+}^+(d) \geq \{\gamma^+(d)e^{i\varpi^+(d)} \wedge \tau^+(h)e^{i\upsilon^+(h)}\} = (\eta^+ \cap \mu^+)(d) > 0$. This indicates that $\eta^+(d)$, $\mu^+(k) > 0$, and $\eta^-(d)$, $\mu^-(d) \leq \{\gamma^-(d)e^{i\varpi^-(d)} \vee \tau^-(h)e^{i\upsilon^-(h)}\} = (\eta^- \cap \mu^-)(d) < 0$. This implies that $\theta^-(d), \eta^-(d) < 0$. This implies that $f \in \eta_\star \cap \mu_\star$. Consequently, $(\eta \cap \mu)_\star \subseteq \eta_\star \cap \mu_\star$. Moreover, $d \in \eta_\star \cap \mu_\star$. This implies that $\eta^+(d), \mu^+(d) > 0$ and $\eta^-(d)$, $\mu^-(d) < 0$. This implies that $\{\gamma^+(d)e^{i\varpi^+(d)} \wedge \tau^+(h)e^{i\upsilon^+(h)}\} > 0$, and $\mu^-(d) \leq \{\gamma^-(d)e^{i\varpi^-(d)} \vee \tau^-(h)e^{i\upsilon^-(h)}\} < 0$. This implies that $(\eta^+ \cap \mu^+)(d) > 0$ and $(\eta^- \cap \mu^-)(d) < 0$. This implies that $d \in (\eta \cap \mu)_\star$. Therefore, $(\eta \cap \mu)_\star \supseteq \eta_\star \cap \mu_\star$. Consequently, $(\eta \cap \mu)_\star = \eta_\star \cap \mu_\star$. This completes the proof. □

Remark 3.16. *Suppose that η and μ are $\mathbb{CBFSR}$s of $\mathfrak{H}$; then, $(\eta + \mu)_\star = \eta_\star + \mu_\star$.*

Definition 3.17. *Assume that η and μ are the $\mathbb{CBFSS}$ of $\mathfrak{H}$, respectively, with $\eta \subseteq \mu$. Then, η is known as a $\mathbb{CBFI}$ of μ if the subsequent assumptions is true:*

1) $\eta^+(d-h) \geq \{\gamma^+(h)e^{i\varpi^+(h)} \wedge \gamma^+(d)e^{i\varpi^+(d)}\}$, $\forall d,\ h \in \mathfrak{H}$,
2) $\eta^+(dh) \geq \{\vee\{\gamma^+(h)e^{i\varpi^+(h)} \wedge \tau^+(d)e^{i\upsilon^+(d)}\}, \{\tau^+(h)e^{i\upsilon^+(h)} \wedge \gamma^+(d)e^{i\varpi^+(d)}\}\}\ \forall\ d,\ h \in \mathfrak{H}$,
3) $\eta^-(d-h) \geq \{\gamma^-(h)e^{i\varpi^-(h)} \vee \gamma^-(d)e^{i\varpi^-(d)}\}$, $\forall d,\ h \in \mathfrak{H}$,
4) $\eta^-(dh) \geq \{\wedge\{\gamma^-(h)e^{i\varpi^-(h)} \vee \tau^+(d)e^{i\upsilon^-(d)}\}, \{\tau^-(h)e^{i\upsilon^-(h)} \vee \gamma^-(d)e^{i\varpi^-(d)}\}\}\ \forall\ d,\ h \in \mathfrak{H}$.

Theorem 3.18. *Suppose that η and μ are the $\mathbb{CBFSR}$ of a ring $\mathfrak{H}$ and η is a $\mathbb{CBFI}$ of μ. Thus $\eta_\star$ is an ideal of ring $\mu_\star$.*

Theorem 3.19. *Suppose that η is a $\mathbb{CBFI}$ and μ is a $\mathbb{CBFSR}$ of $\mathfrak{H}$; then ,$\eta \cap \mu$ is $\mathbb{CBFI}$ of μ.*

Proof. Suppose a component $d, h \in \mathfrak{H}$; then, We have the following:

$$\begin{aligned}
(\eta^+ \cap \mu^+)(d-h) =& \{\eta^P(d-h) \wedge \eta^+(d-h)\},\\
&\geq \wedge \{\{\gamma^+(d)e^{i\varpi^+(d)} \wedge \gamma^+(h)e^{i\varpi^+(h)}\}, \{\tau^+(d)e^{i\upsilon^+(d)} \wedge\ \tau^+(h)e^{i\upsilon^+(h)}\}\}\\
&= \wedge \{\{\gamma^+(d)e^{i\varpi^+(f)} \wedge \tau^+(d)e^{i\upsilon^+(d)}\}, \{\gamma^+(h)e^{i\varpi^+(h)} \wedge \tau^+(h)e^{i\upsilon^+(h)}\}\}\\
&= \wedge \{(\eta^+ \cap \mu^+)(d), (\eta^+ \cap \mu^+)(h)\},\ \forall\ d, h \in \mathfrak{H}.
\end{aligned}$$

Furthermore, we have the following:

$$\begin{aligned}
(\eta^+ \cap \mu^+)(dh) &= \wedge \{\eta^+(dh), \mu^+(dh)\},\\
&\geq \wedge \{\{\gamma^+(d)e^{i\varpi^+(d)} \vee \gamma^+(h)e^{i\varpi^+(h)}\}, \{\tau^+(d)e^{i\upsilon^+(d)} \wedge\ \tau^+(h)e^{i\upsilon^+(h)}\}\}\\
&= \vee \{\wedge\{\gamma^+(d)e^{i\varpi^+(d)}\{\tau^+(d)e^{i\upsilon^+(d)} \wedge\ \tau^+(h)e^{i\upsilon^+(h)}\},\\
&\quad \{\wedge\{\gamma^+(h)e^{i\varpi^+(h)}\{\tau^+(d)e^{i\upsilon^+(d)} \wedge\ \tau^+(h)e^{i\upsilon^+(h)}\}\\
&= \vee \{(\eta^+ \cap \mu^+)(d) \wedge \mu^+(h)\}, \{(\eta^+ \cap \mu^+)(h) \wedge \mu^+(d)\}\}.
\end{aligned}$$

Moreover, we have the following:

$$\begin{aligned}
(\eta^- \cap \mu^-)(d-h) =& \{\eta^P(d-h) \vee \eta^-(d-h)\},\\
&\geq \vee \{\{\gamma^+(d)e^{i\varpi^-(d)} \vee \gamma^-(h)e^{i\varpi^-(h)}\}, \{\tau^-(d)e^{i\upsilon^-(d)} \vee \tau^-(h)e^{i\upsilon^-(h)}\}\}\\
&= \vee \{\{\gamma^-(d)e^{i\varpi^-(d)} \vee \tau^-(d)e^{i\upsilon^-(d)}\}, \{\gamma^-(h)e^{i\varpi^-(h)} \vee \tau^-(h)e^{i\upsilon^-(h)}\}\}\\
&= \vee \{(\eta^- \cap \mu^-)(d), (\eta^- \cap \mu^-)(h)\},\ \forall\ d, h \in \mathfrak{H}.
\end{aligned}$$

Furthermore, we have the following:

$$
\begin{aligned}
(\eta^- \cap \mu^-)(dh) &= \vee\, \{\eta^-(dh), \mu^-(dh)\},\\
&\geq \vee\, \{\{\gamma^-(d)e^{i\varpi^-(d)} \wedge \gamma^-(h)e^{i\varpi^-(h)}\}, \{\tau^-(d)e^{i\upsilon^-(d)} \vee \tau^-(h)e^{i\upsilon^-(h)}\}\}\\
&= \wedge\, \{\vee\{\gamma^-(d)e^{i\varpi^-(d)}\{\tau^-(d)e^{i\upsilon^-(d)} \vee\ \tau^-(h)e^{i\upsilon^-(h)}\},\\
&\quad \{\vee\{\gamma^-(h)e^{i\varpi^-(h)}\{\tau^-(d)e^{i\upsilon^-(d)} \vee \tau^-(h)e^{i\upsilon^-(h)}\}\\
&= \wedge\, \{(\eta^- \cap \mu^-)(f) \vee \mu^-(h)\}, \{(\eta^+ \cap \mu^+)(h) \vee \mu^+(d)\}\}.
\end{aligned}
$$

This complete the proof. □

Remark 3.20. *Suppose that η, μ and ω are $\mathbb{CBFSR}s$ of $\mathfrak{H}$ such that η, and a μ are CBFI of ω; then, $\eta \cap \mu$ is CBFI of ω.*

Theorem 3.21. *Suppose that I is an ideal of a ring $\mathfrak{H}$. If $\eta = \{(f, \eta^+(d), \eta^-(f)) : f \in \mathfrak{H}\}$ is a $\mathbb{CBFSR}$ of $\mathfrak{H}$; then, a $\mathbb{CBFS}$ $\bar{\eta} = \{(f + I, \bar{\eta^+}(f + I), \bar{\eta^-}(f + I)) : f \in \mathfrak{H}\}$ of $\mathfrak{H}/I$ is also a $\mathbb{CBFSR}$ of $\mathfrak{H}/I$, where $\bar{\eta^+}(f + I) = \vee\{\gamma^+(f + c)e^{i\varpi^+(f+c)}|f \in I\}$ and $\bar{\eta^-}(f + I) = \wedge\{\gamma^+(f + c)e^{i\varpi^+(f+c)}|c \in I\}$.*

Proof. First, we demonstrate that $\gamma^+ : \mathfrak{H}/I \to [0, 1]$, $\varpi^+ : \mathfrak{H}/I \to [0, 2\pi]$ and $\gamma^- : \mathfrak{H}/I \to [-1, 0]$, $\varpi^- : \mathfrak{H}/I \to [-2\pi, 0]$ are well defined. Let $f + I = g + I$; then, $g = f + c$ for any $c \in I$.

