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Research article

Existence and nonexistence outcomes for a third-order q-difference equation

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Abstract: In this paper, using the Schauder and the Banach fixed point (FP) theorems, we examine the existence and uniqueness of solutions in the Banach space $C([0, 1])$ for a boundary value problem (BVP) of non-linear third-order q-difference equations with the q-integral boundary condition. Then, we impose the sufficient condition that allows us to deduce a nonexistence result. Furthermore, we offer some examples to support our main outcomes.

Keywords: boundary value problems; q-difference equations; integral-type boundary conditions; Green's function; fixed point theorems
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1. Introduction

The theory of boundary value problems (BVPs) arises in different areas of applied mathematics and physics (see, e.g., [1–4]). Especially the BVPs with integral boundary conditions have been a topic of research for many scholars, such as chemical engineering, groundwater flow, heat conduction, thermo-elasticity, and plasma physics (see, e.g., [5–9]). Over time, a wide range of studies has been conducted on third-order differential equation BVPs, with research focusing on the existence and uniqueness of solutions for third-order differential equations. Several methods have been used to prove the existence and uniqueness of solutions. For example, Boucherif et al. [10] used the method of lower and upper solutions to generate an iterative technique. Xie and Pang [11] obtained some new results by applying the shooting method and provided the existence of at least one positive solution for a third-order differential equation. Cabada and Dimitrov [12] investigated the existence of solutions of the third-order non-linear differential equation using FP index theory. Recently, using the Banach and the Rus FP theorems, Smirnov [13] provided the existence of a unique solution for the non-linear third-order differential equation

$$x'''(l) + \psi(l, x(l)) = 0, \quad l \in [c, d], \tag{1.1}$$

and the integral-type boundary conditions

$$x(c) = 0, \quad x(d) = 0, \quad \int_c^d x(l)\, dl = 0, \tag{1.2}$$

where $\psi \in C([c, d] \times \mathbb{R})$ and $\psi(l, 0) \neq 0$ for all $l \in [c, d]$. Related works can also be found in [14–16].

The topic of q-difference equations arose in the early 20th century and has been applied to different disciplines such as physics and mathematics (see [17, 18]). There are some pioneering works on third-order q-difference equations. For instance, in [19], Ahmad has worked on the existence of solutions for the following non-linear BVP of third-order q-difference equations:

$$D_q^3 x(l) = \psi(l, x(l)), \quad l \in [0, 1],$$

and the boundary conditions

$$x(0) = 0, \quad x(1) = 0, \quad D_q x(0) = 0,$$

where ψ is a given continuous function, applying Leray-Schauder degree theory and some standard FP theorems. In [20], Ahmad and Nieto have studied a nonlocal non-linear BVP of third-order q-difference equations given by

$$D_q^3 x(l) = \psi(l, x(l)), \quad l \in [0, 1]_q,$$

and the boundary conditions

$$x(0) = 0, \quad D_q x(0) = 0, \quad x(1) = \alpha \cdot x(\eta),$$

where $\psi \in C([0, 1]_q \times \mathbb{R})$, $[0, 1]_q = \{q^k : k \in \mathbb{N}\} \cup \{0, 1\}$, $q \in (0, 1)$ is a fixed constant, $\eta \in \{q^k : k \in \mathbb{N}\}$ and $\alpha \neq \frac{1}{\eta^2}$ is a real number. In [21], Yu and Wang have established the existence of positive solutions of the following BVP of the non-linear singular third-order q-difference equation:

$$D_q^3 x(l) + \lambda \cdot \alpha(l) \cdot \psi(x(l)) = 0, \quad l \in [0, 1]_q,$$

and the boundary conditions

$$x(0) = 0, \quad D_q x(0) = 0, \quad \alpha \cdot D_q x(1) + \beta \cdot D_q^2 x(1) = 0,$$

where λ is a positive parameter, by using the Krasnoselskii FP theorem on a cone. For further study, we refer readers to [22]. However, the theory of third-order q-difference equations with integral boundary conditions is not very common in the literature.

