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Research article

On logarithmic coefficients for starlike functions related to nephroid domain

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Abstract: Logarithmic functions are widely used in mathematics and other fields for various applications. As far as we know, no one has used the bounds for the third Hankel determinant using the coefficients of logarithmic functions. This article examined various classes of starlike functions and addressed the problem of the third Hankel determinant concerning logarithmic coefficients for special subclasses related to nephroid functions. Several coefficient estimates were derived, and some of these results were proven to be sharp.

Keywords: analytic functions; Schwarz functions; Hankel determinant; Caratheodory functions; logarithmic coefficients; starlike functions
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1. Introduction

Let A denote the class of functions f given by

$$f(z) = z + \sum_{k=2}^{\infty} b_k z^k \quad (z \in U), \tag{1.1}$$

which are analytic within the open unit disk

$$U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Furthermore, the subclass of functions that are univalent in the set U is represented by $\mathcal{S}$. For two functions h_1 and h_2 belonging to A, we say that h_1 is subordinate to h_2 (written as $h_1 \prec h_2$) if there exists an analytic function w that satisfies

$$|w(z)| \leq 1 \text{ and } \quad w(0) = 0,$$

such that

$$h_1(z) = h_2(w(z)) \qquad (z \in U).$$

Furthermore, if $h_2 \in \mathcal{S}$, the above conditions can be expressed as follows:

$$h_1 \prec h_2 \Leftrightarrow h_1(0) = h_2(0) \text{ and } h_1(U) \subset h_2(U).$$

A famous problem in the theory of univalent functions was solved in 1985 by De Branges [1], who stated the Bieberbach conjecture [2] about coefficient estimates for class $\mathcal{S}$. Geometric function theory was established in the 18th century. This theory introduced a new direction for research in this area, specifically focusing on coefficient bounds. Numerous prominent researchers have examined several subclasses of class $\mathcal{S}$ from various perspectives, focusing on different types of domains and functions. Fekete-Szegö (1933) discovered the Fekete-Szegö inequality for coefficients of univalent analytic functions and related it to the Bieberbach conjecture [2]. In 1992, Ma and Minda [3] established the framework for families of univalent functions:

$$\mathcal{S}^*(\phi) = \left\{ f \in \mathcal{A}\colon \ \frac{zf'(z)}{f(z)} \prec \phi(z) \right\}. \tag{1.2}$$

The analytic function ϕ is such that $\Re(\phi(z))$ exists for every $z \in U$. Suppose that $\phi_{Ne}(z) = 1 + z - \frac{z^3}{3}$. In recent years, several subfamilies of generalized analytic functions, particularly as examples of $\mathcal{S}^*(\phi)$, have been extensively studied. For example, Wani, L. A. Swaminathan [4] studied a specific category of starlike functions associated with nephroid functions, which are described as follows:

$$\mathcal{S}^*_{Ne} = \left\{ f \in \mathcal{A}\colon \ \frac{zf'(z)}{f(z)} \prec 1 + z - \frac{z^3}{3} \right\}.$$

Pommerenke [5, 6] examined the Hankel determinant $H_{q,n}(f)$ for a function $f \in \mathcal{A}$ defined in (1.1). The following is the definition of the Hankel determinant, represented by its symbol $H_{q,k}(f)$:

$$H_{q,k}(f) = \begin{vmatrix} b_k & b_{k+1} & \cdots & b_{k+q-1} \\ b_{k+1} & b_{k+2} & \cdots & b_{k+q} \\ \vdots & \vdots & \cdots & \vdots \\ b_{k+q-1} & b_{k+q} & \cdots & b_{k+2q-2} \end{vmatrix}. \tag{1.3}$$

We derive Hankel determinants with respect to the variation in q and k. For example, the determinant described above takes the following form when $k = 1$ and $q = 2$:

$$\left|H_{2,1}(f)\right| = \begin{vmatrix} b_1 & b_2 \\ b_2 & b_3 \end{vmatrix} = \left|b_3 - b_2^2\right|, \quad \text{where } b_1 = 1.$$

Moreover, we examine the second Hankel determinant $q = k = 2$.

$$\left|H_{2,2}(f)\right| = \begin{vmatrix} b_2 & b_3 \\ b_3 & b_4 \end{vmatrix} = b_2 b_4 - b_3^2,$$

and

$$\left|H_{3,1}(f)\right| = \begin{vmatrix} b_1 & b_2 & b_3 \\ b_2 & b_3 & b_4 \\ b_3 & b_4 & b_5 \end{vmatrix} = 2b_2b_3b_4 - b_3^3 - b_4^2 + b_3b_5 - b_5b_2^2. \tag{1.4}$$

For univalent starlike functions, Pommerenke [7] studied the Hankel determinants in 1966. In 2010, Babalola [8] studied the third Hankel determinant for certain types of univalent functions. Srivastava et al. [9] stated that Gegenbauer polynomials can determine the Fekete-Szegö functional for analytic functions that satisfy a specific subordination condition. The upper and lower estimates for the Hermitian-Toeplitz of the third order, concerning a class of starlike functions related to the cardioid shape region in the right half-plane, were recently calculated by Srivastava et al. [10]. The authors studied the third Hankel determinant that involves the Hohlov operator in [11]. In addition, they explored how to estimate the fourth Hankel determinant for a family of analytic functions using the cardioid domain [12]. We have included the sharp upper bound of a starlike function as given in the Table 1.

Table 1. Sharp estimates for the third Hankel determinant $|H_{3,1}(f)|$ within specific subclasses of starlike functions $\mathcal{S}^*$.

Author(s)	Type of starlike function	Sharp bound	Reference
Deniz et al.	Subordinated by $\frac{z}{\ln(1+z)}$	$\frac{43}{576}$	[21]
Marimuthu et al.	Subordinated by $\cos z$	$\frac{139}{576}$	[22]
Li et al.	Symmetric, connected with exponential function	0.0883	[23]
Tang et al.	Symmetric, subordinated by $1+\frac{4}{5}z+\frac{1}{5}z^4$	0.047	[24]
Riaz et al.	Subordinated by $1+\frac{4}{5}z+\frac{1}{5}z^4$	$\frac{1}{225}$	[26]
Wang et al.	Subordinated by $1+sinh^{-1}(z)$	$\frac{1}{9}$	[25]

The Hankel determinant for inverse functions subordinate to exponential functions was examined by Shi et al. [13] in 2022. For a function $f \in S$, the logarithmic coefficients denoted by $\gamma_k = \gamma_k(f)$ are defined by its series expansion.

