AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(11): 26958–26993.
DOI: 10.3934/math.20251185
Received: 04 July 2025
Revised: 01 November 2025
Accepted: 11 November 2025
Published: 20 November 2025

Research article

Wavelet multipliers for the linear canonical deformed Hankel transform and applications

Saifallah Ghobber[1,*] **and Hatem Mejjaoli**[2]

[1] Department of Mathematics and Statistics, College of Science, King Faisal University, P.O. Box 400, Al-Ahsa 31982, Saudi Arabia

[2] Department of Mathematics, College of Sciences, Taibah University, P.O. Box 30002, Al Madinah Al Munawarah 42353, Saudi Arabia

* **Correspondence:** Email: sghobber@kfu.edu.sa.

Abstract: In this paper, we introduce the notion of a wavelet multiplier in the setting of the linear canonical deformed Hankel transform (LCDHT), which depends on a symbol and two bounded functions. Then, we study the boundedness and compactness of these operators according to the symbol and the bounded functions. We will then show that, under certain assumptions, the wavelet multiplier is equal to the well-known time-frequency restriction operator. Then, we show that a function that is almost time- and band-limited can be approximated by its projection on the subspace spanned by the first eigenfunctions of such an operator, corresponding to the greatest eigenvalues, which are near one. This study for the LCDHT includes, in particular, some known transforms, such as the deformed Hankel, the Fresnel, and the fractional deformed Hankel transforms.

Keywords: linear canonical deformed Hankel transform; wavelet multipliers
Mathematics Subject Classification: 42B15, 44A05

1. Introduction

A fundamental tool for signal and image processing are time-limited and band-limited functions. However, the uncertainty principle informs us that a function cannot at the same time be time- and band-limited. Thus, the set of almost time- and almost band-limited functions is introduced, and has been studied, first by Landau [1], then by Landau-Pollak [2] and Donoho-Stark [3]. In the current work, the eigenvalue problem related to the time-frequency operator is examined. An orthonormal collection of eigenfunctions that meet certain optimality in localization in an area of the time-frequency plane are produced by the resultant operator. Taking a finite linear combination of these eigenfunctions, we derive approximations for functions which are almost time- and band-limited, and we provide a characterization of such functions.

He and Wong were the first to introduce the idea of wavelet multipliers in their seminal paper [4]; this notion was then developed in [5], and detailed by Wong in [6]. Numerous mathematical analysis issues may be resolved with the knowledge of wavelet multipliers, particularly in time-frequency analysis. Furthermore, in the time-frequency plane, wavelet multipliers may be thought of as a filter that time- and band-limit signals. Wavelet multipliers have been studied by several researchers in a variety of settings, see for instance [7–17] and the references therein.

Collins [18] and Moshinsk-Quesne [19] were the first to introduce the linear canonical transform (LCT) to study important problems in paraxial optics and quantum mechanics. This transformatiom is more flexible than other known transformations, and is a suitable and powerful tool for studying deep problems in optics, quantum physics, and signal processing [20–26].

LCT has recently attracted much interest and has been expanded to include a broad class of integral transformations. Moreover, several researchers have been investigating the behavior of the generalized LCT with respect to several problems already studied for the Fourier transform; for instance, uncertainty principles [27, 28], sampling and multiplicative filtering [29], localization operators [30, 31], Gabor transform [32, 33], wavelet transform [34, 35], convolution and correlation [36–38], heat equation [39], Poisson summation formula [40], wave packet frames [41] and so on.

In this paper, our main goal is to extend the notion of wavelet multipliers to the setting of the linear canonical deformed Hankel transform (LCDHT) recently introduced in [28]. This transformation is indeed unitary and provides a unified treatment of generalized Fourier transforms, in the sense that it coincides with several well-known integral transforms, including the deformed Hankel, the fractional deformed Hankel, the Fresnel, and the linear canonical Bessel transforms. To be more precise, let us fix some notation. Let $L^p_{d,\ell}(\mathbb{R})$ be the space of measurable functions f on $\mathbb{R}$ with

$$\|f\|_{L^p_{d,\ell}} = \left(\int_{\mathbb{R}} |f(t)|^p \gamma_{d,\ell}(dt)\right)^{\frac{1}{p}} < \infty, \qquad 1 \leq p < \infty, \tag{1.1}$$

$$\|f\|_{L^\infty_{d,\ell}} = \operatorname*{ess\,sup}_{t\in\mathbb{R}} |f(t)| < \infty, \tag{1.2}$$

where for $\ell \geq \frac{d-1}{d}$, $d \in \mathbb{N}$, we have

$$\gamma_{d,\ell}(dx) := C_{d,\ell}|x|^{2\ell-2+2/d}dx, \qquad C_{d,\ell} = \frac{d^{d(\ell-1/2)}}{2^{1+d(\ell-1/2)}\Gamma(1+d(\ell-1/2))}. \tag{1.3}$$

Then, the deformed Hankel transform (DHT) is defined by [42]

$$\mathfrak{F}_{d,\ell}(f)(\xi) = \int_{\mathbb{R}} f(x)e_{d,\ell}(\xi,x)\gamma_{d,\ell}(dx), \qquad \xi \in \mathbb{R}, \tag{1.4}$$

where $f \in L^1_{d,\ell}(\mathbb{R})$ and $e_{d,\ell}(\xi,x)$ is the deformed Hankel kernel given by

$$e_{d,\ell}(\xi,x) = j_{d\ell-d/2}(d|\xi x|^{\frac{1}{d}}) + (-i)^d(d/2)^d\frac{\Gamma(d\ell-d/2+1)}{\Gamma(d\ell+d/2+1)}\xi x j_{d\ell+d/2}(d|\xi x|^{\frac{1}{d}}), \tag{1.5}$$

and

$$j_\alpha(t) := \Gamma(\alpha+1)\sum_{j=0}^{\infty}\frac{(-1)^j}{j!\,\Gamma(\alpha+j+1)}\left(\frac{t}{2}\right)^{2j}. \tag{1.6}$$

For $M = \begin{pmatrix} a & b \\ a' & b' \end{pmatrix} \in SL(2,\mathbb{R})$ be a matrix, so that $b \neq 0$, the linear canonical deformed Hankel transform (LCDHT) is defined by [28],

$$\mathfrak{F}^M_{d,\ell}(f)(\xi) = \frac{1}{(ib)^{1/d-1/2+\ell}}\int_{\mathbb{R}} D^M_{d,\ell}(\xi,x)f(x)\gamma_{d,\ell}(dx), \tag{1.7}$$

where $f \in L^1_{d,\ell}(\mathbb{R})$ and

$$D^M_{d,\ell}(\xi,x) = e^{\frac{i}{2}\left(\frac{b'}{b}\xi^2+\frac{a}{b}x^2\right)}e_{d,\ell}\left(\xi/b,x\right). \tag{1.8}$$

Let φ_1,φ_2, and ς be measurable functions on $\mathbb{R}$ and $p \in [1,\infty]$. Then, we introduce in this paper the linear canonical deformed Hankel wavelet multiplier operator as follows:

$$\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)(f)(t) = \frac{1}{(-ib)^{1/d-1/2+\ell}}\int_{\mathbb{R}}\varsigma(\xi)\mathfrak{F}^M_{d,\ell}(\varphi_1 f)(\xi)\overline{\varphi_2(t)D^M_{d,\ell}(\xi,t)}\gamma_{d,\ell}(d\xi), \qquad t \in \mathbb{R}. \tag{1.9}$$

Our first aim is to study the L^p-boundedness of $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)$, according to the symbol ς and the functions φ_1,φ_2. More precisely, we prove the following results.

Theorem A:

(1) If the symbol ς belongs to $L^p_{d,\ell}(\mathbb{R})$, $1 \le p \le \infty$, and φ_1,φ_2 are in $L^\infty_{d,\ell}(\mathbb{R}) \cap L^1_{d,\ell}(\mathbb{R})$, then $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma) : L^2_{d,\ell}(\mathbb{R}) \to L^2_{d,\ell}(\mathbb{R})$ is in S_∞, such that

$$\left\|\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)\right\|_{S_\infty} \leqslant c^{1/p}_{\ell,b,d}\left(\|\varphi_2\|_{L^\infty_{d,\ell}}\|\varphi_1\|_{L^\infty_{d,\ell}}\right)^{\frac{p-1}{p}}\|\varsigma\|_{L^p_{d,\ell}}, \tag{1.10}$$

where $S_\infty = B\left(L^2_{d,\ell}(\mathbb{R})\right)$ is the set of bounded operators from $L^2_{d,\ell}(\mathbb{R})$ onto itself and $c_{\ell,b,d} = |b|^{-2\ell-2/d+1}$.

(2) For $\varsigma \in L^1_{d,\ell}(\mathbb{R})$, we have:

(a) If $\varphi_1,\varphi_2 \in L^\infty_{d,\ell}(\mathbb{R}) \cap L^1_{d,\ell}(\mathbb{R})$, then $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma) : L^p_{d,\ell}(\mathbb{R}) \longrightarrow L^p_{d,\ell}(\mathbb{R})$, $1 \le p \le \infty$, is bounded, with

$$\left\|\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)\right\|_{B\left(L^p_{d,\ell}(\mathbb{R})\right)} \leqslant c_{\ell,b,d}\left(\|\varphi_1\|_{L^\infty_{d,\ell}}\|\varphi_2\|_{L^1_{d,\ell}}\right)^{\frac{1}{p}}\left(\|\varphi_1\|_{L^1_{d,\ell}}\|\varphi_2\|_{L^\infty_{d,\ell}}\right)^{\frac{1}{p'}}\|\varsigma\|_{L^1_{d,\ell}}. \tag{1.11}$$

(b) If $\varphi_1 \in L^{p'}_{d,\ell}(\mathbb{R})$, $p \in [1,\infty]$, and $\varphi_2 \in L^p_{d,\ell}(\mathbb{R})$, then $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma) : L^p_{d,\ell}(\mathbb{R}) \longrightarrow L^p_{d,\ell}(\mathbb{R})$ is bounded such that

$$\left\|\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)\right\|_{B\left(L^p_{d,\ell}(\mathbb{R})\right)} \leqslant c_{\ell,b,d}\|\varphi_1\|_{L^{p'}_{d,\ell}}\|\varphi_2\|_{L^p_{d,\ell}}\|\varsigma\|_{L^1_{d,\ell}}. \tag{1.12}$$

(3) If $\varsigma \in L^r_{d,\ell}(\mathbb{R})$, $1 \le r \le 2$, and $\varphi_1,\varphi_2 \in L^1_{d,\ell}(\mathbb{R}) \cap L^\infty_{d,\ell}(\mathbb{R})$, then $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma) : L^p_{d,\ell}(\mathbb{R}) \longrightarrow L^p_{d,\ell}(\mathbb{R})$ is bounded for every $p \in [r,r']$, with

$$\left\|\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)\right\|_{B\left(L^p_{d,\ell}(\mathbb{R})\right)} \leqslant \mathfrak{C}_1^t\mathfrak{C}_2^{1-t}\|\varsigma\|_{L^r_{d,\ell}}, \tag{1.13}$$

where $\mathfrak{C}_1$, $\mathfrak{C}_2$ are constants that depend on φ_1,φ_2 given respectively by (4.14), (4.15) and $\dfrac{1-t}{r'} + \dfrac{t}{r} = \dfrac{1}{p}$.

Next, we study the compactness of $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)$ and show the following results.
Theorem B:

(1) For $\varphi_1,\varphi_2 \in L^\infty_{d,\ell}(\mathbb{R}) \cap L^1_{d,\ell}(\mathbb{R})$, we have:
   (a) If $\varsigma \in L^p_{d,\ell}(\mathbb{R})$, $1 \leqslant p < \infty$, then $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma) : L^2_{d,\ell}(\mathbb{R}) \longrightarrow L^2_{d,\ell}(\mathbb{R})$ is in the Schatten class S_p, such that

$$\|\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)\|_{S_p} \leqslant c^{1/p}_{\ell,b,d}\left(\|\varphi_2\|_{L^\infty_{d,\ell}}\|\varphi_1\|_{L^\infty_{d,\ell}}\right)^{\frac{p-1}{p}}\|\varsigma\|_{L^p_{d,\ell}}. \tag{1.14}$$

   (b) If $\varsigma \in L^r_{d,\ell}(\mathbb{R})$, $1 \le r \le 2$, then $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma) : L^p_{d,\ell}(\mathbb{R}) \longrightarrow L^p_{d,\ell}(\mathbb{R})$ is compact for every $r \le p \le r'$.

(2) If $\varsigma \in L^1_{d,\ell}(\mathbb{R})$, $\varphi_1 \in L^{p'}_{d,\ell}(\mathbb{R})$, $1 \le p \le \infty$, and $\varphi_2 \in L^p_{d,\ell}(\mathbb{R})$, then $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma) : L^p_{d,\ell}(\mathbb{R}) \longrightarrow L^p_{d,\ell}(\mathbb{R})$ is compact for every $1 \leqslant p \le \infty$.

In Section 5, we give several examples of linear canonical deformed Hankel wavelet multipliers. Specifically, if we take $\varphi_1 = \varphi_2 = \mathbf{1}_{R_1}$, we prove that

$$\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma) = Q_{R_1}P^M_{R_2}Q_{R_1}, \qquad \varsigma = \mathbf{1}_{R_2}, \tag{1.15}$$

where for a subset $R \subset \mathbb{R}$ of finite measure, $0 < \gamma_d(R) < \infty$, $Q_R : L^2_{d,\ell}(\mathbb{R}) \longrightarrow L^2_{d,\ell}(\mathbb{R})$ and $P^M_R : L^2_{d,\ell}(\mathbb{R}) \longrightarrow L^2_{d,\ell}(\mathbb{R})$ are the self-adjoint projections defined by

$$Q_R f = \mathbf{1}_R f, \qquad P^M_R f = \mathfrak{F}^{M^{-1}}_{d,\ell}\Big(Q_R\,\mathfrak{F}^M_{d,\ell}(f)\Big). \tag{1.16}$$

In time-frequency analysis, this operator also referred to as the time-frequency limiting operator. Notice that $Q_{R_1}P^M_{R_2}Q_{R_1} = \left(Q_{R_1}P^M_{R_2}\right)\left(Q_{R_1}P^M_{R_2}\right)^*$, where $\left(Q_{R_1}P^M_{R_2}\right)^* = P^M_{R_2}Q_{R_1}$. We show in particular the following theorem.
Theorem C:

(1) The operator $Q_{R_1}P^M_{R_2}$ is bounded and Hilbert-Schmidt, such that

$$\left\|Q_{R_1}P^M_{R_2}\right\|_{S_\infty} \le \left\|Q_{R_1}P^M_{R_2}\right\|_{HS} \le \sqrt{c_{\ell,b,d}\gamma_{d,\ell}(R_1)\gamma_{d,\ell}(R_2)}. \tag{1.17}$$

(2) If $\gamma_{d,\ell}(R_1)\gamma_{d,\ell}(R_2) < |b|^{2\ell+2/d-1}$, then for every $f \in L^2_{d,\ell}(\mathbb{R})$,

$$\|f\|^2_{L^2_{d,\ell}} \le \left(1-\sqrt{c_{\ell,b,d}\gamma_{d,\ell}(R_1)\gamma_{d,\ell}(R_2)}\right)^{-2}\left(\left\|Q_{R_1^c}f\right\|^2_{L^2_{d,\ell}} + \left\|P^M_{R_2^c}f\right\|^2_{L^2_{d,\ell}}\right). \tag{1.18}$$

(3) If $\gamma_{d,\ell}(R_1)\gamma_{d,\ell}(R_2) < |b|^{2\ell+2/d-1}$, then for every $f \in L^2_{d,\ell}(\mathbb{R})$,

$$\|f\|^2_{L^2_{d,\ell}} \le \left(1-\sqrt{c_{\ell,b,d}\gamma_{d,\ell}(R_1)\gamma_{d,\ell}(R_2)}\right)^{-2}\left(\left\|\mathbf{1}_{R_1^c}f\right\|^2_{L^2_{d,\ell}} + \left\|\mathbf{1}_{R_2^c}\,\mathfrak{F}^M_{d,\ell}(f)\right\|^2_{L^2_{d,\ell}}\right). \tag{1.19}$$

Notice that inequality (1.19) can be obtained from (1.18) by using the Plancherel-type formula for the LCDHT. In particular, if supp$f \subset R_1$ and supp $\mathfrak{F}^M_{d,\ell}(f) \subset R_2$, then f is the zero function. This means that (R_1,R_2) is an annihilating pair, according to the terminology of Havin and Jöricke [43].

For finite-measure subsets R_1 and R_2, $0 < \gamma_{d,\ell}(R_1), \gamma_{d,\ell}(R_2) < \infty$, one would want to reach a function $f \in L^2_{d,\ell}(\mathbb{R})$ that is band-limited on $R_2 \subset \mathbb{R}$ (i.e. supp $\mathfrak{F}^M_{d,\ell}(f) \subset R_2$) and time-limited on $R_1 \subset \mathbb{R}$ (i.e. supp$f \subset R_1$). However, there are no such functions since $f = 0$ if supp$f \subset R_1$ and supp$\mathfrak{F}^M_{d,\ell}(f) \subset R_2$. Therefore, it makes sense to substitute concentrated functions by ε-concentrated functions, $0 < \varepsilon < 1$, on an area of the time-frequency plane. That is,

(1) $f \in L^2_{d,\ell}(\mathbb{R})\backslash\{0\}$ is said to be ε-concentrated on $R \subset \mathbb{R}$ if $\|Q_{R^c} f\|_{L^2_{d,\ell}} \leq \varepsilon \|f\|_{L^2_{d,\ell}}$.
(2) $f \in L^2_{d,\ell}(\mathbb{R})\backslash\{0\}$ is said to be ε-concentrated with regard to an operator $\mathcal{L}$ if

$$\|\mathcal{L}f - f\|_{L^2_{d,\ell}} \leq \varepsilon \|f\|_{L^2_{d,\ell}},$$

where $R^c = \mathbb{R}\backslash R$. Note that if $\mathfrak{F}^M_{d,\ell}(f)$ is ε-concentrated on R, then by Eq (2.14), we have

$$\|P^M_{R^c} f\|_{L^2_{d,\ell}} \leq \varepsilon \|f\|_{L^2_{d,\ell}}. \tag{1.20}$$

In this case, it is common to say that the function f is ε-band-limited on R. Notice that if a function is ε-concentrated on R, then it is ε-concentrated with regard to Q_R. Likewise, if a function is ε-band-limited on R, then it is ε-concentrated with regard to P^M_R.

If $\varepsilon = 0$ in (1.20), then $R = \operatorname{supp}\mathfrak{F}^M_{d,\ell}(f) = \left\{\xi \in \mathbb{R} \,:\, \mathfrak{F}^M_{d,\ell}(f)(\xi) \neq 0\right\}$, and if $\varepsilon \in (0,1)$, then $\mathfrak{F}^M_{d,\ell}(f)$ is almost concentrated on R, which will be the *essential* support of $\mathfrak{F}^M_{d,\ell}(f)$.

For $0 < \varepsilon_1, \varepsilon_2 < 1$, let $L^2_{d,\ell}(\varepsilon_1, \varepsilon_2, R_1, R_2)$ be the set containing any function $f \in L^2_{d,\ell}(\mathbb{R})$ that is ε_1-concentrated on R_1 and ε_2-band-limited on R_2. Then, $L^2_{d,\ell}(0,0,R_1,R_2) = \emptyset$, and by Eq (1.18), if $f \in L^2_{d,\ell}(\varepsilon_1, \varepsilon_2, R_1, R_2)$, we prove that

$$\gamma_{d,\ell}(R_1)\gamma_{d,\ell}(R_2) \geq |b|^{2\ell+2/d-1}\left(1 - \sqrt{\varepsilon_1^2 + \varepsilon_2^2}\right)^2. \tag{1.21}$$

This inequality is known as the Donoho-Stark uncertainty principle, which means that the essential supports of f and $\mathfrak{F}^M_{d,\ell}(f)$ cannot be too small.