Motivated by the papers mentioned above, we will consider the following non-linear BVP of the third-order q-difference equation

$$D_q^3 x(l) + \psi(l, x(l)) = 0, \quad l \in [0, 1], \tag{1.3}$$

and the integral-type boundary conditions

$$x(0) = 0, \quad x(1) = 0, \quad \int_0^1 x(l)\, d_q l = 0. \tag{1.4}$$

Here, $\psi \in C([0,1] \times \mathbb{R})$, and $\psi(l,0) \neq 0$ for all $l \in [0,1]$, which ensures that the trivial solution does not occur. This paper aims to obtain some existence and nonexistence outcomes for the problem (1.3)-(1.4). The rest of the paper is organized as follows: In section 2, we recall some necessary definitions and theorems; In section 3, we rewrite the BVP (1.3)-(1.4) as an integral equation and construct the Green function. We also drive an inequality related to the Green function; In Section 4, we establish the main results concerning the existence, uniqueness, and nonexistence of solutions. To complement these findings, we include several examples as supporting evidence.

2. Preliminaries

This section is devoted to a brief review of fundamental notions from q-calculus [23,24].

Definition 1. The q-derivative of a real-valued function x is defined by

$$(D_q x)(l) = \frac{x(l) - x(ql)}{(1-q)l}$$

where $(D_q x)(0) = \lim_{l \to 0} (D_q x)(l)$. Here $\lim_{q \to 1^-} D_q x(l) = x'(l)$. The higher-order q-derivatives are given by

$$D_q^k x(l) = D_q D_q^{k-1} x(l), \qquad k \in \mathbb{N},$$

where $D_q^0 x(l) = x(l)$.

Definition 2. The q-integral of a function x on the interval $[c,d]$ is defined by

$$\int_c^l x(s)\, d_q s = \sum_{k=0}^{\infty} (1-q) \cdot q^k \cdot [l \cdot x(lq^k) - c \cdot x(cq^k)], \qquad l \in [c,d]$$

and for $c = 0$, it is denoted

$$I_q x(l) = \int_0^l x(s)\, d_q s = \sum_{k=0}^{\infty} l \cdot (1-q) \cdot q^k \cdot x(lq^k).$$

Here, the right-side series must satisfy the convergence condition. If $c \in [0,d]$ and the function x is defined in $[0,d]$, in this case the following statement holds true:

$$\int_c^d x(l)\, d_q l = \int_0^d x(l)\, d_q l - \int_0^c x(l)\, d_q l.$$

Similarly, we have

$$I_q^k x(l) = I_q I_q^{k-1} x(l), \qquad k \in \mathbb{N},$$

where $I_q^0 x(l) = x(l)$.

Observe that $D_q I_q x(l) = x(l)$, and if x is continuous at $l = 0$, then $I_q D_q x(l) = x(l) - x(0)$.

The change of order of integration is as

$$\int_0^l \int_0^s x(r)\, d_q r\, d_q s = \int_0^l \int_{qr}^l x(r)\, d_q s\, d_q r.$$

Lemma 1. *[25]*

1. *If x is q-integrable on the interval $[c, d]$, and $\alpha \in [c, d]$, then*

$$\int_c^d x(l)\, d_q l = \int_c^\alpha x(l)\, d_q l + \int_\alpha^d x(l)\, d_q l.$$

2. *If $|x|$ is q-integrable on the interval $[0, d]$, then*

$$\left| \int_0^d x(l)\, d_q l \right| \leq \int_0^d |x(l)|\, d_q l.$$

The existence and uniqueness of solutions are examined through the following two FP theorems. For further information on FP theory, the readers may consult the works in [26–30].

Theorem 1. *(Schauder FP theorem) [31, 32] Let $\mathcal{D}$ be a nonempty, closed, bounded, convex subset of a Banach space Ω, and $\Upsilon : \mathcal{D} \to \mathcal{D}$ be a completely continuous map (i.e., Υ is continuous and $\Upsilon(\mathcal{D})$ is relatively compact). Then Υ has at least one FP in $\mathcal{D}$.*

Theorem 2. *(Banach FP theorem) [33] Let Ω be a Banach space. If $\Upsilon : \Omega \to \Omega$ is a contraction map, that is, there exists $\theta \in [0, 1)$ with*

$$\|\Upsilon(l) - \Upsilon(s)\| \leq \theta \cdot \|l - s\|$$

for any $l, s \in \Omega$, then Υ has a unique FP in Ω.