Assume the following:

$$
\begin{aligned}
\bar{\eta^+}(g + L) &= \vee\{\gamma^+(g + d)e^{i\varpi^+(g+d)}|d \in I\}\\
&= \vee\{\gamma^+(f + c + d)e^{i\varpi^+(g+d)}(f + c + d)|d \in I\}\\
&= \vee\{\gamma^+(f + x)e^{i\varpi^+(g+d)}(f + x)(k + c)|x = c + d \in I\}\\
&= \bar{\eta^+}(f + I),
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{\eta^-}(g + L) &= \wedge\{\gamma^-(g + d)e^{i\varpi^-(g+d)}|d \in I\}\\
&= \wedge\{\gamma^-(f + c + d)e^{i\varpi^-(g+d)}(f + c + d)|d \in I\}\\
&= \wedge\{\gamma^-(f + x)e^{i\varpi^-(g+d)}(f + x)(k + c)|x = c + d \in I\}\\
&= \bar{\eta^-}(f + I).
\end{aligned}
$$

Thus, we have a well-defined $\bar{\eta^+}$ and $\bar{\eta^-}$. Now, we will demonstrate that $\bar{\eta}$ are a $\mathbb{CBFSR}s$ of $\mathfrak{H}/I$. We have $\bar{\eta^+}\{(d + I) - (h + I)\}$,

$$
\begin{aligned}
&= \bar{\eta^+}\{(d - h) + I\}\\
&= \vee\{\gamma^+(d - h + v)e^{i\varpi^+(h-d+v)}|v \in I\}\\
&= \vee\{\gamma^+(d - h + a - b)e^{i\varpi^+(d-h+a-b)}(d - h + v - w)|v = a - b \in I\}\\
&= \vee\{\gamma^+((d + a) - (h + b))e^{i\varpi^+((d+a)-(h+b))}(k - t + v - w)|a, b \in I\}\\
&\geq \vee\{\{\gamma^+(d + a)e^{i\varpi^+(d+a)} \wedge \gamma^+((h + b)e^{i\varpi^+(h+b)}|a, b \in I\}\}\\
&= \wedge \vee \{\gamma^+(d + a)e^{i\varpi^+(d+a)}|a \in I\}, \vee\{\gamma^+((h + b)e^{i\varpi^+(h+b)}|b \in I\}\}
\end{aligned}
$$

since a and b vary independently

$$= \wedge\{\bar{\eta^+}(d + I), \bar{\eta^-}(h + I)\}.$$

Furthermore, we have $\bar{\eta}^{+}\{(d+I)(h+I)\}$,

$$\begin{aligned} &= \bar{\eta}^{+}\{(dh)+I\} \\ &= \vee\{\gamma^{+}(dh+v)e^{i\varpi^{+}(dh+v)}|v\in I\} \\ &\geq \vee\{\{\gamma^{+}(d+a)e^{i\varpi^{+}(d+a)}\wedge\gamma^{+}((h+b)e^{i\varpi^{+}(h+b)}|a,b\in I\}\} \\ &= \wedge\{\vee\{\gamma^{+}(d+a)e^{i\varpi^{+}(d+a)},a\in I\},\vee\{\gamma^{+}((h+b)e^{i\varpi^{+}(h+b)}|b\in I\}\} \\ &= \wedge\{\bar{\eta}^{+}(d+I),\bar{\eta}^{+}(g+I)\}. \end{aligned}$$

Moreover, we have $\bar{\eta}^{-}\{(d+I)-(h+I)\}=\bar{\eta}^{-}\{(d-h)+I\}$,

$$\begin{aligned} &= \bar{\eta}^{-}\{(d-h)+I\} \\ &= \wedge\{\gamma^{-}(d-h+v)e^{i\varpi^{-}(d-h+v)}|v\in I\} \\ &= \wedge\{\gamma^{-}(d-h+a-b)e^{i\varpi^{-}(d-h+a-b)}(d-h+v-w)|v=a-b\in I\} \\ &= \wedge\{\gamma^{-}((d+a)-(h+b))e^{i\varpi^{-}((d+a)-(h+b))}(d-h+v-w)|a,b\in I\} \\ &\leq \vee\{\{\gamma^{-}(d+a)e^{i\varpi^{-}(d+a)}\vee\gamma^{-}((h+b)e^{i\varpi^{-}(h+b)}|a,b\in I\}\} \\ &= \vee\wedge\{\gamma^{-}(d+a)e^{i\varpi^{-}(d+a)}|a\in I\},\vee\{\gamma^{-}((h+b)e^{i\varpi^{-}(h+b)}|b\in I\}\} \end{aligned}$$

since a and b vary independently

$$= \vee\{\bar{\eta}^{-}(d+I),\bar{\eta}^{-}(h+I)\}.$$

Moreover, we have $\bar{\eta}^{-}\{(d+I)(h+I)\}$,

$$\begin{aligned} &= \bar{\eta}^{-}\{(dh)+I\} \\ &= \wedge\{\gamma^{-}(dh+v)e^{i\varpi^{-}(dh+v)}|v\in I\} \\ &\leq \wedge\{\{\gamma^{-}(d+a)e^{i\varpi^{-}(d+a)}\vee\gamma^{-}((h+b)e^{i\varpi^{-}(h+b)}|a,b\in I\}\} \\ &= \vee\{\wedge\{\gamma^{-}(d+a)e^{i\varpi^{-}(f+a)},a\in I\},\wedge\{\gamma^{-}((h+b)e^{i\varpi^{-}(g+b)}|b\in I\}\} \\ &= \vee\{\bar{\eta}^{-}(d+I),\bar{\eta}^{-}(h+I)\}. \end{aligned}$$

Hence, $\bar{\eta}=\{(d+I,\bar{\eta}^{+}(d+I),\bar{\eta}^{-}(d+I)):d\in\mathfrak{H}\}$ is a $\mathbb{CBFSR}$ of $\mathfrak{H}/I$. □

4. The basic results of complex bipolar fuzzy isomorphism of complex bipolar fuzzy subrings and complex bipolar fuzzy homomorphism

In this section, we examine the notion of the $\mathbb{CBF}$ homomorphism of the $\mathbb{CBFSR}$ and demonstrate that the fuzzy addition and fuzzy multiplication of the $\mathbb{CBFSR}$ of ring $\mathfrak{H}$ are preserved by this homomorphism. We examine the concept of a $\mathbb{CBF}$ homomorphism relationship among any two CBFSRs and provide clarification on complex bipolar fuzzy homomorphism for these $\mathbb{CBFSR}s$. In addition, we introduce the $\mathbb{CBF}$ isomorphism theorem for $\mathbb{CBFSR}s$.

Definition 4.1. *Suppose that $d:\mathfrak{H}\to\mathfrak{H}^{\star}$ is complex ring homomorphism from $\mathfrak{H}$ to $\mathfrak{H}^{\star}$. Let η and μ be a $\mathbb{CBFSR}s$ of $\mathfrak{H}$ and $\mathfrak{H}^{\star}$, consequently, the image and pre-image of η and μ are explained as $\psi(\eta)(d)=\{(d,\psi(\gamma^{+}(h)e^{i\varpi^{+}(h)}),\psi(\gamma^{-}(h)e^{i\varpi^{-}(h)}),h\in\mathfrak{H}^{\star}\}$ and $\psi^{-1}(\eta)(d)=\{(d,\psi^{-1}(\gamma^{+}(d)e^{i\varpi^{+}(d)}),\psi^{-1}(\gamma^{-}(d)e^{i\varpi^{-}(d)}),d\in\mathfrak{H}\}$, where we have the following:*

$$\psi(\eta^+)(h) = \begin{cases} \vee\{\gamma^+(d)e^{i\varpi^+(d)}, & d \in \psi^{-1}(h) \neq \emptyset\}, \text{for all } h \in \mathfrak{H}^\star; \\ e^{2\pi}, & \text{otherwise.} \end{cases}$$

$$\psi(\eta^-)(h) = \begin{cases} \wedge\{\gamma^-(d)e^{i\varpi^-(d)}, & d \in \psi^{-1}(h) \neq \emptyset\}, \text{for all } h \in \mathfrak{H}^\star; \\ -e^{-2\pi}, & \text{otherwise.} \end{cases}$$

Thus, the following scenario as valid:

$$\psi^{-1}(\tau^+(d)e^{i\upsilon^+(d)}) = \tau^+(\psi(d))e^{i\upsilon^+(\psi(d))}, \psi^{-1}(\tau^-(d)e^{i\upsilon^-(d)}) = \tau^-(\psi(d))e^{i\upsilon^-(\psi(d))} \quad \forall\, d \in \mathfrak{H},$$

where $\mu^+(d) = \tau^+(d)e^{i\upsilon^+(d)}$ and $\mu^-(d) = \tau^-(d)e^{i\upsilon^-(d)}$. The homomorphism ψ is known as a $\mathbb{CBF}$ homomorphism from η onto μ if $\psi(\eta) = \mu$ and is represented by $\eta \approx \mu$. A homomorphism ψ from a $\mathbb{CBFSR}$ η to μ is called a $\mathbb{CBF}$ isomorphism from η to μ if $\psi(\eta) = \mu$. In this condition, η is a $\mathbb{CBF}$ isomorphic to μ and is denoted by $\eta \cong \mu$. The homomorphism ψ is defined as a weak $\mathbb{CBF}$ homomorphism from η to μ if $\psi(\eta) \subseteq \mu$.

We demonstrate the complex fuzzy homomorphism link between the $\mathbb{CBFSR}$ of a ring and any one of its component rings in the following result.