According to Landau's perspective [1], ρ is a pseudo-eigenvalue of $\mathcal{L}$, if there is a $f \in L^2_{d,\ell}(\mathbb{R})$, satisfying

$$\|\mathcal{L}f - \rho f\|_{L^2_{d,\ell}} \leq \varepsilon \|f\|_{L^2_{d,\ell}}.$$

This means that if $f \in L^2_{d,\ell}(\mathbb{R})$ is ε-concentrated with regard to $\mathcal{L}$, then f is a pseudo-eigenfunction of $\mathcal{L}$ with pseudo-eigenvalue 1. Specifically, in the case of $\varepsilon = 0$, the eigenvalue 1 corresponds to the eigenfunction $f \in L^2_{d,\ell}(\mathbb{R})$ of $\mathcal{L}$ that is ε-concentrated with regard to $\mathcal{L}$. It should be noted that the ε-concentration notion, which originates from the idea of pseudo-spectra of linear operators [1–3], was extensively studied in [44].

We also prove that if $f \in L^2_{d,\ell}(\mathbb{R})$ is ε_1-concentrated with regard to $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\mathbf{1}_{R_1})$ and ε_2-concentrated with regard to $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\mathbf{1}_{R_2})$, then

$$\gamma_{d,\ell}(R_1)\gamma_{d,\ell}(R_2) \geq |b|^{4(\ell+1/d)-2}(1 - \varepsilon_1 - \varepsilon_2), \qquad \varepsilon_1 + \varepsilon_2 < 1, \tag{1.22}$$

where $\varphi_1, \varphi_2 \in L^2_{d,\ell}(\mathbb{R}) \cap L^\infty_{d,\ell}(\mathbb{R})$ are two unit L^2-norm functions, with $\|\varphi_2\|_{L^\infty_{d,\ell}}\|\varphi_1\|_{L^\infty_{d,\ell}} = 1$.

Next, we consider the time-frequency limiting operator

$$\Pi^M_{R_1,R_2} = \left(Q_{R_1}P^M_{R_2}\right)^* \left(Q_{R_1}P^M_{R_2}\right) = P^M_{R_2}Q_{R_1}P^M_{R_2}. \tag{1.23}$$

This operator is self-adjoint and positive. Moreover since by (1.17), the operator $Q_{R_1}P^M_{R_2}$ is Hilbert-Schmidt, then $\Pi^M_{R_1,R_2}$ is compact such that

$$\left\|\Pi^M_{R_1,R_2}\right\|_{S_1} = \left\|Q_{R_1}P^M_{R_2}\right\|^2_{HS} \leq c_{\ell,b,d}\gamma_{d,\ell}(R_1)\gamma_{d,\ell}(R_2). \tag{1.24}$$

On the other hand, since Q_{R_1} and $P^M_{R_2}$ are two orthogonal projections, then $\left\|Q_{R_1}P^M_{R_2}\right\|_{S_\infty} \leq 1$, and then

$$\left\|\Pi^M_{R_1,R_2}\right\|_{S_\infty} \leq 1. \tag{1.25}$$

Thus, $\Pi^M_{R_1,R_2} : L^2_{d,\ell}(\mathbb{R}) \to L^2_{d,\ell}(\mathbb{R})$ satisfies

$$\Pi^M_{R_1,R_2} f = \sum_{i=1}^{\infty} \lambda_i \langle f, \sigma_i\rangle_{L^2_{d,\ell}} \sigma_i, \tag{1.26}$$

where $\{\lambda_i = \lambda_i(R_1, R_2)\}_{i=1}^{\infty}$ is the set of eigenvalues, and $\{\sigma_i = \sigma_i(R_1, R_2)\}_{i=1}^{\infty}$ corresponds to an orthonormal sequence of eigenfunctions, such that

$$0 \leq \cdots \leq \lambda_{i+1} \leq \lambda_i \leq \cdots \leq \lambda_2 \leq \lambda_1 \leq 1. \tag{1.27}$$

In particular,

$$\|\Pi^M_{R_1,R_2}\|_{S_\infty} = \lambda_1. \tag{1.28}$$

Then, the greatest concentration on the set $R_1 \times R_2$ is realized by the first eigenfunction σ_1. Moreover, for all i,

$$\left\|\Pi^M_{R_1,R_2}\,\sigma_i - \sigma_i\right\|_{L^2_{d,\ell}} = \left\langle \sigma_i - \Pi^M_{R_1,R_2}\,\sigma_i, \sigma_i\right\rangle_{L^2_{d,\ell}} = 1 - \lambda_i, \tag{1.29}$$

and

$$\begin{aligned}\left\|\Pi^M_{R_1,R_2}\left(\Pi^M_{R_1,R_2}\,\sigma_i\right) - \Pi^M_{R_1,R_2}\,\sigma_i\right\|_{L^2_{d,\ell}} &= \lambda_i^{-1}\left\langle \Pi^M_{R_1,R_2}\,\sigma_i - \Pi^M_{R_1,R_2}\left(\Pi^M_{R_1,R_2}\,\sigma_i\right), \Pi^M_{R_1,R_2}\,\sigma_i\right\rangle_{L^2_{d,\ell}} \\ &= \lambda_i(1-\lambda_i) = (1-\lambda_i)\left\|\Pi^M_{R_1,R_2}\,\sigma_i\right\|_{L^2_{d,\ell}}. \end{aligned} \tag{1.30}$$

Hence, for every i, σ_i and $\Pi^M_{R_1,R_2}\,\sigma_i$ are $(1-\lambda_i)$-concentrated with respect to $\Pi^M_{R_1,R_2}$. Moreover, if we denote by $\Pi^M_{R_1,R_2}(\varepsilon)$ the subset of $L^2_{d,\ell}(\mathbb{R})$ consisting of functions that are ε-concentrated with regard to $\Pi^M_{R_1,R_2}$, and

$$L^2_{d,\ell}(\varepsilon, R_1, R_2) = \left\{ f \in L^2_{d,\ell}(\mathbb{R}) \;:\; \left\langle f - \Pi^M_{R_1,R_2} f, f\right\rangle_{L^2_{d,\ell}} \leq \varepsilon \|f\|^2_{L^2_{d,\ell}} \right\}, \tag{1.31}$$

then we prove the following comparison:

(1) If $f \in L^2_{d,\ell}(\varepsilon_1, \varepsilon_2, R_1, R_2)$, then $f \in \Pi^M_{R_1,R_2}(\varepsilon_1 + 2\varepsilon_2)$ and is $(\varepsilon_1 + \varepsilon_2)$-concentrated with regard to $Q_{R_1}P^M_{R_2}$.

(2) If $f \in \Pi^M_{R_1,R_2}(\varepsilon)$, then

$$\left\langle f - \Pi^M_{R_1,R_2} f, f\right\rangle_{L^2_{d,\ell}} \leq (\varepsilon + \varepsilon^2)\|f\|^2_{L^2_{d,\ell}}. \tag{1.32}$$

(3) If $f \in L^2_{d,\ell}(\varepsilon, R_1, R_2)$, then $f \in \Pi^M_{R_1,R_2}(\sqrt{\varepsilon})$.

(4) If $f \in L^2_{d,\ell}(\varepsilon_1, \varepsilon_2, R_1, R_2)$, then

$$\left\langle f - \Pi^M_{R_1,R_2} f, f\right\rangle_{L^2_{d,\ell}} < (\varepsilon_1 + 2\varepsilon_2)\|f\|^2_{L^2_{d,\ell}}. \tag{1.33}$$

Notice that if $f \in L^2_{d,\ell}(\varepsilon, R_1, R_2)$, then

$$\left\langle \Pi^M_{R_1,R_2} f, f \right\rangle_{L^2_{d,\ell}} \geq (1-\varepsilon)\|f\|^2_{L^2_{d,\ell}}. \tag{1.34}$$

Therefore, by (1.29) and (1.30),

$$\forall\, i \geq 1, \qquad \sigma_i,\ \Pi^M_{R_1,R_2}\, \sigma_i \in L^2(1-\lambda_i, R_1, R_2). \tag{1.35}$$

Moreover, according to (1.33), if $f \in L^2_{d,\ell}(\varepsilon_1, \varepsilon_2, R_1, R_2)$, then $f \in L^2_{d,\ell}(\varepsilon_1 + 2\varepsilon_2, R_1, R_2)$, and according to (1.32), if $f \in \Pi^M_{R_1,R_2}(\varepsilon)$, then $f \in L^2(2\varepsilon, R_1, R_2)$. Thus, we are willing to resolve this optimization issue:

$$\text{Maximize} \qquad \left\langle \Pi^M_{R_1,R_2} f, f \right\rangle_{L^2_{d,\ell}}, \qquad \|f\|_{L^2_{d,\ell}} = 1, \tag{1.36}$$

which seeks to find orthonormal functions in $L^2_{d,\ell}(\mathbb{R})$, that are almost time- and band-limited to an area of the time-frequency plane. Consequently, the number of eigenfunctions $\Pi^M_{R_1,R_2}$ with eigenvalues that are extremely near to one is an optimal solution of (1.36). This is because, if $\lambda_i \geq 1-\varepsilon$, then

$$\left\langle \Pi^M_{R_1,R_2}\, \sigma_i, \sigma_i \right\rangle_{L^2_{d,\ell}} = \lambda_i \geq 1-\varepsilon. \tag{1.37}$$

Let $N = N(\varepsilon, R_1, R_2)$ be the number of λ_i, that are close to one, i.e.,

$$\lambda_1 \geq \cdots \geq \lambda_N \geq 1-\varepsilon > \lambda_{1+N} \geq \cdots, \tag{1.38}$$

and let $V_N = \operatorname{span}\{\sigma_i\}_{i=1}^N$.

Thus, from (1.35) and (1.37), σ_i and $\Pi^M_{R_1,R_2}\, \sigma_i$ belong to $L^2_{d,\ell}(\varepsilon, R_1, R_2)$, if and only if $1 \leq i \leq N$, and if $f \in V_N$, then

$$\left\langle \Pi^M_{R_1,R_2} f, f \right\rangle_{L^2_{d,\ell}} = \sum_{i=1}^N \lambda_i \left| \langle f, \sigma_i \rangle_{L^2_{d,\ell}} \right|^2 \geq \lambda_N \sum_{i=1}^N \left| \langle f, \sigma_i \rangle_{L^2_{d,\ell}} \right|^2 \geq (1-\varepsilon)\|f\|^2_{L^2_{d,\ell}}.$$

Hence, V_N is the subspace of $L^2_{d,\ell}(\mathbb{R})$ with a maximum dimension, which is in $L^2_{d,\ell}(\varepsilon, R_1, R_2)$.

Finally, we used time-frequency localization operators to approximate time-frequency localized functions with finite combinations of basis functions. These approximations allow for the construction of localized time-frequency frames, which are sets of vectors that can be used to reconstruct the original function. Research focuses on the properties of these operators and frames, especially on how well finite expansions represent functions within a specific region of interest in the time-frequency plane. In fact, we prove the following characterization and approximation inequality.

Theorem D:

(1) A function f belongs to $L^2_{d,\ell}(\varepsilon, R_1, R_2)$, if and only if,

$$\sum_{i=1}^N (\lambda_i + \varepsilon - 1)\left| \langle f, \sigma_i \rangle_{L^2_{d,\ell}} \right|^2 \geq (1-\varepsilon)\|f_{\text{ker}}\|^2_{L^2_{d,\ell}} + \sum_{i=1+N}^{\infty} (1-\varepsilon-\lambda_i)\left| \langle f, \sigma_i \rangle_{L^2_{d,\ell}} \right|^2. \tag{1.39}$$

(2) For any $f \in L^2_{d,\ell}(\varepsilon, R_1, R_2)$,

$$\left\| f - \sum_{i=1}^{N_0} \langle f, \sigma_i \rangle_{L^2_{d,\ell}} \sigma_i \right\|_{L^2_{d,\ell}} \leq \sqrt{\frac{\varepsilon}{\varepsilon_0}}\, \|f\|_{L^2_{d,\ell}}, \tag{1.40}$$

where $f_{\ker}$ is the orthogonal projection of f onto $\mathrm{Ker}\left(\Pi^M_{R_1,R_2}\right)$, and $N_0 = N(\varepsilon_0, R_1, R_2)$ for a fixed $\varepsilon_0 \in (0,1)$. By (1.32) and (1.33), we have:

(1) If $f \in L^2_{d,\ell}(\varepsilon_1, \varepsilon_2, R_1, R_2)$, then

$$\left\| f - \sum_{i=1}^{N_0} \langle f, \sigma_i \rangle_{L^2_{d,\ell}} \sigma_i \right\|_{L^2_{d,\ell}} \leq \sqrt{\frac{2\varepsilon_1 + \varepsilon_2}{\varepsilon_0}} \, \|f\|_{L^2_{d,\ell}}. \tag{1.41}$$

(2) If $f \in \Pi^M_{R_1,R_2}(\varepsilon)$, then

$$\left\| f - \sum_{i=1}^{N_0} \langle f, \sigma_i \rangle_{L^2_{d,\ell}} \sigma_i \right\|_{L^2_{d,\ell}} \leq \sqrt{\frac{2\varepsilon}{\varepsilon_0}} \, \|f\|_{L^2_{d,\ell}}. \tag{1.42}$$

2. Preliminaries

For any $f \in C^1(\mathbb{R})$, the Dunkl operator T_ℓ on $\mathbb{R}$ is defined by

$$T_\ell f(x) := f'(x) + 2\ell \left(\frac{f(x) - f(-x)}{x} \right), \tag{2.1}$$

where the corresponding Dunkl-Laplace operator $\triangle_\ell$ for any $f \in C^2(\mathbb{R})$ is given by

$$\triangle_\ell f(x) := T_\ell^2 f(x) = f''(x) + 2\ell \left(\frac{f'(x)}{x} - \frac{f(x) - f(-x)}{2x^2} \right). \tag{2.2}$$

Consider the operator $\Delta_{d,\ell}$ defined by

$$\Delta_{d,\ell} := \left|x\right|^{2-\frac{2}{d}} \triangle_\ell - \left|x\right|^{\frac{2}{d}}. \tag{2.3}$$

In what follows, we recall some spectral properties of the differential-difference operator $\Delta_{n,k}$.

- $\Delta_{d,\ell}$ is an essentially self-adjoint operator on $L^2_{d,\ell}(\mathbb{R})$.
- There is no continuous spectrum of $\Delta_{d,\ell}$.
- The discrete spectrum of $-\Delta_{d,\ell}$ is $\{\frac{4m}{d} + 2\ell + \frac{2}{d} \pm 1 : m \in \mathbb{N}\}$.

An important motivation to study Dunkl operators originates in their relevance for the analysis of quantum many-body systems of Calogero-Moser-Sutherland type. These describe algebraically integrable systems in one dimension and have gained considerable interest in mathematical physics, especially in conformal field theory. A good bibliography is given in [45]. One of the motivations for the introduction of the deformed Laplace operator $\Delta_{n,k}$ and its generalized Fourier transform is to generalize the previous subjects, which are bound with the physics.

The deformed Hankel transform (DHT) is defined by [42]

$$\mathfrak{F}_{d,\ell}(f)(\xi) = \int_{\mathbb{R}} f(x) e_{d,\ell}(\xi, x) \gamma_{d,\ell}(dx), \qquad \xi \in \mathbb{R}, \tag{2.4}$$

where $f \in L^1_{d,\ell}(\mathbb{R})$ and $e_{d,\ell}(\xi, x)$ is the deformed Hankel kernel given by

$$e_{d,\ell}(\xi, x) = j_{d\ell-d/2}(d|\xi x|^{\frac{1}{d}}) + (-i)^d (d/2)^d \frac{\Gamma(d\ell - d/2 + 1)}{\Gamma(d\ell + d/2 + 1)} \xi x j_{d\ell+d/2}(d|\xi x|^{\frac{1}{d}}), \tag{2.5}$$

and

$$j_\alpha(t) := \Gamma(\alpha + 1) \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\,\Gamma(\alpha + j + 1)} \left(\frac{t}{2}\right)^{2j}. \tag{2.6}$$

Here, we list some important properties of the deformed Hankel kernel:
(i) $e_{d,\ell}(z, t) = e_{d,\ell}(t, z)$, $e_{d,\ell}(z, 0) = 1$, $\overline{e_{d,\ell}(z, t)} = e_{d,\ell}((-1)^d z, t)$, $e_{d,\ell}(\lambda z, t) = e_{d,\ell}(z, \lambda t), \forall z, t, \lambda \in \mathbb{R}$.
(ii) $e_{d,\ell}(\cdot, \cdot)$ solves the following differential-difference equations on $\mathbb{R} \times \mathbb{R}$:

$$|\lambda|^{2-\frac{2}{d}} \triangle^\lambda_\ell e_{d,\ell}(\lambda, x) = -|x|^{\frac{2}{d}} e_{d,\ell}(\lambda, x) \tag{2.7}$$

and

$$|x|^{2-\frac{2}{d}} \triangle^x_\ell e_{d,\ell}(\lambda, x) = -|\lambda|^{\frac{2}{d}} e_{d,\ell}(\lambda, x), \tag{2.8}$$

where the superscript in $\triangle^x_\ell$ denotes the relevant variable.

Let $M = \begin{pmatrix} a & b \\ a' & b' \end{pmatrix} \in SL(2, \mathbb{R})$ be a matrix such that $b \neq 0$. Then, the linear canonical deformed Hankel transform (LCDHT) is defined by [28],

$$\mathfrak{F}^M_{d,\ell}(f)(\xi) = \frac{1}{(ib)^{1/d-1/2+\ell}} \int_{\mathbb{R}} D^M_{d,\ell}(\xi, x) f(x) \gamma_{d,\ell}(dx), \tag{2.9}$$

where $f \in L^1_{d,\ell}(\mathbb{R})$ and

$$D^M_{d,\ell}(\xi, x) = e^{\frac{i}{2}\left(\frac{b'}{b}\xi^2 + \frac{a}{b}x^2\right)} e_{d,\ell}\left(\xi/b, x\right). \tag{2.10}$$

The relation between the LCDHT and DHT is given by

$$\mathfrak{F}^M_{d,\ell}(f)(\xi) = \frac{e^{\frac{i}{2}\frac{b'}{b}\xi^2}}{(ib)^{1/d-1/2+\ell}} \mathfrak{F}_{d,\ell}\left(e^{\frac{i}{2}\frac{a}{b}x^2} f\right)(\xi/b). \tag{2.11}$$

Involving (2.11), and the fact that the DHT of any function $f \in L^1_{d,\ell}(\mathbb{R})$ belongs to $C_0(\mathbb{R})$ (see [28]), we derive that $\mathfrak{F}^M_{d,\ell}(f)$ belongs to $C_0(\mathbb{R})$ such that

$$\left\|\mathfrak{F}^M_{d,\ell}(f)\right\|_{L^\infty_{d,\ell}} \le c^{1/2}_{\ell,b,d} \|f\|_{L^1_{d,\ell}}, \tag{2.12}$$

where $c_{\ell,b,d} = |b|^{-2\ell-2/d+1}$. Moreover, we have the following Plancherel-type results [28]:

(1) For each $f, g \in L^1_{d,\ell}(\mathbb{R})$,

$$\int_{\mathbb{R}} \mathfrak{F}^M_{d,\ell}(f)(\xi)\overline{g(\xi)}\gamma_{d,\ell}(d\xi) = \int_{\mathbb{R}} f(t)\overline{\mathfrak{F}^{M^{-1}}_{d,\ell}(g)(t)}\gamma_{d,\ell}(dt), \tag{2.13}$$

where $M^{-1} := \begin{pmatrix} b' & -b \\ -a' & a \end{pmatrix}$.

(2) If $f \in L^1_{d,\ell}(\mathbb{R}) \cap L^2_{d,\ell}(\mathbb{R})$, then $\mathfrak{F}^M_{d,\ell}(f) \in L^2_{d,\ell}(\mathbb{R})$ with

$$\|\mathfrak{F}^M_{d,\ell}(f)\|_{L^2_{d,\ell}} = \|f\|_{L^2_{d,\ell}}. \tag{2.14}$$

(3) The LCDHT has a unique extension to an isometric isomorphism of $L^2_{d,\ell}(\mathbb{R})$.

(4) For each $f, g \in L^2_{d,\ell}(\mathbb{R})$,

$$\langle \mathfrak{F}^M_{d,\ell}(f), g\rangle_{L^2_{d,\ell}} = \langle f, \mathfrak{F}^{M^{-1}}_{d,\ell}(g)\rangle_{L^2_{d,\ell}}. \tag{2.15}$$

(5) For each $f \in L^1_{d,\ell}(\mathbb{R})$ such that $\mathfrak{F}^M_{d,\ell}(f) \in L^1_{d,\ell}(\mathbb{R})$,

$$\left(\mathfrak{F}^{M^{-1}}_{d,\ell} \circ \mathfrak{F}^M_{d,\ell}\right)(f) = \left(\mathfrak{F}^M_{d,\ell} \circ \mathfrak{F}^{M^{-1}}_{d,\ell}\right)(f) = r_d(f), \qquad a.e, \tag{2.16}$$

where $r_d(f)(t) := f\left((-1)^d t\right)$.