3. Formulation of the Green's function

The target of this section is to rewrite the BVP (1.3)-(1.4) as an equivalent integral equation. Hence, we shall consider the linear equation

$$D_q^3 x(l) + u(l) = 0, \qquad l \in [0, 1], \tag{3.1}$$

where $u \in C([0, 1])$, associated with the boundary conditions (1.4). In order to reach our results, we need the following two preliminary lemmas.

Lemma 2. *The BVP (3.1)-(1.4) has the solution*

$$\begin{aligned} x(l) = & \int_0^l \frac{-q^2(l - q^2 s)(l - qs) + [l^2(1 + q + q^2) - l(1 + q)](1 - qs)(1 - q^2 s)}{q^2(1 + q)} u(s)\, d_q s \\ & - \int_0^l \frac{(l^2 - l)(1 + q)(1 - qs)(1 - q^2 s)(1 - q^3 s)}{q^2(1 + q)} u(s)\, d_q s \\ & + \int_l^1 \frac{[(l^2(1 + q + q^2) - l(1 + q)) - (l^2 - l)(1 + q)(1 - q^3 s)](1 - qs)(1 - q^2 s)}{q^2(1 + q)} u(s)\, d_q s \end{aligned}$$

that we can rewrite as

$$x(l) = \int_0^1 G(l, qs) \cdot u(s)\, d_q s,$$

where

$$G(l, qs) = \begin{cases} \frac{-q^2(l - q^2 s)(l - qs) + [[l^2(1+q+q^2) - l(1+q)] - (l^2 - l)(1+q)(1 - q^3 s)](1 - qs)(1 - q^2 s)}{q^2(1+q)}, & qs \leq l, \\ \frac{[(l^2(1+q+q^2) - l(1+q)) - (l^2 - l)(1+q)(1 - q^3 s)](1 - qs)(1 - q^2 s)}{q^2(1+q)}, & l \leq qs. \end{cases} \tag{3.2}$$

Proof. Taking the q-integral of Eq (3.1) from 0 to l, and then changing the order of q-integration, we get

$$D_q^2 x(l) = -\int_0^l u(s)\,d_q s + A,$$
$$D_q x(l) = -\int_0^l (l - qs)u(s)\,d_q s + Al + B,$$
$$x(l) = -\frac{1}{1+q}\int_0^l [l^2 - lqs(1+q) + q(qs)^2]u(s)\,d_q s + A\frac{l^2}{1+q} + Bl + C. \tag{3.3}$$

Using the first boundary condition in Eq (3.3), we have

$$x(l) = -\frac{1}{1+q}\int_0^l [l^2 - lqs(1+q) + q(qs)^2]u(s)\,d_q s + A\frac{l^2}{1+q} + Bl. \tag{3.4}$$

By writing $l = 1$ in Eq (3.4) and using the second boundary condition, we get

$$B = \frac{1}{1+q}\int_0^1 [1 - qs(1+q) + q(qs)^2]u(s)\,d_q s - \frac{A}{1+q}. \tag{3.5}$$

Taking the q-integral of Eq (3.4) from 0 to 1 and using Eq (3.5) with the integral boundary condition, we obtain

$$\begin{aligned} A =& \frac{1+q+q^2}{q^2}\int_0^1 [1 - qs(1+q) + q(qs)^2]u(s)\,d_q s \\ &- \frac{(1+q+q^2)(1+q)}{q^2}\int_0^1 \left[\frac{1-(qs)^3}{1+q+q^2} + q(qs)^2(1-qs) - qs(1-(qs)^2)\right]u(s)\,d_q s. \end{aligned} \tag{3.6}$$