Theorem 4.2. *Suppose that ψ is a homomorphism from ring $\mathfrak{H}$ to ring $\mathfrak{P}$. Suppose that η and μ are both $\mathbb{CBFS}$s of $\mathfrak{H}$.*

As a result, we have the following circumstance:

$$\begin{aligned} \psi(\eta + \mu) &= \psi(\eta) + \psi(\mu), \\ \psi(\eta \circ \mu) &= \psi(\eta) \circ \psi(\mu). \end{aligned}$$

Proof. Now, for $h \in \mathfrak{P}$, we possess the following:

$$\begin{aligned} \psi(\eta + \mu)(h) &= (\psi(\gamma^+ e^{i\varpi^+} + \tau^+ e^{i\upsilon^+})(h), \psi(\gamma^- e^{i\varpi^-} + \tau^- e^{i\upsilon^-})(h) \\ &= (\psi(\eta^+ + \mu^+)(h), \psi(\eta^- + \mu^-)(h)), \\ \text{and } (\psi(\eta) + \psi(\mu))(h) &= (\psi(\gamma^+ e^{i\varpi^+}) + \psi(\tau^+ e^{i\upsilon^+}))(h), \psi(\gamma^- e^{i\varpi^-}) + \psi(\tau^- e^{i\upsilon^-})(h)) \\ &= (\psi(\eta^+) + \psi(\mu^+)(h), \psi(\eta^-) + \psi(\mu^-)(h)). \end{aligned}$$

Assume the following:
$\psi(\eta^+ + \mu^+)(h) = \vee\{(\gamma^+ e^{i\varpi^+} + \tau^+ e^{i\upsilon^+})(h) : f \in \mathfrak{H}, h = \psi(d)\}$

$$\begin{aligned} &= \vee\{\vee\{\gamma^+(d_1)e^{i\varpi^+(d_1)} \wedge \tau^+(d_2)e^{i\upsilon^+(d_2)} : d_1, d_2 \in \mathfrak{H}, d = d_1 + d_2\}d \in \mathfrak{H}, h = \psi(d)\} \\ &= \vee\{\vee\{\gamma^+(d_1)e^{i\varpi^+(d_1)} \wedge \tau^+(d_2)e^{i\upsilon^+(d_2)} : d_1, d_2 \in \mathfrak{H}, h_1 = \psi(d_1), h_2 = \psi(d_2)\}, h_1, h_2 \in \mathfrak{P}, \\ &\quad h = h_1 + h_2\} \\ &= \vee\{\wedge\{\vee\{\gamma^+(d_1)e^{i\varpi^+(d_1)} : d_1 \in \mathfrak{H}, h_1 = \psi(d_1)\}, \vee\{\tau^+(d_2)e^{i\upsilon^+(d_2)} : d_2 \in \mathfrak{H} : h_2 = \psi(d_2)\}\}\} \\ &= \vee\{\psi(\gamma^+(h_1)e^{i\varpi^+(h_1)} \wedge \psi(\tau^+(h_2)e^{i\upsilon^+(h_2)}\} : h_1, h_2 \in \mathfrak{P}, h = h_1 + h_2\} \\ &= (\psi(\gamma^+ e^{i\varpi^+)} + \psi(\tau^+ e^{i\upsilon^+})(h) \\ &= (\psi(\eta^+) + \psi(\mu^+))(h). \end{aligned}$$

Furthermore, we possess the following:
$\psi(\eta^- + \mu^-)(h) = \wedge\{(\gamma^- e^{i\varpi^+} + \tau^- e^{i\upsilon^-})(d) : d \in \mathfrak{H}, d = \psi(d)\}$

$$= \wedge\{\wedge\{\gamma^-(d_1)e^{i\varpi^-(d_1)} \vee \tau^-(d_2)e^{i\upsilon^-(d_2)} : d_1, d_2 \in \mathfrak{H}, d = d_1 + d_2\}d \in \mathfrak{H}, h = \psi(d)\}$$
$$= \wedge\{\wedge\{\gamma^-(d_1)e^{i\varpi^-(d_1)} \vee \tau^-(d_2)e^{i\upsilon^+(d_2)} : d_1, d_2 \in \mathfrak{H}, d_1 = \psi(d_1), h_2 = \psi(h_2)\}, h_1, h_2 \in \mathfrak{P}, h = h_1 + h_2\}$$
$$= \wedge\{\vee\{\wedge\{\gamma^-(d_1)e^{i\varpi^-(d_1)} : d_1 \in \mathfrak{H}, h_1 = \psi(d_1)\}, \vee\{\tau^-(d_2)e^{i\upsilon^-(d_2)} : d_2 \in \mathfrak{H} : h_2 = \psi(d_2)\}\}\}$$
$$= \wedge\{\psi(\gamma^-(h_1)e^{i\varpi^-(h_1)} \vee \psi(\tau^-(h_2)e^{i\upsilon^-(h_2)}\} : h_1, h_2 \in \mathfrak{P}, h = h_1 + h_2\}$$
$$= (\psi(\gamma^- e^{i\varpi^+)} + \psi(\tau^- e^{i\upsilon^-})(h)$$
$$= (\psi(\eta^-) + \psi(\mu^-))(h).$$

Therefore, $\psi(\eta + \mu) = \psi(\eta) + \psi(\mu)$.

(ii) For $g \in \mathfrak{P}$, we have the following:

$$\begin{aligned} \psi(\eta \circ \mu)(h) &= (\psi(\gamma^+ e^{i\varpi^+} \circ \tau^+ e^{i\upsilon^+})(h), \psi(\gamma^- e^{i\varpi^-} \circ \tau^- e^{i\upsilon^-})(h) \\ &= (\psi(\eta^+ \circ \mu^+)(h), \psi(\eta^- \circ \mu^-)(h)), \\ \text{and } (\psi(\eta) \circ \psi(\mu))(h) &= (\psi(\gamma^+ e^{i\varpi^+}) \circ \psi(\tau^+ e^{i\upsilon^+}))(h), \psi(\gamma^- e^{i\varpi^-}) \circ \psi(\tau^- e^{i\upsilon^-})(h)) \\ &= (\psi(\eta^+) \circ \psi(\mu^+)(h), \psi(\eta^-) \circ \psi(\mu^-)(h)). \end{aligned}$$

Assume the following:
$\psi(\eta^+ \circ \mu^+)(h) = \vee\{(\gamma^+ e^{i\varpi^+} \circ \tau^+ e^{i\upsilon^+})(d) : d \in \mathfrak{H}, h = \psi(d)\}$

$$= \vee\{\vee\{\gamma^+(d_1)e^{i\varpi^+(d_1)} \wedge \tau^+(d_2)e^{i\upsilon^+(d_2)} : d_1, d_2 \in \mathfrak{H}, d = d_1 d_2\}d \in \mathfrak{H}, h = \psi(d)\}$$
$$= \vee\{\vee\{\gamma^+(d_1)e^{i\varpi^+(d_1)} \wedge \tau^+(d_2)e^{i\upsilon^+(d_2)} : d_1, d_2 \in \mathfrak{H}, h_1 = \psi(d_1), h_2 = \psi(d_2)\}, h_1, h_2 \in \mathfrak{P}, h = h_1 h_2\}$$
$$= \vee\{\wedge\{\vee\{\gamma^+(d_1)e^{i\varpi^+(d_1)} : d_1 \in \mathfrak{H}, h_1 = \psi(d_1)\}, \vee\{\tau^+(d_2)e^{i\upsilon^+(d_2)} : d_2 \in \mathfrak{H} : h_2 = \psi(d_2)\}\}\}$$
$$= \vee\{\psi(\gamma^+(h_1)e^{i\varpi^+(h_1)} \wedge \psi(\tau^+(h_2)e^{i\upsilon^+(h_2)}\} : h_1, h_2 \in \mathfrak{P}, h = h_1 h_2\}$$
$$= (\psi(\gamma^+ e^{i\varpi^+}) \circ \psi(\tau^+ e^{i\upsilon^+})(h)$$
$$= (\psi(\eta^+) \circ \psi(\mu^+))(h).$$

Furthermore, we possess the following:
$\psi(\eta^- \circ \mu^-)(h) = \wedge\{(\gamma^- e^{i\varpi^+} \circ \tau^- e^{i\upsilon^-})(d) : d \in \mathfrak{H}, h = \psi(d)\}$

$$= \wedge\{\wedge\{\gamma^-(d_1)e^{i\varpi^-(d_1)} \vee \tau^-(d_2)e^{i\upsilon^-(d_2)} : d_1, d_2 \in \mathfrak{H}, d = d_1 d_2\}d \in \mathfrak{H}, h = \psi(d)\}$$
$$= \wedge\{\wedge\{\gamma^-(d_1)e^{i\varpi^-(d_1)} \vee \tau^-(d_2)e^{i\upsilon^+(d_2)} : d_1, d_2 \in \mathfrak{H}, h_1 = \psi(d_1), h_2 = \psi(d_2)\}, h_1, h_2 \in \mathfrak{P}, g = g_1 g_2\}$$
$$= \wedge\{\vee\{\wedge\{\gamma^-(d_1)e^{i\varpi^-(d_1)} : d_1 \in \mathfrak{H}, h_1 = \psi(d_1)\}, \vee\{\tau^-(d_2)e^{i\upsilon^-(d_2)} : d_2 \in \mathfrak{H} : h_2 = \psi(d_2)\}\}\}$$
$$= \wedge\{\psi(\gamma^-(h_1)e^{i\varpi^-(h_1)} \vee \psi(\tau^-(h_2)e^{i\upsilon^-(h_2)}\} : h_1, h_2 \in \mathfrak{P}, h = h_1 h_2\}$$
$$= (\psi(\gamma^- e^{i\varpi^+)} \circ \psi(\tau^- e^{i\upsilon^-})(h)$$
$$= (\psi(\eta^-) \circ \psi(\mu^-))(h).$$

Therefore, $\psi(\eta \circ \mu) = \psi(\eta) \circ \psi(\mu)$.

□

Theorem 4.3. *Suppose $\phi : \mathfrak{H} \to \mathfrak{H}/I$ is a $\mathbb{CRH}$ from $\mathfrak{H}$ onto $\mathfrak{H}/I$, and I represents an ideal of the ring $\mathfrak{H}$. Suppose η and η_ϱ are a $\mathbb{CBFSR}$s of $\mathfrak{H}$ and $\mathfrak{H}/I$, respectively. Thus, ϕ is a CBFH from η onto η_ϱ.*

Proof. Given that the function ϕ is a homomorphism from $\mathfrak{H}$ onto $\mathfrak{H}/I$, as defined based on the principle of $\phi(c) = c + I$ for each $c \in \mathfrak{H}$. We have $\phi(\eta)(\mathrm{c + I}) = (\phi(\eta^+)(\mathrm{c + I}), \phi(\eta^-)(\mathrm{c + I}))$. Where $\phi(\eta^+(\mathrm{c + I}) = \vee\{\gamma^+(d)e^{i\varpi^+(d)} : d \in \phi^{-1}(\mathrm{c + l})\}$ and $\phi(\eta^-(\mathrm{c + I}) = \wedge\{\gamma^-(d)e^{i\varpi^-(d)} : d \in \phi^{-1}(\mathrm{c + I})\}$. Suppose the following scenario:

$$\begin{aligned}
\phi(\eta^+(\mathrm{c + I}) &= \vee\,\{\gamma^+(d)e^{i\varpi^+(d)} : d \in \phi^{-1}(\mathrm{c + I})\}\\
&= \vee\,\{\gamma^+(d)e^{i\varpi^+(d)} : \phi(d) = \mathrm{c + I}\}\\
&= \vee\,\{\gamma^+(d)e^{i\varpi^+(d)}(f) : d + I = \mathrm{c + I}\}\\
&= \vee\,\{\gamma^+(d)e^{i\varpi^+(d)} : d = c + m, h \in I\}\\
&= \vee\,\{\gamma^+(c + m)e^{i\varpi^+(c+m)} : h \in L\}\\
&= \eta_\varrho^+(c + I).
\end{aligned}$$

It indicates that $\phi(\eta^+) = \eta_\varrho^+$. Further, we have

$$\begin{aligned}
\phi(\eta^-(\mathrm{c + I}) &= \wedge\,\{\gamma^-(d)e^{i\varpi^-(d)} : d \in \phi^{-1}(\mathrm{c + I})\}\\
&= \wedge\,\{\gamma^-(d)e^{i\varpi^-(f)} : \phi(d) = \mathrm{c + I}\}\\
&= \wedge\,\{\gamma^-(d)e^{i\varpi^-(d)}(d) : d + I = \mathrm{c + I}\}\\
&= \wedge\,\{\gamma^-(d)e^{i\varpi^-(d)} : d = c + m, h \in I\}\\
&= \wedge\,\{\gamma^-(c + m)e^{i\varpi^-(c+m)} : h \in I\}\\
&= \eta_\varrho^-(c + I).
\end{aligned}$$

It indicates that $\phi(\eta^-) = \eta_\varrho^-$. Hence, this completes the proof. □

In the following example, we illustrate the $\mathbb{CBF}$ homomorphism from a ring to its $\mathbb{QR}$.