Let $\Delta^M_{d,\ell}$ be the operator defined by

$$\Delta^M_{d,\ell} := |\xi|^{2(1-\frac{1}{d})} \left\{\frac{d^2}{dx^2} + \left(\frac{2\ell}{\xi} - 2i\frac{b'}{b}\xi\right)\frac{d}{dx} - \left(\frac{b'^2}{b^2}\xi^2 + (2\ell+1)i\frac{b'}{b} + \frac{\ell}{\xi^2}(1-s)\right)\right\}, \tag{2.17}$$

where $s(u(\xi)) := u(-\xi)$. Then,

(1) We have

$$e^{-\frac{i}{2}\frac{b'}{b}\xi^2} \circ \Delta^M_{d,\ell} \circ e^{\frac{i}{2}\frac{b'}{b}\xi^2} = \Delta_{d,\ell} + |\xi|^{\frac{2}{d}}, \tag{2.18}$$

where $\Delta_{d,\ell} := |\xi|^{2-\frac{2}{d}}\Delta_\ell - |\xi|^{\frac{2}{d}}$ is the deformed Laplace operator.

(2) For every f, g in the Schwartz space $\mathcal{S}(\mathbb{R})$,

$$\int_{\mathbb{R}} \Delta^M_{d,\ell} f(\xi)\overline{g(\xi)}\gamma_{d,\ell}(d\xi) = \int_{\mathbb{R}} f(\xi)\overline{\Delta^M_{d,\ell} g(\xi)}\gamma_{d,\ell}(d\xi). \tag{2.19}$$

(3) For each $x \in \mathbb{R}$, the kernel $D^M_{d,\ell}(\cdot, x)$ satisfies

$$\begin{cases} \Delta^M_{d,\ell} D^M_{d,\ell}(\cdot, x) = -\left|\frac{x}{b}\right|^{\frac{2}{d}} D^M_{d,\ell}(\cdot, x), \\ D^M_{d,\ell}(0, x) = e^{\frac{i}{2}\frac{a}{b}x^2}. \end{cases} \tag{2.20}$$

(4) For each $x, \xi \in \mathbb{R}$,

$$\left|D^M_{d,\ell}(\xi, x)\right| \le 1. \tag{2.21}$$

Let Π_s, $s \in \mathbb{R}$, and δ_s be the operators defined by

$$\Pi_s f(x) = e^{\frac{is}{2}x^2} f(x), \qquad \delta_s f(x) = \frac{1}{|s|^{\ell-1/2+1/d}} f(x/s), \quad s \neq 0. \tag{2.22}$$

Then,

$$\delta_s \circ \mathfrak{F}_{d,\ell} = \mathfrak{F}_{d,\ell} \circ \delta_{\frac{1}{s}}, \qquad e^{i(\ell-1/2+1/d)\frac{\pi}{2}\operatorname{sgn}(b)}\mathfrak{F}^M_{d,\ell} = \Pi_{\frac{b'}{b}} \circ \delta_b \circ \mathfrak{F}_{d,\ell} \circ \Pi_{\frac{a}{b}}. \tag{2.23}$$

We define the LCDHT on $L^p_{d,\ell}(\mathbb{R})$, $p \in [1, 2]$, by

$$\mathfrak{F}^M_{d,\ell} = e^{-i\operatorname{sgn}(b)(\ell-1/2+1/d)\frac{\pi}{2}}\left(\Pi_{\frac{b'}{b}} \circ \delta_b \circ \mathfrak{F}_{d,\ell} \circ \Pi_{\frac{a}{b}}\right),$$

where $\mathfrak{F}_{d,\ell} : L^p_{d,\ell}(\mathbb{R}) \to L^{p'}_{d,\ell}(\mathbb{R})$ is the DHT on $L^p_{d,\ell}(\mathbb{R})$. Then, we have a Young-type inequality

$$\left\|\mathfrak{F}^M_{d,\ell}(f)\right\|_{L^{p'}_{d,\ell}} \leq c^{-1/p'+1/2}_{\ell,b,d}\|f\|_{L^p_{d,\ell}}, \qquad p \in [1,2], \tag{2.24}$$

where q' is the conjugate exponent of $q \in [1,\infty]$.

The main symbols used in this paper are summarized in the Table 1.

Table 1. List of symbols.

Name	Symbol	Equation
Deformed Hankel transform	$\mathfrak{F}_{d,\ell}(f)$	(2.4)
Deformed Hankel kernel	$e_{d,\ell}$	(2.5)
Linear canonical deformed Hankel transform (LCDHT)	$\mathfrak{F}^M_{d,\ell}$	(2.9)
LCDH kernel	$D^M_{d,\ell}$	(2.10)
Linear canonical deformed Hankel multiplier	$\mathfrak{M}^M_{\varsigma}$	(3.1)
Linear canonical deformed Hankel wavelet multiplier	$\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)$	(3.11)
Time-frequency limiting operator	$Q_{R_2}P^M_R Q_{R_1}$	(5.9)
Generalized projection operators	Q_R and P^M_R	(5.8)
Time-frequency limiting operator	$\Pi^M_{R_1,R_2}$	(1.23)

3. Linear canonical deformed Hankel wavelet multipliers

We introduce the two-wavelet multiplier operator in the linear canonical deformed Hankel setting. Knowing the fact that the study of this operator is both theoretically interesting and practically useful, we investigated several subjects of spectral analysis for the new operator. First, we present a comprehensive analysis of the generalized two-wavelet multiplier operator. Next, we introduce and study the generalized Landau-Pollak-Slepian operator. As applications, some problems of the approximation theory and the uncertainty principles are studied. Finally, we give many results on the boundedness and compactness of the LC deformed Hankel wavelet multipliers on $L^p_{d,\ell}(\mathbb{R})$, $1 \leq p \leq \infty$.

For $\varsigma \in L^\infty_{d,\ell}(\mathbb{R})$, the linear canonical deformed Hankel multiplier $\mathfrak{M}^M_\varsigma : L^2_{d,\ell}(\mathbb{R}) \to L^2_{d,\ell}(\mathbb{R})$ is given by

$$\mathfrak{M}^M_\varsigma(f) = \mathfrak{F}^{M^{-1}}_{d,\ell}\left(\varsigma\mathfrak{F}^M_{d,\ell}(f)\right). \tag{3.1}$$

Then, by Eq (2.14), it is bounded with

$$\left\|\mathfrak{M}^M_\varsigma\right\|_{S_\infty} \leq \|\varsigma\|_{L^\infty_{d,\ell}}. \tag{3.2}$$

In the following we give two examples of the linear canonical deformed Hankel multiplier.

Example 3.1. *Let $r > 0$ and $t \in \mathbb{R}$. Consider the functions*

$$\varsigma(t) = \left(2|b|^{\frac{2d-2}{d}}\right)^{D(\ell,d)} e^{-\frac{d^2}{4r|b|^{2/d}}|t|^{\frac{2}{d}}} \tag{3.3}$$

and

$$f(t) = e^{-i\frac{a}{2b}t^2}e^{-r|t|^{2/d}}, \tag{3.4}$$

where $D(\ell, d) := \frac{(2\ell-1)d+2}{2}$.

First, we recall that

$$\mathfrak{F}_{d,\ell}\left(e^{-r|\cdot|^{2/d}}\right)(y) = \left(\frac{d}{2r}\right)^{D(\ell,d)} e^{-\frac{d^2}{4r}|y|^{2/d}}. \tag{3.5}$$

Then, in view of (2.11) *and* (3.1)*, we obtain*

$$\mathfrak{M}_{\varsigma}^{M}(f)(\lambda) = \left(\frac{d}{r|b|^{2/d}}\right)^{D(\ell,d)} e^{-i\frac{a}{2b}\lambda^2} \mathfrak{F}_{d,\ell}\left(e^{-\frac{d^2}{2r|b|^{2/d}}|\cdot|^{2/d}}\right)\left(\frac{\lambda}{b}\right). \tag{3.6}$$

Thus, by (3.5)*, we derive*

$$\mathfrak{M}_{\varsigma}^{M}(f)(\lambda) = e^{-\frac{r}{2}|\lambda|^{2/d}} e^{-i\frac{a}{2b}\lambda^2}. \tag{3.7}$$

Figure 1:

(a) Real part of Example 3.1 for $r = 2$, $a = 1$, $b = 1$, and $\lambda \in [-5, 5]$, respectively, for $d = 1, 2, 3, 4, 5$.

(b) Imaginary part of $\mathfrak{M}_{\varsigma}^{M}(f)$ of Example 3.1 for $r = 2$, $a = 1$, $b = 1$, and $\lambda \in [-5, 5]$, respectively, for $d = 1, 2, 3, 4, 5$.

(c) Absolute value of (a).

(d) absolute value of (b).

(e) Modulus of $\mathfrak{M}_{\varsigma}^{M}(f)$ of Example 3.1 for $r = 2$, $a = 1$, $b = 1$, and $\lambda \in [-5, 5]$, respectively, for $d = 1, 2, 3, 4, 5$.

(f) Graph of $\mathfrak{M}_{\varsigma}^{M}(f)$ in the complex plane of Example 3.1 for $r = 2$, $a = 1$, $b = 1$, and $\lambda \in [-5, 5]$, respectively, for $d = 1, 2, 3, 4, 5$.

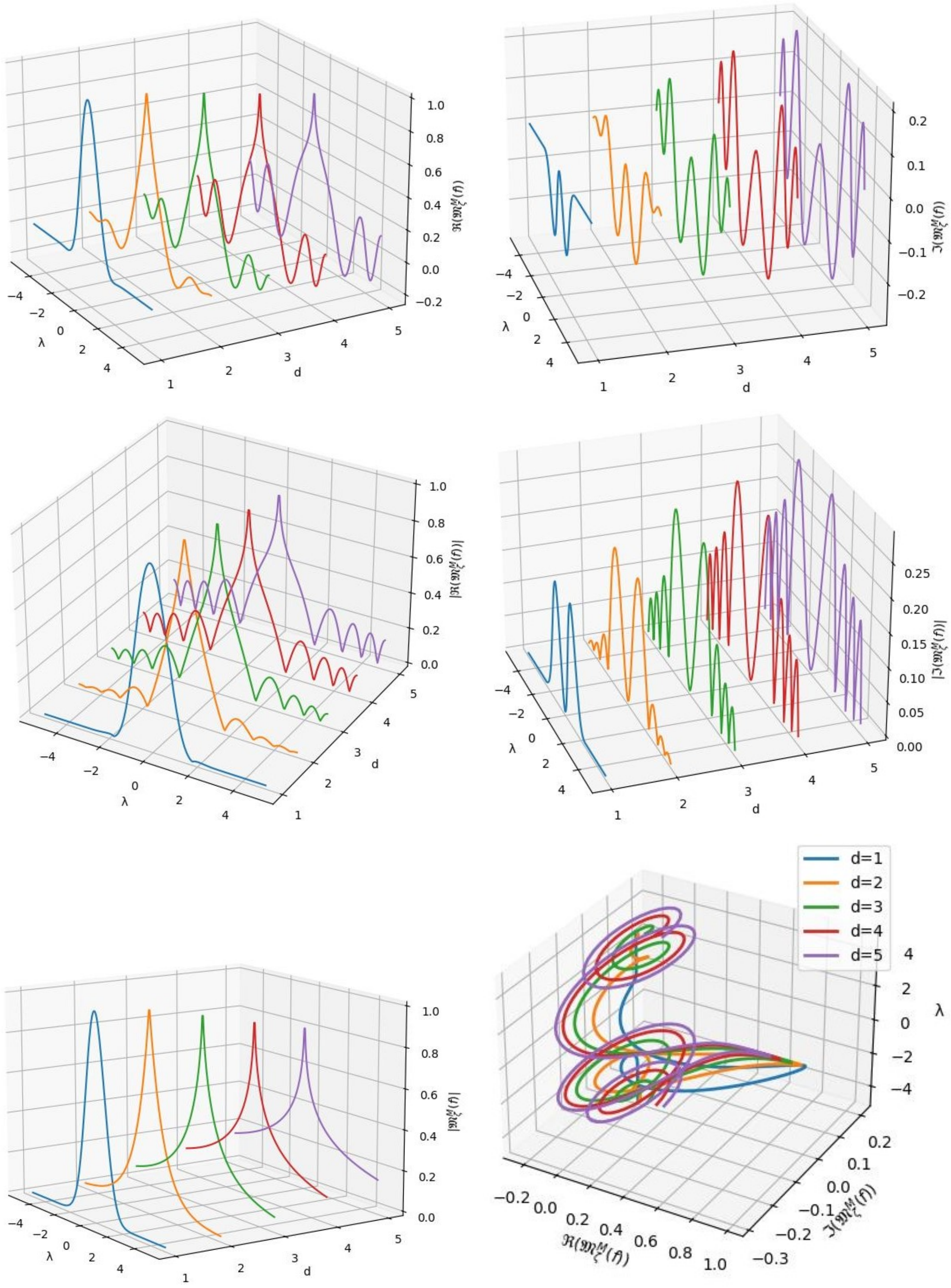

Figure 1. Example 3.1.

Example 3.2. *Let $r > 0$ and $t \in \mathbb{R}$. Consider the functions*

$$\varsigma(t) = \left(|b|^{\frac{2d-2}{d}}\right)^{D(\ell,d)} \Gamma(D(\ell,d))\mathbf{1}_{[-1,1]}(t), \quad \text{and} \quad f(t) = e^{-i\frac{a}{2b}t^2} e^{-r|t|^{2/d}}. \tag{3.8}$$

Involving (3.1) *and* (3.5)*, we get*

$$\mathfrak{M}_{\varsigma}^{M}(f)(\lambda) = \left(\frac{d}{2r|b|^{2/d}}\right)^{D(\ell,d)} \Gamma(D(\ell,d))e^{-i\frac{a}{2b}\lambda^2}\mathfrak{F}_{d,\ell}\left(\mathbf{1}_{[-1,1]}e^{-\frac{d^2}{4r}|.|^{2/d}}\right)\left(\frac{\lambda}{b}\right)$$

$$= C_{d,\ell}\left(\frac{d}{2r|b|^{\frac{2}{d}}}\right)^{D(\ell,d)} \Gamma(D(\ell,d))e^{-i\frac{a}{2b}\lambda^2}\left(\int_{-1}^{1} j_{ld-\frac{d}{2}}\left(d\frac{|\lambda t|^{\frac{1}{d}}}{|b|^{\frac{1}{d}}}\right)e^{-\frac{d^2}{4r|b|^{\frac{2}{d}}}|t|^{\frac{2}{d}}}|t|^{\frac{(2\ell-2)d+2}{d}}dt\right).$$

By a suitable change of variables, we infer that

$$\mathfrak{M}_{\varsigma}^{M}(f)(\lambda) = \mathfrak{I}_{\ell,d,r,b}(\lambda)e^{-i\frac{a}{2b}\lambda^2}, \tag{3.9}$$

where

$$\mathfrak{I}_{\ell,d,r,b}(\lambda) := \int_{0}^{\frac{d}{2\sqrt{r}|b|^{1/d}}} j_{\ell d-\frac{d}{2}}\left(2\sqrt{r}x|\lambda|^{1/d}\right)e^{-x^2}x^{(2\ell-1)d+1}dx. \tag{3.10}$$

Figure 2: (a) Real part and (b) imaginary parts of $\mathfrak{M}_{\varsigma}^{M}(f)$ in Example 3.2 for $r = 2$, $a = 1$, $b = 1$, $\ell = \frac{1}{2}$, and $\lambda \in [-5, 5]$, respectively, for $d = 1, 2, 3, 4, 5$; (c) Absolute value of (a); (d) Absolute value of (b).

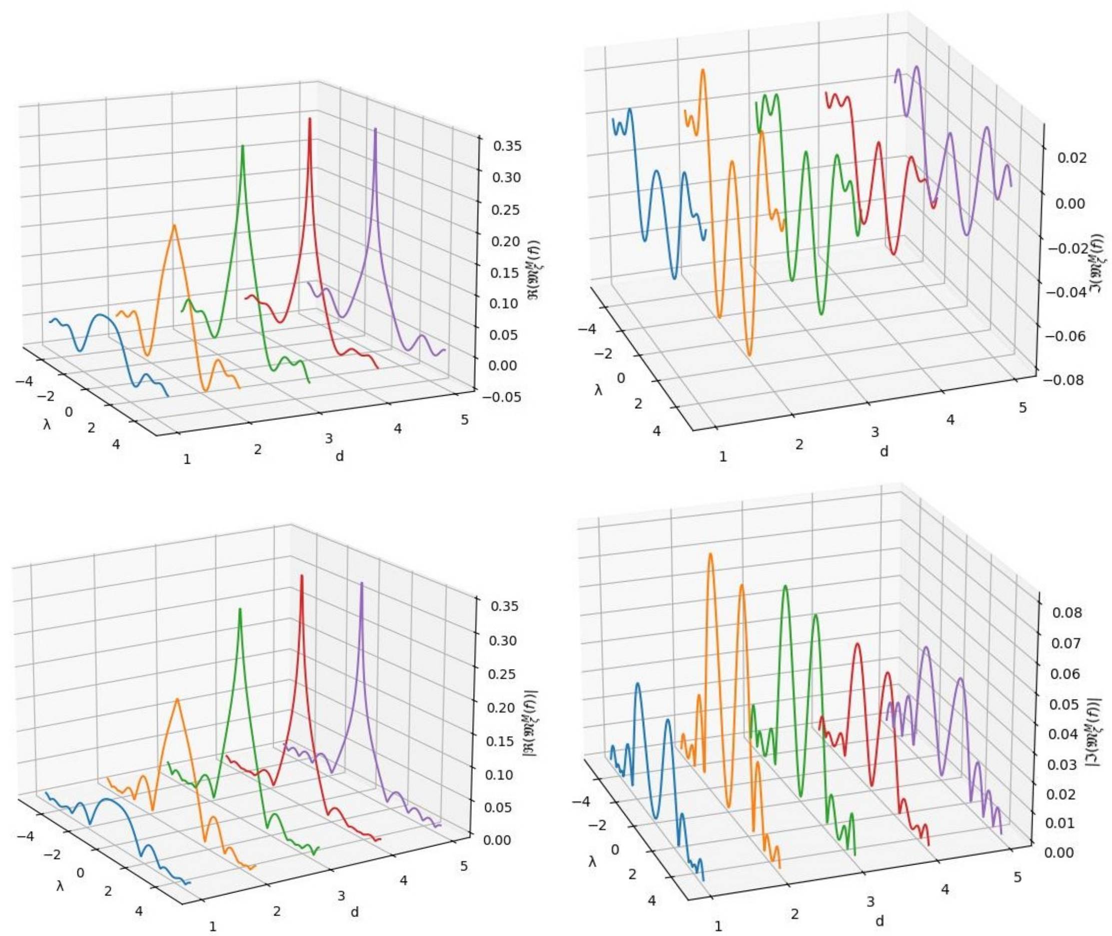

Figure 2. Example 3.2.

Figure 3: (a) Modulus of $\mathfrak{M}_{\varsigma}^{M}(f)$ of Example 3.2 for $r = 2$, $a = 1$, $b = 1$, $\ell = \frac{1}{2}$, and $\lambda \in [-5, 5]$, respectively, for $d = 1, 2, 3, 4, 5$. (b) Modulus of $\mathfrak{M}_{\varsigma}^{M}(f)$ of Example 3.2 for $r = 2$, $a = 1$, $b = 1$, $\ell = \frac{1}{2}$, and $\lambda \in [-5, 5]$, respectively, for $d = 1, 2, 3, 4, 5$. (c) Modulus of $\mathfrak{M}_{\varsigma}^{M}(f)$ of Example 3.2 for $r = 2$, $a = 1$, $b = 1$, $\lambda \in [-5, 5]$, $d = 1$, and $\ell \in [\frac{1}{2}, 5]$. (d) Real part of $\mathfrak{M}_{\varsigma}^{M}(f)$ of Example 3.2 for $r = 2$, $a = 1$, $b = 1$, $\lambda \in [-5, 5]$, $d = 1$, and $\ell \in [\frac{1}{2}, 5]$. (e) Imaginary part of $\mathfrak{M}_{\varsigma}^{M}(f)$ of Example 3.2 for $r = 2$, $a = 1$, $b = 1$, $\lambda \in [-5, 5]$, $d = 1$, and $\ell \in [\frac{1}{2}, 5]$. (f) Modulus of $\mathfrak{M}_{\varsigma}^{M}(f)$ of Example 3.2 for $r = 2$, $a = 1$, $b = 1$, $\lambda \in [-5, 5]$, $\ell = 1$, and $d \in [1, 5]$.