Substituting (3.5) and (3.6) in (3.4), we have

$$\begin{aligned} x(l) =& \frac{-1}{1+q}\int_0^l [l^2 - lqs(1+q) + q(qs)^2]u(s)\,d_q s \\ &+ \frac{lq^2 + (l^2 - l)(1+q+q^2)}{q^2(1+q)}\int_0^1 [1 - qs(1+q) + q(qs)^2]u(s)\,d_q s \\ &- \frac{(l^2 - l)(1+q+q^2)}{q^2}\int_0^1 \left[\frac{1-(qs)^3}{1+q+q^2} + q(qs)^2(1-qs) - qs(1-(qs)^2)\right]u(s)\,d_q s. \end{aligned}$$

Thus, we achieve the desired result. □

Remark 1. In this lemma, we generalize the solution $x(l)$ of the problem (1.1)-(1.2) in [13, Proposition 1] to q-calculus where $c = 0$ and $d = 1$. We also observe that $G(l, qs)$ is reduced to $G(l, s)$ in [13, Proposition 1] when $q \to 1^-$ for $c = 0$ and $d = 1$.

Now, we prove the following inequality involving the Green's function for the integral.

Lemma 3. *The Green's function $G(l, qs)$ in (3.2) satisfies the inequality*

$$\left|\int_0^1 G(l, qs)\,d_q s\right| \le \int_0^1 |G(l, qs)|\,d_q s \le \frac{8q^4 + q^3 + 10q^2 + 2q + 2}{4q^2(1+q)(1+q^2)(1+q+q^2)} =: M_1 \tag{3.7}$$

for all $l \in [0, 1]$.

Proof. For all $l \in [0, 1]$, from Lemma 1 and the Green function (3.2), we have

$$
\begin{aligned}
\left|\int_0^1 G(l, qs)\, d_q s\right| &\leq \int_0^1 |G(l, qs)|\, d_q s \\
&= \int_0^l |G(l, qs)|\, d_q s + \int_l^1 |G(l, qs)|\, d_q s \\
&= \frac{1}{q^2(1+q)} \int_0^l (q^2(l - q^2 s)(l - qs) + [(l - l^2)(1+q) + l^2 q^2](1 - qs)(1 - q^2 s))\, d_q s \\
&\quad + \frac{1}{q^2(1+q)} \int_0^l (l - l^2)(1+q)(1 - qs)(1 - q(qs))(1 - q^2(qs))\, d_q s \\
&\quad + \frac{1}{q^2(1+q)} \int_l^1 [(l - l^2)(1+q) + l^2 q^2](1 - qs)(1 - q^2 s)\, d_q s \\
&\quad + \frac{1}{q^2(1+q)} \int_l^1 (l - l^2)(1+q)(1 - qs)(1 - q(qs))(1 - q^2(qs))\, d_q s \\
&= \frac{q^2 l^3}{q^2(1+q)(1+q+q^2)} + \frac{(l - l^2)(1+q) + l^2 q^2}{q^2(1+q)(1+q+q^2)} + \frac{(l - l^2)}{q^2(1+q)(1+q^2)} \\
&= \frac{q^2 l^3 + l^2 q^2}{q^2(1+q)(1+q+q^2)} + \frac{(l - l^2)[(1+q)(1+q^2) + (1+q+q^2)]}{q^2(1+q)(1+q^2)(1+q+q^2)} \\
&\leq \max_{0 \leq l \leq 1} \left(\frac{q^2 l^3 + l^2 q^2}{q^2(1+q)(1+q+q^2)}\right) + \max_{0 \leq l \leq 1} \left(\frac{(l - l^2)[(1+q)(1+q^2) + (1+q+q^2)]}{q^2(1+q)(1+q^2)(1+q+q^2)}\right) \\
&\leq \frac{2q^2}{q^2(1+q)(1+q+q^2)} + \frac{[(1+q)(1+q^2) + (1+q+q^2)]}{q^2(1+q)(1+q^2)(1+q+q^2)} \left(\frac{1}{2} - \frac{1}{4}\right) \\
&= \frac{8q^4 + q^3 + 10q^2 + 2q + 2}{4q^2(1+q)(1+q^2)(1+q+q^2)}.
\end{aligned}
$$

