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Research article

A stochastic HIV model based on population mobility

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Abstract: This paper established a stochastic HIV model that incorporates population mobility to analyze the role of stochastic noise in HIV dynamics. First, the existence and uniqueness of a globally positive solution for the stochastic system were proven, and sufficient conditions for the stochastic extinction of the disease were derived. Second, by constructing a Lyapunov function, it was demonstrated that the stochastic system possesses a unique ergodic stationary distribution when $\hat{R}_0^s > 1$. Finally, theoretical findings were illustrated through numerical simulations. From an epidemiological perspective, our study reveals that an increased response to infection risks and higher white noise intensity appear advantageous for limiting HIV transmission.

Keywords: HIV infection; population mobility; extinction; stationary distribution
Mathematics Subject Classification: 60G10, 60H10

1. Introduction

Since its initial discovery in the 1980s, the human immunodeficiency virus (HIV) and the resulting acquired immunodeficiency syndrome (AIDS) have evolved into a major global public health threat. According to the latest report from the World Health Organization (WHO) [1], approximately 39.9 million people were living with HIV globally by the end of 2023, with 1.3 million new infections reported that year. AIDS-related deaths reached 630,000 in 2023, bringing the cumulative death toll to 42.3 million throughout history. The persistent epidemic not only causes a significant decline in the health status of infected individuals but also imposes a heavy burden on healthcare systems and socioeconomic development worldwide.

As we now know, AIDS is a highly fatal and malignant infectious disease caused by the HIV. Once the HIV virus invades the human body, it can damage the immune system, gradually causing the infected individual to lose resistance to various diseases, ultimately leading to death. The initial stage after HIV enters the human body is the "acute phase", during which the individual is only referred to as an HIV-infected person, not an AIDS patient. When the individual progresses to the typical AIDS

stage, it marks the final phase of HIV infection [2]. This stage is characterized by three fundamental features: severe cellular immune deficiency, the occurrence of various fatal opportunistic infections, and the development of various malignant tumors. In the terminal stage of AIDS, the immune system collapses, and the patient experiences a variety of severe complications until death.

HIV is primarily transmitted through blood, semen, vaginal secretions, and breast milk. When infected individuals lack awareness of protection and do not receive effective interventions, the virus will continuously damage the immune system, eventually progressing to AIDS. Although there is currently no curative vaccine or definitive medication available, antiretroviral therapy (ART) significantly boosts life expectancy in seropositive patients, enabling survival comparable to uninfected individuals. Studies have shown that when infected individuals consistently receive ART and their viral load is reduced to below the detection limit, the risk of transmitting HIV to their sexual partners becomes negligible [3].

To gain a deeper understanding of the epidemiology of AIDS, the establishment of mathematical models that align with its transmission characteristics based on the disease's developmental stages and infection patterns has been extensively explored. Numerous scholars have conducted research on HIV/AIDS models and their dynamical behaviors. Luo and Chen [4] examined treatment effects on HIV/AIDS transmission dynamics using an epidemic model. Silva and Torres [5] established global stability criteria for an HIV/AIDS model featuring bilinear incidence. Naik [6] examined a fractional-order model for HIV transmission within two distinct gender populations, namely, the general male population and female sex workers.

However, environmental noise modulates infectious disease transmission dynamics in real-world systems. In other words, driven by the proliferation of newly emerging pathogens and regional public health interventions, the number of individuals in each compartment typically fluctuates. Mathematical epidemiology necessitates incorporating the impact of white noise on disease transmission into the analysis. Existing studies have widely employed white noise to construct stochastic infectious disease models [7–9], and have developed various stochastic modeling frameworks (such as [10–13]) to deepen the understanding of stochastic perturbation mechanisms. Liu [14] proposed a high-order stochastic perturbation SICA infectious disease model, establishing sufficient conditions for both disease extinction and the existence of an ergodic stationary distribution. Rao and Luo [15] formulated stochastic HIV/AIDS transmission models where the effective contact rate incorporates stochastic perturbations. Wang [16] established and analyzed a class of stochastic SICA infectious disease models with a standard incidence rate, providing sufficient conditions for the average extinction and persistence of the disease. Zhai and Li [17] developed a stochastic HIV/AIDS model that incorporates protective awareness and fluctuations. Jiao et al. [18] developed a stochastic framework for modeling AIDS transmission dynamics, integrating a log-normal Ornstein-Uhlenbeck process with protective awareness.

The relationship between population mobility and HIV transmission is intricate. Research by [19] indicates that population mobility is one of the key factors driving the spread of HIV, as it facilitates the diffusion of the virus by connecting high-prevalence and low-prevalence areas. Mobile populations, including migrant workers, displaced individuals, and long-distance travelers, often face structural barriers such as limited access to healthcare resources, social marginalization, and economic instability, which significantly increase their vulnerability to HIV infection. Our model focuses on quantifying the core mechanisms of mobility and environmental noise, which provides a foundational framework.

Subsequent research can build upon this foundation to incorporate additional layers of complexity, such as the specific impacts of healthcare access disparities or cultural factors on transmission dynamics.

However, the integration of population mobility and environmental fluctuations into stochastic HIV/AIDS dynamics frameworks remains understudied. Addressing this research void, our study undertakes a systematic exploration of the ways in which demographic mobility and environmental stochasticity shape the transmission dynamics and long-term behavior of the HIV/AIDS model. We organize the paper into the following sections. The stochastic epidemiological model formulation is detailed in Section 2, followed by essential mathematical preliminaries in Section 3. Section 4 establishes extinction thresholds for disease clearance, while Section 5 characterizes stationary distribution properties. Numerical simulations are demonstrated in Section 6. Section 7 synthesizes key findings.

2. Model formulation

The number of contacts an infected individual has with all others per unit time is denoted by C, which typically depends on the total population size N. Let p represent the effective transmission probability per infectious contact. The general form of the infection rate $\lambda(t)$ is then expressed as:

$$\lambda(t) = pC\frac{I}{N},$$

where the term $\frac{CI}{N}$ quantifies the number of infectious contacts. Notably, as the number of infected cases rises, individuals tend to adopt enhanced protective measures during HIV outbreaks, such as modifying sexual behaviors, improving hygiene practices, and maintaining adherence to antiretroviral therapy. This implies that the transmission probability p becomes a decreasing function of I. In this study, we assume p follows the form: $p = \frac{p_0}{1+hI}$ where p_0 and h are positive constants. Substituting this into the infection rate formula yields

$$\lambda(t) = C\frac{p_0 I}{(1 + hI)N}.$$

Assume that m represents population mobility intensity, which can be quantified as the proportional time spent in public venues per unit time. Suppose the contact rate C is a bilinear function of the total population N and population mobility intensity m, expressed as $C = K_1 mN$, where K_1 is a proportionality constant. Consequently, the infection rate becomes:

$$\lambda(t) = \frac{mK_1 p_0 I}{1 + hI}. \tag{2.1}$$

The population mobility intensity m is influenced by both infectious disease risks and economic benefits. Unlike human mobility patterns in physical space, which strictly decrease with distance [20], we assume that in the absence of infectious diseases, m is solely driven by economic benefits and governed by the equation:

$$\frac{dm}{dt} = m(b - am), \tag{2.2}$$

where a and b are constants related to human activities, with $\frac{b}{a}$ defining the maximum tourism activity growth. This maximum economic capacity $\frac{b}{a}$ sets the equilibrium state for traveler volume, and

mobility density follows logistic growth. This logistic growth assumption captures the self-limiting nature of mobility due to congestion or resource constraints, providing a tractable form to analyze the feedback between disease dynamics and behavioral change. However, when infectious diseases emerge, human activity intensity inevitably suffers negative impacts from infection risks. We assume the economic loss rate is proportional to the infection force $\omega\lambda(t)$, where ω quantifies the behavioral response of the population to infection risks affecting travel decisions. Consequently, Eq (2.2) is modified to:

$$\frac{dm}{dt} = m\left(b - am - \frac{\xi I}{1 + hI}\right), \tag{2.3}$$

where $\xi = \omega K_1 p_0$. The modification in Eq (2.3) introduces a key phenomenological mechanism: The reduction in mobility due to perceived infection risk, quantified by ξ. This formulation allows us to isolate and study the core interaction between epidemiology and mobility decisions. It is worth noting that the model framework is conceptually compatible with more granular mobility data or utility-based decision models, which could be integrated in future applications for specific population studies.

Since both K_1 and p_0 are constants, ξ and ω carry identical epidemiological significance in characterizing population responses to infection risk. Defining $\beta := K_1 p_0$ as the maximum transmission probability, Eq (2.1) is rewritten as:

$$\lambda(t) = \frac{\beta m I}{1 + hI}.$$

Fatmawati et al. [21] established a model addressing the transmission dynamics of HIV/AIDS, incorporating the concept of protection awareness. This model partitions the total population N into five sub-classes: susceptible individuals lacking protective awareness (S_u), susceptible individuals with developed protective awareness (S_a), infected individuals not receiving antiretroviral therapy (I), infected individuals undergoing ART (C), and infected individuals who have progressed to AIDS (A). The total population size is expressed as $N = S_u + S_a + I + C + A$.

We have refined the model introduced by Fatmawati et al. [21] by integrating population mobility into the system:

$$\begin{cases} dS_u = \left(\Lambda - \alpha S_u - \frac{\beta m S_u I}{1+hI} - \mu S_u\right) dt, \\ dS_a = \left(\alpha S_u - (1-\varepsilon)\frac{\beta m S_a I}{1+hI} - \mu S_a\right) dt, \\ dI = \left(\frac{\beta m S_u I}{1+hI} + (1-\varepsilon)\frac{\beta m S_a I}{1+hI} + \eta C + \upsilon A - (\rho + \gamma + \mu) I\right) dt, \\ dC = (\rho I - (\eta + \mu) C)\, dt, \\ dA = (\gamma I - (\upsilon + \mu + d) A)\, dt, \\ dm = m\left(b - am - \frac{\xi I}{1+hI}\right) dt. \end{cases} \tag{2.4}$$

Here, Λ represents the recruitment rate, μ denotes the natural mortality rate, and d specifically quantifies AIDS-related mortality. The parameter α signifies the transition rate from S_u to S_a, while ε quantifies the effectiveness of protective measures. The parameters η, υ, γ, and ρ correspond to the transition rates from C to I, A to I, I to A, and I to C, respectively. The intercompartmental transitions are visually depicted in Figure 1, providing a clear schematic of system dynamics. All parameters within system (2.4) are taken to be positive. Through the comparison principle, the positively invariant set of this system is derived as:

$$\Gamma := \{(S_u(t), S_a(t), I(t), C(t), A(t), m(t)) \in \mathbb{R}^6_+ : 0 \le N \le \frac{\Lambda}{\mu}, m \le \frac{b}{a}\}.$$

Based on the next generation matrix method [22], we derive the basic reproduction number for the system as

$$R_0 = \frac{\Lambda\beta b(\mu + (1-\varepsilon)\alpha)(\mu + d + \upsilon)(\mu + \eta)}{\mu a(\alpha + \mu)(\mu(\mu + \eta(\mu + d + \upsilon + \gamma) + \rho(\mu + \eta) + \gamma d) + d\eta\gamma)}.$$

Biologically, R_0 represents the average number of secondary infections produced by a single infected individual in a fully susceptible population. When $R_0 < 1$, following the proof method of Theorem 2 in [21], it can be demonstrated that system (2.4) admits a unique disease-free equilibrium $E_0 = \left(\frac{\Lambda}{\alpha+\mu}, \frac{\Lambda\alpha}{\mu(\alpha+\mu)}, 0, 0, 0, \frac{b}{a}\right)$, and this equilibrium is globally asymptotically stable within the set Γ, that is, the disease cannot sustain itself and will eventually die out.

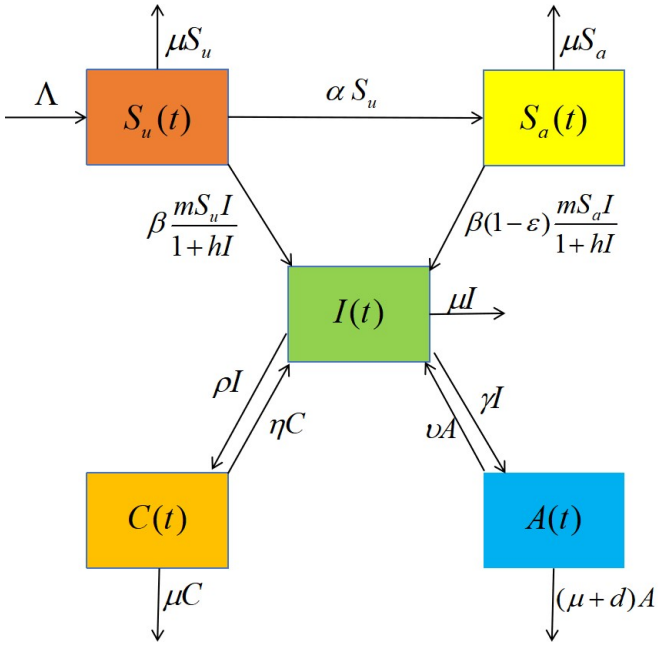

Figure 1. Flow chart of the disease transmission.

Inspired by [17, 23], we incorporate Gauss white noise into the system through combined parameter and system perturbations to characterize continuous minor random disturbances affecting HIV transmission dynamics. Assuming environmental fluctuations cause the incidence coefficient β to oscillate around its mean value according to $\beta dt \to \beta dt + \sigma_1 dB_1(t)$, the deterministic system (2.4) is thus modified into the following stochastic system:

$$\begin{cases} dS_u = & \left(\Lambda - \alpha S_u - \frac{\beta m S_u I}{1+hI} - \mu S_u\right)dt - \sigma_1 \frac{mS_uI}{1+hI} dB_1(t) + \sigma_2 S_u dB_2(t), \\ dS_a = & \left(\alpha S_u - (1-\varepsilon)\frac{\beta m S_a I}{1+hI} - \mu S_a\right)dt - \sigma_1(1-\varepsilon)\frac{mS_aI}{1+hI} dB_1(t) + \sigma_3 S_a dB_3(t), \\ dI = & \left(\frac{\beta m S_u I}{1+hI} + (1-\varepsilon)\frac{\beta m S_a I}{1+hI} + \eta C + \upsilon A - (\rho + \gamma + \mu)I\right)dt \\ & + \left(\sigma_1 \frac{mS_uI}{1+hI} + \sigma_1(1-\varepsilon)\frac{mS_aI}{1+hI}\right)dB_1(t) + \sigma_4 I dB_4(t), \\ dC = & (\rho I - (\eta + \mu)C)\,dt + \sigma_5 C dB_5(t), \\ dA = & (\gamma I - (\upsilon + \mu + d)A)\,dt + \sigma_6 C dB_6(t), \\ dm = & m\left(b - am - \frac{\xi I}{1+hI}\right)dt + \sigma_7 m dB_7(t). \end{cases} \tag{2.5}$$

All parameters are positive constants in the biological context. $B_i(t)(i = 1, 2, 3, 4, 5, 6, 7)$ represent mutually independent Brownian motions defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, where $\sigma_i > 0$ $(i = 1, 2, 3, 4, 5, 6, 7)$ denote the intensities of the white noise.