$$\log\left(\frac{f(z)}{z}\right) = 2\sum_{k=1}^{\infty} \gamma_k z^k.$$

The logarithmic coefficients for the function f given by Eq (1.1) are as follows:

$$\gamma_1 = \frac{1}{2}b_2, \tag{1.5}$$

$$\gamma_2 = \frac{1}{2}\left(b_3 - \frac{1}{2}b_2^2\right), \tag{1.6}$$

$$\gamma_3 = \frac{1}{2}\left(b_4 - b_2b_3 + \frac{1}{3}b_2^3\right), \tag{1.7}$$

$$\gamma_4 = \frac{1}{2}\left(b_5 - b_2b_4 + b_2^2b_3 - \frac{1}{2}b_3^2 - \frac{1}{4}b_2^4\right), \tag{1.8}$$

$$\gamma_5 = \frac{1}{2}\left(b_6 - b_2b_5 - b_3b_4 + b_2b_3^2 + b_2^2b_4 - b_2^3b_3 + \frac{1}{5}b_2^5\right). \tag{1.9}$$

As a result of the concepts mentioned above, we propose investigating the Hankel determinant, in which the parameters are logarithmic coefficients of $f \in S$. With these logarithmic coefficients, we have the following Hankel determinant.

$$H_{q,k}(f) = \begin{vmatrix} \gamma_k & \gamma_{k+1} & \cdots & \gamma_{k+q-1} \\ \gamma_{k+1} & \gamma_{k+2} & \cdots & \gamma_{k+q} \\ \vdots & \vdots & \cdots & \vdots \\ \gamma_{k+q-1} & \gamma_{k+q} & \cdots & \gamma_{k+2q-2} \end{vmatrix}.$$

The purpose of this paper [20] is to provide sharp upper bounds for Hankel determinants involving logarithmic coefficients associated with the hyperbolic tangent function.
In this paper, we aim to determine the sharp estimates for the logarithmic coefficients, Krushkal inequalities, Zalcman inequality, and the third-order Hankel determinants for the class of starlike functions $\mathcal{S}^*_{Ne}$ associated with the nephroid domain.

A set of lemmas

Let $\mathcal{P}$ denote the class of analytic functions p that can be expressed in the following form and are normalized by the condition

$$p(0) = 1 \qquad \text{with} \qquad \Re\,(p(z)) > 0 \qquad (z \in U)\,,$$

and the formula is given by:

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \qquad (z \in U)\,. \tag{1.10}$$

Lemma 1. ([*14*]) *Suppose $p \in \mathcal{P}$. Then, x and δ exist such that* $|x| \le 1$, $|\delta| \le 1$,

$$2c_2 = \left(mx + c^2\right), \tag{1.11}$$

$$4c_3 = \left\{-mx^2c + 2mxc + 2m(1-|x|^2)\delta + c^3\right\}, \tag{1.12}$$

$$8c_4 = \left[\begin{array}{c} \left(4x + \left(x^2 - 3x + 3\right)c^2\right)mx - 4m\left(1 - |x|^2\right) \\ -\rho(1-|\delta|^2) + (x-1)\,\delta c + \delta^2\overline{x} + c^4 \end{array}\right], \tag{1.13}$$

where $m = (4 - c_1^2)$.

Lemma 2. *If $p \in \mathcal{P}$ and is expressed in the form of (1.10), then*

$$|c_k| \le 2 \ \ \text{for}\ k \ge 1, \tag{1.14}$$

$$|c_{n+k} - \mu c_n c_k| < 2 \ \ \text{for}\ \ 0 \le \mu \le 1, \tag{1.15}$$

$$|c_m c_k - c_k c_1| \le 4 \ \ \text{for}\ \ m + k = k + 1, \tag{1.16}$$

$$\left|c_{k+2k} - \mu c_k c_k^2\right| \le 2\,(1 + 2\mu)\,, \ \ \text{for}\ \ \mu \in \mathbb{R}, \tag{1.17}$$

$$\left|c_2 - \frac{c_1^2}{2}\right| \le \left(2 - \frac{\left|c_1^2\right|}{2}\right). \tag{1.18}$$

For a complex number η, we get

$$\left|c_2 - \eta c_1^2\right| < 2\max\{1, |2\eta - 1|\}. \tag{1.19}$$

For inequalities (1.14)–(1.19) , see [6], with (1.19) specifically provided in [15].

Lemma 3. *([16]) If $p \in \mathcal{P}$ and is expressed in the form of (1.10), then*

$$\left|c_1^5 + 3c_1c_2^2 + 3c_1^2c_3 - 4c_1^3c_2 - 2c_1c_4 - 2c_2c_3 + c_5\right| \leq 2, \tag{1.20}$$

$$\left|\begin{array}{c} c_1^6 + 6c_1^2c_2^2 + 4c_1^3c_3 + 2c_1c_5 + 2c_2c_4 + c_3^2 \\ -c_2^3 - 5c_1^4c_2 - 3c_1^2c_4 - 2c_2c_3 - 6c_1c_2c_3 - c_6 \end{array}\right| \leq 2, \tag{1.21}$$

$$\left|c_1^4 + c_2^2 + 2c_1c_3 - 3c_1^2c_2 - c_4\right| \leq 2. \tag{1.22}$$

Lemma 4. ([*17*]) *If $p \in \mathcal{P}$ and is expressed as in the form of* (1.10)*, then*

$$\left|Ac_1^3 - Bc_1c_2 + Cc_3\right| \leq 2|A| + 2|B - 2A| + 2|A - B - C|, \tag{1.23}$$

where A, B, and C are real numbers.

2. Coefficient bounds for $\mathcal{S}_{Ne}^*$

To begin, we can examine the upper bounds through the fifth coefficient b_5 for $f \in \mathcal{S}_{Ne}^*$.

Theorem 1. *If $f \in \mathcal{S}_{Ne}^*$ and follows the form specified in Eq* (1.1)*, then*

$$|b_2| \leq 1, \tag{2.1}$$

$$|b_3| \leq \frac{1}{2}, \tag{2.2}$$

$$|b_4| \leq \frac{1}{3}, \tag{2.3}$$

$$|b_5| \leq \frac{13}{24}. \tag{2.4}$$

These three result are sharp.