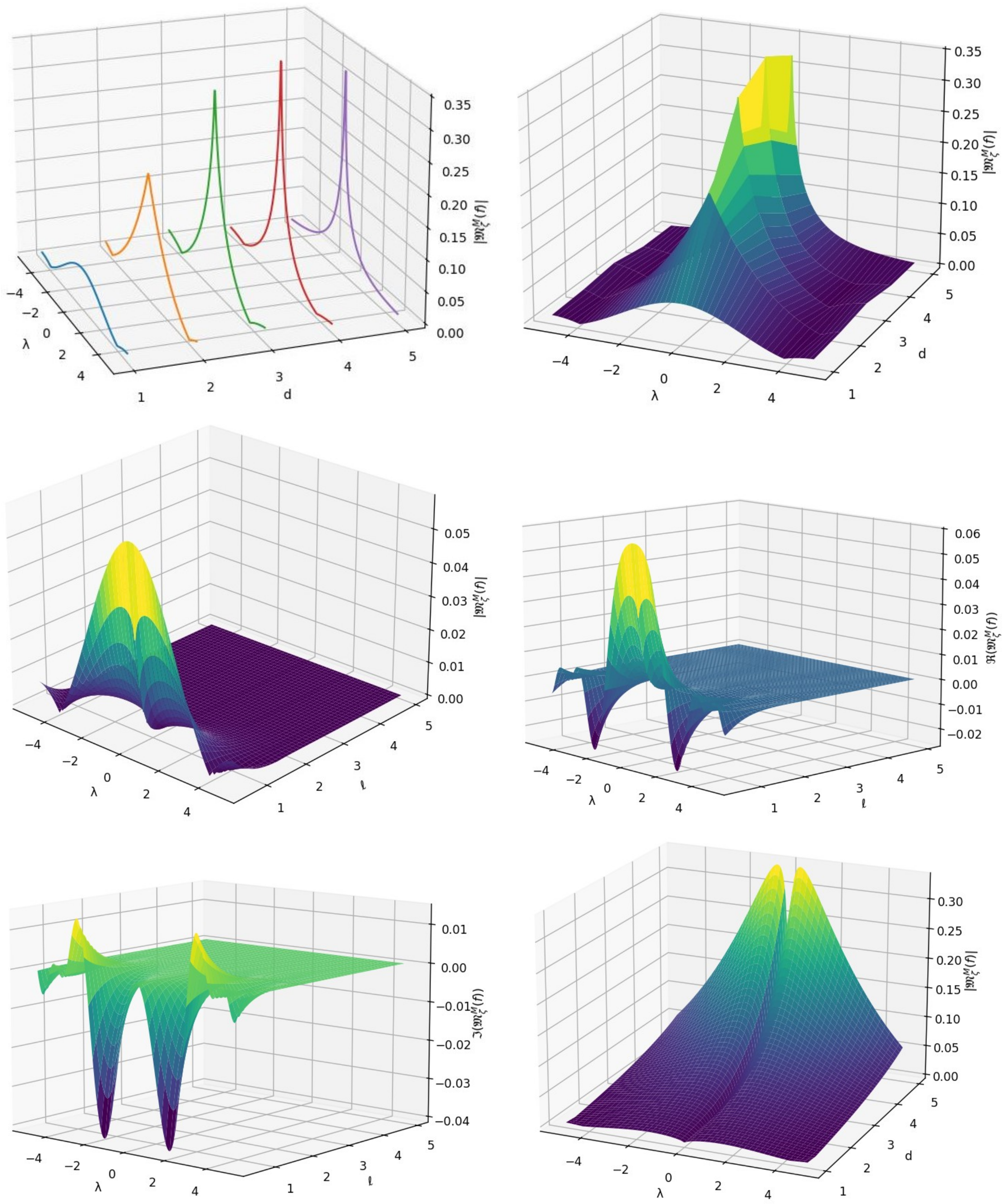

Figure 3. Example 3.2.

Let $p \in [1, \infty]$ and let φ_1, φ_2, and ς be measurable functions on $\mathbb{R}$. Then, the linear canonical deformed Hankel wavelet multiplier operator is defined on $L^p_{d,\ell}(\mathbb{R})$ by

$$\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)(f)(t) = \frac{1}{(-ib)^{1/d-1/2+\ell}} \int_{\mathbb{R}} \varsigma(\xi)\mathfrak{F}^M_{d,\ell}(\varphi_1 f)(\xi)\overline{\varphi_2(t)D^M_{d,\ell}(\xi,t)}\gamma_{d,\ell}(d\xi), \qquad t \in \mathbb{R}. \tag{3.11}$$

To be well defined on $L^p_{d,\ell}(\mathbb{R})$, some conditions on ς, φ_1, and φ_2 are required, and, in a weak sense, it can be written as

$$\langle \mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)(f_1), f_2\rangle_{L^2_{d,\ell}} = \int_{\mathbb{R}} \varsigma(\xi)\mathfrak{F}^M_{d,\ell}(\varphi_1 f_1)(\xi)\overline{\mathfrak{F}^M_{d,\ell}(\varphi_2 f_2)(\xi)}\gamma_{d,\ell}(d\xi), \tag{3.12}$$

for every $f_1 \in L^p_{d,\ell}(\mathbb{R})$ and $f_2 \in L^{p'}_{d,\ell}(\mathbb{R})$. The adjoint $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma) : L^p_{d,\ell}(\mathbb{R}) \to L^p_{d,\ell}(\mathbb{R})$, $p \in [1,\infty)$, is defined on $L^{p'}_{d,\ell}(\mathbb{R})$ onto itself by

$$\left(\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)\right)^* = \mathfrak{P}^M_{\varphi_2,\varphi_1}(\overline{\varsigma}). \tag{3.13}$$

Proposition 1. *Let $\varsigma \in L^1_{d,\ell}(\mathbb{R}) \cup L^\infty_{d,\ell}(\mathbb{R})$ and let $\varphi_1, \varphi_2 \in L^\infty_{d,\ell}(\mathbb{R}) \cap L^2_{d,\ell}(\mathbb{R})$. Then,*

$$\left\langle \mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)(f_1), f_2\right\rangle_{L^2_{d,\ell}} = \left\langle \overline{\varphi}_2 \mathfrak{M}^M_{\varsigma}(\varphi_1 f_1), f_2\right\rangle_{L^2_{d,\ell}}. \tag{3.14}$$

Proof. From (2.13), (3.12), and (3.1), we have

$$\begin{aligned}
\langle \mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)(f_1), f_2\rangle_{L^2_{d,\ell}} &= \int_{\mathbb{R}} \varsigma(\xi)\mathfrak{F}^M_{d,\ell}(\varphi_1 f_1)(\xi)\overline{\mathfrak{F}^M_{d,\ell}(\varphi_2 f_2)(\xi)}\gamma_{d,\ell}(d\xi) \\
&= \int_{\mathbb{R}} \mathfrak{F}^M_{d,\ell}(\mathfrak{M}^M_{\varsigma}(\varphi_1 f_1))(\xi)\overline{\mathfrak{F}^M_{d,\ell}(\varphi_2 f_2)(\xi)}\gamma_{d,\ell}(d\xi) \\
&= \int_{\mathbb{R}} \mathfrak{M}^M_{\varsigma}(\varphi_1 f_1)(x)\overline{(\varphi_2 f_2)(x)}\gamma_{d,\ell}(dx) = \langle \bar{\varphi}_2 \mathfrak{M}^M_{\varsigma}(\varphi_1 f_1), f_2\rangle_{L^2_{d,\ell}},
\end{aligned}$$

as desired. □

In the remainder of this section, we will assume that $\varphi_1, \varphi_2 \in L^2_{d,\ell}(\mathbb{R}) \cap L^\infty_{d,\ell}(\mathbb{R})$ with $\|\varphi_1\|_{L^2_{d,\ell}} = \|\varphi_2\|_{L^2_{d,\ell}} = 1$.

3.1. Study of L^2-boundedness

In this subsection, we study the boundedness of the linear operator $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma) : L^2_{d,\ell}(\mathbb{R}) \to L^2_{d,\ell}(\mathbb{R})$ with respect to the symbol ς.

Proposition 2. *If ς is in $L^1_{d,\ell}(\mathbb{R})$, then the linear canonical deformed Hankel wavelet multiplier $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)$ is in S_∞ such that*

$$\left\|\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)\right\|_{S_\infty} \leqslant c_{\ell,b,d}\|\varsigma\|_{L^1_{d,\ell}}. \tag{3.15}$$

Proof. For every $f_1, f_2 \in L^2_{d,\ell}(\mathbb{R})$, by (3.12), we have

$$\left|\langle \mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)(f_1), f_2\rangle_{L^2_{d,\ell}}\right| \leq \int_{\mathbb{R}} |\varsigma(\xi)| \left|\mathfrak{F}^M_{d,\ell}(\varphi_1 f_1)(\xi)\mathfrak{F}^M_{d,\ell}(\varphi_2 f_2)(\xi)\right| \gamma_{d,\ell}(d\xi)$$

$$\leqslant \|\mathfrak{F}_{d,\ell}^M(\varphi_1 f_1)\|_{L^\infty_{d,\ell}}\|\mathfrak{F}_{d,\ell}^M(\varphi_2 f_2)\|_{L^\infty_{d,\ell}}\|\varsigma\|_{L^1_{d,\ell}}.$$

Then, by (2.12),

$$\|\mathfrak{F}_{d,\ell}^M(\varphi_1 f)\|_{L^\infty_{d,\ell}} \leq c_{\ell,b,d}^{1/2}\|\varphi_1\|_{L^2_{d,\ell}}\|f_1\|_{L^2_{d,\ell}},$$

and

$$\|\mathfrak{F}_{d,\ell}^M(\varphi_2 f_2)\|_{L^\infty_{d,\ell}} \leq c_{\ell,b,d}^{1/2}\|\varphi_2\|_{L^2_{d,\ell}}\|f_2\|_{L^2_{d,\ell}}.$$

Hence,

$$\left|\left\langle \mathfrak{P}_{\varphi_1,\varphi_2}^M(\varsigma)(f_1), f_2\right\rangle_{L^2_{d,\ell}}\right| \leqslant c_{\ell,b,d}\|f_1\|_{L^2_{d,\ell}}\|f_2\|_{L^2_{d,\ell}}\|\varsigma\|_{L^1_{d,\ell}}.$$

Thus, $\left\|\mathfrak{P}_{\varphi_1,\varphi_2}^M(\varsigma)\right\|_{S_\infty} \leqslant c_{\ell,b,d}\|\varsigma\|_{L^1_{d,\ell}}$. □

Proposition 3. *If $\varsigma \in L^\infty_{d,\ell}(\mathbb{R})$, then the linear canonical deformed Hankel wavelet multiplier operator $\mathfrak{P}_{\varphi_1,\varphi_2}^M(\varsigma)$ is in S_∞, with*

$$\left\|\mathfrak{P}_{\varphi_1,\varphi_2}^M(\varsigma)\right\|_{S_\infty} \leqslant \|\varphi_2\|_{L^\infty_{d,\ell}}\|\varphi_1\|_{L^\infty_{d,\ell}}\|\varsigma\|_{L^\infty_{d,\ell}}. \tag{3.16}$$

Proof. We have

$$\begin{aligned}\left|\left\langle \mathfrak{P}_{\varphi_1,\varphi_2}^M(\varsigma)(f_1), f_2\right\rangle_{L^2_{d,\ell}}\right| &\leqslant \int_{\mathbb{R}} |\varsigma(\xi)|\left|\mathfrak{F}_{d,\ell}^M(\varphi_1 f_1)(\xi)\right|\left|\overline{\mathfrak{F}_{d,\ell}^M(\varphi_2 f_2)(\xi)}\right|\gamma_{d,\ell}(d\xi)\\ &\leqslant \|\varsigma\|_{L^\infty_{d,\ell}}\|\mathfrak{F}_{d,\ell}^M(\varphi_1 f_1)\|_{L^2_{d,\ell}}\|\mathfrak{F}_{d,\ell}^M(\varphi_2 f_2)\|_{L^2_{d,\ell}}.\end{aligned}$$

Using Plancherel's formula (2.14), we get

$$|\langle \mathfrak{P}_{\varphi_1,\varphi_2}^M(\varsigma)(f_1), f_2\rangle_{L^2_{d,\ell}}| \leqslant \|\varphi_1\|_{L^\infty_{d,\ell}}\|\varsigma\|_{L^\infty_{d,\ell}}\|\varphi_2\|_{L^\infty_{d,\ell}}\|f_1\|_{L^2_{d,\ell}}\|f_2\|_{L^2_{d,\ell}}.$$

Thus, $\|\mathfrak{P}_{\varphi_1,\varphi_2}^M(\varsigma)\|_{S_\infty} \leqslant \|\varphi_1\|_{L^\infty_{d,\ell}}\|\varsigma\|_{L^\infty_{d,\ell}}\|\varphi_2\|_{L^\infty_{d,\ell}}$. □

Remark 1. *If $\varsigma = \varsigma_1 + \varsigma_\infty \in L^1_{d,\ell}(\mathbb{R}) + L^\infty_{d,\ell}(\mathbb{R})$, then from Propositions 2 and 3, the operator $\mathfrak{P}_{\varphi_1,\varphi_2}^M(\varsigma)$ belongs to S_∞, with*

$$\begin{aligned}\|\mathfrak{P}_{\varphi_1,\varphi_2}^M(\varsigma)\|_{S_\infty} &\leqslant c_{\ell,b,d}\|\varsigma_1\|_{L^1_{d,\ell}} + \|\varphi_1\|_{L^\infty_{d,\ell}}\|\varsigma_\infty\|_{L^\infty_{d,\ell}}\|\varphi_2\|_{L^\infty_{d,\ell}}\\ &\leqslant \max\left\{c_{\ell,b,d}, \|\varphi_2\|_{L^\infty_{d,\ell}}\|\varphi_1\|_{L^\infty_{d,\ell}}\right\}\|\varsigma\|,\end{aligned}$$

where

$$\|\varsigma\| = \inf\left\{\|\varsigma_1\|_{L^1_{d,\ell}} + \|\varsigma_\infty\|_{L^\infty_{d,\ell}},\ \varsigma = \varsigma_1 + \varsigma_\infty,\ \varsigma_1 \in L^1_{d,\ell}(\mathbb{R}),\ \varsigma_\infty \in L^\infty_{d,\ell}(\mathbb{R})\right\}. \tag{3.17}$$

From the previous study, $\mathfrak{P}_{\varphi_1,\varphi_2}^M(\varsigma) : L^2_{d,\ell}(\mathbb{R}) \to L^2_{d,\ell}(\mathbb{R})$ is well defined for each symbol ς in $L^p_{d,\ell}(\mathbb{R})$, $1 \leq p \leq \infty$, and belongs to S_∞.

Theorem 1. *If ς is in $L^p_{d,\ell}(\mathbb{R})$, $1 \leq p \leq \infty$, then $\mathfrak{P}_{\varphi_1,\varphi_2}^M(\varsigma) : L^2_{d,\ell}(\mathbb{R}) \to L^2_{d,\ell}(\mathbb{R})$ is in S_∞, such that*

$$\left\|\mathfrak{P}_{\varphi_1,\varphi_2}^M(\varsigma)\right\|_{S_\infty} \leqslant c_{\ell,b,d}^{1/p}\left(\|\varphi_2\|_{L^\infty_{d,\ell}}\|\varphi_1\|_{L^\infty_{d,\ell}}\right)^{\frac{p-1}{p}}\|\varsigma\|_{L^p_{d,\ell}}. \tag{3.18}$$

Proof. For $f \in L^2_{d,\ell}(\mathbb{R})$, define the operator $\mathcal{T}^M : L^1_{d,\ell}(\mathbb{R}) \cap L^\infty_{d,\ell}(\mathbb{R}) \to L^2_{d,\ell}(\mathbb{R})$ by

$$\mathcal{T}^M(\varsigma) := \mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)(f).$$

Then, by (3.15) and (3.16),

$$\|\mathcal{T}^M(\varsigma)\|_{L^2_{d,\ell}} \leqslant c_{\ell,b,d}\|f\|_{L^2_{d,\ell}}\|\varsigma\|_{L^1_{d,\ell}} \tag{3.19}$$

and

$$\|\mathcal{T}^M(\varsigma)\|_{L^2_{d,\ell}} \leq \|\varphi_2\|_{L^\infty_{d,\ell}}\|\varphi_1\|_{L^\infty_{d,\ell}}\|f\|_{L^2_{d,\ell}}\|\varsigma\|_{L^\infty_{d,\ell}}. \tag{3.20}$$

Therefore, by (3.19), (3.20), and [6, Theorem 2.11], $\mathcal{T}^M$ has a unique extension to a linear operator on $L^p_{d,\ell}(\mathbb{R})$, $p \in [1,\infty]$, and

$$\begin{aligned}\|\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)(f)\|_{L^2_{d,\ell}} &= \|\mathcal{T}^M(\varsigma)\|_{L^2_{d,\ell}} \\ &\leq c^{1/p}_{\ell,b,d}(\|\varphi_2\|_{L^\infty_{d,\ell}}\|\varphi_1\|_{L^\infty_{d,\ell}})^{\frac{p-1}{p}}\|f\|_{L^2_{d,\ell}}\|\varsigma\|_{L^p_{d,\ell}},\end{aligned} \tag{3.21}$$

which implies the result. □

3.2. *Study of L^2-compactness and schatten class*

In this subsection, we study the compactness of the operator $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma) : L^2_{d,\ell}(\mathbb{R}) \to L^2_{d,\ell}(\mathbb{R})$. More precisely, we will show that it belongs to S_p.