□

4. Main outcomes

In this section, we firstly demonstrate the existence and uniqueness of the solutions of the problem (1.3)-(1.4), using some FP theorems. Therefore, we convert this problem into an FP problem, $\Upsilon x = x$, by defining a map Υ. In this case, let $C([0, 1])$ show the Banach space of all continuous functions

$$x : [0, 1] \to \mathbb{R}$$

equipped with the norm

$$\|x\| = \max_{l \in [0,1]} |x(l)|.$$

Thus, by Lemma 2, we can define a map $\Upsilon : C([0, 1]) \to C([0, 1])$ as follows:

$$(\Upsilon x)(l) = \int_0^1 G(l, qs) \cdot \psi(s, x(s))\, d_q s. \tag{4.1}$$

We begin by presenting our first outcome, which is established through the use of the Schauder FP theorem.

Theorem 3. *Let Υ be defined as in (4.1) and let $\psi \in C([0,1] \times \mathbb{R})$ satisfy the following conditions:*

(S1) Suppose that there exist $h, v \in L^1([0,1])$ such that

$$|\psi(l,x)| \leq |h(l)| \cdot |x| + |v(l)|$$

where $(l,x) \in [0,1] \times \mathbb{R}$.

(S2) There exists a constant $L > 0$ such that

$$|\psi(l,x) - \psi(l,y)| \leq L \cdot |x - y|$$

for all $l \in [0,1]$ and $x, y \in \mathbb{R}$.

Then, the problem (1.3)-(1.4) has at least one nontrivial solution $x(l)$ in the Banach space $C([0,1])$.

Proof. We consider the set $\mathcal{D} = \{x \in C([0,1]) : \|x\| \leq \mathcal{K}\}$. It is clear that the set $\mathcal{D}$ is a closed, bounded, and convex subset of $C([0,1])$. Firstly, we prove that $\Upsilon : \mathcal{D} \to \mathcal{D}$. For that purpose, we choose

$$\mathcal{K} \geq \frac{\mathcal{V} \cdot M_1}{1 - M_1 \cdot \mathcal{H}},$$

where $\max_{l\in[0,1]} |h(l)| = \mathcal{H}$ and $\max_{l\in[0,1]} |v(l)| = \mathcal{V}$. For any $x \in \mathcal{D}$, we possess

$$\begin{aligned}
|\Upsilon x(s)| =& \left| \int_0^1 G(l,qs) \cdot \psi(s,x(s))\, d_q s \right| \\
\leq & \int_0^1 |G(l,qs)| \cdot |\psi(s,x(s))|\, d_q s \\
\leq & \int_0^1 |G(l,qs)| \cdot \{|h(s)| \cdot |x(s)| + |v(s)|\}\, d_q s \\
\leq & \int_0^1 |G(l,qs)| \cdot \{\max_{s\in[0,1]} |h(s)| \cdot \max_{s\in[0,1]} |x(s)| + \max_{s\in[0,1]} |v(s)|\}\, d_q s \\
\leq & \int_0^1 |G(l,qs)| \cdot (\mathcal{H} \cdot \mathcal{K} + \mathcal{V})\, d_q s \\
\leq & (\mathcal{H} \cdot \mathcal{K} + \mathcal{V}) \cdot M_1 \leq \mathcal{K}.
\end{aligned} \tag{4.2}$$

Thus, we get $\Upsilon : \mathcal{D} \to \mathcal{D}$. Now we must show that Υ is completely continuous. In order to prove that the map Υ is completely continuous, it is necessary to show that it is continuous and relatively compact. As a first step, we establish the continuity of the map Υ. For this, let $\{x_k\}$ be a sequence in $\mathcal{D}$ that converges to x in $\mathcal{D}$. In this case, for $l \in [0,1]$, we have

$$\begin{aligned}
|\Upsilon[x_k](l) - \Upsilon[x](l)| =& \left| \int_0^1 G(l,qs) \cdot [\psi(s,x_k(s)) - \psi(s,x(s))]\, d_q s \right| \\
\leq & \int_0^1 |G(l,qs)| \cdot |\psi(s,x_k(s)) - \psi(s,x(s))|\, d_q s \\
\leq & L \int_0^1 |G(l,qs)| \cdot |x_k(s) - x(s)|\, d_q s
\end{aligned}$$

$$
\begin{aligned}
&\leq L \int_0^1 |G(l,qs)| \cdot \max_{s\in[0,1]} |x_k(s) - x(s)|\, d_q s \\
&= L \cdot \|x_k - x\| \cdot \int_0^1 |G(l,qs)|\, d_q s. \qquad (4.3)
\end{aligned}
$$