Example 4.4. *Let the factor of the ring $A/2A = \{2A, 1 + 2A\}$, where $\mathfrak{H} = A$ is an integer of the ring and $\mathfrak{P} = 2A = \{2d | d \in A\}$ is the ideal ring A. The definition of a $\mathbb{CBFS}$ of A is as follows:*

$$\eta^+(d) = \begin{cases} 0.6e^{i\pi 0.3}, & \text{if} \quad d \in 2A,\\ 0.3e^{i\pi 0.2}, & \text{if} \quad d \in 1 + 2A, \end{cases}$$

and

$$\eta^+(d) = \begin{cases} -0.7e^{-i\pi 0.5}, & \text{if} \quad d \in 2A,\\ -0.4e^{-i\pi 0.3}, & \text{if} \quad d \in 1 + 2A. \end{cases}$$

Define a $\mathbb{CBFSR}$ η_ϱ of $A/2A$ as follows:

$$\eta_\varrho^+(h) = \begin{cases} 0.6e^{0.3}, & \text{if} \quad h = 2A,\\ 0.3e^{0.2}, & \text{if} \quad h = 1 + 2A, \end{cases}$$

and

$$\eta_{\varrho}^{-}(h)=\begin{cases}-0.7e^{-i\pi 0.5}, & \text{if} \quad h=2A,\\ -0.4e^{-i\pi 0.3}, & \text{if} \quad h=1+2A.\end{cases}$$

The structure defines a natural homomorphism ϕ from A to $A/2A$: $\phi(d)=d+2A$, for all $d \in A$. This indicates that $\phi(\eta^{+})(2A)=\vee\{\gamma^{+}(d)e^{i\varpi^{+}(d)} : d \in 2A\}$, which implies that $\phi(\eta^{+})(2A)=0.6e^{i\pi 0.3}7$ which $\pi(\eta^{-})(2A)=\wedge\{\gamma^{-}(d)e^{i\varpi^{-}(d)} : d \in 2A\ \}$, which implies that $\phi(\eta^{-})(2A)=-0.7e^{-i\pi 0.5}$. Moreover, $\pi(\eta^{+})(1+2A)=\vee\{\gamma^{+}(d)e^{i\varpi^{+}(d)} : d \in 1+2A\ \}$, which implies that $\phi(\eta^{+})(1+2A)=0.3e^{i\pi 0.2}$ and $\phi(\eta^{-})(1+2Z)=\wedge\{\gamma^{-}(d)e^{i\varpi^{-}(d)} : d \in 1+2A\ \}$, and implies that $\phi(\eta^{-})(1+2A)=-0.4e^{-i\pi 0.3}$. Thus, $\phi(\eta)=\eta_{\varrho}$.

Theorem 4.5. *Let η and μ be a $\mathbb{CBFSR}$s of $\mathfrak{H}$ and $\mathfrak{H}^{\star}$ rings, respectively, and ψ is a $\mathbb{CBF}$ homomorphism from η onto μ. Then, a mapping $\phi : \mathfrak{H}/I \rightarrow H^{\star}$ is a $\mathbb{CBF}$ homomorphism from η_{ϱ} onto μ, as η_{ϱ} is a $\mathbb{CBFSR}$ of $\mathfrak{H}/I$.*

Proof. Consider that $\psi(\eta)=\mu$. Additionally, we possess a homomorphism ϕ from $\mathfrak{H}/I$ onto $\mathfrak{H}^{\star}$ as specified by the principal $\phi(f+I)=\psi(d)=h,\ \forall\ d \in \mathfrak{H}$. The following can be utilized to define the representation of η_{ϱ} for the function ϕ:

$$\phi(\eta_{\varrho})(g)=(\phi(\eta_{\rho}^{+})(h),\phi(\eta_{\varrho}^{-})(h)), \forall\ h \in \mathfrak{H}^{\star}.$$

Now, we possess the following:

$$\begin{aligned}\phi(\eta_{\varrho}^{+})(h) &= \vee\ \{\gamma_{\rho}^{+}(d+I)e^{i\varpi_{\rho}^{+}(d+I)} : d+I \in \phi^{-1}(h), h \in \mathfrak{H}^{\star}\}\\ &= \vee\ \{\gamma_{\rho}^{+}(d+I)e^{i\varpi_{\rho}^{+}(d+I)} : \phi(d+I)=h, h \in \mathfrak{H}^{\star}\}\\ &= \vee\ \{\gamma_{\rho}^{+}(d+h)e^{i\varpi_{\rho}^{+}(d+h)} : h \in I, \psi(d)=h\}\\ &= \vee\ \{\gamma_{\rho}^{+}(v)e^{i\varpi_{\rho}^{+}(\mathfrak{v})} : v\in \psi^{-1}(h)\}\\ &=\psi(\eta^{+})(h)\\ &=\mu^{+}(h).\end{aligned}$$

This indicates that $\phi(\eta_{\varrho}^{+})(h)=\mu^{+}(h)\ \forall\ d \in \mathfrak{H}^{\star}$, which show that $\phi(\eta_{\varrho})=\mu$.

Furthermore, we possess the following:

$$\begin{aligned}\phi(\eta_{\varrho}^{-})(h) &= \wedge\ \{\gamma_{\varrho}^{-}(d+I)e^{i\varpi_{\varrho}^{-}(d+I)} : d+I \in \phi^{-1}(h), h \in \mathfrak{H}^{\star}\}\\ &= \wedge\ \{\gamma_{\varrho}^{-}(d+I)e^{i\varpi_{\varrho}^{-}(d+I)} : \phi(d+I)=h, h \in \mathfrak{H}^{\star}\}\\ &= \wedge\ \{\gamma_{\varrho}^{-}(d+h)e^{i\varpi_{\varrho}^{-}(d+h)} : h \in I, \psi(d)=h\}\\ &= \wedge\ \{\gamma_{\varrho}^{-}(v)e^{i\varpi_{\varrho}^{-}(v)} : v\in \psi^{-1}(h)\}\\ &=\psi(\eta^{-})(h)\\ &=\mu^{-}(h).\end{aligned}$$

Thus, $\phi(\eta_{\varrho})=\mu$. This completes the proof. □

The following example depicted Theorem 4.5.

Example 4.6. *Let $\mathbb{M} = \{0, \pm1, \pm2, \pm3, ...\}$ be the ring, and $Z_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ be the modulo of the ring of integers 4. Let $\psi(d) = d$ (mod 4) be the homomorphism definition of $\mathbb{M}$ on $\mathbb{Z}_4$. Then, let M's $\mathbb{CBFS}$ be defined as follows:*

$$\eta^{+}(d) = \begin{cases} 0.8e^{i\pi 0.4}, & \text{if} \quad d \in 2M, \\ 0.6^{0.3i\pi}, & \text{if} \quad d \notin 2M, \end{cases}$$

and

$$\eta^{-}(d) = \begin{cases} -0.8e^{-i\pi 0.6}, & \text{if} \quad d \in 2M, \\ -0.5e^{-i\pi 0.4}, & \text{if} \quad d \notin 2M. \end{cases}$$

The following is the CBFS μ of M_4:

$$\mu^{+}(d) = \begin{cases} 0.7e^{i\pi 0.4}, & \text{if} \quad d \in 2M, \\ 0.5^{i\pi 0.3}, & \text{if} \quad d \notin 2M, \end{cases}$$

and

$$\eta^{-}(d) = \begin{cases} -0.8e^{-i\pi 0.6}, & \text{if} \quad d \in 2M, \\ -0.5e^{-i\pi 0.4}, & \text{if} \quad d \notin 2M. \end{cases}$$

Consider the following scenario:

$$\begin{aligned} \psi(\eta^{+})(0) &= \vee\{\gamma^{+}(v)e^{i\varpi^{+}(v)} : v \in 4M\} = 0.7e^{i\pi 0.4}, \\ \psi(\eta^{+})(1) &= \vee\{\gamma^{+}(v)e^{i\varpi^{+}(v)} : v \in 1 + 4Z\} = 0.6^{i\pi 0.3}. \end{aligned}$$

Similarly, $\psi(\eta^{+})(2) = 0.7e^{i\pi 0.4}$ and $\psi(\eta^{+})(3) = 0.6^{i\pi 0.3}$. In addition, we possess the following:

$$\begin{aligned} \psi(\eta^{-})(0) &= \wedge\{\gamma^{-}(v)e^{i\varpi^{-}(v)} : v \in 4M\} = -0.8e^{-i\pi 0.6}, \\ \psi(\eta^{-})(1) &= \wedge\{\gamma^{-}(v)e^{i\varpi^{-}(v)} : v \in 1 + 4A\} = -0.5e^{-i\pi 0.4}, \\ \psi(\eta^{-})(2) &= -0.8e^{-i\pi 0.6}, \ \psi(\eta^{-})(3) = -0.5e^{-i\pi 0.4}. \end{aligned}$$