Proposition 4. *If $\varsigma \in L^1_{d,\ell}(\mathbb{R})$, then $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)$ is in S_2, with*

$$\left\|\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)\right\|_{S_2} \leqslant c_{\ell,b,d}\|\varsigma\|_{L^1_{d,\ell}}. \tag{3.22}$$

Proof. If $\{\phi_j\}$ is an orthonormal basis for $L^2_{d,\ell}(\mathbb{R})$, then by (3.12) and (3.13),

$$\begin{aligned}&\sum_{j\in\mathbb{N}}\left\|\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)(\phi_j)\right\|^2_{L^2_{d,\ell}} = \sum_{j\in\mathbb{N}}\left\langle\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)(\phi_j), \mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)(\phi_j)\right\rangle_{L^2_{d,\ell}} \\ =\ & c_{\ell,b,d}\sum_{j\in\mathbb{N}}\int_{\mathbb{R}}\varsigma(\xi)\left\langle\phi_j, \overline{\varphi_1 D^M_{d,\ell}(\xi,\cdot)}\right\rangle_{L^2_{d,\ell}}\overline{\left\langle\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)(\phi_j), \overline{\varphi_2 D^M_{d,\ell}(\xi,\cdot)}\right\rangle}_{L^2_{d,\ell}}\gamma_{d,\ell}(d\xi) \\ =\ & c_{\ell,b,d}\int_{\mathbb{R}}\varsigma(\xi)\sum_{j\in\mathbb{N}}\left\langle\left(\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)\right)^*(\overline{\varphi_2 D^M_{d,\ell}(\xi,\cdot)}), \phi_j\right\rangle_{L^2_{d,\ell}}\left\langle\phi_j, \overline{\varphi_1 D^M_{d,\ell}(\xi,\cdot)}\right\rangle_{L^2_{d,\ell}}\gamma_{d,\ell}(d\xi) \\ =\ & c_{\ell,b,d}\int_{\mathbb{R}}\varsigma(\xi)\left\langle\left(\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)\right)^*(\overline{\varphi_2 D^M_{d,\ell}(\xi,\cdot)}), \overline{\varphi_1 D^M_{d,\ell}(\xi,\cdot)}\right\rangle_{L^2_{d,\ell}}\gamma_{d,\ell}(d\xi).\end{aligned}$$

Thus, from (2.21) and (3.15),

$$\sum_{j\in\mathbb{N}}\left\|\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)(\phi_j)\right\|^2_{L^2_{d,\ell}} \leq c_{\ell,b,d}\int_{\mathbb{R}}|\varsigma(\xi)|\left\|(\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma))^*\right\|_{S_\infty}\gamma_{d,\ell}(d\xi) \tag{3.23}$$

$$\leq c^2_{\ell,b,d}\|\varsigma\|^2_{L^1_{d,\ell}} < \infty. \tag{3.24}$$

Hence, by (3.23) and [6, Proposition 2.8], the operator $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma) : L^2_{d,\ell}(\mathbb{R}) \to L^2_{d,\ell}(\mathbb{R})$ is in S_2. □

Proposition 5. *If $\varsigma \in L^p_{d,\ell}(\mathbb{R})$, $1 \leqslant p < \infty$, then $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)$ is compact.*

Proof. Given (ς_j) a sequence in $L^1_{d,\ell}(\mathbb{R}) \cap L^\infty_{d,\ell}(\mathbb{R})$, where $\varsigma_j \to \varsigma$ in $L^p_{d,\ell}(\mathbb{R})$, Then by (3.18),

$$\left\|\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma_j) - \mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)\right\|_{S_\infty} \leq c^{1/p}_{\ell,b,d}(\|\varphi_2\|_{L^\infty_{d,\ell}}\|\varphi_1\|_{L^\infty_{d,\ell}})^{\frac{p-1}{p}}\|\varsigma_j - \varsigma\|_{L^p_{d,\ell}}.$$

Hence, $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma_j) \to \mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)$ in S_∞. Moreover, since by (3.22) $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma_j)$ is compact, then $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)$ is compact. □

Theorem 2. *If $\varsigma \in L^1_{d,\ell}(\mathbb{R})$, then $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma) : L^2_{d,\ell}(\mathbb{R}) \to L^2_{d,\ell}(\mathbb{R})$ is in S_1, with*

$$\frac{2c_{\ell,b,d}}{\|\varphi_1\|^2_{L^\infty_{d,\ell}} + \|\varphi_2\|^2_{L^\infty_{d,\ell}}}\|\widetilde{\varsigma}\|_{L^1_{d,\ell}} \leqslant \|\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)\|_{S_1} \leqslant c_{\ell,b,d}\|\varsigma\|_{L^1_{d,\ell}}, \tag{3.25}$$

where $\widetilde{\varsigma}$ is given by

$$\widetilde{\varsigma}(\xi) = \left\langle \mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)\, D^M_{d,\ell}(\xi,\cdot)\varphi_1,\, D^M_{d,\ell}(\xi,\cdot)\varphi_2 \right\rangle_{L^2_{d,\ell}}, \qquad \xi \in \mathbb{R}. \tag{3.26}$$

Proof. Since $\varsigma \in L^1_{d,\ell}(\mathbb{R})$, then by Proposition 4, $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma) \in S_2$. From [6, Theorem 2.2], there is $\{\psi_j\}$ an orthonormal sequence in $L^2_{d,\ell}(\mathbb{R})$ and an orthonormal basis $\{\phi_j\}$ for the orthogonal complement of the kernel of the operator $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)$, which consists of eigenvectors of $|\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)|$, satisfying

$$\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)(f) = \sum_{j\in\mathbb{N}} e_j\langle f, \phi_j\rangle_{L^2_{d,\ell}}\psi_j. \tag{3.27}$$

Here, $\{e_j\}$ is the sequence of nonnegative singular values of $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)$ which corresponds to $\{\phi_j\}$. It follows that

$$\|\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)\|_{S_1} = \sum_{j\in\mathbb{N}} e_j = \sum_{j\in\mathbb{N}}\langle \mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)(\phi_j), \psi_j\rangle_{L^2_{d,\ell}}. \tag{3.28}$$

Then, by Bessel's inequality and (2.21),

$$\begin{aligned}
\|\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)\|_{S_1} &= \sum_{j\in\mathbb{N}}\langle \mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)(\phi_j), \psi_j\rangle_{L^2_{d,\ell}} \\
&= \sum_{j\in\mathbb{N}}\int_{\mathbb{R}} \varsigma(\xi)\mathfrak{F}^M_{d,\ell}(\varphi_1\phi_j)(\xi)\overline{\mathfrak{F}^M_{d,\ell}(\varphi_2\psi_j)(\xi)}\gamma_{d,\ell}(d\xi) \\
&= c_{\ell,b,d}\int_{\mathbb{R}} \varsigma(\xi)\sum_{j\in\mathbb{N}}\left\langle \phi_j, \overline{\varphi_1 D^M_{d,\ell}(\xi,\cdot)}\right\rangle_{L^2_{d,\ell}}\left\langle \overline{\varphi_2 D^M_{d,\ell}(\xi,\cdot)}, \psi_j\right\rangle_{L^2_{d,\ell}}\gamma_{d,\ell}(d\xi) \\
&\leq c_{\ell,b,d}\int_{\mathbb{R}} |\varsigma(\xi)|\left(\sum_{j\in\mathbb{N}}\left|\left\langle \phi_j, \overline{\varphi_1 D^M_{d,\ell}(\xi,\cdot)}\right\rangle_{L^2_{d,\ell}}\right|^2\right)^{\frac{1}{2}}\left(\sum_{j\in\mathbb{N}}\left|\left\langle \overline{\varphi_2 D^M_{d,\ell}(\xi,\cdot)}, \psi_j\right\rangle_{L^2_{d,\ell}}\right|^2\right)^{\frac{1}{2}}\gamma_{d,\ell}(d\xi) \\
&\leq c_{\ell,b,d}\int_{\mathbb{R}} |\varsigma(\xi)|\left\|\overline{\varphi_1 D^M_{d,\ell}(\xi,\cdot)}\right\|_{L^2_{d,\ell}}\left\|\overline{\varphi_2 D^M_{d,\ell}(\xi,\cdot)}\right\|_{L^2_{d,\ell}}\gamma_{d,\ell}(d\xi) \leq c_{\ell,b,d}\|\varsigma\|_{L^1_{d,\ell}}.
\end{aligned}$$

In addition, $\widetilde{\varsigma}$ belongs to $L^1_{d,\ell}(\mathbb{R})$, and by (3.27),

$$|\widetilde{\varsigma}(\xi)| = \left|\left\langle \mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)\left(D^M_{d,\ell}(\xi,\cdot)\varphi_1\right), D^M_{d,\ell}(\xi,\cdot)\varphi_2\right\rangle_{L^2_{d,\ell}}\right|$$

$$= \left| \sum_{j\in\mathbb{N}} e_j \left\langle D^M_{d,\ell}(\xi,\cdot)\varphi_1, \phi_j \right\rangle_{L^2_{d,\ell}} \left\langle \psi_j, D^M_{d,\ell}(\xi,\cdot)\varphi_2 \right\rangle_{L^2_{d,\ell}} \right|$$
$$\leqslant \frac{1}{2} \sum_{j\in\mathbb{N}} e_j \left(\left| \left\langle D^M_{d,\ell}(\xi,\cdot)\varphi_1, \phi_j \right\rangle_{L^2_{d,\ell}} \right|^2 + \left| \left\langle D^M_{d,\ell}(\xi,\cdot)\varphi_2, \psi_j \right\rangle_{L^2_{d,\ell}} \right|^2 \right).$$

Then, by (2.14),

$$\int_{\mathbb{R}} |\widetilde{\varsigma}(\xi)| \gamma_{d,\ell}(d\xi) = \frac{1}{2} \sum_{j\in\mathbb{N}} e_j \left(\int_{\mathbb{R}} \left| \left\langle D^M_{d,\ell}(\xi,\cdot)\varphi_1, \phi_j \right\rangle_{L^2_{d,\ell}} \right|^2 \gamma_{d,\ell}(d\xi) + \int_{\mathbb{R}} \left| \left\langle D^M_{d,\ell}(\xi,\cdot)\varphi_2, \psi_j \right\rangle_{L^2_{d,\ell}} \right|^2 \gamma_{d,\ell}(d\xi) \right).$$

Thus,

$$\begin{aligned} \int_{\mathbb{R}} |\widetilde{\varsigma}(\xi)| \gamma_{d,\ell}(d\xi) &\leq \frac{\|\varphi_1\|^2_{L^\infty_{d,\ell}} + \|\varphi_2\|^2_{L^\infty_{d,\ell}}}{2c_{\ell,b,d}} \sum_{j\in\mathbb{N}} e_j \\ &= \frac{\|\varphi_1\|^2_{L^\infty_{d,\ell}} + \|\varphi_2\|^2_{L^\infty_{d,\ell}}}{2c_{\ell,b,d}} \left\| \mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma) \right\|_{S_1}. \end{aligned}$$

As desired. □

Corollary 1. *For $\varsigma \in L^1_{d,\ell}(\mathbb{R})$, we have*

$$\operatorname{tr}\left(\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)\right) = c_{\ell,b,d} \int_{\mathbb{R}} \varsigma(\xi) \left\langle \overline{\varphi_2 D^M_{d,\ell}(\xi,\cdot)}, \overline{\varphi_1 D^M_{d,\ell}(\xi,\cdot)} \right\rangle_{L^2_{d,\ell}} \gamma_{d,\ell}(d\xi). \tag{3.29}$$

Proof. From (3.25), $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)$ is in S_1. It follows that,

$$\begin{aligned} \operatorname{tr}\left(\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)\right) &= \sum_{j\in\mathbb{N}} \left\langle \mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)(\phi_j), \phi_j \right\rangle_{L^2_{d,\ell}} \\ &= c_{\ell,b,d} \sum_{j\in\mathbb{N}} \int_{\mathbb{R}} \varsigma(\xi) \left\langle \phi_j, \overline{D^M_{d,\ell}(\xi,\cdot)\varphi_1} \right\rangle_{L^2_{d,\ell}} \overline{\left\langle \phi_j, \overline{D^M_{d,\ell}(\xi,\cdot)\varphi_2} \right\rangle_{L^2_{d,\ell}}} \gamma_{d,\ell}(d\xi) \\ &= c_{\ell,b,d} \int_{\mathbb{R}} \varsigma(\xi) \sum_{j\in\mathbb{N}} \left\langle \phi_j, \overline{D^M_{d,\ell}(\xi,\cdot)\varphi_1} \right\rangle_{L^2_{d,\ell}} \left\langle \overline{D^M_{d,\ell}(\xi,\cdot)\varphi_2}, \phi_j \right\rangle_{L^2_{d,\ell}} \gamma_{d,\ell}(d\xi) \\ &= c_{\ell,b,d} \int_{\mathbb{R}} \varsigma(\xi) \left\langle \overline{D^M_{d,\ell}(\xi,\cdot)\varphi_2}, \overline{D^M_{d,\ell}(\xi,\cdot)\varphi_1} \right\rangle_{L^2_{d,\ell}} \gamma_{d,\ell}(d\xi), \end{aligned}$$

for every orthonormal basis $\{\phi_j\}$ of $L^2_{d,\ell}(\mathbb{R})$. □

Corollary 2. *If $\varsigma \in L^p_{d,\ell}(\mathbb{R})$, $1 \leqslant p \leqslant \infty$, then $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma) : L^2_{d,\ell}(\mathbb{R}) \longrightarrow L^2_{d,\ell}(\mathbb{R})$ belongs to S_p, with*

$$\|\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)\|_{S_p} \leqslant c^{1/p}_{\ell,b,d} \left(\|\varphi_2\|_{L^\infty_{d,\ell}} \|\varphi_1\|_{L^\infty_{d,\ell}} \right)^{\frac{p-1}{p}} \|\varsigma\|_{L^p_{d,\ell}}. \tag{3.30}$$

Proof. Using Eqs (3.16), (3.25), and, interpolation theorems [6, Theorem 2.10 and Theorem 2.11], we obtain (3.30). □

Specifically, if $\varphi_1 = \varphi_2$ and $\varsigma \in L^1_{d,\ell}(\mathbb{R})$ is nonnegative, then $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma) : L^2_{d,\ell}(\mathbb{R}) \to L^2_{d,\ell}(\mathbb{R})$ is nonnegative, and by (3.29),

$$\left\|\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)\right\|_{S_1} = c_{\ell,b,d} \int_{\mathbb{R}} \varsigma(\xi) \left\|D^M_{d,\ell}(\xi,\cdot)\varphi_1\right\|^2_{L^2_{d,\ell}} \gamma_{d,\ell}(d\xi). \tag{3.31}$$

We denote by:

(1) $B(L^p_{d,\ell}(\mathbb{R}))$, $1 \le p \le \infty$, the space of bounded operators from $L^p_{d,\ell}(\mathbb{R})$ into itself.
(2) $CO(L^p_{d,\ell}(\mathbb{R}))$, $1 \le p \le \infty$, the set of all compact operators from $L^p_{d,\ell}(\mathbb{R})$ into itself.

Then, the main results for boundedness and compactness of $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)$ on $L^2_{d,\ell}(\mathbb{R})$ are summarized in the Table 2.

Table 2. Boundedness and compactness of $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)$ on $L^2_{d,\ell}(\mathbb{R})$.

Symbol	Windows		Operator
ς	φ_1	φ_2	$\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)$
$L^p_{d,\ell}(\mathbb{R})$, $p \in [1,\infty]$	$L^2_{d,\ell}(\mathbb{R})$	$L^2_{d,\ell}(\mathbb{R})$	S_∞
$L^p_{d,\ell}(\mathbb{R})$, $p \in [1,\infty]$	$L^2_{d,\ell}(\mathbb{R})$	$L^2_{d,\ell}(\mathbb{R})$	S_p
$L^p_{d,\ell}(\mathbb{R})$, $p \in [1,\infty)$	$L^2_{d,\ell}(\mathbb{R})$	$L^2_{d,\ell}(\mathbb{R})$	$CO(L^2_{d,\ell}(\mathbb{R}))$

4. Study of L^p-boundedness and compactness

In this section, we prove that $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)$ is bounded and compact on $L^p_{d,\ell}(\mathbb{R})$, $p \in [1,\infty]$, for every $\varphi_1 \in L^{p'}_{d,\ell}(\mathbb{R})$, $\varphi_2 \in L^p_{d,\ell}(\mathbb{R})$, and $\varsigma \in L^r_{d,\ell}(\mathbb{R})$, $1 \le r \le 2$.

Proposition 6. *If $\varphi_1 \in L^\infty_{d,\ell}(\mathbb{R})$, $\varphi_2 \in L^1_{d,\ell}(\mathbb{R})$, and $\varsigma \in L^1_{d,\ell}(\mathbb{R})$, then $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma) : L^1_{d,\ell}(\mathbb{R}) \longrightarrow L^1_{d,\ell}(\mathbb{R})$ is bounded such that*

$$\left\|\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)\right\|_{B(L^1_{d,\ell}(\mathbb{R}))} \leqslant c_{\ell,b,d}\|\varphi_2\|_{L^\infty_{d,\ell}}\|\varphi_1\|_{L^1_{d,\ell}}\|\varsigma\|_{L^1_{d,\ell}}. \tag{4.1}$$

Proof. By (2.12), (2.21), and (3.11), for every $f \in L^1_{d,\ell}(\mathbb{R})$,

$$\begin{aligned}\|\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)(f)\|_{L^1_{d,\ell}} &\leqslant c^{1/2}_{\ell,b,d} \int_{\mathbb{R}}\int_{\mathbb{R}} |\varsigma(\xi)| \left|\mathfrak{F}^M_{d,\ell}(\varphi_1 f)(\xi)\right| \left|D^M_{d,\ell}(\xi,y)\varphi_2(y)\right| \gamma_{d,\ell}(d\xi)\gamma_{d,\ell}(dy)\\ &\leqslant c_{\ell,b,d}\|f\|_{L^1_{d,\ell}}\|\varphi_2\|_{L^\infty_{d,\ell}}\|\varphi_1\|_{L^1_{d,\ell}}\|\varsigma\|_{L^1_{d,\ell}}.\end{aligned}$$

Thus, $\left\|\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)\right\|_{B(L^1_{d,\ell}(\mathbb{R}))} \leqslant c_{\ell,b,d}\|\varphi_2\|_{L^\infty_{d,\ell}}\|\varphi_1\|_{L^1_{d,\ell}}\|\varsigma\|_{L^1_{d,\ell}}$. □

Proposition 7. *If $\varsigma \in L^1_{d,\ell}(\mathbb{R})$, $\varphi_1 \in L^1_{d,\ell}(\mathbb{R})$, and $\varphi_2 \in L^\infty_{d,\ell}(\mathbb{R})$, then $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma) : L^\infty_{d,\ell}(\mathbb{R}) \longrightarrow L^\infty_{d,\ell}(\mathbb{R})$ is bounded such that*

$$\left\|\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)\right\|_{B(L^\infty_{d,\ell}(\mathbb{R}))} \leqslant c_{\ell,b,d}\|\varphi_1\|_{L^1_{d,\ell}}\|\varphi_2\|_{L^\infty_{d,\ell}}\|\varsigma\|_{L^1_{d,\ell}}. \tag{4.2}$$

Proof. Let $f \in L^\infty_{d,\ell}(\mathbb{R})$. By (2.12), (2.21), and (3.11),

$$\left|\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)(f)(y)\right| \leqslant c^{1/2}_{\ell,b,d} \int_{\mathbb{R}} |\varsigma(\xi)|\,|\mathfrak{F}^M_{d,\ell}(\varphi_1 f)(\xi)|\,|D^M_{d,\ell}(\xi,y)\varphi_2(y)|\gamma_{d,\ell}(d\xi)$$

$$\leqslant \quad c_{\ell,b,d}\|f\|_{L^\infty_{d,\ell}}\|\varphi_1\|_{L^1_{d,\ell}}\|\varphi_2\|_{L^\infty_{d,\ell}}\|\varsigma\|_{L^1_{d,\ell}},$$

Then, $\left\|\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)\right\|_{B(L^\infty_{d,\ell}(\mathbb{R}))} \leqslant c_{\ell,b,d}\|\varphi_1\|_{L^1_{d,\ell}}\|\varphi_2\|_{L^\infty_{d,\ell}}\|\varsigma\|_{L^1_{d,\ell}}$. □

Notice that we can prove (4.2), by using (3.13) and (4.1).

Theorem 3. *If $\varsigma \in L^1_{d,\ell}(\mathbb{R})$ and $\varphi_1, \varphi_2 \in L^\infty_{d,\ell}(\mathbb{R}) \cap L^1_{d,\ell}(\mathbb{R})$, then $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma) : L^p_{d,\ell}(\mathbb{R}) \longrightarrow L^p_{d,\ell}(\mathbb{R})$, $1 \leq p \leq \infty$ is bounded, with*

$$\left\|\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)\right\|_{B\left(L^p_{d,\ell}(\mathbb{R})\right)} \leqslant c_{\ell,b,d}\|\varphi_1\|^{\frac{1}{p'}}_{L^1_{d,\ell}}\|\varphi_2\|^{\frac{1}{p}}_{L^1_{d,\ell}}\|\varphi_1\|^{\frac{1}{p}}_{L^\infty_{d,\ell}}\|\varphi_2\|^{\frac{1}{p'}}_{L^\infty_{d,\ell}}\|\varsigma\|_{L^1_{d,\ell}}. \tag{4.3}$$

Moreover $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma) : L^p_{d,\ell}(\mathbb{R}) \longrightarrow L^p_{d,\ell}(\mathbb{R})$ is compact for every $1 \leq p \leq \infty$.