Taking the norm of Eq (4.3) for $l \in [0,1]$, it follows that

$$
\|\Upsilon[x_k] - \Upsilon[x]\| \leq M_1 \cdot L \cdot \|x_k - x\|. \qquad (4.4)
$$

This implies that the right side of (4.4) approaches zero as $\{x_k\}$ approaches x. Therefore, $\Upsilon : \mathcal{D} \to \mathcal{D}$ is continuous. Finally, we demonstrate the relatively compactness of the map Υ. $\mathcal{D}$ is bounded and one can easily show that $\Upsilon(\mathcal{D})$ is bounded. On the other hand, let $l_1 < l_2$ for all $l_1, l_2 \in [0,1]$. In this case, we acquire

$$
\begin{aligned}
|\Upsilon[x](l_2) - \Upsilon[x](l_1)| &= \left| \int_0^1 [G(l_2,qs) - G(l_1,qs)] \cdot \psi(s,x(s))\, d_q s \right| \\
&\leq \int_0^1 |G(l_2,qs) - G(l_1,qs)| \cdot \{|h(s)| \cdot |x(s)| + |v(s)|\}\, d_q s \\
&\leq \int_0^1 |G(l_2,qs) - G(l_1,qs)| \cdot \max_{s\in[0,1]} \{|h(s)| \cdot |x(s)| + |v(s)|\}\, d_q s \\
&\leq (\mathcal{K} \cdot \mathcal{H} + \mathcal{V}) \int_0^1 |G(l_2,qs) - G(l_1,qs)|\, d_q s.
\end{aligned}
$$

Because the Green's function is continuous, we have $\|\Upsilon[x](l_2) - \Upsilon[x](l_1)\| \to 0$ as $l_2 \to l_1$, which means that the map Υ is equicontinuous. Thus, Υ is relatively compact since it is bounded and equicontinuous. Consequently, since all the conditions of Theorem 1 are satisfied, the problem (1.3)-(1.4) has at least one nontrivial solution in $\mathcal{D}$. □

Example 1. Consider the following BVP:

$$
\begin{cases}
D^3_{\frac{1}{3}} x(l) + \frac{\sqrt{|x|}}{5(1+4l^2)\sqrt{|x|+1}} + \frac{1}{e^l \cos l} = 0, \ l \in [0,1], \\
x(0) = 0, \ x(1) = 0, \ \int_0^1 x(l)\, d_{\frac{1}{3}} l = 0,
\end{cases}
$$

where $q = \frac{1}{3}$ and

$$
\psi(l,x) = \frac{\sqrt{|x|}}{5(1+4l^2)\sqrt{|x|+1}} + \frac{1}{e^l \cos l}
$$

is continuous in $[0,1] \times \mathbb{R}$, and $\psi(l,0) = \frac{1}{e^l \cos l} \neq 0$ for all $l \in [0,1]$. Clearly, for $(l,x) \in [0,1] \times \mathbb{R}$, we see that

$$
|\psi(l,x)| \leq \frac{|x|}{5(1+4l^2)} + \frac{1}{e^l \cos l}
$$

where $|h(l)| = \frac{1}{5(1+4l^2)}$ and $|v(l)| = \frac{1}{e^l \cos l}$. Now, for every $(l,x), (l,y) \in [0,1] \times \mathbb{R}$, we have

$$
|\psi(l,x) - \psi(l,y)| = \frac{1}{5(1+4l^2)} \left| \frac{\sqrt{|x|}}{\sqrt{|x|+1}} - \frac{\sqrt{|y|}}{\sqrt{|y|+1}} \right|
$$

$$\leq \frac{1}{5(1+4l^2)}\Big||x|-|y|\Big|$$
$$\leq \frac{1}{5}|x-y|$$

where $L = \frac{1}{5} > 0$. As a result, it follows from Theorem 3 that the problem has at least one nontrivial solution $x(l)$ in $C([0,1])$.