Thus, $\psi(\eta) = \mu$. $M/4M = \{4M, 1 + 4M, 2 + 4M, 3 + 4M\}$ represents $\mathbb{QR}$ of $M = \{0, \pm1, \pm2, \pm3, ...\}$, where the integer ring M has an ideal $4M$. For $M/4M$, we define the $\mathbb{CBFS}$ η_{ϱ} as follows:

$$\eta_{\varrho}^{+}(v) = \begin{cases} 0.7e^{i\pi 0.4} : & v \in \{4M, 2 + 4M\}, \\ 0.6^{i\pi 0.3} : & v \in \{1 + 4M, 3 + 4M\}, \end{cases}$$

and

$$\eta_{\varrho}^{-}(v) = \begin{cases} -0.8e^{-i\pi 0.6} : & v \in \{4M, 2 + 4M\}, \\ -0.5e^{-i\pi 0.4} : & v \in \{1 + 4M, 3 + 4M\}. \end{cases}$$

Illustrate a mapping ϕ from $M/4M$ onto M_4, which follows, $\phi(d + 4M) = \psi(d) = d$ (mod 4), for all $d \in M$. Based on the above mentioned data, we possess the following:

$$\phi(\eta_{\varrho}^{+})(0) = \vee\{\gamma_{\varrho}^{+}(d + 4M)e^{i\varpi_{\varrho}^{+}(d+4M)} : f + 4M \in \phi^{-1}(0), \quad d \in M\} = 0.7e^{i\pi 0.4}, \text{ and}$$

$$\begin{aligned} &\phi(\eta_{\varrho}^{-})(0) = \wedge\{\gamma^{-}(f + 4M)e^{i\varpi_{\varrho}^{-(d+4M)}} : d + 4M \in \phi^{-1}(0), d \in M\} = -0.8e^{-i\pi 0.6}, \\ &\phi(\eta_{\varrho}^{+})(1) = 0.6^{i\pi 0.3} = \phi(\eta_{\varrho}^{+})(3), \ \phi(\theta_{\varrho}^{+})(2) = 0.7e^{i\pi 0.4} \text{ and } \phi(\eta_{\varrho}^{-})(1) = \phi(\eta_{\varrho}^{-})(3) \\ &= -0.5e^{-i\pi 0.4}, \phi(\theta_{\rho}^{-})(2) = -0.8e^{-i\pi 0.6}. \text{ Therefore, } \phi(\eta_{\varrho}) = \mu. \end{aligned}$$

Remark 4.7. *Let η and μ be $\mathbb{CBFSR}$s of rings $\mathfrak{H}$ and $\mathfrak{H}^\star$, respectfully, and h be a $\mathbb{CBF}$ homomorphism from η onto μ with $\mathfrak{P} = \{h \in \mathfrak{H}, \psi(h) = 0_{\mathfrak{H}^\star}\}$ such as a kernel of h. Thus, the mapping χ from $\mathfrak{H}/\mathfrak{P}$ to $\mathfrak{H}^\star$ is a $\mathbb{CBF}$ homomorphism from $\eta^{\mathfrak{P}}$ onto μ, where $\eta^{\mathfrak{P}}$ is a $\mathbb{CBFSR}$ of $\mathfrak{H}/\mathfrak{P}$.*

As a consequence of the following result, we demonstrate a crucial connection between any factor ring and the $\mathbb{CBFSR}$*s* of a ring $\mathfrak{H}^\star$.

Theorem 4.8. *Suppose that η and μ are $\mathbb{CBFSR}$ of $\mathfrak{H}$ and $\mathfrak{H}^\star$, consequently. Let ψ be a $\mathbb{CBF}$ homomorphism from η onto μ and the inherent homomorphism ϕ from $\mathfrak{H}^\star$ onto $\mathfrak{H}^\star/I^\star$ be a $\mathbb{CBF}$ homomorphism from μ onto $\eta_{\varrho^\star}$, whereas $\eta_{\varrho^\star}$ is a $\mathbb{CBFSR}$ of $\mathfrak{H}^\star/I^\star$. Then $\chi = \phi \circ \psi$ is a $\mathbb{CBF}$ homomorphism from η onto $\eta_{\varrho^\star}$, where I is an ideal of $\mathfrak{H}$ with $\psi(I) = I^\star$.*

Proof. In this case, ϕ is the inherent homomorphism between $\mathfrak{H}$ and $\mathfrak{H}^\star/I^\star$.

For any $c^\star + I^\star \in \mathfrak{H}^\star/I^\star$, we have $(\phi \circ \psi)(\eta)(\mathrm{c}^\star + I^\star) = ((\phi \circ \psi)(\mu^+)(\mathrm{c}^\star + I^\star), (\phi \circ \psi)$ $(\eta^-)(\mathrm{c}^\star + L^\star))$ in the following scenarios:

$$(\phi \circ \psi)(\eta^+)(\mathrm{c}^\star + I^\star) = \vee\{\gamma^+(v)e^{i\varpi^+(v)} : v \in (\phi \circ \psi)^{-1}(\mathrm{c}^\star + I^\star)\},$$

and

$$(\phi \circ \psi)(\eta^-)(\mathrm{c}^\star + I^\star) = \wedge\{\gamma^-(v)e^{i\varpi^-(v)} : v \in (\phi \circ \psi)^{-1}(\mathrm{c}^\star + I^\star)\}.$$

Assume the following:

$$\begin{aligned}
(\phi \circ \psi)(\eta^+)(\mathrm{c}^\star + I^\star) &= \vee\{\gamma^+(v)e^{i\varpi^+(v)} : v \in (\phi \circ \psi)^{-1}(\mathrm{c}^\star + I^\star)\} \\
&= \vee\{\gamma^+(v)e^{i\varpi^+(v)} : v \in \psi^{-1}(\phi^{-1}(\mathrm{c}^\star + I^\star))\} \\
&= \psi(\eta^+)(\phi^{-1}(\mathrm{c}^\star + I^\star)) \\
&= \mu^{P=+}(\phi^{-1}(\mathrm{c}^\star + I^\star)) \\
&= (\phi^{-1})^{-1}(\mu^+)(\mathrm{c}^\star + I^\star) \\
&= \phi(\mu^+)(\mathrm{c}^\star + I^\star) = \eta^+_{\varrho^\star}(\mathrm{c}^\star + I^\star). \\
&= \eta^+_{\varrho^\star}(\mathrm{c}^\star + I^\star). \\
&\Rightarrow \chi(\eta^+) = \eta^+_{\varrho^\star}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
(\phi \circ \psi)(\eta^-)(\mathrm{c}^\star + I^\star) &= \wedge\{\gamma^-(v)e^{i\varpi^-(v)} : v \in (\phi \circ \psi)^{-1}(\mathrm{c}^\star + I^\star)\} \\
&= \wedge\{\gamma^-(v)e^{i\varpi^-(v)} : v \in \psi^{-1}(\phi^{-1}(\mathrm{c}^\star + I^\star))\} \\
&= \psi(\eta^-)(\phi^{-1}(\mathrm{c}^\star + I^\star)) \\
&= \mu^{P=+}(\phi^{-1}(\mathrm{c}^\star + I^\star)) \\
&= (\phi^{-1})^{-1}(\mu^-)(\mathrm{c}^\star + I^\star) \\
&= \phi(\mu^-)(\mathrm{c}^\star + I^\star) = \eta^-_{\varrho^\star}(\mathrm{c}^\star + I^\star). \\
&= \eta^-_{\varrho^\star}(\mathrm{c}^\star + I^\star). \\
&\Rightarrow \chi(\eta^-) = \eta^-_{\varrho^\star}.
\end{aligned}$$

Hence, this proof is completed. □

Theorem 4.9. *Let η and μ be $\mathbb{CBFSR}$s of $\mathfrak{H}$ and $\mathfrak{H}^\star$, respectfully, and ψ is a $\mathbb{CBF}$ homomorphism from η onto μ. Consider $\phi : \mathfrak{H}^\star \to \mathfrak{H}^\star/\psi^\star$ an possess essential homomorphism and $I = \{d \in \mathfrak{H} : \psi(d) \in I^\star\}$. Thus, a mapping $\rho : \mathfrak{H}/I \to \mathfrak{H}^\star/I^\star$ is a $\mathbb{CBF}$ homeomorphism from η_ϱ onto $\eta_{\varrho^\star}$, whereas η_ϱ and $\eta_{\varrho^\star}$ are $\mathbb{CBFSR}$s of $\mathfrak{H}/I$ and $\mathfrak{H}^\star/I^\star$, respectfully.*

Proof. From Theorem 4.8, we determine a mapping $\varphi : \mathfrak{H} \to \mathfrak{H}^\star/I^\star$, which means that the φ composition is made up by visualizing ψ and ϕ which means that $\varphi(\eta) = (\phi\circ\psi)(\eta) = c^\star + I^\star, \forall\, c^\star \in \mathfrak{H}^\star$. Moreover, $\varphi(I) = (\phi \circ \psi)(I) = \phi(\psi(I)) = \phi(I^\star) = I^\star$. Consider the $\mathbb{CBFSR}$ η_ϱ of $\mathfrak{H}/I$ as: $\eta_\varrho(d + I) = (\eta^+_\varrho(d + I), \eta^-_\varrho(d + I)))$, where we possess the following:

$$\eta^+_\varrho(d + I) = \vee\{\gamma^+(v)e^{i\varpi^+(v)} : v \in d + I\},$$

and

$$\eta^-_\varrho(d + I) = \wedge\{\gamma^+(v)e^{i\varpi^+(v)} : v \in d + I\}.$$

This demonstrates that φ is a $\mathbb{CBF}$ homeomorphism using $ker(\varphi) = I$. Express a visualization ρ from $\mathfrak{H}/I$ to $\mathfrak{H}^\star/I^\star$, that is, $\rho(c + I) = c^\star + I^\star, c \in \mathfrak{H}, c^\star \in \mathfrak{H}^\star$, where $\rho(\eta_\varrho) \to \eta_{\varrho\star}$ is defined by the following rule:

$$\rho(\eta_\varrho)(\mathrm{c}^\star + I^\star) = (\rho(\eta^+_\varrho)(\mathrm{c}^\star + I^\star), \rho(\eta^-_\varrho)(\mathrm{c}^\star + I^\star)),$$

where

$$\rho(\eta^+_\varrho)(\mathrm{c}^\star + I^\star) = \vee\{\gamma^+_\varrho(d + I)e^{i\varpi^+_\varrho(d+I)} : d + I \in \rho^{-1}(\mathrm{c}^\star + I^\star)\},$$

and

$$\rho(\eta^-_\varrho)(\mathrm{c}^\star + I^\star) = \vee\{\gamma^-_\varrho(d + I)e^{i\varpi^-_\varrho(d+I)} : d + I \in \rho^{-1}(\mathrm{c}^\star + I^\star)\}.$$