Proof. The boundedness (4.3) follows by interpolating (4.1) and (4.2). On the other hand, to prove the compactness, we first prove that $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma) : L^1_{d,\ell}(\mathbb{R}) \longrightarrow L^1_{d,\ell}(\mathbb{R})$ is compact. For this, let $\{f_j\} \in L^1_{d,\ell}(\mathbb{R})$ be a sequence that converges to zero weakly in $L^1_{d,\ell}(\mathbb{R})$. Then, there is $C > 0$ such that $\|f_j\|_{L^1_{d,\ell}} \leq C$. Moreover, we have

$$\left\|\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)(f_j)\right\|_{L^1_{d,\ell}} \leqslant c_{\ell,b,d}\int_{\mathbb{R}^2}|\varsigma(\xi)|\left|\left\langle f_j, D^M_{d,\ell}(\xi,\cdot)\varphi_1\right\rangle_{L^2_{d,\ell}}\right|\left|D^M_{d,\ell}(-\xi,y)\varphi_2(y)\right|\gamma_{d,\ell}(d\xi)\gamma_{d,\ell}(dy). \tag{4.4}$$

Therefore, for every $\xi, y \in \mathbb{R}$,

$$\lim_{j\to\infty}|\varsigma(\xi)|\left|\left\langle f_j, D^M_{d,\ell}(\xi,\cdot)\varphi_1\right\rangle_{L^2_{d,\ell}}\right|\left|D^M_{d,\ell}(\xi,y)\varphi_2(y)\right| = 0, \tag{4.5}$$

and then

$$|\varsigma(\xi)|\left|\left\langle f_j, D^M_{d,\ell}(\xi,\cdot)\varphi_1\right\rangle_{L^2_{d,\ell}}\right|\left|D^M_{d,\ell}(\xi,y)\varphi_2(y)\right| \leq C|\varsigma(\xi)|\,\|\varphi_1\|_{L^\infty_{d,\ell}}\,|\varphi_2(y)|. \tag{4.6}$$

Now, by (2.21),

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}}|\varsigma(\xi)|\left|\left\langle f_j, D^M_{d,\ell}(\xi,\cdot)\varphi_1\right\rangle_{L^2_{d,\ell}}\right|\left|D^M_{d,\ell}(\xi,y)\varphi_2(y)\right|\gamma_{d,\ell}(d\xi)\gamma_{d,\ell}(dy)\\
\leq\quad& C\|\varphi_1\|_{L^\infty_{d,\ell}}\int_{\mathbb{R}}|\varsigma(\xi)|\int_{\mathbb{R}}|\varphi_2(y)|\gamma_{d,\ell}(dy)\gamma_{d,\ell}(d\xi)\\
\leq\quad& C\|\varphi_1\|_{L^\infty_{d,\ell}}\|\varphi_2\|_{L^1_{d,\ell}}\|\varsigma\|_{L^1_{d,\ell}} < \infty.
\end{aligned} \tag{4.7}$$

Hence, from (4.4)–(4.7) we obtain $\lim_{n\to\infty}\|\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)(f_n)\|_{L^1_{d,\ell}(\mathbb{R})} = 0$, which implies the compactness of $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma) : L^1_{d,\ell}(\mathbb{R}) \longrightarrow L^1_{d,\ell}(\mathbb{R})$. Then, $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma) : L^\infty_{d,\ell}(\mathbb{R}) \longrightarrow L^\infty_{d,\ell}(\mathbb{R})$ becomes compact since it is the adjoint of $\mathfrak{P}^M_{\varphi_2,\varphi_1}(\overline{\varsigma}) : L^1_{d,\ell}(\mathbb{R}) \longrightarrow L^1_{d,\ell}(\mathbb{R})$. Finally, by interpolation [46, pages 202 and 203], we get the desired result. □

In order to give another version of L^p-boundedness, we need the next proposition.

Proposition 8. *Let $\varsigma \in L^1_{d,\ell}(\mathbb{R})$, $\varphi_1 \in L^{p'}_{d,\ell}(\mathbb{R})$, $1 < p \leq \infty$, and $\varphi_2 \in L^p_{d,\ell}(\mathbb{R})$. Then, $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma) : L^p_{d,\ell}(\mathbb{R}) \longrightarrow L^p_{d,\ell}(\mathbb{R})$ is bounded such that*

$$\left\|\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)\right\|_{B\left(L^p_{d,\ell}(\mathbb{R})\right)} \leqslant c_{\ell,b,d}\|\varphi_1\|_{L^{p'}_{d,\ell}}\|\varphi_2\|_{L^p_{d,\ell}}\|\varsigma\|_{L^1_{d,\ell}}. \tag{4.8}$$

Proof. For $f_1 \in L^p_{d,\ell}(\mathbb{R})$, let $\mathcal{I}^M_{f_1} : L^{p'}_{d,\ell}(\mathbb{R}) \to \mathbb{C}$ given by $\mathcal{I}^M_{f_1}(f_2) = \left\langle f_2, \mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)(f_1)\right\rangle_{L^2_{d,\ell}}$. Then, from (3.12),

$$\left|\left\langle \mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)(f_1), f_2\right\rangle_{L^2_{d,\ell}}\right| \leqslant \int_{\mathbb{R}} |\varsigma(\xi)| \left|\mathfrak{F}^M_{d,\ell}(\varphi_1 f_1)(\xi)\right| \left|\mathfrak{F}^M_{d,\ell}(\varphi_2 f_2)(\xi)\right| \gamma_{d,\ell}(d\xi)$$
$$\leqslant \|\varsigma\|_{L^1_{d,\ell}} \left\|\mathfrak{F}^M_{d,\ell}(\varphi_1 f_1)\right\|_{L^\infty_{d,\ell}} \left\|\mathfrak{F}^M_{d,\ell}(\varphi_2 f_2)\right\|_{L^\infty_{d,\ell}}.$$

Using (2.9) and (2.21),

$$\left|\left\langle \mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)(f_1), f_2\right\rangle_{L^2_{d,\ell}}\right| \leqslant c_{\ell,b,d}\|\varsigma\|_{L^1_{d,\ell}}\|\varphi_1\|_{L^{p'}_{d,\ell}}\|\varphi_2\|_{L^p_{d,\ell}}\|f_1\|_{L^p_{d,\ell}}\|f_2\|_{L^{p'}_{d,\ell}}. \tag{4.9}$$

Then, $\mathcal{I}^M_{f_1}$ is continuous on $L^{p'}_{d,\ell}(\mathbb{R})$, such that

$$\left\|\mathcal{I}^M_{f_1}\right\|_{B(L^{p'}_{d,\ell}(\mathbb{R}))} \leqslant c_{\ell,b,d}\|\varphi_1\|_{L^{p'}_{d,\ell}}\|\varphi_2\|_{L^p_{d,\ell}}\|f_1\|_{L^p_{d,\ell}}\|\varsigma\|_{L^1_{d,\ell}}. \tag{4.10}$$

As $\mathcal{I}^M_{f_1}(f_2) = \left\langle f_2, \mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)\right\rangle_{L^2_{d,\ell}}$, then by the Riesz representation theorem,

$$\|\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)(f_1)\|_{L^p_{d,\ell}(\mathbb{R})} = \|\mathcal{I}^M_{f_1}\|_{B(L^{p'}_{d,\ell}(\mathbb{R}))} \leq c_{\ell,b,d}\|\varphi_1\|_{L^{p'}_{d,\ell}}\|\varphi_2\|_{L^p_{d,\ell}}\|f_1\|_{L^p_{d,\ell}}\|\varsigma\|_{L^1_{d,\ell}}, \tag{4.11}$$

as desired. □

Theorem 4. *If $\varsigma \in L^1_{d,\ell}(\mathbb{R})$, $\varphi_1 \in L^{p'}_{d,\ell}(\mathbb{R})$, $p \in [1,\infty]$, and $\varphi_2 \in L^p_{d,\ell}(\mathbb{R})$, then $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma) : L^p_{d,\ell}(\mathbb{R}) \longrightarrow L^p_{d,\ell}(\mathbb{R})$ is bounded such that*

$$\left\|\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)\right\|_{B\left(L^p_{d,\ell}(\mathbb{R})\right)} \leqslant c_{\ell,b,d}\|\varphi_1\|_{L^{p'}_{d,\ell}}\|\varphi_2\|_{L^p_{d,\ell}}\|\varsigma\|_{L^1_{d,\ell}}. \tag{4.12}$$

Moreover, $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma) : L^p_{d,\ell}(\mathbb{R}) \longrightarrow L^p_{d,\ell}(\mathbb{R})$ is compact for every $1 \leq p \leq \infty$.

Proof. The boundedess follows by combining Eqs (4.1) and (4.8). In addition, the compactness follows by interpolation [46, pages 202 and 203], Corollary 2, and the fact that $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma) : L^1_{d,\ell}(\mathbb{R}) \longrightarrow L^1_{d,\ell}(\mathbb{R})$ is compact. □

Theorem 5. *If $\varsigma \in L^r_{d,\ell}(\mathbb{R})$, $1 \leq r \leq 2$, and $\varphi_1, \varphi_2 \in L^1_{d,\ell}(\mathbb{R}) \cap L^\infty_{d,\ell}(\mathbb{R})$, then $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma) : L^p_{d,\ell}(\mathbb{R}) \longrightarrow L^p_{d,\ell}(\mathbb{R})$ is bounded for every $p \in [r, r']$, with*

$$\left\|\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)\right\|_{B\left(L^p_{d,\ell}(\mathbb{R})\right)} \leqslant \mathfrak{C}_1^t \mathfrak{C}_2^{1-t}\|\varsigma\|_{L^r_{d,\ell}}, \tag{4.13}$$

where

$$\mathfrak{C}_1 = \left(c_{\ell,b,d}\|\varphi_2\|_{L^\infty_{d,\ell}}\|\varphi_1\|_{L^1_{d,\ell}}\right)^{\frac{2}{r}-1}\left(c_{\ell,b,d}\|\varphi_2\|_{L^\infty_{d,\ell}}\|\varphi_1\|_{L^\infty_{d,\ell}}\right)^{\frac{1}{r'}}, \tag{4.14}$$

$$\mathfrak{C}_2 = \left(c_{\ell,b,d}\|\varphi_2\|_{L^\infty_{d,\ell}}\|\varphi_1\|_{L^\infty_{d,\ell}}\right)^{\frac{1}{r'}}\left(c_{\ell,b,d}\|\varphi_2\|_{L^\infty_{d,\ell}}\|\varphi_1\|_{L^1_{d,\ell}}\right)^{\frac{2}{r}-1}, \tag{4.15}$$

and

$$\frac{1-t}{r'} + \frac{t}{r} = \frac{1}{p}. \tag{4.16}$$

Moreover, $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma) : L^p_{d,\ell}(\mathbb{R}) \longrightarrow L^p_{d,\ell}(\mathbb{R})$ is compact for every $r \leq p \leq r'$.

Proof. Let $\mathcal{I}^M_{d,\ell}:\left(L^1_{d,\ell}(\mathbb{R})\cap L^2_{d,\ell}(\mathbb{R})\right)\times\left(L^1_{d,\ell}(\mathbb{R})\cap L^2_{d,\ell}(\mathbb{R})\right)\to L^1_{d,\ell}(\mathbb{R})\cap L^2_{d,\ell}(\mathbb{R})$, given by

$$\mathcal{I}^M_{d,\ell}(\varsigma,f)=\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)(f).$$

Then, by (3.18) and (4.1),

$$\|\mathcal{I}^M_{d,\ell}(\varsigma,f)\|_{L^1_{d,\ell}}\leqslant c_{\ell,b,d}\|\varphi_2\|_{L^\infty_{d,\ell}}\|\varphi_1\|_{L^1_{d,\ell}}\|f\|_{L^1_{d,\ell}}\|\varsigma\|_{L^1_{d,\ell}} \tag{4.17}$$

and

$$\|\mathcal{I}^M_{d,\ell}(\varsigma,f)\|_{L^2_{d,\ell}}\leq\sqrt{c_{\ell,b,d}\|\varphi_2\|_{L^\infty_{d,\ell}}\|\varphi_1\|_{L^\infty_{d,\ell}}}\|\varsigma\|_{L^2_{d,\ell}}\|f\|_{L^2_{d,\ell}}. \tag{4.18}$$

Thus, by (4.17), (4.18), and [47, Section 10.1], there exists a unique linear bounded operator $\mathcal{I}^M_{d,\ell}(\varsigma,f)$: $L^r_{d,\ell}(\mathbb{R})\times L^r_{d,\ell}(\mathbb{R})\to L^r_{d,\ell}(\mathbb{R})$, with

$$\|\mathcal{I}^M_{d,\ell}(\varsigma,f)\|_{L^r_{d,\ell}}\leq\mathfrak{C}_1\|f\|_{L^r_{d,\ell}}\|\varsigma\|_{L^r_{d,\ell}}, \tag{4.19}$$

where

$$\mathfrak{C}_1=\left(c_{\ell,b,d}\|\varphi_2\|_{L^\infty_{d,\ell}}\|\varphi_1\|_{L^\infty_{d,\ell}}\right)^{\frac{1-\theta}{2}}\left(c_{\ell,b,d}\|\varphi_2\|_{L^\infty_{d,\ell}}\|\varphi_1\|_{L^1_{d,\ell}}\right)^{\theta},\qquad\frac{1-\theta}{2}+\theta=\frac{1}{r}.$$

It follows that

$$\left\|\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)\right\|_{B(L^r_{d,\ell}(\mathbb{R}))}\leqslant\mathfrak{C}_1\|\varsigma\|_{L^r_{d,\ell}}.$$

Since $\left(\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)\right)^*=\mathfrak{P}^M_{\varphi_2,\varphi_1}(\overline{\varsigma})$, then $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)$ is bounded on $L^{r'}_{d,\ell}(\mathbb{R})$, such that

$$\left\|\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)\right\|_{B(L^{r'}_{d,\ell}(\mathbb{R}))}=\|\mathfrak{P}^M_{\varphi_2,\varphi_1}(\overline{\varsigma})\|_{B(L^r_{d,\ell}(\mathbb{R}))}\leq\mathfrak{C}_2\|\varsigma\|_{L^r_{d,\ell}}, \tag{4.20}$$

where

$$\mathfrak{C}_2=\left(c_{\ell,b,d}\|\varphi_2\|_{L^\infty_{d,\ell}}\|\varphi_1\|_{L^\infty_{d,\ell}}\right)^{\frac{1-\theta}{2}}\left(c_{\ell,b,d}\|\varphi_1\|_{L^1_{d,\ell}}\|\varphi_2\|_{L^\infty_{d,\ell}}\right)^{\theta}.$$

Interpolating Eqs (4.19) and (4.20), for every $p\in[r,r']$, we obtain

$$\left\|\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)\right\|_{B\left(L^p_{d,\ell}(\mathbb{R})\right)}\leqslant\mathfrak{C}_1^t\mathfrak{C}_2^{1-t}\|\varsigma\|_{L^r_{d,\ell}},$$

with $\frac{t}{r}+\frac{1-t}{r'}=\frac{1}{p}$.

On the other hand, the compactness follows by interpolation [46, pages 202 and 203], Corollary 2, and the fact that $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma):L^1_{d,\ell}(\mathbb{R})\longrightarrow L^1_{d,\ell}(\mathbb{R})$ is compact. □

The main results for boundedness of $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)$ on $L^p_{d,\ell}(\mathbb{R})$, $1\leq p\leq\infty$, are summarized in the Table 3.

Table 3. Boundedness of $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)$ on $L^p_{d,\ell}(\mathbb{R})$, $p\in[1,\infty]$.

ς	Windows		Operator
	φ_1	φ_2	$\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)$
$L^1_{d,\ell}(\mathbb{R})$	$L^1_{d,\ell}(\mathbb{R})\cap L^\infty_{d,\ell}(\mathbb{R})$	$L^1_{d,\ell}(\mathbb{R})\cap L^\infty_{d,\ell}(\mathbb{R})$	$B(L^p_{d,\ell}(\mathbb{R}))$, $p\in[1,\infty]$
$L^1_{d,\ell}(\mathbb{R})$	$L^{p'}_{d,\ell}(\mathbb{R})$, $p\in[1,\infty]$	$L^p_{d,\ell}(\mathbb{R})$	$B(L^p_{d,\ell}(\mathbb{R}))$
$L^r_{d,\ell}(\mathbb{R})$, $r\in[1,2]$	$L^1_{d,\ell}(\mathbb{R})\cap L^\infty_{d,\ell}(\mathbb{R})$	$L^1_{d,\ell}(\mathbb{R})\cap L^\infty_{d,\ell}(\mathbb{R})$	$B(L^p_{d,\ell}(\mathbb{R}))$, $p\in[r,r']$.

The main results for compactness of $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)$ on $L^p_{d,\ell}(\mathbb{R})$, $1 \le p \le \infty$, are summarized in the Table 4.

Table 4. Compactness of $\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)$ on $L^p_{d,\ell}(\mathbb{R})$, $p \in [1,\infty]$

ς	Windows		Operator
	φ_1	φ_2	$\mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma)$
$L^1_{d,\ell}(\mathbb{R})$	$L^1_{d,\ell}(\mathbb{R}) \cap L^\infty_{d,\ell}(\mathbb{R})$	$L^1_{d,\ell}(\mathbb{R}) \cap L^\infty_{d,\ell}(\mathbb{R})$	$CO(L^p_{d,\ell}(\mathbb{R}))$, $p \in [1,\infty]$
$L^r_{d,\ell}(\mathbb{R})$, $r \in [1,2]$	$L^1_{d,\ell}(\mathbb{R}) \cap L^\infty_{d,\ell}(\mathbb{R})$	$L^1_{d,\ell}(\mathbb{R}) \cap L^\infty_{d,\ell}(\mathbb{R})$	$CO(L^p_{d,\ell}(\mathbb{R}))$, $p \in [r,r']$

5. Examples and applications

In this section, we will formulate certain typical examples of the linear canonical deformed Hankel wavelet multipliers and give certain applications.

5.1. Examples

By equality (3.11), it is enough to formulate some examples of the LCDHT. Then, inspired by [28], we state the following four examples.

(1) If $M := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, then $\Delta^M_{d,\ell} = \Delta_{d,\ell}$ and the LCDHT is equal with a constant multiplier to the DHT.

(2) For $\alpha \in \mathbb{R}$, let $M = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$. Then, the LCDHT is the Fresnel transform related to the DHT given by:

$$\mathcal{W}^\alpha_{d,\ell} f(\xi) = \begin{cases} \dfrac{1}{(i\alpha)^{1/d-1/2+\ell}} \displaystyle\int_{\mathbb{R}} E^\alpha_{d,\ell}(\xi,t) f(t)\gamma_{d,\ell}(dt), & \alpha \neq 0, \\ f(\xi), & \alpha = 0, \end{cases} \tag{5.1}$$

where

$$E^\alpha_{d,\ell}(\xi,t) = e^{\frac{i}{2\alpha}(\xi^2+t^2)} e_{d,\ell}\left(\frac{\xi}{\alpha}, t\right). \tag{5.2}$$

(3) For $\beta \in \mathbb{R}$, let $M = \begin{pmatrix} \cosh(\beta) & \sinh(\beta) \\ \sinh(\beta) & \cosh(\beta) \end{pmatrix}$. Then, the LCDHT is given by

$$\mathcal{V}^\beta_{d,\ell} f(\xi) = \begin{cases} \dfrac{1}{(i\sinh(\beta))^{1/d-1/2+\ell}} \displaystyle\int_{\mathbb{R}} R^\beta_{d,\ell}(\xi,t) f(t)\gamma_{d,\ell}(dt), & \beta \neq 0, \\ f(\xi), & \beta = 0, \end{cases} \tag{5.3}$$

where

$$R^\beta_{d,\ell}(\xi,t) = e^{\frac{i}{2}\coth(\beta)(\xi^2+t^2)} e_{d,\ell}\left(\frac{\xi}{\sinh(\beta)}, t\right). \tag{5.4}$$

(4) For $\eta \in \mathbb{R}$, let $M = \begin{pmatrix} \cos(\eta) & \sin(\eta) \\ -\sin(\eta) & \cos(\eta) \end{pmatrix}$. Then, the LCDHT is the fractional deformed Hankel transform given by:

$$\mathfrak{F}^{\eta}_{d,\ell} f(\xi) = \begin{cases} c(d,\eta) \displaystyle\int_{\mathbb{R}} \mathcal{K}^{\eta}_{d,\ell}(\xi, t) f(t) \gamma_{d,\ell}(dt), & (2q-1)\pi < \eta < (2q+1)\pi, \\ f(\xi), & \eta = 2q\pi, \\ f(-\xi), & \eta = (2q+1)\pi, \end{cases} \tag{5.5}$$

where

$$c(d,\eta) = \frac{e^{i(\ell - 1/2 + 1/d)\left(\eta - 2d\pi - \frac{\pi}{2}\operatorname{sgn}(\sin(\eta))\right)}}{|\sin(\eta)|^{\ell - 1/2 + 1/d}} \tag{5.6}$$

and

$$\mathcal{K}^{\eta}_{d,\ell}(\xi, t) = e^{-\frac{i}{2}\cot(\eta)(\xi^2 + t^2)} e_{d,\ell}\left(\frac{\xi}{\sin(\eta)}, t\right). \tag{5.7}$$

(5) We introduce the well-known time-frequency limiting operator $Q_{R_2} P^M_R Q_{R_1} : L^2_{d,\ell}(\mathbb{R}) \longrightarrow L^2_{d,\ell}(\mathbb{R})$, where for a subset $R \subset \mathbb{R}$ of finite measure, $0 < \gamma_d(R) < \infty$, the self-adjoint projections $Q_R : L^2_{d,\ell}(\mathbb{R}) \longrightarrow L^2_{d,\ell}(\mathbb{R})$ and $P^M_R : L^2_{d,\ell}(\mathbb{R}) \longrightarrow L^2_{d,\ell}(\mathbb{R})$ are defined by

$$Q_R f = \mathbf{1}_R f, \qquad P^M_R f = \mathfrak{F}^{M^{-1}}_{d,\ell}\Big(Q_R\, \mathfrak{F}^M_{d,\ell}(f)\Big). \tag{5.8}$$

Then, if we take $\varphi_1 = \mathbf{1}_{R_1}$ and $\varphi_2 = \mathbf{1}_{R_2}$, by some calculations we prove that

$$Q_{R_2} P^M_R Q_{R_1} = \mathfrak{P}^M_{\varphi_1,\varphi_2}(\varsigma), \qquad \varsigma = \mathbf{1}_R. \tag{5.9}$$

Indeed, by (2.15), for all $u, v \in L^2_{d,\ell}(\mathbb{R})$,

$$\begin{aligned} \left\langle Q_{R_2} P^M_R Q_{R_1} u, v \right\rangle_{L^2_{d,\ell}} &= \left\langle P^M_R Q_{R_1} u, \varphi_2 v \right\rangle_{L^2_{d,\ell}} \\ &= \left\langle \mathfrak{F}^M_{d,\ell}\left(P^M_R Q_{R_1} u\right), \mathfrak{F}^M_{d,\ell}(\varphi_2 v) \right\rangle_{L^2_{d,\ell}} \\ &= \left\langle \mathbf{1}_R\, \mathfrak{F}^M_{d,\ell}(\varphi_1 u), \mathfrak{F}^M_{d,\ell}(\varphi_2 v) \right\rangle_{L^2_{d,\ell}} \\ &= \left\langle \overline{\varphi}_2 \mathfrak{F}^{M^{-1}}_{d,\ell}\left(\mathbf{1}_R\, \mathfrak{F}^M_{d,\ell}(\varphi_1 u)\right), \varphi_2 v \right\rangle_{L^2_{d,\ell}}. \end{aligned}$$

Thus, by (3.14), we have the desired result.