The second outcome is derived by employing the Banach FP theorem.

Theorem 4. *Assume that (S2) holds and $L \cdot M_1 < 1$, where M_1 is given by (3.7). Then the BVP (1.3)-(1.4) has a unique nontrivial solution in the Banach space $C([0,1])$.*

Proof. Suppose that $\max_{l\in[0,1]} |\psi(l,0)| = M$, and choose a constant R satisfying

$$R \geq \frac{M \cdot M_1}{1-L}.$$

First, we will evidence that $\Upsilon(\mathcal{D}) \subset \mathcal{D}$, where $\mathcal{D} = \{x \in C([0,1]) : \|x\| \leq R\}$. Then, for any $x \in \mathcal{D}$, by following similar steps as in (4.2), we obtain the following inequality:

$$\|\Upsilon x\| \leq (L \cdot R + M)M_1 \leq R.$$

Therefore, $\Upsilon(\mathcal{D}) \subset \mathcal{D}$. Now, we will prove that Υ is a contraction. For all $x, y \in C([0,1])$ and $l \in [0,1]$, we get

$$\begin{aligned}|(\Upsilon x)(l) - (\Upsilon y)(l)| &\leq \left|\int_0^1 G(l,qs)\cdot[\psi(s,x(s)) - \psi(s,y(s))]\,d_qs\right| \\ &\leq \int_0^1 |G(l,qs)| \cdot L \cdot \max_{s\in[0,1]} |x(s)-y(s)|\,d_qs \\ &\leq L \cdot M_1 \cdot \|x-y\|. \end{aligned} \tag{4.5}$$

Taking the maximum of both sides of (4.5) over $[0,1]$, we obtain

$$\|\Upsilon x - \Upsilon y\| \leq L \cdot M_1 \cdot \|x-y\|$$

for all $x, y \in C([0,1])$, where $L \cdot M_1 < 1$. Hence, by Theorem 2, we achieve the unique solution in $\mathcal{D}$. □

Example 2. Consider the following BVP:

$$\begin{cases} D_{\frac{1}{2}}^3 x(l) + \frac{l^2+9+\sin x}{9} = 0, \ l \in [0,1], \\ x(0) = 0, \ x(1) = 0, \ \int_0^1 x(l)\,d_{\frac{1}{2}}l = 0, \end{cases}$$

where $q = \frac{1}{2}$ and

$$\psi(l,x) = \frac{l^2+9+\sin x}{9}$$

is continuous in $[0,1]\times\mathbb{R}$, and $\psi(l,0) = \frac{l^2}{9}+1 \neq 0$ for all $l \in [0,1]$. Now, for every $(l,x),(l,y) \in [0,1]\times\mathbb{R}$, we have

$$\begin{aligned}|\psi(l,x)-\psi(l,y)| =& \frac{1}{9}|\sin x - \sin y|\\ \leq& \frac{1}{9}|x-y|\end{aligned}$$

where $L = \frac{1}{9}$. On the other hand, if we write $q = \frac{1}{2}$ in (3.7), we get $M_1 = \frac{28}{15}$. From here, we conclude that $L \cdot M_1 = \frac{1}{9} \cdot \frac{28}{15} = \frac{28}{135} < 1$. Thus, by Theorem 4, there exists a unique nontrivial solution $x(l)$ in $C([0,1])$. In this case, the inequality $L \cdot M_1 \cong 0.2 < 1$ holds for $q = \frac{1}{2}$. Similarly, for $q = \frac{1}{3}$ and $q = \frac{1}{4}$, $L \cdot M_1 < 1$ (see Table 1), and thus the conditions of Theorem 4 are also satisfied. Therefore, the problem has a unique nontrivial solution for these values of q.

In the following theorem, we establish a condition that guarantees the integral equation (4.1) admits no nontrivial solution in $C([0,1])$.