Proof. Using the definition of subordinations, if $f \in \mathcal{S}_{Ne}^*$, then $w(z)$ is a Schwartz function with the following attributes:

$$w(0) = 0, \qquad w(z) < 1,$$

$$\frac{zf'(z)}{f(z)} = 1 + w(z) - \frac{(w(z))^3}{3}. \tag{2.5}$$

Given that $p \in P$, it follows that

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1z + c_2z^2 + c_3z^3 + \cdots.$$

When expanded, we obtain

$$\begin{aligned}\frac{zf^{'}(z)}{f(z)} &= 1+b_2z+\left(2b_3-b_2^2\right)z^2+\left(3b_4-3b_2b_3+b_2^3\right)z^3\\ &\quad+\left(4b_5-4b_4b_2-2b_3^2+4b_3b_2^2-b_2^4\right)z^4. \end{aligned} \tag{2.6}$$

Also,

$$\begin{aligned}1+w(z)-\frac{(w(z))^3}{3} &= 1+\frac{1}{2}c_1z+\left(\frac{1}{2}c_2-\frac{1}{4}c_1^2\right)z^2+\left(\frac{1}{2}c_3-\frac{1}{2}c_1c+\frac{1}{12}c_1^3\right)z^3\\ &\quad+\left(\frac{1}{2}c_1c_3+\frac{1}{4}c_1^2c_2+\frac{1}{2}c_2-\frac{1}{4}c_2^2\right)z^4+\cdots. \end{aligned} \tag{2.7}$$

Comparing (2.6) and (2.7), we get

$$b_2=\frac{1}{2}c_1, \tag{2.8}$$

$$b_3=\frac{1}{4}c_{2,} \tag{2.9}$$

$$b_4=\frac{1}{6}c_3-\frac{1}{24}c_1c_2+\left(\frac{-1}{72}\right)c_1^3, \tag{2.10}$$

$$b_5=\frac{1}{8}c_4-\frac{1}{24}c_1c_3-\frac{1}{32}c_2^2-\frac{1}{48}c_2c_1^2+\frac{5}{576}c_1^4. \tag{2.11}$$

By using (1.14) in (2.8), we have

$$|b_2|\leq 1.$$

Using (1.14) in (2.9), we obtain

$$|b_3|\leq\frac{1}{2}.$$

We can write (2.10) to get

$$|b_4|=\frac{1}{6}\left|c_3-2\left(\frac{1}{8}\right)c_1c_2+\left(\frac{-1}{12}c_1^3\right)\right|.$$

$$B=\frac{1}{8}\quad\text{and}\quad D=\frac{-1}{12}.$$

Clearly, $0\leq B\leq 1$ and $B\geq D$, so

$$B\,(2B-1)=\frac{-3}{32}\leq D.$$

From (1.23), we obtain

$$|b_4|\leq\frac{1}{3}.$$

Using (1.14)–(1.15) and (1.19) in (2.11), we have

$$|b_5|\leq\frac{1}{8}\left|\left(c_4-\frac{1}{3}c_1c_3\right)\right|+\frac{\left|c_1^2\right|}{48}\left|\left(c_2-\frac{5}{12}c_1^2\right)\right|+\frac{1}{32}\left|c_2^2\right|,$$

$$|b_5| \leq \frac{13}{24}.$$

These three results are sharp, with equality attained by the function $f_0(z)$ described below:

$$\frac{zf_0'(z)}{f_0(z)} = z + z^2 + z^3.$$

□

3. Krushkal inequalities

This section presents a significant result and offers direct proof of the inequality

$$\left|b_n^p - b_2^{p(n-1)}\right| \leq 2^{p(n-1)} - n^p.$$

Specifically, within class $f \in S_{Ne}^*$ using the upcoming parameter values such as $n = 4$ and $p = 1$, Krushkal examined this result and provided its proof for the entire class of univalent functions in his article [18].

Theorem 2. *If* $f \in \mathcal{S}_{Ne}^*$ *and is defined by Eq* (1.1) , *then*

$$\left|b_4 - b_2^3\right| \leq \frac{1}{3}.$$

The result is sharp for the function $f_0(z)$ given by

$$f_0(z) = z\exp\left(\int_0^z \left(1 + \left(t - \frac{t^3}{3}\right)\right) dt\right) = z + z^2 + z^3 + \frac{2}{3}z^4 + \frac{1}{3}z^5.$$

Proof. Substituting Eqs (2.10) and (2.8), we obtain

$$\begin{aligned} \left|b_4 - b_2^3\right| &= \left|\frac{1}{6}c_3 - \left(\frac{1}{24}\right)c_1c_2 - \frac{5}{36}c_1^3\right| \\ &= 6\left(\frac{1}{6}c_3 - \left(\frac{1}{24}\right)c_1c_2 - \frac{5}{36}c_1^3\right) \\ &= c_3 - \frac{1}{4}c_1c_2 - \frac{5}{6}c_1^3. \end{aligned}$$

We can compare this with the lemma:

$$\begin{aligned} \left|c_3 - 2Bc_1c_2 + Dc_1^3\right| &\leq 2, \\ 2B &= \frac{1}{4}, \quad D = \frac{-5}{6}. \end{aligned}$$

According to (1.23), we consider the following:

$$B = \frac{1}{8} \text{ and } D = \frac{-5}{6}.$$

Let $0 \leq B \leq 1$, and suppose $B \geq D$:

$$B\,(2B-1) = \frac{-1}{8} \geq D$$
$$= \frac{1}{6}\left|c_3 - 2\left(\frac{1}{8}\right)c_1c_2 - \frac{5}{6}c_1^3\right| \leq 2.$$
$$\left|b_4 - b_2^3\right| \leq \frac{1}{3}.$$

□

4. Zalcman inequality

In 1960, Zalcman formulated a conjecture about univalent functions. He claimed that any function $f \in \mathcal{S}$ of the form (1.1) satisfies the inequality

$$\left|b_n^2 - b_{2n-1}\right| \leq (n-1)^2\,, n \geq 2. \tag{4.1}$$

In 1999, Ma [19] established a generalized form of the Zalcman conjecture, demonstrating that every univalent function $f \in \mathcal{S}$ satisfies the inequality

$$|b_n b_i - b_{n+i-1}| \leq (n-1)\,(i-1), \quad \forall\, i, n \in N, \quad n \geq 2, \quad i \geq 2. \tag{4.2}$$

Theorem 3. *If the function $f \in \mathcal{S}^*_{Ne}$ is represented by the series in Eq* (1.1)*, then*

$$|b_4 - b_2b_3| \leq \frac{1}{3}. \tag{4.3}$$

The result is sharp for the function $f_0(z)$ given by

$$f_0(z) = z\exp\left(\int_0^z \left(1 + \left(t - \frac{t^3}{3}\right)\right)dt\right) = z + z^2 + z^3 + \frac{2}{3}z^4 + \frac{1}{3}z^5.$$

Proof. From (2.10)–(2.8), we have

$$|b_4 - b_2b_3| = \left|\frac{1}{6}c_3 - \frac{1}{6}c_1c_2 - \frac{1}{72}c_1^3\right|.$$

It can be easily shown that

$$|b_4 - b_2b_3| = \frac{1}{6}\left|c_3 - 2\left(\frac{1}{2}\right)c_1c_2 - \frac{1}{12}c_1^3\right|,$$

and then

$$B = \frac{1}{2}$$

and

$$D = -\frac{1}{12}.$$

Clearly, $0 \leq B \leq 1$ and $B \geq D$.