The main extensions and new results, which have not been studied in the literature, that arise from this article are summarized in Table 5.

Table 5. Generalization for the wavelet multipliers for the known integral transforms.

Matrix M	Parameters		Conclusion and References
	d	ℓ	
$M \in SL(2,\mathbb{R})$	$d=1$	$\ell=0$	$\mathfrak{F}^M_{d,\ell} \equiv \mathfrak{F}^M$. Our results cover on $\mathbb{R}$ the wavelet multipliers for the LCT.
$M=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$	$d \in \mathbb{N}$	$\ell \geq \frac{1}{2}$	$\mathfrak{F}^M_{d,\ell} \equiv \mathfrak{F}_{d,\ell}$. Our results generalize on $\mathbb{R}$ the wavelet multipliers for the $(\ell, \frac{2}{d})$-generalized Fourier transform [11, 14].
$M=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$	$d=2$	$\ell \geq \frac{1}{2}$	$\mathfrak{F}^M_{d,\ell} \equiv \mathfrak{F}_{2,\ell}$. Our results generalize the wavelet multipliers for the Bessel transform [10, 12].
$M=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$	$d=1$	$\ell \geq 0$	$\mathfrak{F}^M_{d,\ell} \equiv D_\ell$. Our results generalize on $\mathbb{R}$ the wavelet multipliers for the Dunkl transform [7].
$M=\begin{pmatrix}1 & \alpha\\ 0 & 1\end{pmatrix}, \alpha \in \mathbb{R}$	$d \in \mathbb{N}$	$\ell \geq \frac{1}{2}$	$\mathfrak{F}^M_{d,\ell} \equiv \mathcal{W}^\alpha_{d,\ell}$. Our results cover the wavelet multipliers for the generalized Fresnel transform.
$M=\begin{pmatrix}1 & \alpha\\ 0 & 1\end{pmatrix}, \alpha \in \mathbb{R}$	$d=1$	$\ell=0$	$\mathfrak{F}^M_{d,\ell} \equiv \mathcal{W}^\alpha_{d,\ell}$. 0ur results cover the wavelet multipliers theory for the usual Fresnel transform.
$M=\begin{pmatrix}\cos(\eta) & -\sin(\eta)\\ \sin(\eta) & \cos(\eta)\end{pmatrix}, \eta \in \mathbb{R}$	$d \in \mathbb{N}$	$\ell \geq \frac{1}{2}$	$\mathfrak{F}^M_{d,\ell} \equiv \mathfrak{F}^\eta_{d,\ell}$. Our results cover the wavelet multipliers for the deformed fractional Fourier transform.
$M=\begin{pmatrix}\cos(\eta) & -\sin(\eta)\\ \sin(\eta) & \cos(\eta)\end{pmatrix}, \eta \in \mathbb{R}$	$d=1$	$\ell \geq 0$	$\mathfrak{F}^M_{d,\ell} \equiv \mathfrak{F}^\eta_{d,\ell}$. Our results cover the wavelet multipliers for the fractional Dunkl transform.
$M=\begin{pmatrix}\cos(\eta) & -\sin(\eta)\\ \sin(\eta) & \cos(\eta)\end{pmatrix}, \eta \in \mathbb{R}$	$d=2$	$\ell \geq \frac{1}{2}$	$\mathfrak{F}^M_{d,\ell} \equiv \mathfrak{F}^\eta_{d,\ell}$. Our results cover the wavelet multipliers for the fractional Bessel transform.
$M=\begin{pmatrix}\cos(\eta) & -\sin(\eta)\\ \sin(\eta) & \cos(\eta)\end{pmatrix}, \eta \in \mathbb{R}$	$d=1$	$\ell=0$	$\mathfrak{F}^M_{d,\ell} \equiv \mathfrak{F}^\eta_{d,\ell}$. Our results cover the wavelet multipliers for the usual fractional Fourier transform.

5.2. Uncertainty and approximation relations

The uncertainty principle fundamentally defines the limits of simultaneous time and frequency localization. In signal reconstruction, this is not a crippling barrier, but a critical design parameter. By respecting its limitations, engineers and researchers have developed sophisticated techniques like compressed sensing, wavelet analysis, and graph-based signal processing. These methods exploit the inherent trade-offs to achieve robust and efficient reconstruction, even in the presence of incomplete data and noise.

We recall in the following points how the uncertainty principle applies:

(1) Trade-off between domains: The principle mathematically represents the fundamental trade-off in signal processing: A signal cannot be both very short in time and very narrow in frequency. The more concentrated a signal is in one domain (e.g., time), the more spread out it must be in the other (e.g., frequency) [48].

(2) Guarantees for recovery: In signal reconstruction, uncertainty principles are used to establish conditions under which a unique and accurate reconstruction can be achieved from incomplete data. For example, Donoho and Stark showed that the generalized uncertainty principle can prove that a signal can be recovered from missing information if it is sparse in a certain basis [49].

(3) Foundation for compressed sensing: This is particularly relevant for compressed sensing, where a signal is sampled at a rate lower than the Nyquist rate. The uncertainty principle, in the context of sparse signals, provides the theoretical basis for why a signal can still be perfectly reconstructed from these few samples through optimization [50].

(4) Design of transforms and dictionaries: The principle is also crucial for designing transforms, such as the Fourier transform or more advanced transforms like the linear canonical transform, that are used to represent signals. The goal is often to find "dictionaries" of basis functions that are well-localized in both time and frequency, and the uncertainty principle helps to quantify the best possible localization achievable [51].

(5) Graph signal processing: The application extends to more complex data structures, such as graphs. In graph signal reconstruction, uncertainty principles are used to define the properties of signals on graphs and to guide the development of reconstruction algorithms that can recover signals even when only a small subset of vertices is known [52].

Profiting off our study in the first part of this paper, in the following theorem we give some uncertainty principles for the LCDHT.

Let $\ell_{R_1}^M := \mathfrak{P}_{\varphi_1,\varphi_2}^M(\mathbf{1}_{R_1})$, $\ell_{R_2}^M := \mathfrak{P}_{\varphi_1,\varphi_2}^M(\mathbf{1}_{R_2})$, where $\varphi_1, \varphi_2 \in L^2_{d,\ell}(\mathbb{R}) \cap L^\infty_{d,\ell}(\mathbb{R})$ are two unit L^2-norm functions, with

$$\|\varphi_2\|_{L^\infty_{d,\ell}} \|\varphi_1\|_{L^\infty_{d,\ell}} = 1. \tag{5.10}$$

Theorem 6.

(1) Assume that $\varepsilon_1 + \varepsilon_2 < 1$. If $f \in L^2_{d,\ell}(\mathbb{R})$ is ε_1-concentrated with regard to $\ell_{R_1}^M$ and ε_2-concentrated with regard to $\ell_{R_2}^M$, then

$$\gamma_{d,\ell}(R_1)\gamma_{d,\ell}(R_2) \geq |b|^{4(\ell+1/d)-2}(1 - \varepsilon_1 - \varepsilon_2). \tag{5.11}$$

(2) If $\gamma_{d,\ell}(R_1)\gamma_{d,\ell}(R_2) < |b|^{2\ell+2/d-1}$, then for every $f \in L^2_{d,\ell}(\mathbb{R})$,

$$\|f\|^2_{L^2_{d,\ell}} \leq \left(1 - \sqrt{c_{\ell,b,d}\gamma_{d,\ell}(R_1)\gamma_{d,\ell}(R_2)}\right)^{-2} \left(\left\|Q_{R_1^c} f\right\|^2_{L^2_{d,\ell}} + \left\|P^M_{R_2^c} f\right\|^2_{L^2_{d,\ell}} \right). \tag{5.12}$$

Specifically, if $f \in L^2_{d,\ell}(\varepsilon_1, \varepsilon_2, R_1, R_2)$, then

$$\gamma_{d,\ell}(R_1)\gamma_{d,\ell}(R_2) \geq |b|^{2\ell+2/d-1}\left(1 - \sqrt{\varepsilon_1^2 + \varepsilon_2^2}\right)^2. \tag{5.13}$$

Proof. By (3.16) and (5.10),

$$\begin{aligned} \|f - \ell^M_{R_2}\ell^M_{R_1} f\|_{L^2_{d,\ell}} &\leq \|f - \ell^M_{R_2} f\|_{L^2_{d,\ell}} + \|\ell^M_{R_2} f - \ell^M_{R_2}\ell^M_{R_1} f\|_{L^2_{d,\ell}} \\ &\leq \|\ell^M_{R_2} f - f\|_{L^2_{d,\ell}} + \|\ell^M_{R_2}\|_{S_\infty}\|\ell^M_{R_1} f - f\|_{L^2_{d,\ell}} \\ &\leq (\varepsilon_1 + \varepsilon_2)\|f\|_{L^2_{d,\ell}}. \end{aligned}$$

Then,

$$\|\ell^M_{R_2}\ell^M_{R_1} f\|_{L^2_{d,\ell}} \geq \|f\|_{L^2_{d,\ell}} - \|f - \ell^M_{R_2}\ell^M_{R_1} f\|_{L^2_{d,\ell}} \geq (1 - \varepsilon_1 - \varepsilon_2)\|f\|_{L^2_{d,\ell}}.$$

Therefore, by (3.15),

$$\begin{aligned} 1 - \varepsilon_1 - \varepsilon_2 &\leq \|\ell^M_{R_2}\ell^M_{R_1}\|_{S_\infty} \leq \|\ell^M_{R_1}\|_{S_\infty}\|\ell^M_{R_2}\|_{S_\infty} \\ &\leq c^2_{\ell,b,d}\gamma_{d,\ell}(R_1)\gamma_{d,\ell}(R_2). \end{aligned}$$

This proves the first part. On the other hand, we will use this well-known inequality [53, Lemma 4.1], that is, if $\left\|Q_{R_1} P^M_{R_2}\right\|_{L^2_{d,\ell}} < 1$, then for every $f \in L^2_{d,\ell}(\mathbb{R})$,

$$\|f\|^2_{L^2_{d,\ell}} \leq \left(1 - \left\|Q_{R_1} P^M_{R_2}\right\|_{S_\infty}\right)^{-2}\left(\left\|Q_{R_1^c} f\right\|^2_{L^2_{d,\ell}} + \left\|P^M_{R_2^c} f\right\|^2_{L^2_{d,\ell}}\right). \tag{5.14}$$

Indeed,

$$\begin{aligned} Q_{R_1} P^M_{R_2}(y) &= \frac{1}{(-ib)^{1/d-1/2+\ell}}\mathbf{1}_{R_1}(y)\int_{\mathbb{R}} \mathbf{1}_{R_2}(\eta)\mathfrak{F}^M_{d,\ell}(f)(\eta)\overline{D^M_{d,\ell}((-1)^d\eta, y)}\gamma_{d,\ell}(d\eta) \\ &= \frac{1}{b^{2/d-1+2\ell}}\mathbf{1}_{R_1}(y)\int_{\mathbb{R}} \mathbf{1}_{R_2}(\eta)\left(\int_{\mathbb{R}} f(x)D^M_{d,\ell}(\eta, x)\gamma_{d,\ell}(dx)\right)\overline{D^M_{d,\ell}((-1)^d\eta, y)}\gamma_{d,\ell}(d\eta) \\ &= \int_{\mathbb{R}} f(x)K(x, y)\gamma_{d,\ell}(dx), \end{aligned}$$

where

$$\begin{aligned} K(x, y) &= \frac{1}{b^{2/d-1+2\ell}}\mathbf{1}_{R_1}(y)\int_{\mathbb{R}} \mathbf{1}_{R_2}(\eta)D^M_{d,\ell}(\eta, x)\overline{D^M_{d,\ell}((-1)^d\eta, y)}\gamma_{d,\ell}(d\eta) \\ &= \frac{e^{\frac{i}{2}\frac{a-b'}{b}x^2}}{b^{2/d-1+2\ell}}\mathbf{1}_{R_1}(y)\int_{\mathbb{R}} \mathbf{1}_{R_2}(\eta)e^{\frac{i}{2}\frac{b'-a}{b}\eta^2}D^M_{d,\ell}(x, \eta)\overline{D^M_{d,\ell}((-1)^d\eta, y)}\gamma_{d,\ell}(d\eta) \\ &= \frac{e^{\frac{i}{2}\frac{a-b'}{b}x^2}}{(-ib)^{1/d-1/2+\ell}}\mathbf{1}_{R_1}(y)\mathfrak{F}^M_{d,\ell}\left(e^{\frac{i}{2}\frac{b'-a}{b}|\cdot|^2}\mathbf{1}_{R_2}(\cdot)\overline{D^M_{d,\ell}((-1)^d\,.\,, y)}\right)(x). \end{aligned}$$

Then, by (2.14) and (2.21),

$$\|Q_{R_1} P^M_{R_2}\|^2_{HS} = \|K\|^2_{L^2_{d,\ell}(\mathbb{R})\otimes L^2_{d,\ell}(\mathbb{R})}$$

$$
\begin{aligned}
&= c_{\ell,b,d}\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\mathbf{1}_{R_1}(y)\right|^2\left|\mathfrak{F}^M_{d,\ell}\left(e^{\frac{i}{2}\frac{b'-a}{b}|\cdot|^2}\mathbf{1}_{R_2}(\cdot)\overline{D^M_{d,\ell}((-1)^d\,.\,,y)}\right)(x)\right|^2\gamma_{d,\ell}(dx)\gamma_{d,\ell}(dy)\\
&= c_{\ell,b,d}\int_{\mathbb{R}}\left|\mathbf{1}_{R_1}(y)\right|\int_{\mathbb{R}}\left|e^{\frac{i}{2}\frac{b'-a}{b}x^2}\mathbf{1}_{R_2}(x)\overline{D^M_{d,\ell}((-1)^d x,y)}\right|^2\gamma_{d,\ell}(dx)\gamma_{d,\ell}(dy)\\
&\le c_{\ell,b,d}\int_{\mathbb{R}}\left|\mathbf{1}_{R_1}(y)\right|\int_{\mathbb{R}}\left|\mathbf{1}_{R_2}(x)\right|\gamma_{d,\ell}(dx)\gamma_{d,\ell}(dy)\\
&\le c_{\ell,b,d}\gamma_{d,\ell}(R_1)\gamma_{d,\ell}(R_2).
\end{aligned}
$$

Now, since

$$
\|Q_{R_1}P^M_{R_2}\|_{S_\infty}\le\|Q_{R_1}P^M_{R_2}\|_{HS}\le\sqrt{c_{\ell,b,d}\gamma_{d,\ell}(R_1)\gamma_{d,\ell}(R_2)},\tag{5.15}
$$

then by (5.14) we have the result. Inequality (5.13) is a simple consequence. □

Notice that (5.12) can be written by Plancherel formula (2.14): If $\gamma_{d,\ell}(R_1)\gamma_{d,\ell}(R_2)<|b|^{2\ell+2/d-1}$, then, for every $f\in L^2_{d,\ell}(\mathbb{R})$,

$$
\|f\|^2_{L^2_{d,\ell}}\le\left(1-\sqrt{c_{\ell,b,d}\gamma_{d,\ell}(R_1)\gamma_{d,\ell}(R_2)}\right)^{-2}\left(\left\|\mathbf{1}_{R_1^c}f\right\|^2_{L^2_{d,\ell}}+\left\|\mathbf{1}_{R_2^c}\mathfrak{F}^M_{d,\ell}(f)\right\|^2_{L^2_{d,\ell}}\right).\tag{5.16}
$$

In particular, if supp$f\subset R_1$ and supp $\mathfrak{F}^M_{d,\ell}(f)\subset R_2$, then f is the zero function. This means that (R_1,R_2) is an annihilating pair.

Proposition 9.

(1) If $f\in L^2_{d,\ell}(\varepsilon_1,\varepsilon_2,R_1,R_2)$, then f belongs to $\Pi^M_{R_1,R_2}(\varepsilon_1+2\varepsilon_2)$ and is $(\varepsilon_1+\varepsilon_2)$-concentrated with regard to $Q_{R_1}P^M_{R_2}$.

(2) If $f\in\Pi^M_{R_1,R_2}(\varepsilon)$, then

$$
\left\langle f-\Pi^M_{R_1,R_2}f,f\right\rangle_{L^2_{d,\ell}}\le(\varepsilon+\varepsilon^2)\|f\|^2_{L^2_{d,\ell}}.\tag{5.17}
$$

(3) If $f\in L^2_{d,\ell}(\varepsilon,R_1,R_2)$, then $f\in\Pi^M_{R_1,R_2}(\sqrt{\varepsilon})$.