It should be noted that the parameters within system (2.5), including the transmission rate β, mortality rates, and transition rates, are held constant. This assumption facilitates a theoretical analysis of the system's long-term behavior but may not reflect the temporal variability or state-dependent nature of some parameters observed in real-world epidemics. Furthermore, while antiretroviral therapy (ART) is represented through the transition of individuals from the infected class (I) to the treated class (C) and the subsequent viral suppression, the model does not explicitly model dynamic ART coverage rates, adherence levels, or their potential stochastic fluctuations. A more granular representation of ART dynamics, including its interaction with mobility patterns and environmental noise, represents a valuable direction for future research to more accurately assess the impact of treatment programs on disease outcomes.

3. Main results

Let $B(t) = (B_1(t), ..., B_m(t))^T (t \geq 0)$ denote an m-dimensional standard Brownian motion constructed on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Consider the time interval $0 \leq t_0 < T < \infty$, and let x_0 represent an $\mathbb{R}^d$-valued random variable measurable with respect to $\{\mathcal{F}_{t_0}\}$, and satisfying $E[x_0^2] < \infty$. Suppose $f : \mathbb{R}^d \times [t_0, T] \to \mathbb{R}^d$ and $g : \mathbb{R}^d \times [t_0, T] \to \mathbb{R}^{d\times m}$ are Borel-measurable functions. We consider the n-dimensional stochastic equation subject to the initial condition $x(t_0) = x_0$ [24]:

$$dX(t) = f(X(t), t)dt + g(X(t), t)dB(t), t_0 \leq t \leq T. \tag{3.1}$$

Define an operator L associated with Eq (3.1) as follows:

$$L = \frac{\partial}{\partial t} + \sum_{i=1}^{d} f_i(X, t)\frac{\partial}{\partial X_i} + \frac{1}{2}\sum_{i,j=1}^{d}[g^T(X, t)g(X, t)]_{ij}\frac{\partial^2}{\partial X_i \partial X_j}.$$

If the operator L acts on a function$V \in C^{2,1}(\mathbb{R}^d \times [t_0, T] \to \mathbb{R}_+)$, then

$$LV(X, t) = V_t(X, t) + V_X(X, t)f(X, t) + \frac{1}{2}trace[g^T(X, t)V_{XX}(X, t)g(X, t)].$$

The operator L is uniquely determined by the functions f and g, and is referred to as the differential operator or diffusion operator associated with Eq (3.1), where $trace(\cdot)$ denotes the matrix trace.

The process $V(X(t), t)$ is called an Itô process and satisfies the following stochastic differential equation:

$$dV(X(t), t) = LV(X(t), t)dt + V_X(X(t), t)g(X(t), t)dB(t),$$

where $V_t = \frac{\partial V}{\partial t}$, $V_X = (\frac{\partial V}{\partial X_1}, \ldots, \frac{\partial V}{\partial X_d})$, $V_{XX} = \left(\frac{\partial^2 V}{\partial X_i \partial X_j}\right)_{d\times d}$.

We recall some fundamental results in stochastic analysis that will be used in our proofs. For the existence and uniqueness of a stationary distribution, we refer to Has'minskii's theorem [25]. For the geometric ergodicity of Markov processes, we refer to the criteria established in [26, 27]. The strong law of large numbers for martingales can be found in [28].

Before we begin to study the dynamical behavior of the stochastic system (2.5), the existence of a global positive solution is essential. Subsequently, we demonstrate that for any given initial value, there exists a unique global positive solution to the stochastic system (2.5).

Theorem 1. *For any given initial condition* $(S_u(0), S_a(0), I(0), C(0), A(0), m(0)) \in \mathbb{R}^6_+$*, the stochastic system (2.5) admits a globally unique positive solution* $(S_u(t), S_a(t), I(t), C(t), A(t), m(t))$ *that remains within* $\mathbb{R}^6_+$ *almost surely.*

Proof. It is straightforward to verify that the right-hand side of system (2.5) satisfies a local Lipschitz condition. Therefore, for any given initial value $X(0) \in \mathbb{R}^6_+$, where $X = (S_u, S_a, I, C, A, m)$, system (2.5) admits a unique maximal local solution $X(0)$ defined on $t \in [0, \tau_e)$, with τ_e denoting the solution's explosion time.

To prove that the solution is global, we now show that $\tau_e = \infty$ a.s.

For every integer n satisfying $n \geq n_0$, we introduce a stopping time:

$$\tau_n = \inf\{t \in [0, \tau_e): S_u(t) \notin (\frac{1}{n}), S_a(t) \notin (\frac{1}{n}), I(t) \notin (\frac{1}{n}), C(t) \notin (\frac{1}{n}), A(t) \notin (\frac{1}{n}), m(t) \notin (\frac{1}{n})\}.$$

Throughout this paper, we always assume $\inf \phi = \infty$ (where ϕ denotes the empty set). Clearly, τ_n is monotonically increasing in n.

Let $\tau_\infty = \lim_{n\to\infty} \tau_n$. Then $\tau_\infty < \tau_e$ a.s. If $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ a.s., and for all $t \geq 0$, $(S_u(t), S_a(t), I(t), C(t), A(t), m(t)) \in \mathbb{R}^6_+$ a.s. Otherwise, there exist constants $T > 0$ and $\varepsilon_0 \in (0, 1)$ such that $\mathbb{P}\{\tau_\infty \leq T\} > \varepsilon_0$. Consequently, there exists a positive integer $n_1 \geq n_0$ such that for all $n \geq n_1$,

$$\mathbb{P}\{\tau_\infty \leq T\} \geq \varepsilon_0. \tag{3.2}$$

Assume $N(t) = S_u(t) + S_a(t) + I(t) + C(t) + A(t)$. From system (2.5), we have

$$\frac{dN(t)}{dt} = \Lambda - \mu N(t) - dA \leq \Lambda - \mu N(t),$$

that is,

$$N(t) \leq \frac{\Lambda}{\mu} - \left|\frac{\Lambda}{\mu} - N(0)\right| e^{-\mu t} = \begin{cases} \frac{\Lambda}{\mu} - \left(\frac{\Lambda}{\mu} - N(0)\right) e^{-\mu t}, \frac{\Lambda}{\mu} \geq N(0), \\ \frac{\Lambda}{\mu} - \left(N(0) - \frac{\Lambda}{\mu}\right) e^{-\mu t}, \frac{\Lambda}{\mu} < N(0). \end{cases}$$

Therefore, for all $t \geq 0$, $N(t) \leq \max\{\frac{\Lambda}{\mu}, N(0)\} = \varsigma$.

From the last inequality in system (2.5), we have $dm \leq m(b - am)dt$, $m(t) \leq \max\{\frac{b}{a}, m(0)\} = \iota$. Thus $m(t) \leq \max\{\frac{b}{a}, m(0)\} = \iota$.

Define a C^2-function $V_1 : \mathbb{R}^6_+ \to \mathbb{R}_+$ as:

$$\begin{aligned} V_1(S_u, S_a, I, C, A, m) =& (S_u - 1 - \ln S_u) + (S_a - 1 - \ln S_a) + (I - 1 - \ln I) \\ &+ (C - 1 - \ln C) + (A - 1 - \ln A) + (m - 1 - \ln m). \end{aligned}$$

Since $u - 1 - \ln u \geq 0$ holds for all $u > 0$, the function V_1 is non-negative.

By Itô's formula, we have:

$$\begin{aligned} dV_1 =& LV_1 dt + (S_u - 1)(-\frac{\sigma_1 mI}{1+hI})dB_1(t) + (S_u - 1)\sigma_2 dB_2(t) \\ &+ (S_a - 1)(-\frac{\sigma_1(1-\varepsilon)mI}{1+hI})dB_1(t) + (S_a - 1)\sigma_3 dB_3(t) \\ &+ (I-1)\left(\frac{\sigma_1 mS_u}{1+hI} + \frac{\sigma_1(1-\varepsilon)mS_a}{1+hI}\right)dB_1(t) + (I-1)\sigma_4 dB_4(t) \\ &+ (C-1)\sigma_5 dB_5(t) + (A-1)\sigma_6 dB_6(t) + (m-1)\sigma_7 dB_7(t), \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} LV_1 =& (1-\frac{1}{S_u})\left(\Lambda - \alpha S_u - \frac{\beta m S_u I}{1+hI} - \mu S_u\right) + \frac{\sigma_1^2 m^2 I^2}{2(1+hI^2)^2} + \frac{\sigma_2^2}{2} \\ &+ (1-\frac{1}{S_a})\left(\alpha S_u - (1-\varepsilon)\frac{\beta m S_a I}{1+hI} - \mu S_a\right) + \frac{\sigma_1^2(1-\varepsilon)^2 m^2 I^2}{2(1+hI^2)^2} + \frac{\sigma_3^2}{2} \\ &+ (1-\frac{1}{I})\left(\frac{\beta m S_u I}{1+hI} + (1-\varepsilon)\frac{\beta m S_a I}{1+hI} + \eta C + \upsilon A - (\rho+\gamma+\mu)I\right) \\ &+ \frac{\sigma_1^2 m^2 S_u^2}{2(1+hI^2)^2} + \frac{\sigma_1^2(1-\varepsilon)^2 m^2 S_a^2}{2(1+hI^2)^2} + \frac{\sigma_4^2}{2} + (1-\frac{1}{C})(\rho I - (\eta+\mu)C) + \frac{\sigma_5^2}{2} \\ &+ (1-\frac{1}{A})(\gamma I - (\upsilon+\mu+d)A) + \frac{\sigma_6^2}{2} + (1-\frac{1}{m})m\left(b - am - \frac{\xi I}{1+hI}\right) + \frac{\sigma_7^2}{2} \\ =& \Lambda - \mu(S_u + S_a + I + C + A) - dA + bm - am^2 - \frac{\xi I}{1+hI}m - b + am + \frac{\xi I}{1+hI} \\ &+ \frac{\sigma_2^2+\sigma_3^2+\sigma_4^2+\sigma_5^2+\sigma_6^2+\sigma_7^2}{2} + \frac{\sigma_1^2 m^2 I^2}{2(1+hI)^2}(1+(1-\varepsilon)^2) \\ &+ \frac{\sigma_1^2 m^2}{2(1+hI)^2}(S_u^2 + (1-\varepsilon)^2 S_a^2) \\ &- \frac{\Lambda}{S_u} - \alpha + \frac{\beta m I}{1+hI} + \mu - \frac{\alpha S_u}{S_a} + (1-\varepsilon)\frac{\beta m I}{1+hI} + \mu - \frac{\beta m S_u}{1+hI} - (1-\varepsilon)\frac{\beta m S_a}{1+hI} \\ &- \frac{\eta C + \upsilon A}{I} + \rho + \gamma + \mu - \frac{\rho I}{C} + \eta + \mu - \frac{\gamma I}{A} + \upsilon + \mu + d \\ \leq& \Lambda + 5\mu + \alpha + \rho + \gamma + \eta + \upsilon + d + (a+b)\iota + \xi \\ &+ \frac{\sigma_2^2+\sigma_3^2+\sigma_4^2+\sigma_5^2+\sigma_6^2+\sigma_7^2}{2} \\ &+ \sigma_1^2 \iota^2 \varsigma^2 (1+(1-\varepsilon)^2) + \iota\beta(2-\varepsilon) := M, \end{aligned}$$

where M is a positive constant. Substituting the above into Eq (3.3) yields:

$$\begin{aligned} dV_1 \leq& Mdt + \left((S_u-1)(-\frac{\sigma_1 mI}{1+hI}) + (S_a-1)(-\frac{\sigma_1(1-\varepsilon)mI}{1+hI}) + \frac{\sigma_1 mS_u}{1+hI} + \frac{\sigma_1(1-\varepsilon)mS_a}{1+hI}\right) \\ &+ (S_u-1)\sigma_2 dB_2(t) + (S_a-1)\sigma_3 dB_3(t) + (I-1)\sigma_4 dB_4(t) + (C-1)\sigma_5 dB_5(t) \\ &+ (A-1)\sigma_6 dB_6(t) + +(m-1)\sigma_7 dB_7(t). \end{aligned}$$

Integrating the above from 0 to $\tau_e \wedge T$ and taking the expectation, we obtain:

$$\begin{aligned} &EV_1(S_u(\tau_e \wedge T), S_a(\tau_e \wedge T), I(\tau_e \wedge T), C(\tau_e \wedge T), A(\tau_e \wedge T), m(\tau_e \wedge T)) \\ &\quad \leq EV_1(S_u(0), S_a(0), I(0), C(0), A(0), m(0)) + ME(\tau_e \wedge T) \\ &\quad \leq V_1(S_u(0), S_a(0), I(0), C(0), A(0), m(0)) + MT. \end{aligned}$$

When $n > n_1$, define $\Omega_n = \{\tau_n \leq T\}$. Then from (3.2), we have: $\mathbb{P}(\Omega_n) \geq \varepsilon_0$.

Note that for any sample path $\omega_0 \in \Omega_n$, at least one of the following: $S_{u\tau_n}(\omega)$, $S_{a\tau_n}(\omega)$, $I_{\tau_n}(\omega)$, $C_{\tau_n}(\omega)$, $A_{\tau_n}(\omega)$, $m_{\tau_n}(\omega)$ hits or exceeds the boundary n or $\frac{1}{n}$. Hence,

$$V_1(S_{u\tau_n}(\omega), S_{a\tau_n}(\omega), I_{\tau_n}(\omega), C_{\tau_n}(\omega), A_{\tau_n}(\omega), m_{\tau_n}(\omega)) \geq (n - 1 - \ln n) \wedge (\frac{1}{n} - 1 - \ln \frac{1}{n}).$$

Therefore, we obtain

$$\begin{aligned} V_1(S_u(0), S_a(0), I(0), C(0), A(0), m(0)) + MT &\geq E[1_{\Omega_n}(\omega) \vee (S_{u\tau_n}, S_{a\tau_n}, I_{\tau_n}, C_{\tau_n}, A_{\tau_n}, m_{\tau_n})] \\ &\geq \varepsilon_0(n - 1 - \ln n) \wedge (\frac{1}{n} - 1 - \ln \frac{1}{n}). \end{aligned}$$

Here $1_{\Omega_n}(\omega)$ denotes the indicator function of Ω_n. As $n \longrightarrow \infty$,

$$\infty < V_1(S_u(0), S_a(0), I(0), C(0), A(0), m(0)) + MT = \infty \quad a.s.$$

yields a contradiction. Therefore we must have $\tau_e = \infty$ a.s. The theorem has been proved. □

Next, we prove the V-geometric ergodicity of the Markov process $Y(t) = (S_u(t), S_a(t), I(t), C(t), A(t), m(t))$ for system (2.5).