Theorem 5. *Assume that $0 \leq \psi(l,x) < \frac{1}{M_1}x$ for every $0 \leq l \leq 1$, where M_1 is given by (3.7). Then the problem (1.3)-(1.4) has no nontrivial solution in the Banach space $C([0,1])$.*

Proof. Assume that there exists $x \in C([0,1])$ with $x = \Upsilon x$. Let $l_0 \in [0,1]$ be such that $\|x\| = x(l_0)$. In this case,

$$\begin{aligned}\|x\| = \|\Upsilon x\| =& \max_{l\in[0,1]}\left|\int_0^1 G(l,qs)\cdot\psi(s,x(s))\,d_qs\right|\\ \leq& \max_{l\in[0,1]}\int_0^1 |G(l,qs)|\cdot|\psi(s,x(s))|\,d_qs\\ =& \int_0^1 |G(l_0,qs)|\cdot\psi(s,x(s))\,d_qs\\ <& \frac{1}{M_1}\int_0^1 |G(l_0,qs)|\cdot x(s)\,d_qs\\ \leq& \frac{1}{M_1}\|x\|\int_0^1 |G(l_0,qs)|\,d_qs\\ \leq& \|x\|,\end{aligned}$$

which is a contradiction. □

Example 3. Consider (1.3)-(1.4) with $\psi(l,x) = \frac{lx}{100}$, $x > 0$, and $q = \frac{1}{10}$. Then, from (3.7), we have $\frac{1}{M_1} \cong 0,02$ such that

$$M_1 = \frac{8q^4+q^3+10q^2+2q+2}{4q^2(1+q)(1+q^2)(1+q+q^2)} \cong 46,66.$$

Consequently, we get $0 \leq \psi(l,x) = \frac{lx}{100} \leq \frac{x}{100} < \frac{2x}{100} = \frac{1}{M_1}x$. Thus, by Theorem 5, the problem (1.3)-(1.4) has no nontrivial solution in $C([0,1])$. Also, when $q = \frac{1}{6}$, we obtain $\frac{1}{M_1} = 0.06$ and $0 \leq \psi(l,x) = \frac{lx}{100} \leq \frac{x}{100} < \frac{6x}{100} = \frac{1}{M_1}x$. Therefore, the conditions of Theorem 5 are satisfied for $q = \frac{1}{6}$ as well. The values of $\frac{1}{M_1}$ corresponding to different choices of q are presented in Table 1.

Table 1. Some numerical results ($L = \frac{1}{9}$).

	$q = \frac{1}{2}$	$q = \frac{1}{3}$	$q = \frac{1}{4}$	$q = \frac{1}{5}$	$q = \frac{1}{6}$	$q = \frac{1}{10}$
M_1	1.86	4.11	7.27	11.39	16.47	46.66
$L \cdot M_1$	0.2	0.45	0.8	1.26	1.83	5.18
$1/M_1$	0.53	0.24	0.13	0.08	0.06	0.02

5. Conclusions

In this study, we establish the existence and uniqueness of solutions for the problem (1.3)-(1.4) by employing the Schauder and Banach FP theorems, respectively. We also determine a sufficient condition under which the considered problem admits no solution. Furthermore, several numerical examples are presented to demonstrate the applicability and effectiveness of the theoretical results obtained. It is worth noting that Lemma 2 reduces to Proposition 1 in [13] when $q \to 1^-$ for $c = 0$ and $d = 1$.

This paper contributes to the growing body of research on BVPs involving q-difference equations, particularly by addressing problems with integral boundary conditions, which have received relatively less attention. The present approach, based primarily on the Banach and Schauder FP theorems, can be further extended by employing alternative FP principles, such as Krasnoselskii's FP theorem, to derive more general existence and uniqueness criteria or to study problems with hybrid and nonlocal boundary conditions.

As a further direction, the present problem may be generalized to the (p, q)-calculus framework, whose fundamental theory is given in [34]. For more details on (p, q)-difference equations, we refer readers to [35,36]. In addition, recent studies such as Yu et al. [37] and Yu et al. [38] have investigated non-linear q-difference equations with the p-Laplacian and non-linear fractional q-integro-difference systems with nonlocal boundary conditions. These works provide valuable insights and may serve as a basis for further extensions of the present problem.

Author contributions

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Use of Generative AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools to create this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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