(4) If $f\in L^2_{d,\ell}(\varepsilon_1,\varepsilon_2,R_1,R_2)$, then

$$
\left\langle f-\Pi^M_{R_1,R_2}f,f\right\rangle_{L^2_{d,\ell}}<(\varepsilon_1+2\varepsilon_2)\|f\|^2_{L^2_{d,\ell}}.\tag{5.18}
$$

Proof. Since $\|Q_{R_1}\|_{S_\infty}=\|P^M_{R_2}\|_{S_\infty}=1$, then

$$
\begin{aligned}
\|Q_SP^M_{R_2}f-f\|_{L^2_{d,\ell}}&\le\|P^M_{R_2}f-f\|_{L^2_{d,\ell}}+\|Q_SP^M_{R_2}f-P^M_{R_2}f\|_{L^2_{d,\ell}}\\
&\le\|P^M_{\Sigma^c}f\|_{L^2_{d,\ell}}+\|P^M_{R_2}\|_{S_\infty}\|Q_{S^c}f\|_{L^2_{d,\ell}}\\
&\le(\varepsilon_1+\varepsilon_2)\|f\|_{L^2_{d,\ell}}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\|\Pi^M_{R_1,R_2}f-f\|_{L^2_{d,\ell}}&\le\|P^M_{R_2}Q_SP^M_{R_2}f-P^M_{R_2}f\|_{L^2_{d,\ell}}+\|P^M_{R_2}f-f\|_{L^2_{d,\ell}}\\
&\le\|P^M_{R_2}\|_{S_\infty}\|Q_SP^M_{R_2}f-f\|_{L^2_{d,\ell}}+\|P^M_{R_2}f-f\|_{L^2_{d,\ell}}\\
&\le(\varepsilon_1+2\varepsilon_2)\|f\|_{L^2_{d,\ell}}.
\end{aligned}
$$

On the other hand, if $f \in L^2_{d,\ell}(\varepsilon, R_1, R_2)$,

$$\begin{aligned} 2\left\langle f - \Pi^M_{R_1,R_2} f, f \right\rangle_{L^2_{d,\ell}} &= \left\|\Pi^M_{R_1,R_2} f - f\right\|^2_{L^2_{d,\ell}} + \|f\|^2_{L^2_{d,\ell}} - \left\|\Pi^M_{R_1,R_2} f\right\|^2_{L^2_{d,\ell}} \\ \leq& \left\|\Pi^M_{R_1,R_2} f - f\right\|^2_{L^2_{d,\ell}} + \left(\left\|\Pi^M_{R_1,R_2} f - f\right\|_{L^2_{d,\ell}} + \left\|\Pi^M_{R_1,R_2} f\right\|_{L^2_{d,\ell}}\right)^2 - \left\|\Pi^M_{R_1,R_2} f\right\|^2_{L^2_{d,\ell}} \\ =& 2\left\|\Pi^M_{R_1,R_2} f - f\right\|^2_{L^2_{d,\ell}} + 2\left\|\Pi^M_{R_1,R_2} f - f\right\|_{L^2_{d,\ell}} \left\|\Pi^M_{R_1,R_2} f\right\|_{L^2_{d,\ell}}, \end{aligned}$$

and as $\left\|\Pi^M_{R_1,R_2}\right\|_{S_\infty} \leq 1$, then

$$\left\langle f - \Pi^M_{R_1,R_2} f, f \right\rangle_{L^2_{d,\ell}} \leq \left\|\Pi^M_{R_1,R_2} f - f\right\|^2_{L^2_{d,\ell}} + \left\|\Pi^M_{R_1,R_2} f - f\right\|_{L^2_{d,\ell}} \|f\|_{L^2_{d,\ell}} \leq (\varepsilon^2 + \varepsilon)\|f\|^2_{L^2_{d,\ell}}, \tag{5.19}$$

and the second result follows.

Now, since

$$\left\langle \left(\Pi^M_{R_1,R_2}\right)^2 f, f \right\rangle_{L^2_{d,\ell}} \leq \left\langle \Pi^M_{R_1,R_2} f, f \right\rangle_{L^2_{d,\ell}}, \tag{5.20}$$

then

$$\left\|\Pi^M_{R_1,R_2} f - f\right\|^2_{L^2_{d,\ell}} = \left\langle \left(I - \Pi^M_{R_1,R_2}\right)^2 f, f \right\rangle_{L^2_{d,\ell}} \leq \left\langle (I - \Pi^M_{R_1,R_2}) f, f \right\rangle_{L^2_{d,\ell}} \leq \varepsilon \|f\|^2_{L^2_{d,\ell}}. \tag{5.21}$$

Finally, since

$$\left\langle f - \Pi^M_{R_1,R_2} f, f \right\rangle_{L^2_{d,\ell}} = \left\langle P^M_{R_2^c} f, f \right\rangle_{L^2_{d,\ell}} + \left\langle P^M_{R_2} f, Q_{R_1^c} f \right\rangle_{L^2_{d,\ell}} + \left\langle Q_{R_1} P^M_{R_2} f, P^M_{R_2^c} f \right\rangle_{L^2_{d,\ell}},$$

then we have the fourth inequality. □

Then, inspired by [54], we derive the following result characterizing functions in $L^2_{d,\ell}(\varepsilon, R_1, R_2)$.

Proposition 10. *A function f belongs to $L^2_{d,\ell}(\varepsilon, R_1, R_2)$ if and only if*

$$\sum_{i=1}^{N} (\lambda_i + \varepsilon - 1) \left|\langle f, \sigma_i \rangle_{L^2_{d,\ell}}\right|^2 \geq (1 - \varepsilon)\|f_{\text{ker}}\|^2_{L^2_{d,\ell}} + \sum_{i=1+N}^{\infty} (1 - \varepsilon - \lambda_i) \left|\langle f, \sigma_i \rangle_{L^2_{d,\ell}}\right|^2,$$

where f_{ker} is the orthogonal projection of f onto $\operatorname{Ker}\left(\Pi^M_{R_1,R_2}\right)$.

Proof. If $f \in L^2_{d,\ell}(\mathbb{R})$, then

$$f = \sum_{i=1}^{\infty} \langle f, \sigma_i \rangle_{L^2_{d,\ell}} \sigma_i + f_{\text{ker}}. \tag{5.22}$$

Therefore,

$$\left\langle \Pi^M_{R_1,R_2} f, f \right\rangle_{L^2_{d,\ell}} = \sum_{i=1}^{\infty} \lambda_i \left|\langle f, \sigma_i \rangle_{L^2_{d,\ell}}\right|^2. \tag{5.23}$$

Hence, $f \in L^2_{d,\ell}(\varepsilon, R_1, R_2)$ if and only if

$$\sum_{i=1}^{\infty} \lambda_i \left|\langle f, \sigma_i \rangle_{L^2_{d,\ell}}\right|^2 \geq (1 - \varepsilon) \left(\|f_{\text{ker}}\|^2_{L^2_{d,\ell}} + \sum_{i=1}^{\infty} \left|\langle f, \sigma_i \rangle_{L^2_{d,\ell}}\right|^2\right), \tag{5.24}$$

and the conclusion follows. □

Moreover, we have the following approximation result.

Proposition 11. *Fix* $\varepsilon_0 \in (0,1)$ *and let* $N_0 = N(\varepsilon_0, R_1, R_2)$. *Then, for each* $f \in L^2_{d,\ell}(\varepsilon, R_1, R_2)$, *we have*

$$\left\| f - \sum_{i=1}^{N_0} \langle f, \sigma_i \rangle_{L^2_{d,\ell}} \sigma_i \right\|_{L^2_{d,\ell}} \leq \sqrt{\frac{\varepsilon}{\varepsilon_0}} \, \|f\|_{L^2_{d,\ell}}. \tag{5.25}$$

Proof. Let $\mathcal{P}$ be the orthogonal projection onto V_{N_0}. Then,

$$\|\mathcal{P}f\|^2_{L^2_{d,\ell}} \geq (1 - \varepsilon/\varepsilon_0)\|f\|^2_{L^2_{d,\ell}}. \tag{5.26}$$

Therefore,

$$\|f - \mathcal{P}f\|^2_{L^2_{d,\ell}} = \|f\|^2_{L^2_{d,\ell}} - \|\mathcal{P}f\|^2_{L^2_{d,\ell}} \leq \|f\|^2_{L^2_{d,\ell}} - (1 - \varepsilon/\varepsilon_0)\|f\|^2_{L^2_{d,\ell}} = \varepsilon/\varepsilon_0 \|f\|^2_{L^2_{d,\ell}},$$

as desired. □

Then, by using Proposition 9, we obtain the following corollary.

Corollary 3.

(1) If $f \in L^2_{d,\ell}(\varepsilon_1, \varepsilon_2, R_1, R_2)$, *then*

$$\left\| f - \sum_{i=1}^{N_0} \langle f, \sigma_i \rangle_{L^2_{d,\ell}} \sigma_i \right\|_{L^2_{d,\ell}} \leq \sqrt{\frac{2\varepsilon_1 + \varepsilon_2}{\varepsilon_0}} \, \|f\|_{L^2_{d,\ell}}. \tag{5.27}$$

(2) If $f \in \Pi^M_{R_1,R_2}(\varepsilon)$, *then*

$$\left\| f - \sum_{i=1}^{N_0} \langle f, \sigma_i \rangle_{L^2_{d,\ell}} \sigma_i \right\|_{L^2_{d,\ell}} \leq \sqrt{\frac{2\varepsilon}{\varepsilon_0}} \, \|f\|_{L^2_{d,\ell}}. \tag{5.28}$$

6. Conclusions and perspectives

In the present paper, we accomplished three major objectives. First, we introduced the linear canonical deformed Hankel two-wavelet multipliers and investigated their trace-class and Schatten-von Neumann class properties. Then, we illustrated the boundedness and compactness of the generalized two-wavelet multipliers associated with the LCDHT. Finally, typical examples and some applications for the new generalized two-wavelet multipliers are discussed. In future work, we will study topics from the perspective of numerical analysis within the framework of the LCDHT.

Author contributions

Saifallah Ghobber: Conceptualization, validation, writing–review and editing, project administration, funding acquisition; Hatem Mejjaoli: Methodology, formal analysis, investigation, writing–original draft. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors are deeply indebted to the referees for providing constructive comments and helps in improving the contents of this article. The second author thanks Professors Khalifa Trimèche, Man Wah Wong and Adnane Hamiaz for their helps.

Funding

This work was supported by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia [Grant No. KFU254038].

Conflict of interest

The authors declare no conflicts of interest.

References

1. H. J. Landau, On Szegö's eigenvalue distribution theorem and non-Hermitian kernels, *J. Anal. Math.*, **28** (1975), 335–357. https://doi.org/10.1007/BF02786820
2. H. J. Landau, H. O. Pollak, Prolate spheroidal wave functions, Fourier analysis and uncertainty–III: The dimension of the space of essentially time- and band-limited signals, *Bell Syst. Tech. J.*, **41** (1962), 1295–1336. https://doi.org/10.1002/j.1538-7305.1962.tb03279.x
3. D. L. Donoho, P. B. Stark, Uncertainty principles and signal recovery, *SIAM J. Appl. Math.*, **49** (1989), 906–931. https://doi.org/10.1137/0149053
4. Z. He, M. W. Wong, Wavelet multipliers and signals, *J. Aust. Math. Soc. Ser. B*, **40** (1999), 437–446. https://doi.org/10.1017/S0334270000010523
5. J. Du, M. W. Wong, Traces of wavelet multipliers, *Math. Rep. Acad. Sci*, **23** (2001), 148–152.
6. M. W. Wong, *Wavelet transforms and localization operators*, Birkhäuser Basel, 2002. https://doi.org/10.1007/978-3-0348-8217-0
7. H. Mejjaoli, Boundedness and compactness of Dunkl two-wavelet multipliers, *Int. J. Wavelets Multi.*, **15** (2017), 1750048. https://doi.org/10.1142/S0219691317500485
8. S. Ghobber, Fourier-like multipliers and applications for integral operators, *Complex Anal. Oper. Theory*, **13** (2019), 1059–1092. https://doi.org/10.1007/s11785-018-0839-9
9. V. Catană, M. G. Scumpu, Localization operators and wavelet multipliers involving two-dimensional linear canonical curvelet transform, *J. Pseudo-Differ. Oper. Appl.*, **14** (2023), 53. https://doi.org/10.1007/s11868-023-00547-1

10. S. Ghobber, A variant of the Hankel multiplier, *Banach J. Math. Anal.*, **12** (2018), 144–166. https://doi.org/10.1215/17358787-2017-0051
11. H. Mejjaoli, Spectral theorems associated with the (k, a)-generalized wavelet multipliers, *J. Pseudo-Differ. Oper. Appl.*, **9** (2018), 735–762. https://doi.org/10.1007/s11868-018-0260-1
12. H. Mejjaoli, New results for the Hankel two-wavelet multipliers, *J. Taibah Univ. Sci.*, **13**(1) 1750048 (2019), 32-40. https://doi.org/10.1080/16583655.2018.1521711
13. H. Mejjaoli, K. Trimèche, Two-wavelet multipliers on the dual of the Laguerre hypergroup and applications, *Mediterr. J. Math.*, **16** (2019), 126. https://doi.org/10.1007/s00009-019-1389-8
14. H. Mejjaoli, Wavelet-multipliers analysis in the framework of the k-Laguerre theory, *Linear Multilinear A.*, **67** (2019), 70–93. https://doi.org/10.1080/03081087.2017.1410093
15. H. Mejjaoli, S. Omri, Spectral theorems associated with the directional short-time Fourier transform, *J. Pseudo-Differ. Oper. Appl.*, **11** (2020), 15–54. https://doi.org/10.1007/s11868-019-00308-z
16. P. Shkula, S. K. Upadhyay, Wavelet multiplier associated with the Watson transform, *Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat.*, **117** (2023), 15. https://doi.org/10.1007/s13398-022-01342-1
17. H. M. Srivastava, P. Shukla, S. K. Upadhyay, The localization operator and wavelet multipliers involving the Watson transform, *J. Pseudo-Differ. Oper. Appl.*, **13** (2022), 46. https://doi.org/10.1007/s11868-022-00477-4
18. S. A. Collins, Lens-system diffraction integral written in terms of matrix optics, *J. Opt. Soc. Am.*, **60** (1970), 1168–1177. https://doi.org/10.1364/JOSA.60.001168
19. M. Moshinsky, C. Quesne, Linear canonical transformations and their unitary representations, *J. Math. Phys.*, **12** (1971), 1772–1780. https://doi.org/10.1063/1.1665805
20. A. Bultheel, H. Martínez-Sulbaran, Recent development in the theory of the fractional Fourier and linear canonical transforms, *Bull. Belg. Math. Soc. Simon Stevin*, **13** (2007), 971–1005. https://doi.org/10.36045/bbms/1170347822
21. B. Barshan, M. A. Kutay, H. M. Ozaktas, Optimal filtering with linear canonical transformations, *Opt. Commun.*, **135** (1997), 32–36. https://doi.org/10.1016/S0030-4018(96)00598-6
22. B. M. Hennelly, J. T. Sheridan, Fast numerical algorithm for the linear canonical transform, *J. Opt. Soc. Amer. A*, **22** (2005), 928–937. https://doi.org/10.1364/JOSAA.22.000928
23. J. J. Healy, M. A. Kutay, H. M. Ozaktas, J. T. Sheridan, *Linear canonical transforms*, New York: Springer, 2016. https://doi.org/10.1007/978-1-4939-3028-9
24. H. M. Ozaktas, Z. Zalevsky, M. A. Kutay, *The fractional Fourier transform with applications in optics and signal processing*, New York: John Wiley & Sons, 2001.
25. K. B. Wolf, Canonical transforms. II. Complex radial transforms, *J. Math. Phys.*, **15** (1974), 2102–2111. https://doi.org/10.1063/1.1666590
26. T. Z. Xu, B. Z. Li, *Linear canonical transform and its applications*, Beijing: Science Press, 2013.
27. S. Ghazouani, F. Bouzeffour, Heisenberg uncertainty principle for a fractional power of the deformed Hankel transform on the real line, *J. Comput. Appl. Math.*, **294** (2016), 151–176. https://doi.org/10.1016/j.cam.2015.06.013

28. H. Mejjaoli, S. Negzaoui, Linear canonical deformed Hankel transform and the associated uncertainty principles, *J. Pseudo-Differ. Oper. Appl.*, **14** (2023), 29. https://doi.org/10.1007/s11868-023-00518-6

29. F. A. Shah, A. Y. Tantary, Multi-dimensional linear canonical transform with applications to sampling and multiplicative filtering, *Multidim. Syst. Sign. Process.*, **33** (2022), 621–650. https://doi.org/10.1007/s11045-021-00816-6

30. F. A. Shah, A. Y. Tantary, Linear canonical ripplet transform: Theory and localization operators, *J. Pseudo-Differ. Oper. Appl.*, **13** (2022), 45. https://doi.org/10.1007/s11868-022-00476-5

31. S. Ghobber, H. Mejjaoli, Localization operators for the linear canonical Dunkl windowed transformation, *Axioms*, **14** (2025), 262. https://doi.org/10.3390/axioms14040262

32. S. Ghobber, H. Mejjaoli, Novel Gabor-type transform and weighted uncertainty principles, *Mathematics*, **13** (2025), 1109. https://doi.org/10.3390/math13071109

33. H. Yang, Q. Feng, X. Wang, D. Urynbassarova, A. A. Teali, Reduced biquaternion windowed linear canonical transform: Properties and applications, *Mathematics*, **12** (2024), 743. https://doi.org/10.3390/math12050743

34. M. Bahri, S. A. A. Karim, Novel uncertainty principles concerning linear canonical wavelet transform, *Mathematics*, **10** (2022), 3502. https://doi.org/10.3390/math10193502

35. S. Ghobber, H. Mejjaoli, A new wavelet transform and its localization operators, *Mathematics*, **13** (2025), 1771. https://doi.org/10.3390/math13111771

36. J. Shi, X. Liu, N. Zhang, Generalized convolution and product theorems associated with linear canonical transform, *SIViP*, **8** (2014), 967–974. https://doi.org/10.1007/s11760-012-0348-7

37. D. Urynbassarova, A. A. Teali, Convolution, correlation, and uncertainty principles for the quaternion offset linear canonical transform, *Mathematics*, **11** (2023), 2201. https://doi.org/10.3390/math11092201

38. M. Bahri, S. A. A Karim, B. A. S., M. Nur, N. Nurwahidah, A new form of convolution theorem for one-dimensional quaternion linear canonical transform and application, *Symmetry*, **17** (2025), 1004. https://doi.org/10.3390/sym17071004

39. S. Ghazouani, E. A. Soltani, A. Fitouhi, A unified class of integral transforms related to the deformed Hankel transform, *J. Math. Anal. Appl.*, **449** (2017), 1797–1849. https://doi.org/10.1016/j.jmaa.2016.12.054

40. J. F. Zhang, S. P. Hou, The generalization of the Poisson sum formula associated with the linear canonical transform, *J. Appl. Math.*, **2012** (2012), 102039. https://doi.org/10.1155/2012/102039

41. A. A. Teali, F. A. Shah, Wave packet frames in linear canonical domains: Construction and perturbation, *J. Pseudo-Differ. Oper. Appl.*, **15** (2024), 74. https://doi.org/10.1007/s11868-024-00645-8

42. S. B. Saïd, T. Kobayashi, B. Ørsted, Laguerre semigroup and Dunkl operators, *Compos. Math.*, **148** (2012), 1265–1336. https://doi.org/10.1112/S0010437X11007445

43. V. Havin, B. Jöricke, *The uncertainty principle in harmonic analysis*, Berlin: Springer-Verlag, 1994. https://doi.org/10.1007/978-3-642-78377-7

44. L. D. Abreu, J. M. Pereira, Measures of localization and quantitative Nyquist densities, *Appl. Comput. Harmon. Anal.*, **38** (2015), 524–534. https://doi.org/10.1016/j.acha.2014.08.002

45. J. F. Diejen, L. Vinet, *Calogero-Moser-Sutherland models*, New York: Springer, 2000. https://doi.org/10.1007/978-1-4612-1206-5

46. C. Bennett, R. Sharpley, *Interpolation of operators*, Academic Press, 1988.

47. J. Calderon, Intermediate spaces and interpolation, the complex method, *Studia Math.*, **24** (1964), 113–190. https://doi.org/10.4064/sm-24-2-113-190

48. H. Wang, Compressed sensing: Theory and applications, *J. Phys. Conf. Ser.*, **2419** (2023), 012042. https://doi.org/10.1088/1742-6596/2419/1/012042

49. Y. An, Z. Xue, J. Ou, Deep learning-based sparsity-free compressive sensing method for high accuracy structural vibration response reconstruction, *Mech. Syst. Signal Pr.*, **211** (2024), 111168. https://doi.org/10.1016/j.ymssp.2024.111168

50. M. Doi, M. Ohzeki, Phase transition in binary compressed sensing based on L^1-norm minimization, *J. Phys. Soc. Jpn.*, **93** (2024), 084003. https://doi.org/10.7566/JPSJ.93.084003

51. L. R. Chandran, I. Karuppasamy, M. G. Nair, H. Sun, P. K. Krishnakumari, Compressive sensing in power engineering: A comprehensive survey of theory and applications, and a case study, *J. Sens. Actuator Netw.*, **14** (2025), 28. https://doi.org/10.3390/jsan14020028

52. L. Zhao, Y. Zhang, X. Wang, J. Zhang, H. Bai, A. Wang, A survey on image compressive sensing: From classical theory to the latest explicable deep learning, *Pattern Recogn.*, **170** (2026), 112022. https://doi.org/10.1016/j.patcog.2025.112022

53. S. Ghobber, P. Jaming, Strong annihilating pairs for the Fourier-Bessel transform, *J. Math. Anal. Appl.*, **377** (2011), 501–515. https://doi.org/10.1016/j.jmaa.2010.11.015

54. G. A. M. Velasco, M. Dörfler, Sampling time-frequency localized functions and constructing localized time-frequency frames, *Eur. J. Appl. Math.*, **28** (2017), 854–876. https://doi.org/10.1017/S095679251600053X