Theorem 2. *For $Y(0) \in \mathbb{R}^6_+$, the Markov process $Y(t) = (S_u(t), S_a(t), I(t), C(t), A(t), m(t))$ of system (2.5) is V-geometrically ergodic.*

Proof. Let $X(t) = S_u(t) + S_a(t) + I(t) + C(t) + A(t) + m(t)$. For any state $Y(t) \in \mathbb{R}^6_+$, define

$$V(Y(t)) = X(t) + \frac{1}{X(t)}. \tag{3.4}$$

It is clear that $V(Y(t)) \to \infty$ implies $|V(Y(t))| \to \infty$. Applying the Itô formula yields

$$\begin{aligned} LV(Y(t)) =& \Lambda - \mu(S_u + S_a + I + C + A) - dA + m(b - am - \frac{\xi I}{1 + hI}) - \frac{\Lambda}{X^2} + \frac{dA}{X^2} \\ &+ \frac{\mu(S_u + S_a + I + C + A)}{X^2} - \frac{m(b - am - \frac{\xi I}{1+hI})}{X^2} + \frac{2\sigma_1^2\left(\frac{mS_uI}{1+hI}\right)^2}{X^3} \\ &+ \frac{2\sigma_1^2(1-\varepsilon)^2\left(\frac{mS_aI}{1+hI}\right)^2}{X^3} + \frac{2\sigma_1^2(1-\varepsilon)S_uS_a\left(\frac{mI}{1+hI}\right)^2}{X^3} \\ &+ \frac{\sigma_2^2S_u^2 + \sigma_3^2S_a^2 + \sigma_4^2I^2 + \sigma_5^2C^2 + \sigma_6^2A^2 + \sigma_7^2m^2}{X^3} \\ \leq& \Lambda - (\mu \wedge b)X - dA + 2bm - \frac{\Lambda}{X^2} + \frac{(\mu \vee \xi)X}{X^2} + \frac{dA}{X^2} - \frac{m(b - am)}{X^2} \end{aligned}$$

$$
\begin{aligned}
&+ \frac{2\sigma_1^2 m^2 S_u^2}{X^3} + \frac{2\sigma_1^2(1-\varepsilon)^2 m^2 S_a^2}{X^3} + \frac{2\sigma_1^2(1-\varepsilon) m^2 S_u S_a}{X^3} \\
&+ \frac{(\sigma_2^2+\sigma_3^2+\sigma_4^2+\sigma_5^2+\sigma_6^2+\sigma_7^2)X^2}{X^3} \\
\leq & -(\mu \wedge b)\left(X + \frac{1}{X}\right) + \frac{\mu \wedge b}{X} + \frac{\mu \vee \xi}{X} + 2b\iota + \Lambda - \frac{\Lambda}{X^2} + \frac{d}{X} + \frac{2\sigma_1^2\iota^2}{X} \\
&+ \frac{2\sigma_1^2(1-\varepsilon)^2\iota^2}{X} + \frac{2\sigma_1^2(1-\varepsilon)\iota^2}{X} + \frac{\sigma_2^2+\sigma_3^2+\sigma_4^2+\sigma_5^2+\sigma_6^2+\sigma_7^2}{X} \\
= & -(\mu \wedge b)\left(X + \frac{1}{X}\right) + 2b\iota + \Lambda - \frac{\Lambda}{X^2} + \frac{(\mu \wedge b) + (\mu \vee \xi) + d}{X} \\
&+ \frac{2\sigma_1^2\iota^2(2+\varepsilon+(1-\varepsilon)^2)}{X} + \frac{\sigma_2^2+\sigma_3^2+\sigma_4^2+\sigma_5^2+\sigma_6^2+\sigma_7^2}{X} \\
:= & -DV(Y) + B, \qquad (3.5)
\end{aligned}
$$

where

$$D = \mu \wedge b,$$

$$B = 2b\iota + \Lambda + \frac{1}{4\Lambda}\left[(\mu \wedge b) + (\mu \vee \xi) + d + 2\sigma_1^2\iota^2(2+\varepsilon+(1-\varepsilon)^2) + \sigma_2^2+\sigma_3^2+\sigma_4^2 + \sigma_5^2+\sigma_6^2+\sigma_7^2\right]^2.$$

Therefore, the Lyapunov condition [27] is satisfied.

Since system (2.5) is uniformly elliptic, Proposition 11.1 in [29] guarantees the existence of a function $\psi : \mathbb{R} + \times\mathbb{R}^6 + \times\mathbb{R}_+^6 \to (0,\infty)$ such that ψ is uniformly continuous, strictly positive for all (t, Y_0, Z), and satisfies the following condition for all measurable sets C:

$$\int_C \psi_t(Y_0, Z)dZ = 1.$$

It follows that for any $\varrho > 0$, there exists a constant $\epsilon = \epsilon(\varrho, t) > 0$ such that $\inf \psi t(Y_0, Z) : Y_0, Z \in \mathbb{R}^6+, |Y_0|, |Z| \leq \varrho \geq \epsilon$. Consequently, for any measurable set C, the following inequality holds:

$$\psi_t(Y_0, Z) = \int_C \psi_t(Y_0, Z)dZ > \epsilon Leb(C \cap \mathcal{B}_\varrho(0)) = \epsilon Leb(\mathcal{B}_\varrho(0))\nu(C),$$

where *Leb* denotes the Lebesgue measure, and $\nu(C) = Leb(C \cap \mathcal{B}\rho(0))$ / $Leb(\mathcal{B}\rho(0))$ (a probability measure). Therefore, the minorization condition [26] is satisfied. The proof is complete. □

Theorem 3. *For any given initial value $Y(0) \in \mathbb{R}_+^6$, the solution of system (2.5) is stochastically ultimately bounded.*

Proof. Applying the Itô formula and combining Eqs (3.4) and (3.5), we obtain

$$
\begin{aligned}
\mathbb{E}[e^{At}V(t)] =& \mathbb{E}[V(0)] + \mathbb{E}\left[\int_0^t e^{Ds}(DV(s) + LV(s))ds\right] \\
&\leq \mathbb{E}[V(0)] + B\mathbb{E}\left[\int_0^t e^{Ds}ds\right] = \mathbb{E}[V(0)] + \frac{B}{D}\left(e^{Dt} - 1\right),
\end{aligned}
$$

that is,

$$\mathbb{E}[V(t)] \leq e^{-Dt}\mathbb{E}[V(0)] + \frac{B}{D}\left(1 - e^{-Dt}\right) \leq \mathbb{E}[V(0)] + \frac{B}{D} := H.$$

Assume there exists a constantϖ sufficiently large such that $\frac{H}{\varpi} < 1$. Applying Chebyshev's inequality yields:

$$\mathbb{P}\left\{X + \frac{1}{X} > \varpi\right\} \leq \frac{1}{\varpi}\mathbb{E}\left[X + \frac{1}{X}\right] \leq \frac{H}{\varpi} := \vartheta,$$

and then

$$1 - \vartheta \leq \mathbb{P}\left\{X + \frac{1}{X} > \varpi\right\} \leq \mathbb{P}\left\{\frac{1}{\varpi} \leq X \leq \varpi\right\}.$$

By the relation $X^2 \leq 3|Y|^2 \leq 3|X|^2$, it follows that:

$$\mathbb{P}\left\{\frac{1}{\sqrt{3}\varpi} \leq \frac{X}{\sqrt{3}} \leq |Y| \leq X \leq \varpi\right\} \geq 1 - \vartheta.$$

Therefore, the solution of system (2.5) is stochastically ultimately bounded. This completes the proof. □

4. Disease extinction

In this section, we will investigate sufficient conditions for disease extinction under the influence of stochastic factors. First, we define the threshold:

$$R_0^s = \frac{\beta^2}{2\sigma_1^2\left(\mu + \frac{1}{3}\left(\frac{\sigma_4^2}{2} \wedge \frac{\sigma_5^2}{2} \wedge (\frac{\sigma_6^2}{2} + d)\right)\right)}.$$

Theorem 4. *Let $(S_u(t), S_a(t), I(t), C(t), A(t), m(t))$ be a solution to system (2.5) with the initial value $(S_u(0), S_a(0), I(0), C(0), A(0), m(0)) \in \Gamma$. If $R_0^s < 1$, then:*

$$\lim_{t\to\infty} I(t) = 0, \lim_{t\to\infty} C(t) = 0, \lim_{t\to\infty} A(t) = 0 \quad a.s.$$

That is, the disease goes extinct almost surely.

Proof. Define the function $F(t) = I(t) + A(t) + C(t)$. Applying the Itô formula yields:

$$\begin{aligned}
d\ln F(t) =& \frac{1}{I+C+A}\left(\frac{\beta m S_u I}{1+hI} + (1-\varepsilon)\frac{\beta m S_a I}{1+hI} - \mu I - \mu C - (\mu+d)A\right)dt \\
&+ \frac{1}{I+C+A}\left(\sigma_1\frac{mI}{1+hI}(S_u + (1-\varepsilon)S_a)dB_1(t) + \sigma_4 I dB_4(t) + \sigma_5 C dB_5(t) + \sigma_6 A dB_6(t)\right) \\
&- \frac{1}{2(I+C+A)^2}\left(\sigma_1^2(\frac{mI}{1+hI})^2(S_u + (1-\varepsilon)S_a)^2 + \sigma_4^2 I^2 + \sigma_5^2 C^2 + \sigma_6^2 A^2\right)dt \\
\leq& \left(\frac{\beta m I}{(I+C+A)(1+hI)}(S_u + (1-\varepsilon)S_a) - \frac{\sigma_1^2 m^2 I^2}{2(I+C+A)^2(1+hI)^2}(S_u + (1-\varepsilon)S_a)^2\right. \\
&\left. -\mu - \frac{\frac{1}{3}\left(\frac{\sigma_4^2}{2} \wedge \frac{\sigma_5^2}{2} \wedge (\frac{\sigma_6^2}{2} + d)\right)(I^2 + C^2 + A^2)}{2(I+C+A)^2}\right)dt + \frac{1}{I+C+A}\sigma_4 I dB_4(t)
\end{aligned}$$

$$+\frac{1}{I+C+A}\left(\sigma_1\frac{mI}{1+hI}(S_u+(1-\varepsilon)S_a)dB_1(t)+\sigma_5CdB_5(t)+\sigma_6AdB_6(t)\right).$$

By the Cauchy-Schwarz inequality, we have $(I+C+A)^2 \le 3(I^2+C^2+A^2)$, and thus

$$\begin{aligned} d\ln F(t) \le & \left(-\left(\frac{\sigma_1 mI}{\sqrt{2}(I+C+A)(1+HI)}-\frac{\sqrt{2}\beta}{2\sigma_1}\right)^2+\frac{\beta^2}{2\sigma_1^2}-\mu-\frac{1}{3}\left(\frac{\sigma_4^2}{2}\wedge\frac{\sigma_5^2}{2}\wedge(\frac{\sigma_6^2}{2}+d)\right)\right)dt \\ & +\frac{1}{I+C+A}\left(\sigma_1\frac{mI}{1+hI}(S_u+(1-\varepsilon)S_a)dB_1(t)+\sigma_4IdB_4(t)+\sigma_5CdB_5(t)+\sigma_6AdB_6(t)\right) \\ \le & \left(\mu+\frac{1}{3}\left(\frac{\sigma_4^2}{2}\wedge\frac{\sigma_5^2}{2}\wedge(\frac{\sigma_6^2}{2}+d)\right)\right)(R_0^s-1)dt \\ & +\frac{1}{I+C+A}\left(\sigma_1\frac{mI}{1+hI}(S_u+(1-\varepsilon)S_a)dB_1(t)+\sigma_4IdB_4(t)+\sigma_5CdB_5(t)+\sigma_6AdB_6(t)\right). \end{aligned} \tag{4.1}$$

By integrating Eq (4.1) from 0 to t and dividing by t, we obtain

$$\frac{\ln F(t)}{t} \le \frac{\ln F(0)}{t}+\left(\mu+\frac{1}{3}\left(\frac{\sigma_4^2}{2}\wedge\frac{\sigma_5^2}{2}\wedge(\frac{\sigma_6^2}{2}+d)\right)\right)(R_0^s-1)+\frac{M_1(t)}{t}+\frac{M_2(t)}{t}, \tag{4.2}$$

where

$$\begin{aligned} M_1(t) &= \int_0^t \frac{\sigma_1 mI(S_u+(1-\varepsilon)S_a)}{(I+C+A)(1+hI)}dB_1(s), \\ M_2(t) &= \int_0^t \frac{\sigma_4 I}{I+C+A}dB_4(s)+\int_0^t \frac{\sigma_5 C}{I+C+A}dB_5(s)+\int_0^t \frac{\sigma_6 A}{I+C+A}dB_6(s). \end{aligned}$$

By the strong law of large numbers, it follows that

$$\lim_{t\to\infty}\frac{M_1(t)}{t}=0, \lim_{t\to\infty}\frac{M_2(t)}{t}=0.$$

Hence

$$\limsup_{t\to\infty}\frac{\ln F(t)}{t} \le \left(\mu+\frac{1}{3}\left(\frac{\sigma_4^2}{2}\wedge\frac{\sigma_5^2}{2}\wedge(\frac{\sigma_6^2}{2}+d)\right)\right)(R_0^s-1)<0.$$

That is, $\lim_{t\to\infty} I(t)=0, \lim_{t\to\infty} C(t)=0, \lim_{t\to\infty} A(t)=0$ a.s. □

5. Stationary distribution

Define the threshold

$$\hat{R}_0^s = \frac{\Lambda k_1 k_2 \beta(\mu+(1-\varepsilon)\alpha)}{\left(\mu+\frac{\sigma_3^2}{2}\right)\left(\alpha+\mu+\frac{\sigma_2^2}{2}\right)\left(k_1k_2\left(\rho+\gamma+\mu+\frac{\sigma_4^2}{2}\right)-\rho\eta\left(\upsilon+\mu+d+\frac{\sigma_6^2}{2}\right)-\gamma\upsilon\left(\eta+\mu+\frac{\sigma_5^2}{2}\right)\right)},$$

where $k_1=\mu+d+\upsilon$, $k_2=\mu+\eta$.