Assume the following:

$$\begin{aligned}
\rho(\eta^+_\varrho)(\mathrm{c}^\star + I^\star) &= \vee\, \{\gamma^+_\varrho(c + I)e^{i\varpi^+_\varrho(c+I)} : \mathrm{c} + I \in \rho^{-1}(\mathrm{c}^\star + I^\star)\} \\
&= \vee\, \{\vee\{\gamma^+(c + u)e^{i\varpi^-(c+u)}(\mathrm{c} + u) : u \in I, \mathrm{c} \in \mathfrak{H}, \psi(\mathrm{c})\}= \mathrm{c}^\star, \rho(\mathrm{c} + I) = \psi(\mathrm{c}) + I^\star\} \\
&= \vee\, \{\gamma^+(c + u)e^{i\varpi^-(c+u)}(\mathrm{c} + u) : u \in I, \mathrm{c} \in \mathfrak{H}, \psi(\mathrm{c}) = \mathrm{c}^\star\} \\
&= \vee\, \{\gamma^+(u)e^{i\varpi^-(u)}(\mathrm{c} + u) : u \in \psi^{-1}(\mathrm{c}^\star)\} \\
&=\psi(\eta^+)(\mathrm{c}^\star) \\
&=\mu^+(\mathrm{c}^\star), \quad \mathrm{c}^\star \in \mathfrak{H}^\star \\
&= \vee\, \{\mu^+(\mathrm{c}^\star) : \phi(\mathrm{c}^\star) = \mathrm{c}^\star +^I \star\} \\
&= \vee\, \{\mu^+(\mathrm{c}^\star) : \mathrm{c}^\star \in \pi^{-1}(\mathrm{c}^\star + I^\star)\} \\
&=\phi(\mu^+)(\mathrm{c}^\star + I^\star) \\
&=\eta^+_{\varrho^\star}(\mathrm{c}^\star + I^\star).
\end{aligned}$$

This indicates that $\rho(\eta^+_\varrho) = \eta^+_{\varrho^\star}$. In addition, we have the following:

$$\begin{aligned}
\rho(\eta^-_\varrho)(\mathrm{c}^\star + I^\star) &= \wedge\, \{\gamma^-_\varrho(c + I)e^{i\varpi^+_\varrho(c+I)} : \mathrm{c} + I \in \rho^{-1}(\mathrm{c}^\star + I^\star)\} \\
&= \wedge\, \{\wedge\{\gamma^-(c + u)e^{i\varpi^-(c+u)}(\mathrm{c} + u) : u \in I, \mathrm{c} \in \mathfrak{H}, \psi(\mathrm{c})\}= \mathrm{c}^\star, \rho(\mathrm{c} + I) = \psi(\mathrm{c}) + I^\star\} \\
&= \wedge\, \{\gamma^-(c + u)e^{i\varpi^-(c+u)}(\mathrm{c} + u) : u \in I, \mathrm{c} \in \mathfrak{H}, \psi(\mathrm{c}) = \mathrm{c}^\star\}
\end{aligned}$$

$= \wedge \{\gamma^{-}(u)e^{i\varpi^{-}(u)}(\mathrm{c}+u) : u \in \psi^{-1}(\mathrm{c}^{\star})\}$
$=\psi(\eta^{-})(\mathrm{c}^{\star})$
$=\mu^{-}(\mathrm{c}^{\star}), \quad \mathrm{c}^{\star} \in \mathfrak{H}^{\star}$
$= \wedge \{\mu^{-}(\mathrm{c}^{\star}) : \phi(\mathrm{c}^{\star}) = \mathrm{c}^{\star} +^{I} \star\}$
$= \vee \{\mu^{-}(\mathrm{c}^{\star}) : \mathrm{c}^{\star} \in \pi^{-1}(\mathrm{c}^{\star} + I^{\star})\}$
$=\phi(\mu^{-})(\mathrm{c}^{\star} + I^{\star})$
$=\eta^{-}_{\varrho^{\star}}(\mathrm{c}^{\star} + I^{\star}).$

This indicates that $\rho(\eta^{-}_{\varrho}) = \eta^{-}_{\varrho^{\star}}$. Hence, the proof is completed. □

Remark 4.10. *The $\mathbb{CBF}$ homomorphism has many potential applications. For instance, the picture is positioned using a $\mathbb{CBF}$ homomorphism. When someone is photographed, it is actually their homomorphic picture, which provides evidence for all of their true characteristics, including height, weight, and gender. The application of a $\mathbb{CBF}$ homomorphism is explored in Figure 2.*

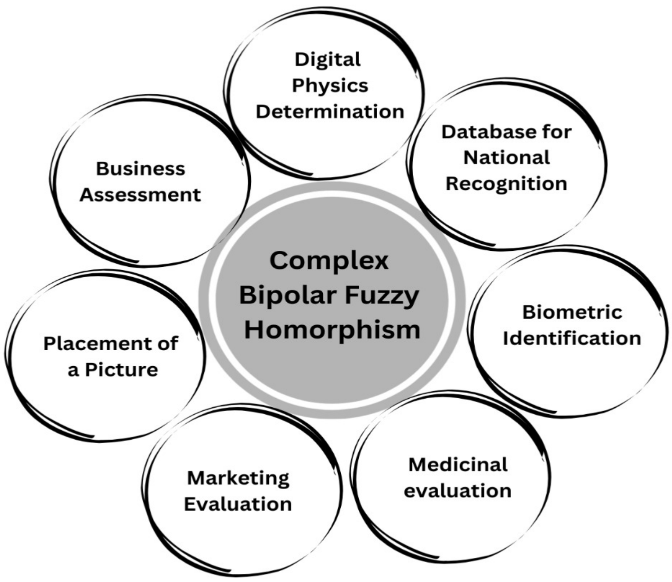

Figure 2. Applications of a complex bipolar fuzzy homomorphism in various fields.

Lemma 4.11. *Suppose that η and μ are both $\mathbb{CBFSR}$s of a $\mathbb{CBFR}$s $\mathfrak{H}$ and $\mathfrak{H}^{\star}$, respectfully; additionally, ψ is an epimorphism from $\mathfrak{H}$ to $\mathfrak{H}^{\star}$ which means $\psi(\eta) = \mu$, where η and μ are $\mathbb{CBFSR}$s of $\mathfrak{H}$ and $\mathfrak{H}^{\star}$, respectfully. Thus, $\psi(\eta_{\star}) = \mu_{\star}$.*

Proof. As stated $\psi(\eta) = \mu$. Suppose that $u \in \psi(\eta) \Rightarrow u = \psi(c)$, for any $c \in \eta_{\star}$. Consider $\psi(\eta^{+})(u) = \vee\{\eta^{+}(c), c \in \psi^{-1}(u)\} \geq \eta^{+}(c) > 0$, $\vee\{\gamma^{+}(c)e^{i}\varphi^{+}(c), c \in \psi^{-1}(u)\} \geq \eta^{+}(c) > 0$ and $\psi(\eta^{-})(u) = \wedge\{\eta^{-}(c), c \in \psi^{-1}(u)\} \geq \eta^{-}(c) > 0$, $\wedge\{\gamma^{-}(c)e^{i}\varphi^{-}(c), c \in \psi^{-1}(u)\} \geq \eta^{-}(c) < 0$. Therefore, $u \in \mu_{\star}$. Thus, $\psi(\eta_{\star}) \subseteq \mu_{\star}$. Furthermore, throughout the existence of Definition 4.1, the epi-morphism g develops as, $\psi(\eta_{\star}) \supseteq \mu_{\star}$. Thus, the proof is established. □

Theorem 4.12. *(First $\mathbb{CBF}$ Isomorphism Theorem): Suppose that η and μ are $\mathbb{CBFSR}$s of $\mathfrak{H}$ and $\mathfrak{H}^{\star}$, respectively, and J is a $\mathbb{CBF}$ homomorphism from η onto μ, whereas ker $J = G_1$ is the $\mathbb{CF}$ homomorphism's kernel. Thus, $\eta/\omega \approx \mu$, since ω is a $\mathbb{CBF}$ isomorphism of η.*

Proof. Suppose that J represents a $\mathbb{CBF}$ homomorphism from η to μ. Let the $\mathbb{CBFSR}s$ ω of H be as follows:

$$\omega^{+}(d) = \begin{cases} \eta^{+}(d), & if\ d \in G_1, \\ e^{2\pi}, & if\ d \notin G_1, \end{cases}$$

and

$$\omega^{-}(d) = \begin{cases} \eta^{-}(f), & if\ d \in G_1, \\ -e^{-2\pi}, & if\ d \notin G_1. \end{cases}$$

Obviously, $\omega \subseteq \eta$. Moreover, for any $f \in G_1$ *and* $g \in \mathfrak{H}$, consider $\omega^{+}(d+h) = \eta^{+}(d+h) \geq \{\eta^{+}(d) \wedge \eta^{+}(d)\} \geq \{\gamma^{+}(d)e^{\varpi^{+}(d)} \wedge \gamma^{+}(h)e^{\varpi^{+}(h)} \geq \{\omega^{+}(d) \wedge \eta^{+}(h)\}$. Similarly, $\omega^{-}(d+h) \leq \{\omega^{-}(d) \vee \eta^{-}(d)\}$. If $n \notin N_1$; then, $\omega^{+}(d) = e^{2\pi}$ and $\omega^{-}(d) = -e^{-2\pi}$. This demonstrates that ω is a $\mathbb{CBF}$ ideal of η. Considering $\eta \approx \mu \Rightarrow J(\eta) = \mu$. In view of Lemma 4.11, $J(\eta_{*}) = \mu_{\star}$. Let $\varphi = J^{\star}_{\eta_{\star}}$; then, $\varphi : \eta_{\star} \to \mu_{\star}$ is a homomorphism with the kernal $\varphi = \omega$. Thus, an isomorphism exists φ from $\eta_{\star}/\omega_{\star}$ to $\mu_{\star}$, which is characterized by $\varphi(f + \omega_{\star}) = v = \varphi(f) = J(d)) \ \forall\ d \in \eta_{\star}$. We have $\varphi(\eta/\omega)(v) = (\varphi(\eta^{+}/\omega^{+})(v), \varphi(\mu^{-}/\varphi^{-})(v))$. Let the following hold:

$$\begin{aligned}
\varphi(\eta^{+}/\omega^{+})(v) &= \vee \{(\eta^{+}/\omega^{+})(d+\omega_{\star}) : d \in \eta_{\star}, \varphi(d+\omega_{\star}) = v\} \\
&= \vee \{(\eta^{+}/\omega^{+})(u)e^{i(\eta^{+}/\omega^{+})(u)}, u \in d+\omega_{\star} : d \in \eta_{\star}, \varphi(u) = v\} \\
&= \vee \{(\eta^{+}/\omega^{+})(u)e^{i(\eta^{+}/\omega^{+})(u)} : u \in \eta_{\star}, \varphi(u) = v\} \\
&= \vee \{(\eta^{+}/\omega^{+})(u)e^{i(\eta^{+}/\omega^{+})(u)} : u \in \mathfrak{H}, J(u) = v\} \\
&= J(\eta^{+})(v) \\
&= \mu^{+}(v), \ \forall\ v \in \mu_{\star}.
\end{aligned}$$