$$B\,(2B-1) = 0 \geq D = -\frac{1}{12}.$$

From (1.23), we obtain

$$|b_4 - b_2b_3| \leq \frac{1}{3}.$$

□

5. Logarithmic coefficients for the class $\mathcal{S}^*_{Ne}$

Theorem 4. *If the function $f \in \mathcal{S}^*_{Ne}$ is represented by the series in Eq* (1.1)*, then*

$$|\gamma_1| \leq \frac{1}{2}, \tag{5.1}$$

$$|\gamma_2| \leq \frac{1}{4}, \tag{5.2}$$

$$|\gamma_3| \leq \frac{1}{6}, \tag{5.3}$$

$$|\gamma_4| \leq \frac{1}{4}, \tag{5.4}$$

$$|\gamma_5| \leq \frac{3}{5}. \tag{5.5}$$

The following functions demonstrate the sharpness of the first five inequalities mentioned above.

$$f_0(z) = z\exp\left(\int_0^z \left(1 + \left(t - \frac{t^3}{3}\right)\right) dt\right) = z + z^2 + \cdots,$$

$$f_1(z) = z\exp\left(\int_0^z \left(1 + \left(t - \frac{t^3}{3}\right)\right) dt\right) = z + z^2 + z^3 + \frac{2}{3}z^4 + \frac{1}{5}z^5 + \cdots,$$

$$f_2(z) = z\exp\left(\int_0^z \left(1 + \left(t - \frac{t^3}{3}\right)^2\right) dt\right) = z + z^2 + \frac{1}{2}z^3 + \frac{1}{2}z^4 + \frac{3}{8}z^5 + \cdots,$$

$$f_3(z) = z\exp\left(\int_0^z \left(1 + 2\left(t - \frac{t^3}{3}\right)^3\right) dt\right) = z + z^2 + \frac{1}{2}z^3 + \frac{1}{6}z^4 + \frac{13}{24}z^5 + \cdots,$$

$$f_4(z) = z\exp\left(\int_0^z \left(1 + 6\left(t - \frac{t^3}{3}\right)^4\right) dt\right) = z + z^2 + \frac{1}{2}z^3 + \frac{1}{6}z^4 + \frac{1}{24}z^5 + \frac{29}{24}z^6 + \cdots.$$

Proof. Now, from (1.5)–(1.8) and (2.8)–(2.11) , we have

$$\gamma_1 = \frac{1}{4}c_{1,} \tag{5.6}$$

$$\gamma_2 = \frac{1}{8}c_2 - \frac{1}{16}c_1^2, \tag{5.7}$$

$$\gamma_3 = \frac{1}{12}c_3 - \frac{1}{12}c_1c_2 + \frac{1}{72}c_1^3, \tag{5.8}$$

$$\gamma_4 = \frac{1}{16}c_4 - \frac{1}{16}c_1c_3 - \frac{1}{32}c_2^2 + \frac{1}{32}c_2c_1^2, \tag{5.9}$$

$$\gamma_5 = \frac{1}{40}c_1c_2^2 + \frac{1}{40}c_3c_1^2 - \frac{1}{20}c_1c_4 - \frac{1}{320}c_1^5 - \frac{1}{20}c_3c_2 + \frac{1}{20}c_5. \tag{5.10}$$

Applying (1.14) to (5.6) , we obtain

$$|\gamma_1| \leq \frac{1}{2}.$$

From (1.18) and using (1.14), we get

$$|\gamma_2| = \frac{1}{8}\left(c_2 - \frac{1}{2}c_1^2\right) \leq \frac{1}{8}\left(c_2 - \frac{|c_1^2|}{2}\right) = G(c_1).$$

It is clear that $G(c_1)$ is a decreasing function that reaches its maximum value when $c_1 = 0$, and we have

$$|\gamma_2| \leq \frac{1}{4}.$$

Using (1.23) in Eq (5.8), we have

$$|\gamma_3| \leq \frac{1}{6}.$$

Furthermore, applying (1.15) and (1.19) to (5.9), we obtain the following:

$$|\gamma_4| \leq \frac{1}{4}.$$

Using (1.14) and (1.15) in Eq (5.10), we get

$$|\gamma_5| \leq \frac{3}{5}.$$

□

Theorem 5. *If $f \in \mathcal{S}^*_{Ne}$ and it has the form given in* (1.1)*, then*

$$\left|\gamma_2 - \delta\gamma_1^2\right| \leq \max\left\{\frac{1}{4}, \ \frac{|\delta|}{4}\right\}. \tag{5.11}$$

Therefore, the Fekete–Szegö functional achieves the optimal result.

Proof. From (1.6)–(1.5) , we achieve

$$\left|\gamma_2 - \delta\gamma_1^2\right| = \frac{1}{8}\left|c_2 - \frac{1}{2}c_1^2 - \frac{1}{2}\delta c_1^2\right|.$$

Using (1.19), we have

$$\left|\gamma_2 - \delta\gamma_1^2\right| \leq \frac{2}{8}\max\left\{1, \ |1 + \delta - 1|\right\}.$$

After the calculation, we have the final result:

$$\left|\gamma_2 - \delta\gamma_1^2\right| \leq \max\left\{\frac{1}{4}, \ \frac{|\delta|}{4}\right\}.$$

□

Theorem 6. *If the function $f \in \mathcal{S}^*_{Ne}$ and is defined by Eq* (1.1) , *then*

$$|\gamma_1\gamma_2 - \gamma_3| \leq \frac{1}{6}. \tag{5.12}$$

The result is sharp for the function $f_0(z)$ given by

$$f_2(z) = z\exp\left(\int_0^z \left(1 + \left(t - \frac{t^3}{3}\right)^2\right) dt\right) = z + z^2 + \frac{1}{2}z^3 + \frac{1}{2}z^4 + \frac{3}{8}z^5.$$

Proof. Using (5.6)–(5.8), we obtain

$$|\gamma_1\gamma_2 - \gamma_3| = \frac{1}{12}\left|c_3 - 2\left(\frac{11}{16}\right)c_1c_2 + \frac{17}{48}c_1^3\right|.$$

$$B = \frac{11}{16} \quad \text{and} \quad D = \frac{17}{48}.$$

Clearly, $0 \leq B \leq 1$ and $B \geq D$.

$$B(2B - 1) = \frac{33}{128} \leq D.$$

From (1.23) , we obtain

$$|\gamma_1\gamma_2 - \gamma_3| \leq \frac{1}{6}.$$

□

Theorem 7. *If the function $f \in \mathcal{S}^*_{Ne}$ is represented by the series in Eq* (1.1), *then*

$$\left|\gamma_1\gamma_3 - \gamma_2^2\right| \leq \frac{1}{16}. \tag{5.13}$$

The result is sharp for the function $f_0(z)$ given by

$$f_0(z) = z\exp\left(\int_0^z \left(1 + \left(t - \frac{t^3}{3}\right)\right) dt\right) = z + z^2 + z^3 + \frac{2}{3}z^4 + \frac{1}{3}z^5.$$