Theorem 5. *If $\hat{R}_0^s > 1$ and $b > \frac{\sigma_7^2}{2}$, then system (2.5) admits a unique ergodic stationary distribution $\pi(\cdot)$.*

Proof. To verify this theorem, we need to prove that conditions (i) and (ii) in Has'minskii's theorem [25] hold. First, we establish condition (i). The diffusion matrix of system (2.5) is

$$B = \begin{pmatrix} B_{11} & B_{12} & B_{13} & 0 & 0 & 0 \\ B_{21} & B_{22} & B_{23} & 0 & 0 & 0 \\ B_{31} & B_{32} & B_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_5^2 C^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_6^2 A^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_7^2 m^2 \end{pmatrix},$$

where

$$\begin{aligned}
B_{11} =& \sigma_1^2 \left(\frac{mS_u I}{1+hI}\right)^2 + \sigma_2^2 S_u^2, \\
B_{12} =& B_{21} = \sigma_1^2 \left(\frac{mI}{1+hI}\right)^2 S_u(1-\varepsilon)S_a, \\
B_{13} =& B_{31} = -\sigma_1^2 \left(\frac{mI}{1+hI}\right)^2 S_u(S_u + (1-\varepsilon)S_a), \\
B_{22} =& \sigma_1^2(1-\varepsilon)^2 \left(\frac{mS_a I}{1+hI}\right)^2 + \sigma_3^2 S_a^2, \\
B_{23} =& B_{32} = -\sigma_1^2 \left(\frac{mI}{1+hI}\right)^2 (1-\varepsilon)S_a(S_u + (1-\varepsilon)S_a), \\
B_{33} =& \sigma_1^2 \left(\frac{mI}{1+hI}\right)^2 (S_u + (1-\varepsilon)S_a)^2 + \sigma_4^2 I^2.
\end{aligned}$$

Let

$$Q = \min(S_u, S_a, I, C, A, m) \in \bar{U}_\delta \subset \mathbb{R}_+^6 \{\sigma_2^2 S_u^2, \sigma_3^2 S_a^2, \sigma_4^2 I^2, \sigma_5^2 C^2, \sigma_6^2 A^2, \sigma_7^2 m^2\}.$$

It can be deduced that

$$\begin{aligned}
\sum_{i,j=1}^{6} b_{ij}(S_u, S_a, I, C, A, m)\zeta_i\zeta_j =& \left(\sigma_1^2 \left(\frac{mS_u I}{1+hI}\right)^2 + \sigma_2^2 S_u^2\right)\zeta_1^2 + 2\sigma_1^2 \left(\frac{mI}{1+hI}\right)^2 S_u(1-\varepsilon)S_a\zeta_1\zeta_2 \\
& - 2\sigma_1^2 \left(\frac{mI}{1+hI}\right)^2 S_u(S_u + (1-\varepsilon)S_a)\zeta_1\zeta_3 \\
& + \left(\sigma_1^2(1-\varepsilon)^2 \left(\frac{mS_a I}{1+hI}\right)^2 + \sigma_3^2 S_a^2\right)\zeta_2^2 + \sigma_5^2 C^2 \zeta_4^2 \\
& - 2\sigma_1^2 \left(\frac{mI}{1+hI}\right)^2 (1-\varepsilon)S_a(S_u + (1-\varepsilon)S_a)\zeta_2\zeta_3 + \sigma_6^2 A^2 \zeta_5^2 \\
& + \left(\sigma_1^2 \left(\frac{mI}{1+hI}\right)^2 (S_u + (1-\varepsilon)S_a)^2 + \sigma_4^2 I^2\right)\zeta_3^2 + \sigma_7^2 m^2 \zeta_6^2 \\
=& \sigma_1^2 \left(\frac{mI}{1+hI}\right)^2 (S_u\zeta_1 + (1-\varepsilon)S_a\zeta_2 - (S_u + (1-\varepsilon)S_a)\zeta_3)^2
\end{aligned}$$

$$
\begin{aligned}
&+\sigma_2^2 S_u^2\zeta_1^2+\sigma_3^2 S_a^2\zeta_2^2+\sigma_4^2 I^2\zeta_3^2+\sigma_5^2 C^2\zeta_4^2+\sigma_6^2 A^2\zeta_5^2+\sigma_7^2 m^2\zeta_6^2\\
\geq&\sigma_2^2 S_u^2\zeta_1^2+\sigma_3^2 S_a^2\zeta_2^2+\sigma_4^2 I^2\zeta_3^2+\sigma_5^2 C^2\zeta_4^2+\sigma_6^2 A^2\zeta_5^2+\sigma_7^2 m^2\zeta_6^2\\
\geq&Q|\zeta|^2,
\end{aligned}
$$

Since $(S_u, S_a, I, C, A, m) \in \bar{U}_\delta$ and $\zeta = (\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6) \in \mathbb{R}_+^6$, condition (i) of Has'minskii's theorem is satisfied.

To verify that condition (ii) of Has'minskii's theorem holds, we define a non-negative C^2 function V_1: $\mathbb{R}_+^6 \to \mathbb{R}_+$ as follows:

$$
V_1 = -c_1 \ln S_u - c_2 \ln S_a + c_3 k_1 k_2 \beta \frac{1}{\alpha} S_a - c_3 k_1 k_2 \ln I + c_3 \rho\eta \ln A + c_3 \gamma\upsilon \ln C.
$$

By applying the Itô formula in conjunction with system (2.5), it can be deduced that

$$
\begin{aligned}
LV_1 =& - c_1\left(\frac{\Lambda}{S_u} - \frac{\beta m I}{1+hI} - \frac{\sigma_1^2}{2}\left(\frac{mI}{1+hI}\right)^2 - \left(\alpha+\mu+\frac{\sigma_2^2}{2}\right)\right)\\
&- c_2\left(\frac{\alpha S_u}{S_a} - (1-\varepsilon)\frac{\beta m I}{1+hI} - \frac{\sigma_1^2}{2}(1-\varepsilon)^2\left(\frac{mI}{1+hI}\right)^2 - \left(\mu+\frac{\sigma_3^2}{2}\right)\right)\\
&+ c_3 k_1 k_2 \beta\frac{1}{\alpha}\left(\alpha S_u - (1-\varepsilon)\frac{\beta m S_a I}{1+hI} - \mu S_a\right) + c_3 k_1 k_2\left(\rho+\gamma+\mu+\frac{\sigma_4^2}{2}\right)\\
&- c_3 k_1 k_2\left(\frac{\beta m S_u}{1+hI} + (1-\varepsilon)\frac{\beta m S_a}{1+hI} + \frac{\eta C+\upsilon A}{I} - \frac{\sigma_1^2}{2}\left(\frac{mS_u}{1+hI} + (1-\varepsilon)\frac{mS_a}{1+hI}\right)^2\right)\\
&+ c_3\rho\eta\left(\frac{\gamma I}{A} - \left(\upsilon+\mu+d+\frac{\sigma_6^2}{2}\right)\right) + c_3\gamma\upsilon\left(\frac{\rho I}{C} - \left(\eta+\mu+\frac{\sigma_5^2}{2}\right)\right)\\
\leq& - \left(c_1\frac{\Lambda}{S_u} + c_2\frac{\alpha S_u}{S_a} + c_3 k_1 k_2\beta\left(\frac{\mu}{\alpha}+1-\varepsilon\right)S_a\right) + c_1\left(\alpha+\mu+\frac{\sigma_2^2}{2}\right) + c_2\left(\mu+\frac{\sigma_3^2}{2}\right)\\
&+ c_3\left(k_1 k_2\left(\rho+\gamma+\mu+\frac{\sigma_4^2}{2}\right) - \rho\eta\left(\upsilon+\mu+d+\frac{\sigma_6^2}{2}\right) - \gamma\upsilon\left(\eta+\mu+\frac{\sigma_5^2}{2}\right)\right)\\
&+ \left(c_1\frac{\beta m}{1+hI} + c_2(1-\varepsilon)\frac{\beta m}{1+hI} + c_3\rho\gamma\left(\frac{\upsilon}{C}+\frac{\eta}{A}\right)\right)I + \frac{\sigma_1^2}{2}\left(\frac{mI}{1+hI}\right)^2(c_1 + c_2(1-\varepsilon)^2)\\
&+ c_3 k_1 k_2(S_u + (1-\varepsilon)S_a)\left(\beta + \frac{\sigma_1^2}{2}\left(\frac{m}{1+hI}\right)^2(S_u + (1-\varepsilon)S_a)\right),
\end{aligned}
$$

where

$$
\begin{aligned}
c_1 =& \frac{1}{\alpha+\mu+\frac{\sigma_2^2}{2}},\quad c_2 = \frac{1}{\mu+\frac{\sigma_3^2}{2}},\\
c_3 =& \frac{1}{k_1 k_2\left(\rho+\gamma+\mu+\frac{\sigma_4^2}{2}\right) - \rho\eta\left(\upsilon+\mu+d+\frac{\sigma_6^2}{2}\right) - \gamma\upsilon\left(\eta+\mu+\frac{\sigma_5^2}{2}\right)},\\
h_1 =& c_1\beta\iota + c_2(1-\varepsilon)\beta\iota + c_3\rho\gamma(\upsilon+\eta),\\
h_2 =& \frac{\sigma_1^2}{2}\iota^2(c_1 + c_2(1-\varepsilon)^2) + c_3 k_1 k_2(2-\varepsilon)\varsigma\left(\beta + \frac{\sigma_1^2}{2}\iota^2(2-\varepsilon)\varsigma\right),
\end{aligned}
$$

$$\tilde{P} =(\mu \wedge b) - \frac{p}{2}((2-\varepsilon+(1-\varepsilon)^2)2\varsigma^2\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2 \vee \sigma_7^2).$$

Thus

$$LV_1 \leq -3(\sqrt[3]{\hat{R}_0^s}-1) + h_1 I + h_2. \tag{5.1}$$

Furthermore, we construct a C^2 function $H : \mathbb{R}_+^6 \to \mathbb{R}_+$ as follows:

$$\begin{aligned} H(S_u, S_a, I, C, A, m) =& K\left(-c_1 \ln S_u - c_2 \ln S_a + c_3 k_1 k_2 \beta \frac{1}{\alpha} S_a - c_3 k_1 k_2 \ln I + c_3 \rho \eta \ln A + c_3 \gamma \upsilon \ln C\right) \\ &+ \frac{1}{p+1}(S_u + S_a + I + C + A + m)^{p+1} - \ln S_u - \ln S_a - \ln C - \ln A + \ln m \\ :=& KV_1 + V_2 + V_3 + V_4 + V_5 + V_6 + V_7, \end{aligned}$$

where $K > 0$ and satisfies the condition

$$\begin{aligned} &-3K(\sqrt[3]{\hat{R}_0^s}-1) + h_1 K\varepsilon_1 + h_2 K + B + h_2 K + \alpha + \mu + \frac{\sigma_2^2}{2} + \mu + \frac{\sigma_3^2}{2} + k_2 + \frac{\sigma_5^2}{2} \\ &+ k_1 + \frac{\sigma_6^2}{2} + b - \frac{\sigma_7^2}{2} + \beta\iota(2-\varepsilon) + \frac{\sigma_1^2}{2}\iota^2(1+(1-\varepsilon)^2) \leq -2, \end{aligned} \tag{5.2}$$

and p is a constant and satisfies the condition

$$(\mu \wedge b) - \frac{p}{2}((2-\varepsilon+(1-\varepsilon)^2)2\varsigma^2\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2 \vee \sigma_4^2 \vee \sigma_5^2 \vee \sigma_6^2 \vee \sigma_7^2) > 0.$$

Clearly,

$$\lim_{n\to\infty} \inf_{(S_u,S_a,I,C,A,m)\in\mathbb{R}_+^6\setminus U_n} H(S_u, S_a, I, C, A, m) = +\infty,$$

where

$$U_k = (\frac{1}{n}, n) \times (\frac{1}{n}, n) \times (\frac{1}{n}, n) \times (\frac{1}{n}, n) \times (\frac{1}{n}, n) \times (\frac{1}{n}, n).$$

Since $H(S_u, S_a, I, C, A, m)$ is a continuous function, it must attain a minimum value at some point $(\tilde{S_u}, \tilde{S_a}, \tilde{I}, \tilde{C}, \tilde{A}, \tilde{m})$.