This indicates that $\varphi(\eta^{+}/\omega^{+}) = \mu^{+}$, alongside the following case:

$$\begin{aligned}
\varphi(\eta^{-}/\omega^{-})(v) &= \wedge \{(\eta^{-}/\omega^{-})(f+\omega_{\star}) : f \in \eta_{\star}, \varphi(f+\omega_{\star}) = v\} \\
&= \wedge \{(\eta^{-}/\omega^{-})(u)e^{i(\eta^{-}/\omega^{-})(u)}, u \in f+\omega_{\star} : f \in \eta_{\star}, \varphi(u) = v\} \\
&= \wedge \{(\eta^{-}/\omega^{-})(u)e^{i(\eta^{-}/\omega^{-})(u)} : u \in \eta_{\star}, \varphi(u) = v\} \\
&= \vee \{(\eta^{-}/\omega^{-})(u)e^{i(\eta^{-}/\omega^{-})(u)} : u \in \mathfrak{H}, J(u) = v\} \\
&= J(\eta^{-})(v) \\
&= \mu^{-}(v), \ \forall\ v \in \mu_{\star}.
\end{aligned}$$

This indicates that $\varphi(\eta^{-}/\omega^{-}) = \mu^{-}$. Thus, $\varphi(\eta/\omega) = \mu$. Hence, $(\eta/\omega) \approx \eta$. □

Theorem 4.13. *(2nd Complex $\mathbb{CBF}$ Isomorphism Theorem) Suppose that η is a $\mathbb{CBF}$ isomorphism, and μ is a $\mathbb{CBFSR}$ of a ring $\mathfrak{H}$ for the case $\eta \subseteq \mu$. Subsequently, $\mu/(\eta \cap \mu) \subseteq (\eta + \mu)/\eta$.*

Proof. Regarding Remark 3.16, there is a reality in which $\eta \subseteq \mu$; then, few may obtain a $\mathbb{QR}s$ $\mu_{\star}/(\eta_{\star} \cap \mu_{\star})$ and $(\eta_{\star} + \mu_{\star})/\eta_{\star}$. Thus, we may derive the following using the second basic result of a conventional complex isomorphism of thering on these particular ring-like structures:

$$\mu_{\star}/(\eta_{\star} \cap \mu_{\star}) \cong (\eta_{\star} + \mu_{\star})/\eta_{\star}.$$

We may conclude that there is a complex ring isomorphism based on the results mentioned above J from $\mu_\star/((\eta\cap\mu))_\star$ to $(\eta_\star+\mu_\star)/\eta_\star$, which may be explained in the following way:

$$J(d+(\eta\cap\mu)^\star)=d+\theta_\star,\quad \forall d\in\mu_\star.$$

Let $J(\mu^+/(\eta^+\cap\mu^+))(d+\eta_\star)$,

$$\begin{aligned}
&=(\mu^+/(\eta^+\cap\mu^+))(d+(\eta\cap\mu)_\star)\\
&=\vee\{\mu^+(v)e^{i\mu^+(v)}:v\in(d+(\eta\cap\mu)_\star)\}\\
&\leq\vee\{(\eta^++\mu^+)(v)e^{i(\eta^++\mu^+)(v)}:v\in(d+(\eta\cap\mu)_\star)\}\\
&\leq\vee\{(\eta^++\mu^+)(v)e^{i(\eta^++\mu^+)(v)}:v\in d+\eta_\star\}\\
&=((\eta^++\mu^+)/\eta^+)(f+\eta_\star),\forall f\in\mu_\star,
\end{aligned}$$

which indicates the following $J(\mu^+/(\eta^+\cap\mu^+))(d+\eta_\star)\leq((\eta^++\mu^+)/\eta^+)(d+\eta_\star),\forall d\in\mu_\star$.

Furthermore, $J(\mu^-/(\eta^-\cap\mu^-))(d+\eta_\star)$

$$\begin{aligned}
&=(\mu^-/(\eta^-\cap\mu^-))(d+(\eta\cap\mu)_\star)\\
&=\wedge\{\mu^-(v)e^{i\mu^-(v)}:v\in(d+(\eta\cap\mu)_\star)\}\\
&\geq\wedge\{(\eta^-+\mu^-)(v)e^{i(\eta^-+\mu^-)(v)}:v\in(d+(\eta\cap\mu)_\star)\}\\
&\geq\wedge\{(\eta^-+\mu^-)(v)e^{i(\eta^-+\mu^-)(v)}:v\in d+\eta_\star\}\\
&=((\eta^-+\mu^-)/\eta^-)(d+\eta_\star),\forall d\in\mu_\star,
\end{aligned}$$

which indicates $J(\mu^-/(\eta^-\cap\mu^-))(d+\eta_\star)\geq((\eta^-+\mu^-)/\eta^-)(d+\eta_\star),\forall d\in\mu_\star$. Thus, $J(\mu/(\eta\cap\mu))\subseteq(\eta+\mu)/\eta$. Consequently, we develop a weak $\mathbb{CBF}$ isomorphism between $(\mu/(\eta\cap\mu))$ and $(\eta+\mu)/\eta$. □

Theorem 4.14. *(Third $\mathbb{CBF}$ isomorphism Theorem): Suppose that η, μ, and ω are $\mathbb{CBFSR}$s of $\mathfrak{H}$ when η and μ are $\mathbb{CBFI}$s of ω, where $\eta\subseteq\mu$. Subsequently, $(\omega/\eta)/(\mu/\eta)\cong(\omega/\mu)$;*

Proof. Regarding Remark 3.16, there is a reality in which η and μ are $\mathbb{CBFI}$*s* of the ω that $\eta\subseteq\mu$ then, few may obtain a $\mathbb{QR}$*s* $(\omega_\star/\eta_\star)/(\mu_\star/\eta_\star)$ and $(\omega_\star/\mu_\star)$. Thus, we may derive the following by using the 3rd basic result of a conventional complex isomorphism of the ring on these particular ring-like structures:

$$(\omega_\star/\eta_\star)/(\mu_\star/\eta_\star)\cong(\omega_\star/\mu_\star).$$

We conclude based on the facts mentioned above that a complex ring isomorphism exists $\mathfrak{H}$ from $(\omega_\star/\eta_\star)/(\mu_\star/\eta_\star)$ to $(\omega_\star/\mu_\star)$, which may be explained in the following way:

$$J(d+\eta_\star+(\mu_\star/\eta_\star))=d+\mu_\star,\quad\forall d\in\omega_\star.$$

Consider $J((\omega^+/\eta^+)/(\mu^+/\eta^+))(d+\mu_\star)$,

$$\begin{aligned}
&=((\omega^+/\eta^+)/(\mu^+/\eta^+))(d+\eta_\star+(\mu_\star/\eta_\star))\\
&=\vee\{(\omega^+/\eta^+)(d+\eta_\star):d\in\omega_\star,d+\eta_\star\in(d+\eta_\star+(\mu_\star/\eta_\star))\}\\
&=\vee\{\vee\{\omega^+(v)e^{i\omega^+(v)}:v\in d+\eta_\star\}:d\in\omega_\star,d+\eta_\star\in(d+\eta_\star+(\mu_\star/\eta_\star))\}
\end{aligned}$$

$$= \vee \{\omega^+(v)e^{i\omega^+(v)} : v \in \omega_\star, v + \eta_\star \in (d + \eta_\star + (\mu_\star/\eta_\star))\}$$
$$= \vee \{\omega^+(v)e^{i\omega^+(v)} : v \in (d + \eta_\star + (\mu_\star/\eta_\star))\}$$
$$= \vee \{\omega^+(v)e^{i\omega^+(v)} : v \in \omega_\star, J(v) \in d + \mu_\star\}$$
$$=(\omega^+/\mu^+)(d + \mu_\star), \forall\ d \in \omega_\star.$$

This indicates that $h((\omega^+/\eta^+)/(\mu^+/\eta^+))(d + \mu_\star) = (\omega^+/\mu^+)(d + \mu_\star), \forall\ f \in \omega_\star$. Moreover, $J((\omega^-/\eta^-)/(\mu^-/\eta^-))(d + \mu_\star)$,

$$=((\omega^-/\eta^-)/(\mu^-/\eta^-))(d + \eta_\star + (\mu_\star/\eta_\star))$$
$$= \wedge \{(\omega^-/\eta^-)(d + \eta_\star) : f \in \omega_\star, d + \eta_\star \in (d + \eta_\star + (\mu_\star/\eta_\star))\}$$
$$= \wedge \{\wedge\{\omega^-(v)e^{i\omega^-(v)} : v \in d + \eta_\star\} : f \in \omega_\star, d + \eta_\star \in (d + \eta_\star + (\mu_\star/\eta_\star))\}$$
$$= \wedge \{\omega^-(v)e^{i\omega^-(v)} : v \in \omega_\star, v + \eta_\star \in (f + \eta_\star + (\mu_\star/\eta_\star))\}$$
$$= \wedge \{\omega^-(v)e^{i\omega^-(v)} : v \in (d + \eta_\star + (\mu_\star/\eta_\star))\}$$
$$= \wedge \{\omega^-(v)e^{i\omega^-(v)} : v \in \omega_\star, J(v) \in d + \mu_\star\}$$
$$=(\omega^-/\mu^-)(d + \mu_\star), \forall\ d \in \omega_\star.$$

This indicates that $h((\omega^-/\eta^-)/(\mu^-/\eta^-))(d + \mu_\star) = (\omega^-/\mu^-)(d + \mu_\star), \forall\ f \in \omega_\star$. Thus, $J(\omega/\eta)/(\mu/\eta) = (\omega/\mu)$. $(\omega/\eta)/(\mu/\eta) \cong (\omega/\mu)$. □

The algorithm illustrated in Figure 3 offers a structured foundation to use the CBFSR concept in real-world decision-making scenarios. The technique essentially shows how abstract algebraic ideas can be turned into a methodical procedure for practical evaluations.