Proof. From (5.6)–(5.8), we have

$$\gamma_1\gamma_3 - \gamma_2^2 = \frac{1}{48}c_1c_3 - \frac{1}{192}c_1^2c_2 - \frac{13}{2304}c_1^4 - \frac{1}{64}c_2^2.$$

By using Eqs (1.11) and (1.12), we express c_2 and c_3 in terms of $c_1 = c \in [0, 2]$, and we obtain

$$\begin{aligned}\gamma_1\gamma_3 - \gamma_2^2 \;&=\; -\frac{1}{576}c^4 - \frac{1}{192}c^2(4-c^2)\alpha^2 - \frac{1}{256}\alpha^2(4-c^2)^2 + \frac{1}{96}c(4-c^2)\delta \\ &\quad - \frac{1}{96}c(4-c^2)\delta\,|\alpha|^2\,.\end{aligned}$$

Applying the triangle inequality and considering that $|\delta| \leq 1$ and $|\alpha| = y \leq 1$, we obtain the following:

$$\left|\gamma_1\gamma_3 - \gamma_2^2\right| \;\leq\; \frac{1}{576}c^4 + \frac{1}{192}c^2(4-c^2)y^2 + \frac{1}{256}y^2(4-c^2)^2 + \frac{1}{96}c(4-c^2)$$

$$
\begin{aligned}
& + \frac{1}{96}c(4-c^2)y^2 \\
:= \ & H(c,y).
\end{aligned}
$$

By partially differentiating for y, we obtain

$$\frac{\partial H(c,y)}{\partial y} = \frac{1}{96}c^2(4-c^2)y + \frac{1}{48}c(4-c^2)y + \frac{1}{128}y(4-c^2)^2.$$

For a constant c, it is obvious that $\frac{\partial H(c,y)}{\partial y} > 0$, and that $H(c,y)$ is increasing in y. Because of this, $H(c,y)$ reaches its highest value at $y = 1$, so,

$$
\begin{aligned}
H(c,y) \ &\leq \ H(c,1) = \frac{1}{576}c^4 + \frac{1}{192}c^2(4-c^2) + \frac{1}{48}c(4-c^2) + \frac{1}{256}(4-c^2)^2 \\
&= \ \frac{1}{2304}c^4 - \frac{1}{96}c^2 + \frac{1}{12}c - \frac{1}{48}c^3 + \frac{1}{16}.
\end{aligned}
$$

Now, differentiating for c, we have

$$H^{'}(c,1) = \frac{1}{576}c^3 - \frac{1}{48}c + \frac{1}{12} - \frac{1}{16}c^2.$$

It is clear that $H^{'}(c,1) \geq 0$, and it is a decreasing function. Therefore, the highest value is achieved at $c = 0$, which is

$$\left|\gamma_1\gamma_3 - \gamma_2^2\right| \leq \frac{1}{16}.$$

□

Theorem 8. *Suppose $f \in \mathcal{S}^*_{Ne}$, and then*

$$\left|\gamma_2\gamma_4 - \gamma_3^2\right| \leq \frac{65}{432}. \tag{5.14}$$

Proof. From (5.7)–(5.9), we have

$$\gamma_2\gamma_4 - \gamma_3^2 = \begin{array}{c} \frac{1}{128}c_2c_4 + \frac{7}{1152}c_2c_1c_3 - \frac{1}{256}c_2^3 - \frac{5}{4608}c_2^2c_1^2 - \frac{1}{256}c_1^2c_4 \\ + \frac{11}{6912}c_1^3c_3 + \frac{5}{13824}c_1^4c_2 - \frac{1}{144}c_3^2 - \frac{1}{5184}c_1^6. \end{array}$$

Regrouping, we get

$$\left|\gamma_2\gamma_4 - \gamma_3^2\right| = \left| \begin{array}{c} \frac{1}{128}c_2\left(c_4 - \frac{1}{2}c_2^2\right) + \frac{7}{1152}c_1c_2\left(c_3 - \frac{5}{28}c_1c_2\right) + \frac{1}{256}c_1^2\left(c_4 - \frac{11}{27}c_1c_3\right) \\ + \frac{5}{13824}c_1^4\left(c_2 - \frac{8}{15}c_1^2\right) + \frac{1}{144}c_3^2 \end{array} \right|.$$

By using the triangle inequality, we have

$$
\begin{aligned}
\left|\gamma_2\gamma_4 - \gamma_3^2\right| \ \leq \ & \frac{1}{128}\left|c_2\right|\left|c_4 - \frac{1}{2}c_2^2\right| + \frac{7}{1152}\left|c_1\right|\left|c_2\right|\left|c_3 - \frac{5}{28}c_1c_2\right| \\
& + \frac{1}{256}\left|c_1^2\right|\left|c_4 - \frac{11}{27}c_1c_3\right| + \frac{5}{13824}\left|c_1^4\right|\left|c_2 - \frac{8}{15}c_1^2\right|
\end{aligned}
$$

$$+\frac{1}{144}\left|c_3^2\right|.$$

Using (1.14) and (1.15), we obtain

$$\left|\gamma_2\gamma_4-\gamma_3^2\right|\leq\frac{65}{432}.$$

□

Theorem 9. *If $f\in\mathcal{S}^*_{Ne}$ and it has the form given in* (1.1)*, then*

$$|\gamma_1\gamma_4-\gamma_2\gamma_3|\leq\frac{11}{48}. \tag{5.15}$$

The result is sharp for the function $f_0(z)$ given by

$$f_3(z)=z\exp\left(\int_0^z\left(1+11\left(t-\frac{t^3}{3}\right)\right)dt\right)=z+z^2+6z^3+\frac{17}{3}z^4+17z^5\cdots.$$

Proof. From (5.6)–(5.9), we obtain

$$\gamma_1\gamma_4-\gamma_2\gamma_3=\frac{1}{64}c_1c_4-\frac{1}{96}c_1^2c_3+\frac{1}{384}c_1c_2^2+\frac{1}{1152}c_2c_1^3-\frac{1}{96}c_2c_3+\frac{1}{1152}c_1^5.$$

Regrouping, we get

$$|\gamma_1\gamma_4-\gamma_2\gamma_3|=\left|\frac{-1}{96}c_2\left(c_3-\frac{1}{12}c_1^3\right)-\frac{1}{96}c_1^2\left(c_3-\frac{1}{12}c_1^3\right)+\frac{1}{64}c_1c_4+\frac{1}{384}c_1c_2^2\right|.$$