Define a non-negative function $C^2 - V : \mathbb{R}_+^6 \to \mathbb{R}_+$ as follows:

$$V(S_u, S_a, I, C, A, m) = H(S_u, S_a, I, C, A, m) - \tilde{H}(\tilde{S_u}, \tilde{S_a}, \tilde{I}, \tilde{C}, \tilde{A}, \tilde{m}).$$

By applying the Itô formula, we obtain

$$\begin{aligned} LV_2 =& (S_u + S_a + I + C + A + m)^p \left(\Lambda - \mu(S_u + S_a + I + C + A) - dA + m\left(b - am - \frac{\xi I}{1+hI}\right)\right) \\ &+ \frac{p}{2}(S_u + S_a + I + C + A + m)^{p-1}\left(2\sigma_1^2\left(\frac{mS_uI}{1+hI}\right)^2 + 2\sigma_1^2(1-\varepsilon)^2\left(\frac{mS_aI}{1+hI}\right)^2\right. \\ &\left.+2\sigma_1^2(1-\varepsilon)^2\left(\frac{mI}{1+hI}\right)^2 S_uS_a + \sigma_2^2S_u^2 + \sigma_3^2S_a^2 + \sigma_4^2I^2 + \sigma_5^2C^2 + \sigma_6^2A^2 + \sigma_7^2m^2\right) \\ \leq& (S_u + S_a + I + C + A + m)^p(\Lambda - (\mu \wedge b)(S_u + S_a + I + C + A + m) + 2mb) \end{aligned}$$

$$
\begin{aligned}
&+\frac{p}{2}(S_u+S_a+I+C+A+m)^{p-1}(2\sigma_1^2m^2\varsigma^2+2\sigma_1^2(1-\varepsilon)^2m^2\varsigma^2+2\sigma_1^2(1-\varepsilon)m^2\varsigma^2\\
&+\sigma_2^2S_u^2+\sigma_3^2S_a^2+\sigma_4^2I^2+\sigma_5^2C^2+\sigma_6^2A^2+\sigma_7^2m^2)\\
\leq&(\Lambda+2\iota b)(S_u+S_a+I+C+A+m)^p-\tilde{P}(S_u+S_a+I+C+A+m)^{p+1}\\
\leq&B-\frac{1}{2}\tilde{P}(S_u+S_a+I+C+A+m)^{p+1}\\
\leq&B-\frac{1}{2}\tilde{P}(S_u^{p+1}+S_a^{p+1}+I^{p+1}+C^{p+1}+A^{p+1}+m^{p+1}),
\end{aligned}
\tag{5.3}
$$

where

$$
B=\sup_{(S_u,S_a,I,C,A,m)\in\mathbb{R}_+^6}\left\{(\Lambda+2\iota b)(S_u+S_a+I+C+A+m)^p-\frac{1}{2}\tilde{P}(S_u+S_a+I+C+A+m)^{p+1}\right\}<\infty.
$$

Furthermore

$$
LV_3=-\frac{\Lambda}{S_u}+\frac{\beta mI}{1+hI}+\frac{\sigma_1^2}{2}\left(\frac{mI}{1+hI}\right)^2+\left(\alpha+\mu+\frac{\sigma_2^2}{2}\right),\tag{5.4}
$$

$$
LV_4=-\frac{\alpha S_u}{S_a}+(1-\varepsilon)\frac{\beta mI}{1+hI}+\frac{\sigma_1^2}{2}(1-\varepsilon)^2\left(\frac{mI}{1+hI}\right)^2+\left(\mu+\frac{\sigma_3^2}{2}\right),\tag{5.5}
$$

$$
LV_5=-\frac{\rho I}{C}+\left(\eta+\mu+\frac{\sigma_5^2}{2}\right),\tag{5.6}
$$

$$
LV_6=-\frac{\gamma I}{A}+\left(\upsilon+\mu+d+\frac{\sigma_6^2}{2}\right),\tag{5.7}
$$

$$
LV_7=b-am-\frac{\xi I}{1+hI}-\frac{\sigma_7^2}{2}.\tag{5.8}
$$

From Eqs (5.1) and (5.3)–(5.8), we derive the following:

$$
\begin{aligned}
LV\leq&-3K(\sqrt[3]{\hat{R}_0^s}-1)+h_1KI+h_2K+B-\frac{\Lambda}{S_u}-\frac{\alpha S_u}{S_a}-\frac{\rho I}{C}-\frac{\gamma I}{A}-am-\frac{\xi I}{1+hI}\\
&-\frac{1}{2}\tilde{P}(S_u^{p+1}+S_a^{p+1}+I^{p+1}+C^{p+1}+A^{p+1}+m^{p+1})+\alpha+\mu+\frac{\sigma_2^2}{2}+\mu+\frac{\sigma_3^2}{2}\\
&+k_2+\frac{\sigma_5^2}{2}+k_1+\frac{\sigma_6^2}{2}+b-\frac{\sigma_7^2}{2}+\frac{\beta mI}{1+hI}+\frac{\sigma_1^2}{2}\left(\frac{mI}{1+hI}\right)^2+(1-\varepsilon)\frac{\beta mI}{1+hI}\\
&+\frac{\sigma_1^2}{2}(1-\varepsilon)^2\left(\frac{mI}{1+hI}\right)^2\\
\leq&-3K(\sqrt[3]{\hat{R}_0^s}-1)+h_1KI+h_2K+B-\frac{\Lambda}{S_u}-\frac{\alpha S_u}{S_a}-\frac{\rho I}{C}-\frac{\gamma I}{A}-am+k_2+\frac{\sigma_5^2}{2}\\
&-\frac{1}{2}\tilde{P}(S_u^{p+1}+S_a^{p+1}+I^{p+1}+C^{p+1}+A^{p+1}+m^{p+1})+\alpha+\mu+\frac{\sigma_2^2}{2}+\mu+\frac{\sigma_3^2}{2}\\
&+k_1+\frac{\sigma_6^2}{2}+b-\frac{\sigma_7^2}{2}+\beta\iota(2-\varepsilon)+\frac{\sigma_1^2}{2}\iota^2(1+(1-\varepsilon)^2).
\end{aligned}
\tag{5.9}
$$

Define a bounded closed set

$$G = \Big\{ \varepsilon_1 \le S_u(t) \le \frac{1}{\varepsilon_1}, \varepsilon_1^2 \le S_a(t) \le \frac{1}{\varepsilon_1^2}, \varepsilon_1 \le I(t) \le \frac{1}{\varepsilon_1}, \varepsilon_1^2 \le C(t) \le \frac{1}{\varepsilon_1^2},$$
$$\varepsilon_1^2 \le A(t) \le \frac{1}{\varepsilon_1^2}, \varepsilon_1 \le m(t) \le \frac{1}{\varepsilon_1} \Big\},$$

where ε_1 is a sufficiently small positive constant satisfying

$$-\frac{\Lambda}{\varepsilon_1} + E \le -1, \tag{5.10}$$
$$-\frac{\alpha}{\varepsilon_1} + E \le -1, \tag{5.11}$$
$$-\frac{\rho}{\varepsilon_1} + E \le -1, \tag{5.12}$$
$$-\frac{\gamma}{\varepsilon_1} + E \le -1, \tag{5.13}$$
$$-\frac{a}{\varepsilon_1} + E \le -1, \tag{5.14}$$
$$-3K(\sqrt[3]{\hat{R}_0^s} - 1) + h_1 K\varepsilon_1 + E - e \le -1, \tag{5.15}$$
$$-\frac{1}{2}\tilde{P}\frac{1}{\varepsilon_1^{p+1}} + E \le -1, \tag{5.16}$$
$$-\frac{1}{2}\tilde{P}\frac{1}{\varepsilon_1^{p+1}} + E \le -1, \tag{5.17}$$
$$-\frac{1}{2}\tilde{P}\frac{1}{\varepsilon_1^{2p+2}} + E \le -1, \tag{5.18}$$
$$-\frac{1}{4}\tilde{P}\frac{1}{\varepsilon_1^{2p+2}} + E \le -1. \tag{5.19}$$

Denote

$$E := B + e + h_2 K + \alpha + \mu + \frac{\sigma_2^2}{2} + \mu + \frac{\sigma_3^2}{2} + k_2 + \frac{\sigma_5^2}{2} + k_1 + \frac{\sigma_6^2}{2} + b - \frac{\sigma_7^2}{2}$$
$$+ \beta\iota(2-\varepsilon) + \frac{\sigma_1^2}{2}\iota^2(1 + (1-\varepsilon)^2),$$
$$e = \sup_{I\in(0,\infty)} \Big\{ -\frac{1}{4}\tilde{P}I^{p+1} + h_1 KI \Big\} < \infty.$$

According to Eq (5.9), the domain $\mathbb{R}_+^6 \setminus G$ is partitioned into the following 12 regions:

$$G_1 = \{(S_u, S_a, I, C, A, m) \in \mathbb{R}_+^6 : 0 < S_u < \varepsilon_1\},$$
$$G_2 = \{(S_u, S_a, I, C, A, m) \in \mathbb{R}_+^6 : S_u \ge \varepsilon_1, 0 < S_a < \varepsilon_1^2\},$$
$$G_3 = \{(S_u, S_a, I, C, A, m) \in \mathbb{R}_+^6 : I \ge \varepsilon_1, 0 < C < \varepsilon_1^2\},$$
$$G_4 = \{(S_u, S_a, I, C, A, m) \in \mathbb{R}_+^6 : I \ge \varepsilon_1, 0 < A < \varepsilon_1^2\},$$

$$
\begin{aligned}
G_5 =&\{(S_u, S_a, I, C, A, m) \in \mathbb{R}_+^6 : m \geq \varepsilon_1\},\\
G_6 =&\{(S_u, S_a, I, C, A, m) \in \mathbb{R}_+^6 : 0 < I < \varepsilon_1\},\\
G_7 =&\{(S_u, S_a, I, C, A, m) \in \mathbb{R}_+^6 : S_u \geq \frac{1}{\varepsilon_1}\},\\
G_8 =&\{(S_u, S_a, I, C, A, m) \in \mathbb{R}_+^6 : S_a \geq \frac{1}{\varepsilon_1^2}\},\\
G_9 =&\{(S_u, S_a, I, C, A, m) \in \mathbb{R}_+^6 : I \geq \frac{1}{\varepsilon_1}\},\\
G_{10} =&\{(S_u, S_a, I, C, A, m) \in \mathbb{R}_+^6 : C \geq \frac{1}{\varepsilon_1^2}\},\\
G_{11} =&\{(S_u, S_a, I, C, A, m) \in \mathbb{R}_+^6 : A \geq \frac{1}{\varepsilon_1^2}\},\\
G_{12} =&\{(S_u, S_a, I, C, A, m) \in \mathbb{R}_+^6 : m \geq \frac{1}{\varepsilon_1}\}.
\end{aligned}
$$

Scenario 1. When $(S_u, S_a, I, C, A, m) \in G_1$, then it can be deduced that

$$
\begin{aligned}
LV \leq& -\frac{\Lambda}{S_u} - 3K(\sqrt[3]{\hat{R}_0^s} - 1) + h_1 KI + h_2 K + B - \frac{1}{2}\tilde{P}(S_u^{p+1} + S_a^{p+1} + I^{p+1} + C^{p+1} + A^{p+1} + m^{p+1})\\
&+ \alpha + \mu + \frac{\sigma_2^2}{2} + \mu + \frac{\sigma_3^2}{2} + k_2 + \frac{\sigma_5^2}{2} + k_1 + \frac{\sigma_6^2}{2} + b - \frac{\sigma_7^2}{2} + \beta\iota(2-\varepsilon) + \frac{\sigma_1^2}{2}\iota^2(1 + (1-\varepsilon)^2)\\
&- \frac{\alpha S_u}{S_a} - \frac{\rho I}{C} - \frac{\gamma I}{A} - am\\
\leq& -\frac{\Lambda}{S_u} + E \leq -\frac{\Lambda}{\varepsilon_1} + E \leq -1.
\end{aligned}
$$

Scenario 2. When $(S_u, S_a, I, C, A, m) \in G_2$, then it can be deduced that

$$
\begin{aligned}
LV \leq& -\frac{\alpha S_u}{S_a} - 3K(\sqrt[3]{\hat{R}_0^s} - 1) + h_1 KI + h_2 K + B - \frac{1}{2}\tilde{P}(S_u^{p+1} + S_a^{p+1} + I^{p+1} + C^{p+1} + A^{p+1} + m^{p+1})\\
&+ \alpha + \mu + \frac{\sigma_2^2}{2} + \mu + \frac{\sigma_3^2}{2} + k_2 + \frac{\sigma_5^2}{2} + k_1 + \frac{\sigma_6^2}{2} + b - \frac{\sigma_7^2}{2} + \beta\iota(2-\varepsilon) + \frac{\sigma_1^2}{2}\iota^2(1 + (1-\varepsilon)^2)\\
&- \frac{\Lambda}{S_u} - \frac{\rho I}{C} - \frac{\gamma I}{A} - am\\
\leq& -\frac{\alpha S_u}{S_a} + E \leq -\frac{\alpha}{\varepsilon_1} + E \leq -1.
\end{aligned}
$$

Scenario 3. When $(S_u, S_a, I, C, A, m) \in G_3$, then it can be deduced that

$$
\begin{aligned}
LV \leq& -\frac{\rho I}{C} - 3K(\sqrt[3]{\hat{R}_0^s} - 1) + h_1 KI + h_2 K + B - \frac{1}{2}\tilde{P}(S_u^{p+1} + S_a^{p+1} + I^{p+1} + C^{p+1} + A^{p+1} + m^{p+1})\\
&+ \alpha + \mu + \frac{\sigma_2^2}{2} + \mu + \frac{\sigma_3^2}{2} + k_2 + \frac{\sigma_5^2}{2} + k_1 + \frac{\sigma_6^2}{2} + b - \frac{\sigma_7^2}{2} + \beta\iota(2-\varepsilon) + \frac{\sigma_1^2}{2}\iota^2(1 + (1-\varepsilon)^2)\\
&- \frac{\Lambda}{S_u} - \frac{\alpha S_u}{S_a} - \frac{\gamma I}{A} - am
\end{aligned}
$$

$$\leq -\frac{\rho I}{C} + E \leq -\frac{\rho}{\varepsilon_1} + E \leq -1.$$

Scenario 4. When $(S_u, S_a, I, C, A, m) \in G_4$, then it can be deduced that

$$\begin{aligned} LV \leq & -\frac{\gamma I}{A} - 3K(\sqrt[3]{\hat{R}_0^s} - 1) + h_1 KI + h_2 K + B - \frac{1}{2}\tilde{P}(S_u^{p+1} + S_a^{p+1} + I^{p+1} + C^{p+1} + A^{p+1} + m^{p+1}) \\ & + \alpha + \mu + \frac{\sigma_2^2}{2} + \mu + \frac{\sigma_3^2}{2} + k_2 + \frac{\sigma_5^2}{2} + k_1 + \frac{\sigma_6^2}{2} + b - \frac{\sigma_7^2}{2} + \beta\iota(2-\varepsilon) + \frac{\sigma_1^2}{2}\iota^2(1 + (1-\varepsilon)^2) \\ & - \frac{\Lambda}{S_u} - \frac{\alpha S_u}{S_a} - \frac{\rho I}{C} - am \\ \leq & -\frac{\gamma I}{A} + E \leq -\frac{\gamma}{\varepsilon_1} + E \leq -1. \end{aligned}$$

Scenario 5. When $(S_u, S_a, I, C, A, m) \in G_5$, then it can be deduced that

$$\begin{aligned} LV \leq & -am - 3K(\sqrt[3]{\hat{R}_0^s} - 1) + h_1 KI + h_2 K + B - \frac{1}{2}\tilde{P}(S_u^{p+1} + S_a^{p+1} + I^{p+1} + C^{p+1} + A^{p+1} + m^{p+1}) \\ & + \alpha + \mu + \frac{\sigma_2^2}{2} + \mu + \frac{\sigma_3^2}{2} + k_2 + \frac{\sigma_5^2}{2} + k_1 + \frac{\sigma_6^2}{2} + b - \frac{\sigma_7^2}{2} + \beta\iota(2-\varepsilon) + \frac{\sigma_1^2}{2}\iota^2(1 + (1-\varepsilon)^2) \\ & - \frac{\Lambda}{S_u} - \frac{\alpha S_u}{S_a} - \frac{\rho I}{C} - \frac{\gamma I}{A} \\ \leq & -am + E \leq -a\varepsilon_1 + E \leq -1. \end{aligned}$$