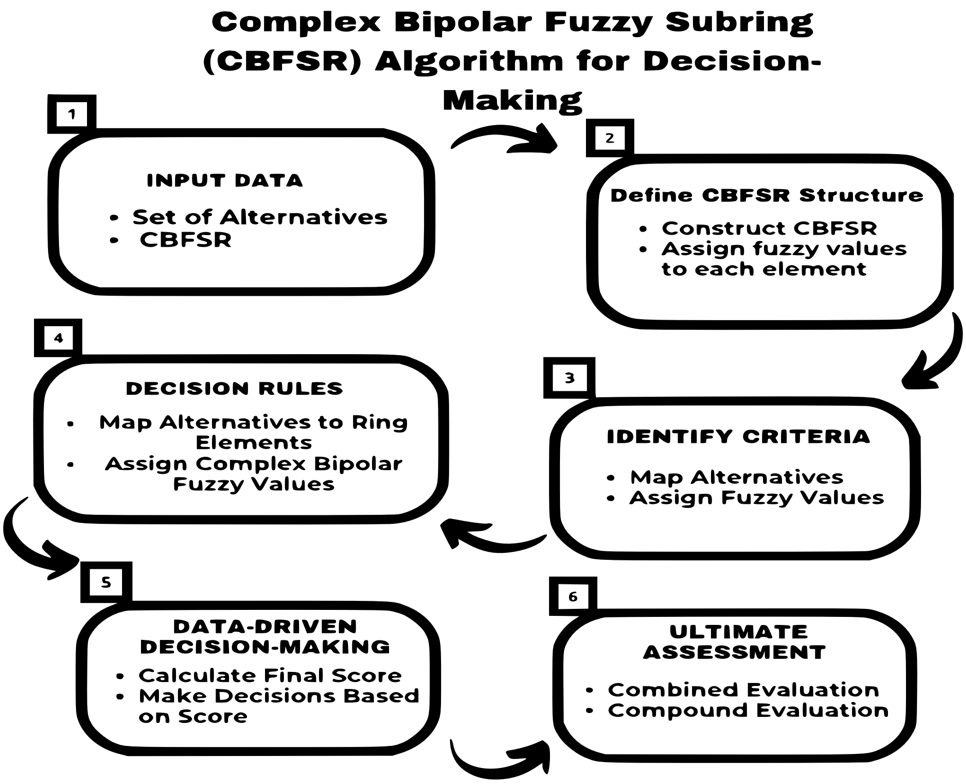

Figure 3. Algorithm of the proposed model.

5. Application

We develop a basic algorithm to demonstrate how the concept of CBFSR may be applied to practical issues.

The goal of the mid-sized, worldwide based Global Crypto Investment Company (GCIC) is to put the finest possible investment plan into action. To choose which investments to add to its portfolio, the company needs a method to evaluate investment prospects based on a number of factors. To assess the interactions of various investment characteristics and their combined impact on investment quality, they chose to use a CBFSR technique.

Objective

To use a CBFSR decision-making (DM) model to rank and select the best group of assets for the GCIC portfolio.

Input

- Consider $\{C_1 = \text{Bitcoin}, C_2 = \text{Ethereum}, C_3 = \text{Solana}, C_4 = \text{Ripple}\}$ as the considered cryptocurrency investment alternatives.
- Assume a ring under modulo 5:

$$R = \mathbb{Z}_5 = \{0, 1, 2, 3, 4\} \mod 5.$$

Output

- Find out about each cryptocurrency's decision.
- Obtain the combined and compounded evaluations for the selected cryptocurrencies.

Step 1: The CBFSR over R is given as:

- For each value in R, assign a fuzzy value in the following form:
  (RP, IP, RN, IN), where,
  - RP: real part (positive support), IP: imaginary part (positive support), RN: real part (negative resistance), IN: imaginary part (negative resistance).

$0 \to (0.84e^{\iota 0.76\pi}, -0.14e^{-\iota 0.19\pi})$,
$1 \to (0.44e^{\iota 0.34\pi}, -0.26e^{-\iota 0.39\pi})$,
$2 \to (0.44e^{\iota 0.34\pi}, -0.44e^{-\iota 0.49\pi})$,
$3 \to (0.63e^{\iota 0.53\pi}, -0.46e^{-\iota 0.53\pi})$,
$4 \to (0.63e^{\iota 0.53\pi}, -0.46e^{-\iota 0.53\pi})$.

Here, the first value is the amplitude (magnitude), which shows the strength of a the positive membership or supportive evidence toward inclusion and second imaginary value represents the phase angle, which introduces an oscillatory/periodic component with respect to time that models uncertainty or hesitation for inclusion in the positive evaluationṠimilarly, the first negative value is the amplitude of a negative membership (rejection/contradiction strength). The minus sign means that this is the bipolar negative side, which shows how strongly the element does not belong. The imaginary negative value tells the phase angle which adds a complex/periodic uncertainty with respect to time to

the negative degree.
Step 2: GCI indicates three essential standards to evaluate investments: Risk level (R L): 0 to 4 (low to very high).
Return potential (R P): 0 to 4 (low to very high).
Market Trend (M T): 0 to 4 (low to very high).

The following formula is used to obtain the overall quality of the investment:

$$(RP + (4 - RL) + MT) \mod 5.$$

Step 3: Assessment of the cryptocurrencies: C_1: $RL = 2, RP = 3, MT = 4 \rightarrow (3 + (4 - 2) + 4) \mod 5 = 9 \mod 5 = 4$.
C_2: $RL = 1, RP = 2, MT = 3 \rightarrow (2 + (4 - 1) + 3) \mod 5 = 8 \mod 5 = 3$.
C_3:$RL = 5, RP = 3, MT = 2 \rightarrow (3 + (4 - 5) + 2) \mod 5 = 4 \mod 5 = 4$.
C_4: $RL = 2, RP = 1, MT = 3 \rightarrow (1 + (4 - 2) + 3) \mod 5 = 6 \mod 5 = 1$.
Step 4: Calculate the final score

- Use the following formula to obtain the score S:

$$S = \frac{1}{2} + \frac{RP + RN}{4} + \frac{IP + IN}{4\pi}.$$

- This gives a value between 0 and 1.

The CBFSR values corresponding to the results are as follows:

- S(4) =0.5425 $\rightarrow C_4$.
- S(3) = 0.5425 $\rightarrow C_3$.
- S(2) = 0.4625 $\rightarrow C_2$.
- S(1) = 0.5325 $\rightarrow C_1$.

Step 5: Make decisions based on score

- If $S(k) \geq 0.6 \rightarrow$ Include in portfolio.
- If $0.5 \leq S(k) < 0.6 \rightarrow$ Consider for inclusion.
- If $S(k) < 0.5 \rightarrow$ Exclude from portfolio.

Thresholds for inclusion as follows:

- $S(k) \geq 0.6 \rightarrow$ Include in portfolio.
- $0.45 \leq S(k) < 0.6 \rightarrow$ Consider for inclusion.
- $S(k) < 0.5 \rightarrow$ Exclude from portfolio.

Step 6: Decisions

- C_1 = Bitcoin : 0.5325 $\rightarrow$ Consider for inclusion.
- C_2 = Ethereum : 0.4625 $\rightarrow$ Exclude.
- C_3 = Solana : 0.5425 $\rightarrow$ Consider for inclusion.

Step 7: Combined effect of including C_1 and C_3:

- For two investments C_1 and C_3:

$$z_{\text{combined}} = (z_1 + z_3) \mod 5.$$

- Get the fuzzy value for z_{combined} and calculate its score

$$4 + 4 \mod 5 = 3 \quad \Rightarrow \quad S(3) = 0.5425.$$

Step 8: Compound effect of including C_1 and C_3:

- For the same two investments:

$$z_{\text{compound}} = (z_1 \times z_2) \mod 5.$$

- Get the fuzzy value for z_{compound} and calculate its score

$$4 \times 4 \mod 5 = 1 \quad \Rightarrow \quad S(1) = 0.5325.$$

The CBFSR model has been instrumental in guiding GCIC decision-making regarding cryptocurrency investments. It integrates the risk level, return potential, and market trend into a ring framework. Among the four cryptocurrencies evaluated, Bitcoin C_1 and Solana C_3 were considered for inclusion in the portfolio. Ethereum is excluded on the basis of the current thresholds. Interestingly, the combined and compound impacts of C_1 and C_3 show moderate synergy, thus reinforcing their status as favorable candidates for further analysis and potential inclusion.

6. Conclusions

The concept of $\mathbb{CBFS}$ is a pragmatic extension of the traditional $\mathbb{BFS}$ that more accurately analyzes the uncertain and unclear nature of an ambiguous reality. We came up with the idea of a $\mathbb{CBFSR}$ and showed that the $(\delta, \alpha; \sigma, \beta)$-cut of a $\mathbb{CBFSR}$ can be used to show a subring of a certain ring. We determined the $\mathbb{CBFSR}$ of the $\mathbb{QR}$, and demonstrated that the combination of two $\mathbb{CBFSR}$ is the subring. This is a fundamentally important change to the natural rings homomorphism. Additionally, we established an intense relationship with a specified surjective homomorphism between two $\mathbb{CBFSR}s$ of the $\mathbb{QR}s$ and demonstrated further fundamental $\mathbb{CBF}$ homomorphism theorems with respect to this particular $\mathbb{CBFSR}$. Lastly, the three basic proofs of the $\mathbb{CBF}$ isomorphic relationships of $\mathbb{CBFSR}s$ are examined and applied to the $\mathbb{CBFSR}$ model in decision making. In future research, we will extend this idea to various algebraic structures and subsequently use it in a variety of fields of ring theory, modules, BCK/BCI algebra, and various decision making problems.

Authors contributions

Kholood Alnefaie: Methodology, original draft preparation; Sarka Hoskova-Mayerova: Conceptualization, supervision, modified and verified the result; Muhammad Haris Mateen: Conceptualization, supervision, methodology, original draft preparation; Bijan Davvaz: Conceptualization, supervision, modified and verified the result. All authors have read and approved the final version of the article.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors thank the Ministry of Defence of the Czech Republic for the support via grant VAROPS

Conflicts of interest

There are not any conflicts of interest.

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