By using the triangle inequality, we have

$$\begin{aligned}|\gamma_1\gamma_4-\gamma_2\gamma_3| \quad\leq\quad & \frac{1}{96}|c_2|\left|c_3-\frac{1}{12}c_1^3\right|+\frac{1}{96}\left|c_1^2\right|\left|c_3-\frac{1}{12}c_1^3\right| \\ & +\frac{1}{64}|c_1||c_4|+\frac{1}{384}|c_1|\left|c_2^2\right|.\end{aligned}$$

Using (1.14) and (1.17), we obtain

$$|\gamma_1\gamma_4-\gamma_2\gamma_3|\leq\frac{11}{48}.$$

□

6. The third Hankel determinant based on logarithmic coefficients for the class $\mathcal{S}^*_{Ne}$

Theorem 10. *If $f\in\mathcal{S}^*_{Ne}$ and it has the form given in* (1.1)*, then*

$$\left|H_{3,1}(f)\right|\leq\frac{2297}{25920}\simeq 0.088619.$$

Proof. Let

$$\begin{aligned}\left|H_{3,1}(f)\right| \quad=\quad & \begin{vmatrix}\gamma_1 & \gamma_2 & \gamma_3\\ \gamma_2 & \gamma_3 & \gamma_4\\ \gamma_3 & \gamma_4 & \gamma_5\end{vmatrix} \\ \leq\quad & |\gamma_3|\left|\gamma_2\gamma_4-\gamma_3^2\right|+|\gamma_4||\gamma_1\gamma_4-\gamma_2\gamma_3|+|\gamma_5|\left|\gamma_1\gamma_3-\gamma_2^2\right|.\end{aligned}$$

From the values of (5.2)–(5.5) and (5.13)–(5.15), we achieve the required result. □

7. The third-order Hankel determinant associated with the class $\mathcal{S}^*_{Ne}$

Theorem 11. *If* $f \in \mathcal{S}^*_{Ne}$ *and it has the form given in* (1.1)*, then*

$$\left|H_{3,1}(f)\right| \leq \frac{1}{9}.$$

The result is sharp for the function $f_0(z)$ given by

$$f_0 = z\exp\left(\int_0^z \left(1+\left(t-\frac{t^3}{3}\right)\right)dt\right) = z+z^2+z^3+\frac{2}{3}z^4+\frac{1}{3}z^5.$$

Proof. Substituting the expressions (1.11)–(1.13) into (1.4) and putting $c_1 = c$, we have

$$H_{3,1}(f) = \frac{1}{L}\begin{pmatrix} 936cc_2c_3 - 198c^2c_2^2 + 57c^4c_2 - 486c_2^3 - 576c_3^2 \\ +312c_3c^3 - 49c^6 + 648c_2c_4 - 648c^2c_4 \end{pmatrix}. \tag{7.1}$$

Given $L = 20,736$, we now set $c_1 = c$, and define $m = \left(4-c^2\right)$, leading to the following result.

$$\begin{aligned} 2c_2 &= \left(m\alpha + c^2\right), \\ 4c_3 &= \left\{-m\alpha^2 c + 2m\alpha c + 2m(1-|\alpha|^2)\delta + c^3\right\}, \\ 8c_4 &= \begin{bmatrix} \left(4\alpha + \left(\alpha^2 - 3\alpha + 3\right)c^2\right)m\alpha - 4m(1-|\alpha|^2) \\ -\rho(1-|\delta|^2) + (\alpha-1)\,\delta c + \delta^2\overline{\alpha} + c^4 \end{bmatrix}. \end{aligned}$$

Using (7.1) , we obtain

$$H_{3,1}(f) = \frac{1}{L}\left\{\begin{array}{c} -54c\alpha(4-c^2)^2(1-|\alpha|^2)\delta + 144c(4-c^2)^2\alpha^2(1-|\alpha|^2)\delta \\ +246c^3(4-c^2)(1-|\alpha|^2)\delta - 162\alpha(4-c^2)^2(1-|\alpha|^2) \\ -144(4-c^2)^2(1-|\alpha|^2)^2\delta^2 + 162c^2(4-c^2)(1-|\alpha|^2) \\ -\frac{49}{4}c^6 - \frac{81}{2}c^2\rho\,|\delta|^2 - \frac{81}{2}c^3\alpha\delta - \frac{81}{2}c^2\alpha\delta^2 + \frac{81}{2}c^3\delta \\ +\frac{81}{2}c^2\rho + \frac{81}{2}\alpha(4-c^2)\rho\,|\delta|^2 + \frac{81}{2}\alpha^2(4-c^2)\delta c \\ -\frac{81}{2}\alpha(4-c^2)\delta c - \frac{81}{2}\alpha(4-c^2)\rho - \frac{3}{2}c^4(4-c^2)\alpha^2 \\ +\frac{117}{4}c^4(4-c^2)\alpha - \frac{189}{2}\alpha^3(4-c^2)^2c^2 - \frac{81}{4}\alpha^2(4-c^2)^2c^2 \\ +\frac{9}{2}c^2(4-c^2)^2\alpha^4 - 162\alpha^2c^2(4-c^2) - \frac{81}{2}c^4\alpha^3(4-c^2) \\ +\frac{81}{2}\alpha^2(4-c^2)\delta^2 + 162\alpha^3(4-c^2)^2 - \frac{243}{4}\alpha^3(4-c^2)^3 \end{array}\right\}.$$

Since $m = \left(4-c^2\right)$, it follows that

$$H_{3,1}(f) = \frac{1}{L}\left(m_0(c,\alpha) + m_1(c,\alpha)\delta + m_2(c,\alpha)\delta^2 + \Pi(c,\alpha,\delta)\rho\right),$$

where

$$m_0(c,\alpha) = -\frac{49}{6}c^6 + \left(4-c^2\right)\begin{bmatrix} -\frac{81}{4}\alpha^2(4-c^2)c^2 + \frac{117}{4}c^4\alpha - \frac{243}{4}\alpha^3(4-c^2)^2 - \frac{3}{2}c^4\alpha^2 \\ -\frac{189}{2}\alpha^3(4-c^2)c^2 + \frac{9}{2}c^2(4-c^2)\alpha^4 - 162\alpha(4-c^2)(1-|\alpha|^2) \\ -162\alpha^2c^2 - \frac{81}{2}c^4\alpha^3 + 162\alpha^3(4-c^2) + 162c^2(1-|\alpha|^2) \end{bmatrix},$$

$$m_1(c,\alpha) = -\frac{81}{2}c^3\alpha + \frac{81}{2}c^3 + \left(4-c^2\right)\left[\begin{array}{c} 246c^3\left(1-\left|\alpha^2\right|\right) - 54c\alpha(4-c^2)(1-|\alpha|^2)\delta \\ +\,144c(4-c^2)\alpha^2(1-|\alpha|^2) + \frac{81}{2}\alpha^2 c - \frac{81}{2}\alpha c \end{array}\right],$$

$$m_2(c,\alpha) = -\frac{81}{2}c^2\alpha + (4-c^2)\left[\frac{81}{2}\alpha^2 - 144(4-c^2)(1-|\alpha|^2)^2\right],$$

and

$$\Pi(c,\alpha,\delta) = \frac{81}{2}c^2\left(1-|\delta|^2\right) + \frac{81}{2}\alpha(4-c^2)\left[|\delta|^2-1\right].$$