Scenario 6. When $(S_u, S_a, I, C, A, m) \in G_6$, then it can be deduced that

$$\begin{aligned} LV \leq & -3K(\sqrt[3]{\hat{R}_0^s} - 1) + h_1 KI + h_2 K + B - \frac{1}{2}\tilde{P}(S_u^{p+1} + S_a^{p+1} + I^{p+1} + C^{p+1} + A^{p+1} + m^{p+1}) \\ & + \alpha + \mu + \frac{\sigma_2^2}{2} + \mu + \frac{\sigma_3^2}{2} + k_2 + \frac{\sigma_5^2}{2} + k_1 + \frac{\sigma_6^2}{2} + b - \frac{\sigma_7^2}{2} + \beta\iota(2-\varepsilon) + \frac{\sigma_1^2}{2}\iota^2(1 + (1-\varepsilon)^2) \\ & - \frac{\Lambda}{S_u} - \frac{\alpha S_u}{S_a} - \frac{\rho I}{C} - \frac{\gamma I}{A} - am \\ \leq & 3K(\sqrt[3]{\hat{R}_0^s} - 1) + h_1 KI + E - e \leq 3K(\sqrt[3]{\hat{R}_0^s} - 1) + h_1 K\varepsilon_1 + E - e \leq -1. \end{aligned}$$

Scenario 7. When $(S_u, S_a, I, C, A, m) \in G_7$, then it can be deduced that

$$\begin{aligned} LV \leq & -\frac{1}{2}\tilde{P}S_u^{p+1} + h_2 K + B - \frac{1}{2}\tilde{P}(S_a^{p+1} + I^{p+1} + C^{p+1} + A^{p+1} + m^{p+1}) \\ & + \alpha + \mu + \frac{\sigma_2^2}{2} + \mu + \frac{\sigma_3^2}{2} + k_2 + \frac{\sigma_5^2}{2} + k_1 + \frac{\sigma_6^2}{2} + b - \frac{\sigma_7^2}{2} + \beta\iota(2-\varepsilon) + \frac{\sigma_1^2}{2}\iota^2(1 + (1-\varepsilon)^2) \\ & - \frac{\Lambda}{S_u} - \frac{\alpha S_u}{S_a} - \frac{\rho I}{C} - \frac{\gamma I}{A} - am - 3K(\sqrt[3]{\hat{R}_0^s} - 1) + h_1 KI \\ \leq & -\frac{1}{2}\tilde{P}S_u^{p+1} + E \leq -\frac{1}{2}\tilde{P}\frac{1}{\varepsilon_1^{p+1}} + E \leq -1. \end{aligned}$$

Scenario 8. When $(S_u, S_a, I, C, A, m) \in G_8$, then it can be deduced that

$$\begin{aligned} LV \leq & -\frac{1}{2}\tilde{P}S_a^{p+1} + h_2K + B - \frac{1}{2}\tilde{P}(S_u^{p+1} + I^{p+1} + C^{p+1} + A^{p+1} + m^{p+1}) \\ & + \alpha + \mu + \frac{\sigma_2^2}{2} + \mu + \frac{\sigma_3^2}{2} + k_2 + \frac{\sigma_5^2}{2} + k_1 + \frac{\sigma_6^2}{2} + b - \frac{\sigma_7^2}{2} + \beta\iota(2-\varepsilon) + \frac{\sigma_1^2}{2}\iota^2(1+(1-\varepsilon)^2) \\ & - \frac{\Lambda}{S_u} - \frac{\alpha S_u}{S_a} - \frac{\rho I}{C} - \frac{\gamma I}{A} - am - 3K(\sqrt[3]{\hat{R}_0^s} - 1) + h_1KI \\ \leq & -\frac{1}{2}\tilde{P}S_a^{p+1} + E \leq -\frac{1}{2}\tilde{P}\frac{1}{\varepsilon_1^{2p+2}} + E \leq -1. \end{aligned}$$

Scenario 9. When $(S_u, S_a, I, C, A, m) \in G_9$, then it can be deduced that

$$\begin{aligned} LV \leq & -\frac{1}{4}\tilde{P}I^{p+1} + h_2K + B - \frac{1}{2}\tilde{P}(S_u^{p+1} + S_a^{p+1} + C^{p+1} + A^{p+1} + m^{p+1}) \\ & + \alpha + \mu + \frac{\sigma_2^2}{2} + \mu + \frac{\sigma_3^2}{2} + k_2 + \frac{\sigma_5^2}{2} + k_1 + \frac{\sigma_6^2}{2} + b - \frac{\sigma_7^2}{2} + \beta\iota(2-\varepsilon) + \frac{\sigma_1^2}{2}\iota^2(1+(1-\varepsilon)^2) \\ & - \frac{\Lambda}{S_u} - \frac{\alpha S_u}{S_a} - \frac{\rho I}{C} - \frac{\gamma I}{A} - am - 3K(\sqrt[3]{\hat{R}_0^s} - 1) + h_1KI \\ \leq & -\frac{1}{4}\tilde{P}I^{p+1} + E \leq -\frac{1}{4}\tilde{P}\frac{1}{\varepsilon_1^{p+1}} + E \leq -1. \end{aligned}$$

Scenario 10. When $(S_u, S_a, I, C, A, m) \in G_{10}$, then it can be deduced that

$$\begin{aligned} LV \leq & -\frac{1}{2}\tilde{P}C^{p+1} + h_2K + B - \frac{1}{2}\tilde{P}(S_u^{p+1} + S_a^{p+1} + I^{p+1} + A^{p+1} + m^{p+1}) \\ & + \alpha + \mu + \frac{\sigma_2^2}{2} + \mu + \frac{\sigma_3^2}{2} + k_2 + \frac{\sigma_5^2}{2} + k_1 + \frac{\sigma_6^2}{2} + b - \frac{\sigma_7^2}{2} + \beta\iota(2-\varepsilon) + \frac{\sigma_1^2}{2}\iota^2(1+(1-\varepsilon)^2) \\ & - \frac{\Lambda}{S_u} - \frac{\alpha S_u}{S_a} - \frac{\rho I}{C} - \frac{\gamma I}{A} - am - 3K(\sqrt[3]{\hat{R}_0^s} - 1) + h_1KI \\ \leq & -\frac{1}{2}\tilde{P}C^{p+1} + E \leq -\frac{1}{2}\tilde{P}\frac{1}{\varepsilon_1^{2p+2}} + E \leq -1. \end{aligned}$$

Scenario 11. When $(S_u, S_a, I, C, A, m) \in G_{11}$, then it can be deduced that

$$\begin{aligned} LV \leq & -\frac{1}{2}\tilde{P}A^{p+1} + h_2K + B - \frac{1}{2}\tilde{P}(S_u^{p+1} + S_a^{p+1} + I^{p+1} + C^{p+1} + m^{p+1}) \\ & + \alpha + \mu + \frac{\sigma_2^2}{2} + \mu + \frac{\sigma_3^2}{2} + k_2 + \frac{\sigma_5^2}{2} + k_1 + \frac{\sigma_6^2}{2} + b - \frac{\sigma_7^2}{2} + \beta\iota(2-\varepsilon) + \frac{\sigma_1^2}{2}\iota^2(1+(1-\varepsilon)^2) \\ & - \frac{\Lambda}{S_u} - \frac{\alpha S_u}{S_a} - \frac{\rho I}{C} - \frac{\gamma I}{A} - am - 3K(\sqrt[3]{\hat{R}_0^s} - 1) + h_1KI \\ \leq & -\frac{1}{2}\tilde{P}A^{p+1} + E \leq -\frac{1}{2}\tilde{P}\frac{1}{\varepsilon_1^{2p+2}} + E \leq -1. \end{aligned}$$

Scenario 12. When $(S_u, S_a, I, C, A, m) \in G_{12}$, then it can be deduced that

$$
\begin{aligned}
LV \leq & -\frac{1}{2}\tilde{P}m^{p+1} + h_2K + B - \frac{1}{2}\tilde{P}(S_u^{p+1} + S_a^{p+1} + I^{p+1} + C^{p+1} + A^{p+1}) \\
& + \alpha + \mu + \frac{\sigma_2^2}{2} + \mu + \frac{\sigma_3^2}{2} + k_2 + \frac{\sigma_5^2}{2} + k_1 + \frac{\sigma_6^2}{2} + b - \frac{\sigma_7^2}{2} + \beta\iota(2-\varepsilon) + \frac{\sigma_1^2}{2}\iota^2(1+(1-\varepsilon)^2) \\
& - \frac{\Lambda}{S_u} - \frac{\alpha S_u}{S_a} - \frac{\rho I}{C} - \frac{\gamma I}{A} - am - 3K(\sqrt[3]{\hat{R}_0^s} - 1) + h_1KI \\
\leq & -\frac{1}{2}\tilde{P}m^{p+1} + E \leq -\frac{1}{2}\tilde{P}\frac{1}{\varepsilon_1^{p+1}} + E \leq -1.
\end{aligned}
$$

Therefore, for sufficiently small ε_1, if $(S_u, S_a, I, C, A, m) \in \mathbb{R}_+^6 \setminus G$, we have

$$LV(S_u, S_a, I, C, A, m) \leq -1.$$

This implies that condition (ii) of Has'minskii's theorem is satisfied. Putting these together, system (2.5) admits a unique ergodic stationary distribution. This completes the proof.

□

6. Numerical simulation

This section employs computer simulations to validate the theoretical results and analyze the impacts of environmental noise and human mobility on the transmission dynamics of HIV/AIDS. Utilizing the Milstein method [30], the discretized equations for system (2.5) are formulated as follows:

$$
\begin{cases}
\begin{aligned}
S_u(i+1) = \; & S_u(i) + \left(\Lambda - \alpha S_u(i) - \tfrac{\beta m(i) S_u(i) I(i)}{1+hI(i)} - \mu S_u(i)\right) \triangle t - \sigma_1 \tfrac{m(i) S_u(i) I(i)}{1+hI(i)} \omega_{1,i} \sqrt{\triangle t} \\
& + \tfrac{1}{2}\sigma_1^2 S_u(i) \left(\tfrac{m(i)I(i)}{1+hI(i)}\right)^2 (\omega_{1,i}^2 - 1) \triangle t + \sigma_2 S_u(i) \omega_{2,i} \sqrt{\triangle t} + \tfrac{1}{2}\sigma_2^2 S_u(i)(\omega_{2,i}^2 - 1) \triangle t, \\
S_a(i+1) = \; & S_a(i) + \left(\alpha S_u(i) - (1-\varepsilon)\tfrac{\beta m(i) S_a(i) I(i)}{1+hI(i)} - \mu S_a(i)\right) \triangle t - \sigma_1 (1-\varepsilon) \tfrac{m(i) S_a(i) I(i)}{1+hI(i)} \omega_{1,i} \sqrt{\triangle t} \\
& + \tfrac{1}{2}\sigma_1^2 S_a(i) \left((1-\varepsilon)\tfrac{m(i)I(i)}{1+hI(i)}\right)^2 (\omega_{1,i}^2 - 1) \triangle t + \sigma_3 S_a(i) \omega_{3,i} \sqrt{\triangle t} + \tfrac{1}{2}\sigma_3^2 S_a(i)(\omega_{3,i}^2 - 1) \triangle t, \\
I(i+1) = \; & I(i) + \left(\tfrac{\beta m(i) S_u(i) I(i)}{1+hI(i)} + (1-\varepsilon)\tfrac{\beta m(i) S_a(i) I(i)}{1+hI(i)} + \eta C(i) + \upsilon A(i) - (\rho + \gamma + \mu) I(i)\right) \triangle t \\
& + \sigma_1 \left(\tfrac{m(i) S_u(i) I(i)}{1+hI(i)} + (1-\varepsilon)\tfrac{m(i) S_a(i) I(i)}{1+hI(i)}\right) \omega_{1,i} \sqrt{\triangle t} + \tfrac{1}{2}\sigma_4^2 I(i)(\omega_{4,i}^2 - 1) \triangle t \\
& + \tfrac{1}{2}\sigma_1^2 \tfrac{(m(i)(S_u(i) + (1-\varepsilon) S_a(i)))^2}{(1+hI(i))^3} I(i)(\omega_{1,i}^2 - 1) \triangle t + \sigma_4 I(i) \omega_{4,i} \sqrt{\triangle t}, \\
C(i+1) = \; & C(i) + (\rho I(i) - (\eta + \mu) C(i)) \triangle t + \sigma_5 C(i) \omega_{5,i} \sqrt{\triangle t} + \tfrac{1}{2}\sigma_5^2 C(i)(\omega_{5,i}^2 - 1) \triangle t, \\
A(i+1) = \; & A(i) + (\gamma I(i) - (\upsilon + \mu + d) A(i)) \triangle t + \sigma_6 A(i) \omega_{6,i} \sqrt{\triangle t} + \tfrac{1}{2}\sigma_6^2 A(i)(\omega_{6,i}^2 - 1) \triangle t, \\
m(i+1) = \; & m(i) + m(i) \left(b - am(i) - \tfrac{\xi I(i)}{1+hI(i)}\right) \triangle t + \sigma_7 m(i) \omega_{7,i} \sqrt{\triangle t} + \tfrac{1}{2}\sigma_7^2 m(i)(\omega_{7,i}^2 - 1) \triangle t.
\end{aligned}
\end{cases}
\tag{6.1}
$$

Here, $\omega_{k,i}$ for $k = 1, 2, \ldots, 7$ denote mutually independent Gaussian random variables each adhering to the standard normal distribution $N(0, 1)$.

This section employs computer simulations to validate the theoretical results and analyze the impacts of environmental noise and human mobility on HIV transmission dynamics. While the current study focuses on theoretical analysis and numerical verification, we acknowledge that empirical validation against real-world HIV transmission data would significantly enhance the model's practical relevance. Future work should aim to calibrate the model parameters using epidemiological

surveillance data from regions with varying mobility patterns and HIV prevalence rates to establish the model's predictive capability in practical public health contexts.

The parameters listed in Table 1 are held constant to establish clear baseline dynamics and identify fundamental thresholds like R_0 and R_0^s. The assumption of constant parameters represents a simplification for theoretical analysis. In practical application, certain parameters, such as the transmission rate β or the recruitment rate Λ, could be modeled as time-varying functions or stochastic processes to reflect seasonal trends, intervention programs, or other temporal influences, further enhancing the model's descriptive power for specific epidemiological contexts.

Table 1. Parameter descriptions and values.

Parameter	Description	Values	Unit	References
α	transition rate from S_u to S_a	0.23	day^{-1}	[17]
h	psychological effect	0.02	dimensionless	Assumed
a	constants related to population activity	0.1	dimensionless	Assumed
b	constants related to population activity	0.2	dimensionless	Assumed
ξ	public behavioral response to infection risk	0.1	dimensionless	Assumed
ε	effectiveness of protective measures	[0.72,0.36]	dimensionless	[17]
d	AIDS mortality rate	0.9	day^{-1}	[16]
μ	natural mortality rate	0.15	day^{-1}	Assumed
γ	transition rate from I to A	0.1	day^{-1}	[31]
υ	transition rate from A to I	0.33	day^{-1}	[31]
Λ	recruitment rate	0.75	Individuals/day	Assumed
β	incidence rate coefficient	[0.03,0.26]	$contact^{-1}\cdot day^{-1}$	Assumed
η	transition rate from C to I	0.18	day^{-1}	[17]
ρ	transition rate from I to C	[0.15,0.32]	day^{-1}	Assumed

First, select parameter values $\beta = 0.03$, $\varepsilon = 0.72$, $\rho = 0.15$, and other parameter values listed in Table 1. Calculation yields the basic reproduction number $R_0 = 0.5752 < 1$ for the deterministic system (2.4). Under this condition, all solution trajectories of system (2.4) converge to the disease-free equilibrium $E_0 = (1.9734, 3.0263, 0, 0, 0, 2)$, which is globally asymptotically stable. Next, set the noise intensities as $\sigma_1 = 0.07$, $\sigma_2 = 0.02$, $\sigma_3 = 0.04$, $\sigma_4 = 0.06$, $\sigma_5 = 0.04$, $\sigma_6 = 0.05$, $\sigma_7 = 0.06$. The stochastic reproduction number is computed as $R_0^s = 0.2843 < 1$. According to Theorem 4, $I(t)$, $C(t)$, and $A(t)$ in the stochastic system (2.5) go extinct almost surely, a conclusion validated by numerical simulations in Figure 2.

Next, we select parameter values $\beta = 0.26$, $\varepsilon = 0.36$, $\rho = 0.32$, and other parameters listed in Table 1. For these parameter values, the basic reproduction number of the deterministic system is $R_0 = 5.4731$, indicating the existence of an endemic equilibrium $E^* = (1.5476, 1.6406, 0.7315, 0.7093, 0.0530, 0.5582)$ for the deterministic system (2.4). This equilibrium is globally asymptotically stable, as demonstrated in Figure 3. We set the white noise intensities to $\sigma_1 = 0.1$, $\sigma_2 = 0.1$, $\sigma_3 = 0.1$, $\sigma_4 = 0.1$, $\sigma_5 = 0.1$, $\sigma_6 = 0.1$, $\sigma_7 = 0.1$. The calculations yield $\hat{R}_0^s = 2.5860 > 1$ and $b = 0.2 > \frac{\sigma_7^2}{2} = 0.0018$, confirming that the disease will persist and the stochastic system (2.5) admits a unique ergodic stationary distribution. As shown in Figure 3, the population sizes of $S_u(t)$, $S_a(t)$, $I(t)$, $C(t)$, and $A(t)$ fluctuate around the deterministic steady-state

values E^*, indicating that the disease will almost surely persist. Figure 4 displays the probability density functions of $S_u(t)$, $S_a(t)$, $I(t)$, $C(t)$, and $A(t)$ obtained from 100,000 sampled trajectories of the stochastic system (2.5), confirming the existence of a unique ergodic stationary distribution.

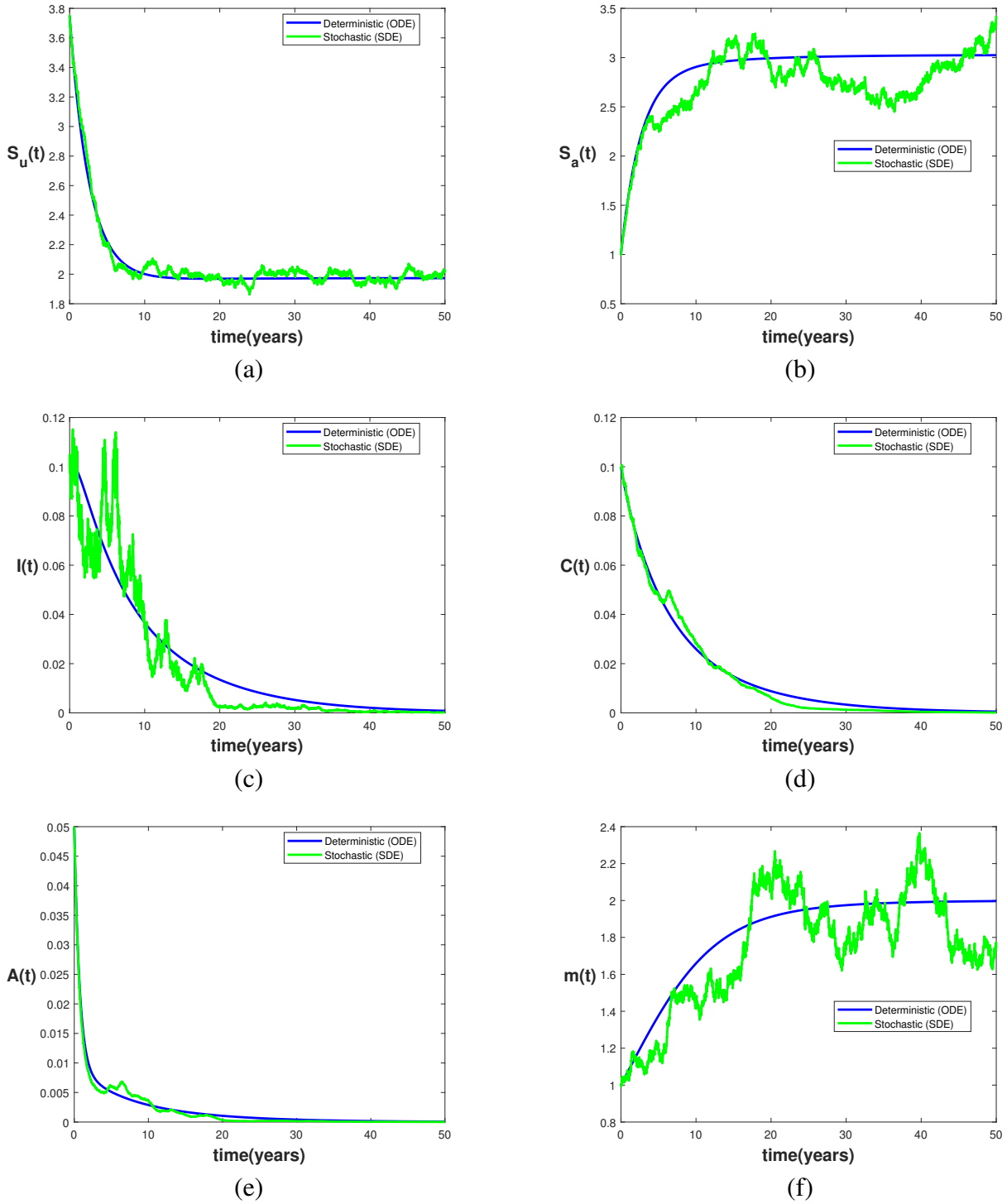

Figure 2. When $R_0 = 0.5752 < 1$ and $R_0^s = 0.2843 < 1$, the time series plots of the stochastic system (2.5) and its corresponding deterministic system (2.4) are presented.

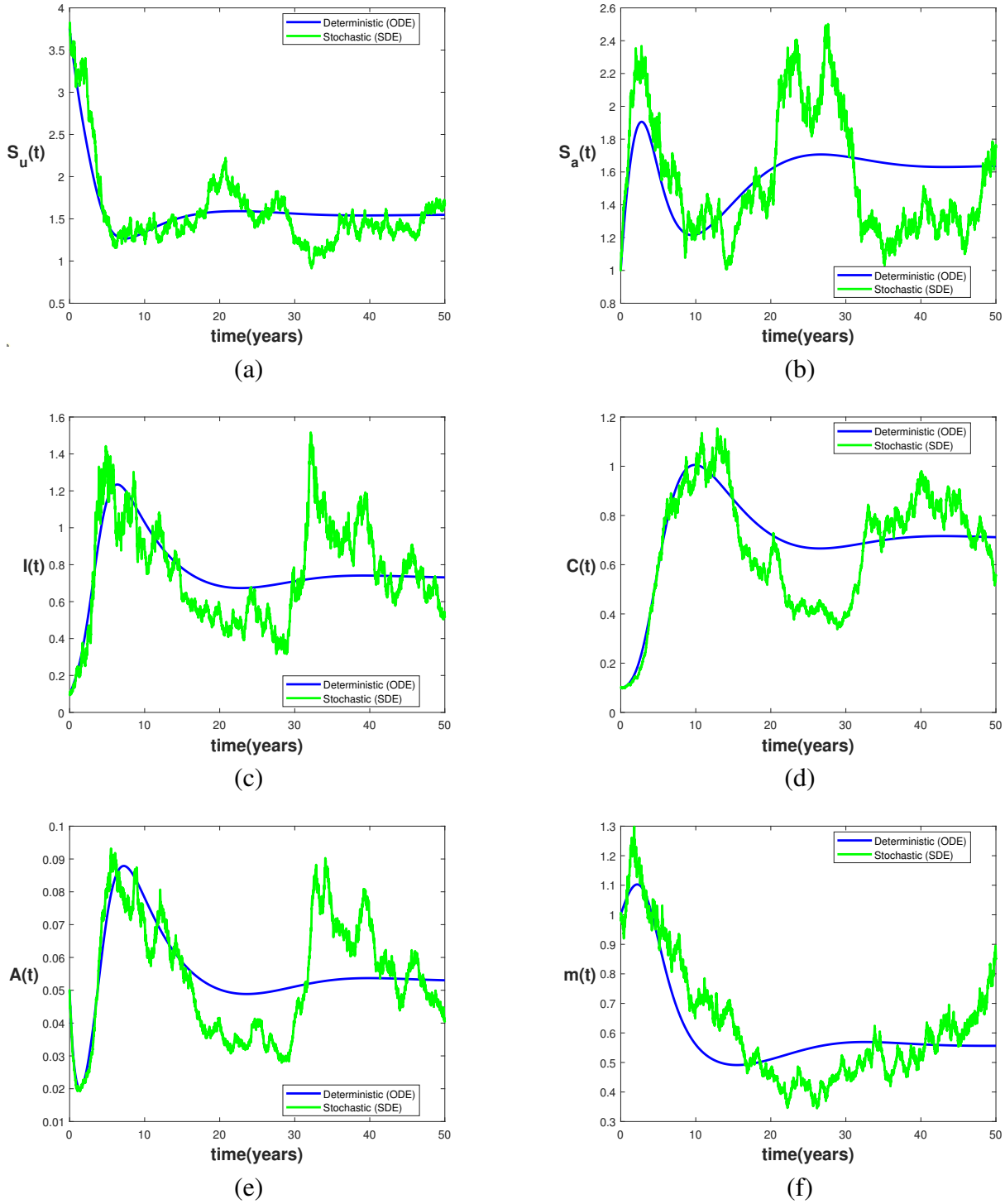

Figure 3. When $R_0 = 5.4731 > 1$ and $\hat{R}_0^s = 2.5860 > 1$, the time series plots of the stochastic system (2.5) and its corresponding deterministic system (2.4) are presented.

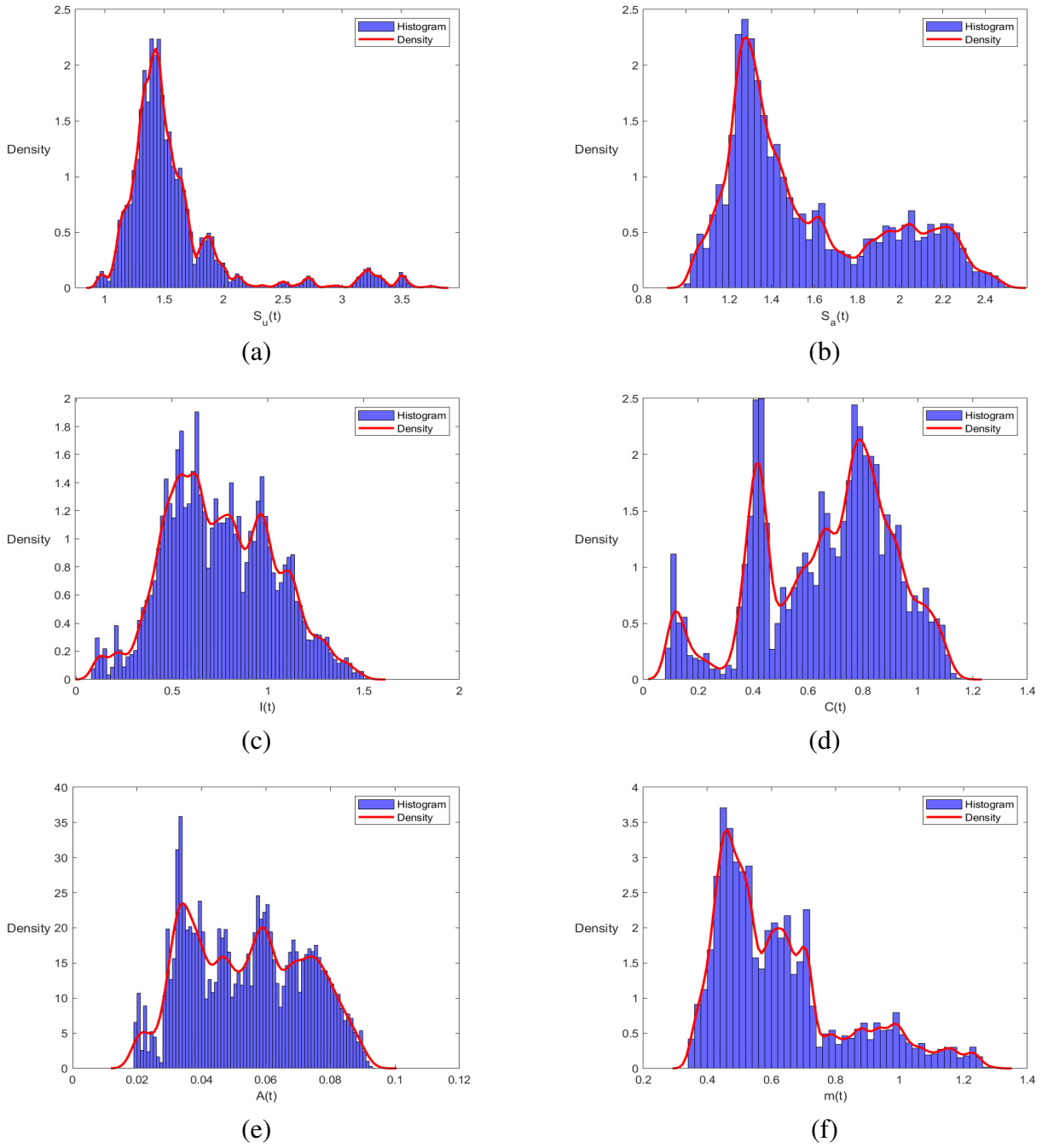

Figure 4. When $R_0 = 5.4731 > 1$ and $\hat{R}_0^s = 2.5860 > 1$, the histograms and probability density function plots of the stochastic system (2.5) at T = 30 are presented.