By taking $|\rho| \leq 1$ and substituting $|\delta|$ with β and $|\alpha|$ with α, one obtains the following.

$$\begin{aligned} \left|H_{3,1}(f)\right| &\leq \frac{1}{L}\left(|m_0(c,\alpha)| + |m_1(c,\alpha)|\beta + |m_2(c,\alpha)|\beta^2 + |\Pi(c,\alpha,\delta)|\right) \\ &\leq \frac{1}{L}\left(f(c,\alpha,\beta)\right), \end{aligned} \tag{7.2}$$

where

$$f(c,\alpha,\beta) = \left(n_0(c,\alpha) + n_1(c,\alpha)\beta + n_2(c,\alpha)\beta^2 + n_3(c,\alpha)(1-\beta^2)\right), \tag{7.3}$$

with

$$n_0(c,\alpha) = \frac{49}{6}c^6 + \left(4-c^2\right)\left[\begin{array}{c} \frac{81}{4}\alpha^2(4-c^2)c^2 + \frac{117}{4}c^4\alpha + \frac{243}{4}\alpha^3(4-c^2)^2 + \frac{3}{2}c^4\alpha^2 \\ +\,\frac{189}{2}\alpha^3(4-c^2)c^2 + \frac{9}{2}c^2(4-c^2)\alpha^4 + 162\alpha(4-c^2)(1-|\alpha|^2) \\ +\,162\alpha^2c^2 + \frac{81}{2}c^4\alpha^3 + 162\alpha^3(4-c^2) + 162c^2(1-|\alpha|^2) \end{array}\right],$$

$$n_1(c,\alpha) = \frac{81}{2}c^3\alpha + \frac{81}{2}c^3 + \left(4-c^2\right)\left[\begin{array}{c} 246c^3\left(1-\left|\alpha^2\right|\right) + 54c\alpha(4-c^2)(1-|\alpha|^2)\delta \\ +144c(4-c^2)\alpha^2(1-|\alpha|^2) + \frac{81}{2}\alpha^2 c + \frac{81}{2}\alpha c \end{array}\right],$$

$$n_2(c,\alpha) = \frac{81}{2}c^2\alpha + (4-c^2)\left[\frac{81}{2}\alpha^2 + 144(4-c^2)(1-|\alpha|^2)^2\right],$$

$$n_3(c,\alpha) = \frac{81}{2}c^2\left(1-\beta^2\right) + \frac{81}{2}\alpha(4-c^2)\left[\beta^2-1\right].$$

When (7.3) is partially differentiated with respect to the parameter β, the result is

$$\begin{aligned} \frac{\partial f}{\partial \beta} &= 246c^3(4-c^2)(1-\alpha^2) + \frac{81}{2}\alpha^2(4-c^2)c + \frac{81}{2}\alpha c(4-c^2) \\ &\quad + 54cx(4-c^2)^2(1-\alpha^2) + 144c(4-c^2)^2\alpha^2(1-\alpha^2) \\ &\quad + 288(4-c^2)^2(1-\alpha^2)^2\beta + 81\alpha(4-c^2)\beta + \frac{81}{2}c^3\alpha \\ &\quad + 81c^2\alpha\beta + 81\alpha^2(4-c^2)\beta + 81c^2\beta + \frac{81}{2}c^3. \end{aligned}$$

Taking $\frac{\partial f}{\partial \beta} = 0$, we have

$$\beta = \frac{1}{6}\frac{c\left(\begin{array}{c} 1536\alpha^3 - 768c^2\alpha^3 + 96c^4\alpha^3 - 960\alpha^2 + 480c^2\alpha^2 \\ -\,60c^4\alpha^2 - 200c^4\alpha - 684\alpha + 971c^2\alpha - 683c^2 + 164c^4 \end{array}\right)}{\left(\begin{array}{c} 512\alpha^3 + 32c^4\alpha^3 - 256c^2\alpha^3 - 512\alpha^2 - 32c^4\alpha^2 + 256c^2\alpha^2 \\ +\,247c^2\alpha - 476\alpha - 32c^4\alpha + 512 + 32c^4 - 247c^2 \end{array}\right)} := \beta_{0.}$$

For β to be within the interval is $(0,1)$, it needs to satisfy

$$\begin{pmatrix} 1536c\alpha^3 - 768c^3\alpha^3 + 96c^5\alpha^3 - 960c\alpha^2 + 480c^3\alpha^2 \\ - 60c^5\alpha^2 - 200c^5\alpha - 684c\alpha + 971c^3\alpha - 683c^3 + 164c^5 \end{pmatrix} < \begin{pmatrix} 3072\alpha^3 + 192c^4\alpha^3 - 1536c^2\alpha^3 - 3072\alpha^2 - 192c^4\alpha^2 + 1536c^2\alpha^2 \\ + 1428c^2\alpha - 2856\alpha - 192c^4\alpha + 3072 + 192c^4 - 1482c^2 \end{pmatrix}, \tag{7.4}$$

and

$$c^2 > \frac{\begin{pmatrix} 1536c\alpha^3 - 768c^3\alpha^3 + 96c^5\alpha^3 - 960c\alpha^2 + 480c^3\alpha^2 \\ - 60c^5\alpha^2 - 200c^5\alpha - 684c\alpha + 971c^3\alpha - 683c^3 + 164c^5 \\ - 3072\alpha^3 - 192c^4\alpha^3 + 3072\alpha^2 + 192c^4\alpha - 3072 - 192c^4 \end{pmatrix}}{(-1536\alpha^4 + 1536\alpha^2 - 1482\alpha - 1482)}. \tag{7.5}$$

Suppose

$$g(\alpha) = \frac{\begin{pmatrix} 1536c\alpha^3 - 768c^3\alpha^3 + 96c^5\alpha^3 - 960c\alpha^2 + 480c^3\alpha^2 \\ - 60c^5\alpha^2 - 200c^5\alpha - 684c\alpha + 971c^3\alpha - 683c^3 + 164c^5 \\ - 3072\alpha^3 - 192c^4\alpha^3 + 3072\alpha^2 + 192c^4\alpha - 3072 - 192c^4 \end{pmatrix}}{(-1536\alpha^4 + 1536\alpha^2 - 1482\alpha - 1482)}.$$

This gives

$$\begin{aligned} g'(\alpha) &= 4608c\alpha^2 - 2304c^3\alpha^2 + 288c^5\alpha^2 - 1920c\alpha + 960c^3\alpha - 120c^5\alpha \\ &\quad - 200c^5 - 684c + 971c^3 - 9216x^2 - 576c^4x^2 + 6144\alpha + 384\alpha c^4 \\ &\quad + 2856 + 192c^4. \end{aligned}$$

This implies that $g(\alpha)$ decreases over the interval $(0,1)$. Thus, $c^2 > \frac{-683c^3+164c^5-3072-192c^4}{-1482}$ and f has no critical point in $(0,2)\times(0,1)\times(0,1)$. As a result, (7.4) is not valid for any α in the interval $(0,1)$.