To investigate the impact of noise intensity on stochastic dynamics, we set three groups of noise parameters σ_i $(i = 1, 2, 3, 4, 5, 6, 7)$ to 0.009, 0.04, and 0.5, respectively, while keeping other parameters consistent with Figure 3. The corresponding stochastic thresholds $\hat{R}_0^s$ were calculated as 2.7353, 2.7116, and 0.7054 (numerical results shown in Figure 5). Figure 5(a,c,e) displays the sample paths of infected individuals $I(t)$ for the stochastic system (2.5) under these three noise regimes, along with the trajectories of the corresponding deterministic system (2.4). The results demonstrate that

$I(t)$ exhibits oscillatory behavior around the deterministic trajectory, with both the irregularity and amplitude of stochastic fluctuations increasing with σ_i. Notably, strong noise ($\sigma_i = 0.5$) can suppress HIV transmission, as evidenced in Figure 5(e).

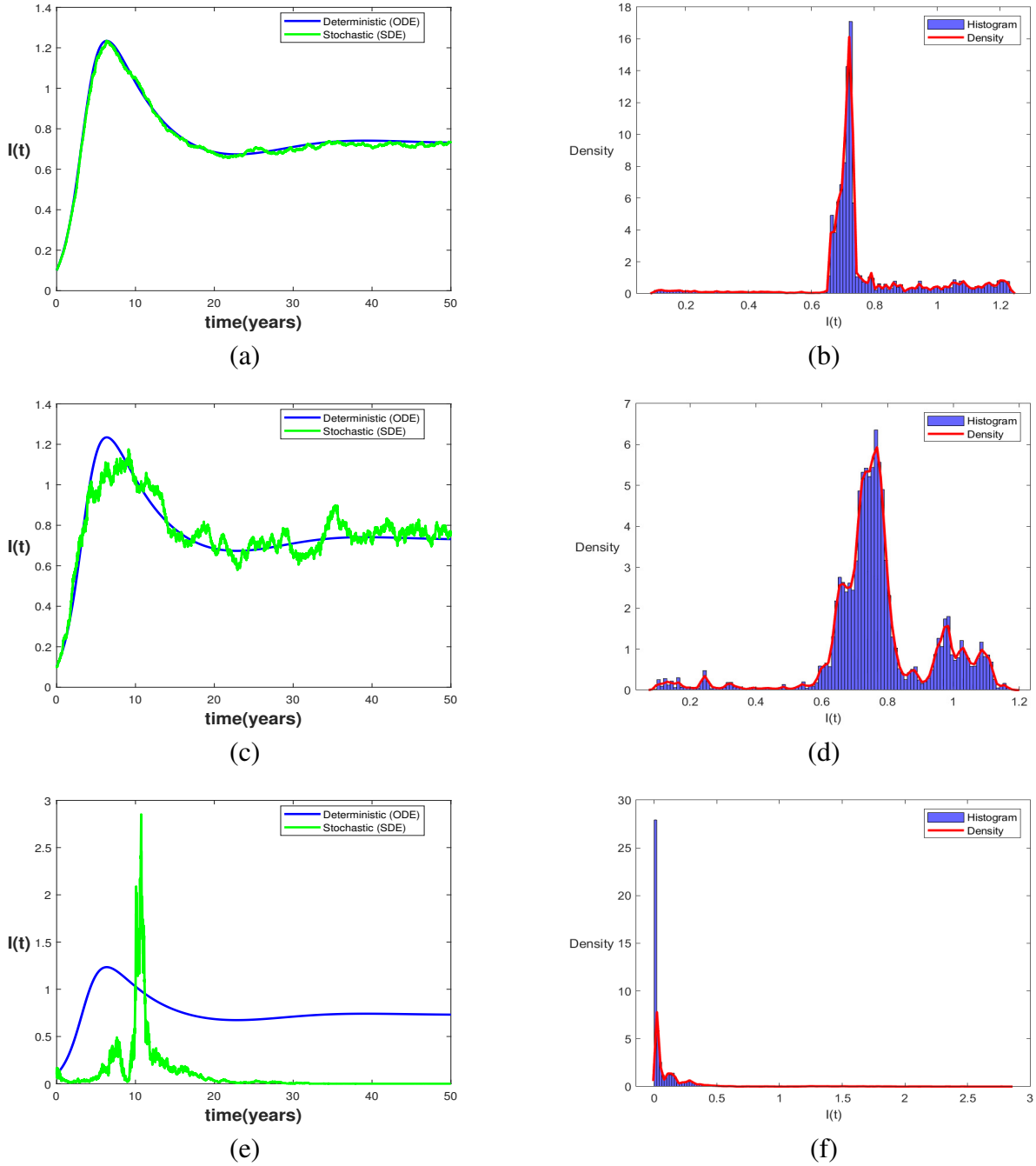

Figure 5. (a), (c), (e): Time series plots of $I(t)$ for the stochastic system (2.5) and its corresponding deterministic system (2.4). (b), (d), (f): Histograms and probability density function (PDF) plots of $I(t)$ for the stochastic system (2.5).

Finally, we employ numerical methods to investigate the impact of behavioral response to infection risk (characterized by ξ) on HIV dynamics. To further explore the effects of stochasticity on HIV transmission, we conduct comparative analyses between the deterministic system (2.4) and the stochastic system (2.5). Notably, while ξ is proportional to β, the basic reproduction number R_0 does not explicitly depend on ξ, implying that ξ does not directly influence R_0. To study the effect of ξ on HIV transmission dynamics, we select values $\xi = 0, 0.05, 0.1, 0.2$ while keeping other parameters consistent with Figure 3, maintaining $R_0 = 5.4731$. Figure 6 provides an overview of the dynamical behavior in the deterministic system. As ξ increases, we observe a clear trend of decreasing infected populations and increasing susceptible populations, indicating that elevated behavioral response (ξ) significantly reduces HIV prevalence. Consequently, reduced disease prevalence correlates with increased population mobility.

Similar to the deterministic system, ξ does not directly affect the stochastic threshold $\hat{R}_0^s$ of system (2.5). To further investigate the mechanistic impact of ξ on the dynamical behavior of the stochastic system, we retain the parameter values from Figure 3 where $\hat{R}_0^s = 2.5860 > 1$. Figure 7 illustrates the dynamical behavior of system (2.5) for $\xi = 0, 0.05, 0.1, 0.2$. As shown, under all these conditions, the sample paths exhibit oscillatory behavior around the trajectory of the deterministic system, with no observed extinction events. This suggests that while increased behavioral response (ξ) reduces HIV prevalence in the deterministic framework (as demonstrated in Figure 6), its primary stochastic effect manifests as modulated fluctuations around the persistent endemic state rather than altering the extinction threshold.

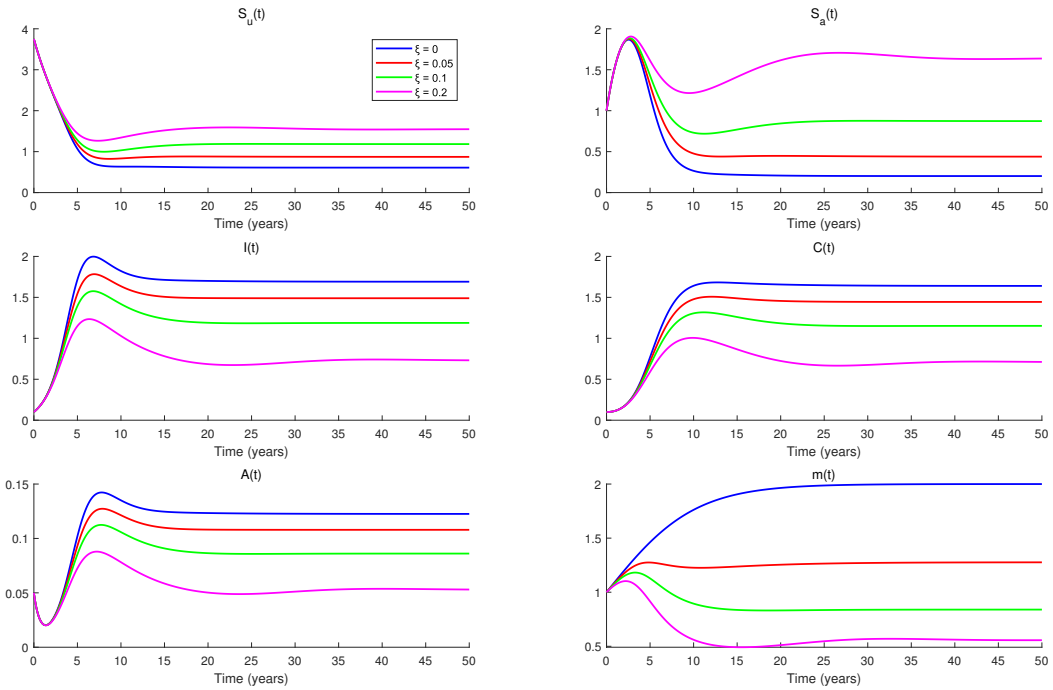

Figure 6. The impact of ξ on the transmission dynamics of the deterministic model (2.4).

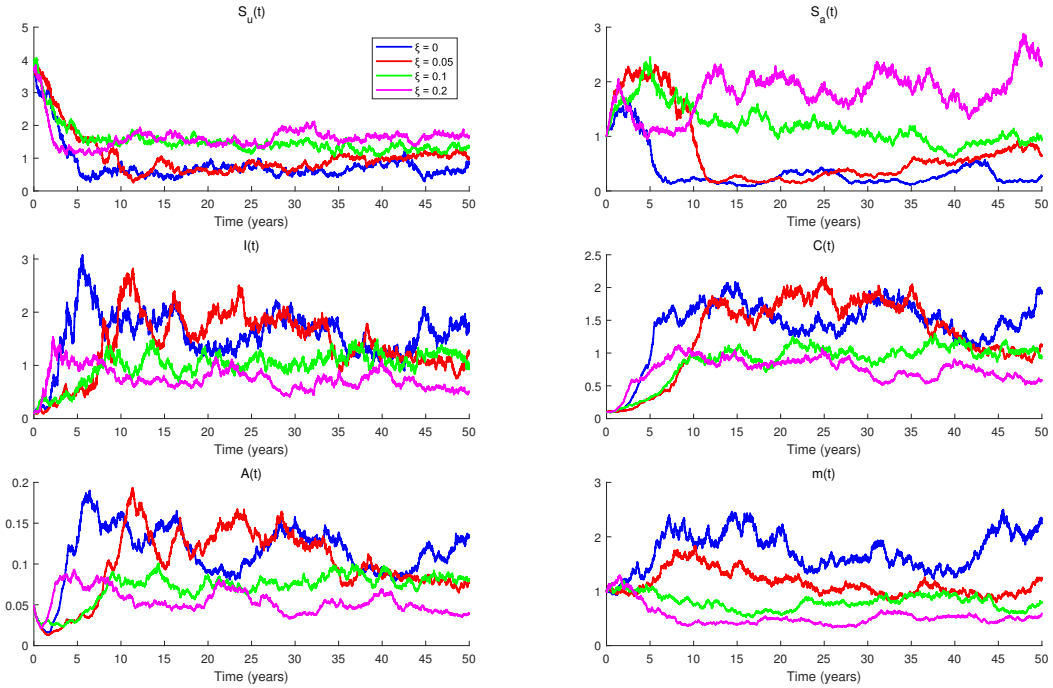

Figure 7. The impact of ξ on the transmission dynamics of the stochastic model (2.5).

7. Conclusions

This study incorporates Gauss white noise into the stochastic HIV model (2.5) that accounts for population mobility, using a combination of parameter perturbation and system disturbance to characterize stochastic environmental fluctuations. First, by employing Lyapunov analysis methods and contradiction arguments, we prove that for any initial value, system (2.5) admits a unique globally positive solution. Second, when $R_0^s < 1$, the disease ultimately undergoes extinction (see Theorem 4 and Figure 2). Finally, if $\hat{R}_0^s > 1$, the solution of the system admits a unique ergodic stationary distribution, implying persistent disease prevalence (see Theorem 5 and Figures 3 and 4).

Through combined theoretical analysis and numerical simulations, we demonstrate that the extinction dynamics and existence of stationary distributions for HIV transmission are closely associated with these stochastic fluctuations. It can be concluded that even when HIV persists in deterministic systems, the disease may still undergo extinction under Gaussian white noise disturbances. High-intensity Gaussian white noise can suppress HIV transmission, suggesting that the disease could be effectively controlled in environments with such strong stochastic perturbations. These simulation results are illustrated in Figure 5.

Although the stochastic threshold $\hat{R}_0^s$ for system (2.5) does not explicitly involve the parameter ξ, ξ is proportional to β, establishing an implicit relationship between $\hat{R}_0^s$ and ξ. As demonstrated in Figures 6 and 7, enhanced behavioral responses to infection risk significantly reduce disease transmission. Furthermore, improving the effectiveness of protective measures and reducing HIV transmission coefficients—such as strengthening AIDS awareness campaigns, implementing post-exposure isolation protocols within 72 hours, and ensuring timely antiretroviral therapy for infected

individuals—can effectively prevent and control disease spread.

While the present parameter perturbation analysis exclusively examines the stochastic effects on the HIV transmission rate β, future investigations should extend to other critical epidemiological parameters—such as AIDS-related mortality, the progression rate from I to A, and population risk perception sensitivity. Furthermore, moving beyond the white noise assumption to incorporate more complex stochastic processes like colored noise or Lévy jumps [14, 32, 33] would better reflect real-world transmission dynamics. Advancing the numerical framework with techniques tailored for complex stochastic systems [34–36] will be crucial for these endeavors, all of which are prioritized in our subsequent work to deepen the understanding of HIV transmission dynamics.

Author contributions

J. H. Yan: Validation and Writing—Review and editing, Formal analysis, Writing—Original draft, Conceptualization. W. Q. Wu: Methodology, Software, Writing—Review and editing, and Project administration. X. W. Tan: Project administration, Visualization, and Writing—Review and editing. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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