The interior of all six faces of the cuboid

Putting $c = 0$ in (7.3), we obtain the following:

$$\begin{aligned} k_1(\alpha,\beta) &= f(0,\alpha,\beta) = 162\alpha + 6480\alpha^3 + 2304(1-\alpha^2)^2\beta^2 \\ &\quad +2592\alpha(1-\alpha^2) + 162\alpha\beta^2 + 162\alpha^2\beta^2. \end{aligned}$$

When $c = 2$ in (7.3), we get

$$f(2,\alpha,\beta) = 946 + 324\beta + 162\beta^2 + 324\beta\alpha + 162\alpha\beta^2. \tag{7.6}$$

When $\alpha = 0$ in (7.3), we get

$$\begin{aligned} k_2(c,\beta) &= f(c,0,\beta) = 246c^3(4-c^2)\beta + \frac{81}{2}c^2 + \frac{49}{4}c^6 + 162c^2(4-c^2) \\ &\quad + 144(4-c^2)^2\beta^2 + \frac{81}{2}c^2\beta^2 + \frac{81}{2}c^3\beta. \end{aligned}$$

The critical points are determined by solving $\frac{\partial k_2}{\partial \beta} = 0$. This gives

$$\beta = \frac{1}{6}\frac{c^3\left(-683 + 164c^2\right)}{(512 - 247c^2 + 32c^4)}.$$

The above function k_2 has no critical point.

Put $\alpha = 1$ in (7.3) , we get

$$\begin{aligned} k_3(c,\beta) &= f(1,\alpha,\beta) = 162 + \frac{243}{4}(4-c^2)^3 + \frac{477}{4}c^2(4-c^2)^2 + 162(4-c^2)^2 \\ &+ \frac{49}{4}c^6 + 162c^2(4-c^2) + 81c^2\beta^2 + 81c^3\beta + 81(4-c^2)\beta^2 \\ &+ \frac{285}{4}c^4(4-c^2) + 81(4-c^2)\beta c. \end{aligned}$$

The above function k_3 has no critical point.

When $\beta = 0$ in (7.3), we get

$$\begin{aligned} k_4(c,\alpha) &= f(c,\alpha,0) = \frac{81}{2}c^2 + \frac{49}{4}c^6 + \frac{81}{2}\alpha(4-c^2) + 162\alpha^3(4-c^2)^2 + \frac{243}{4}\alpha^3(4-c^2)^3 \\ &+ 162\alpha(4-c^2)^2(1-\alpha^2) + 162c^2(4-c^2)(1-\alpha^2) + \frac{117}{4}c^4(4-c^2)\alpha \\ &+ \frac{3}{2}c^4(4-c^2)\alpha^2 + \frac{9}{2}c^2(4-c^2)^2\alpha^4 + \frac{189}{2}\alpha^3(4-c^2)^2c^2 \\ &+ \frac{81}{4}\alpha^2(4-c^2)c^2 + \frac{81}{2}c^4\alpha^3(4-c^2) + 162\alpha^2(4-c^2)c^2. \end{aligned}$$

When $\beta = 1$ in (7.3), we get

$$\begin{aligned} k_5(\alpha,c) &= f(c,\alpha,1) = \frac{81}{2}c\alpha^2(4-c^2) + \frac{81}{2}c\alpha(4-c^2) + \frac{81}{2}\alpha^2(4-c^2) \\ &+ 246c^3(4-c^2)(1-\alpha^2) + 81c^2 + \frac{81}{2}c^3 + \frac{49}{4}c^6 + 81\alpha(4-c^2) \\ &+ 162\alpha^3(4-c^2)^2 + \frac{243}{4}\alpha^3(4-c^2)^3 + 144(4-c^2)^2(1-\alpha^2)^2 \\ &+ 54c\alpha(4-c^2)^2(1-\alpha^2) + \frac{81}{2}c^2\alpha + 144c(4-c^2)^2\alpha^2(1-\alpha^2) \\ &+ \frac{81}{2}c^3\alpha + 162x(4-c^2)^2(1-x^2) + 162c^2(4-c^2)(1-\alpha^2) \\ &+ \frac{117}{4}c^4(4-c^2)\alpha + \frac{3}{2}c^4(4-c^2)\alpha^2 + \frac{9}{2}c^2(4-c^2)^2\alpha^4 \\ &+ \frac{189}{2}\alpha^3(4-c^2)^2c^2 + \frac{81}{4}\alpha^2(4-c^2)^2c^2 + \frac{81}{2}c^4\alpha^3(4-c^2) \\ &+ 162\alpha^2(4-c^2)c^2. \end{aligned}$$

A calculation shows that there is no solution for the equations.

$$\frac{\partial k_5}{\partial c} = 0, \ \frac{\partial k_5}{\partial \alpha} = 0$$

within the interval $(0,2) \times (0,1)$.

On the edges of a cuboid

Put $\alpha = \beta = 0$ in (7.3), and we get

$$k_6(c) = f(c,0,0) = \frac{81}{2}c^2 + \frac{49}{4}c^6 + 162c^2(4-c^2).$$

By using $\frac{\partial k_6}{\partial c} = 0$, we determine that $c_0 \approx 1.88$, which represents the critical point, where k_6 reaches its maximum value.

$$f(c,0,0) = 950.5910041.$$

When $c = 0$ and $\beta = 1$ in (7.3), we obtain

$$k_7(\alpha) = f(0,\alpha,1) = 324\alpha + 2304(1-\alpha^2)^2 + 6480\alpha^3 + 162\alpha^2 + 2592\alpha(1-\alpha^2).$$

The function reaches its maximum value at $\alpha = 1$:

$$f(0,\alpha,1) = 6966.$$

When $\alpha = 1$ and $\beta = 0$ in (7.3), we have

$$\begin{aligned} k_8(c) &= f(c,1,0) = 162 + \frac{243}{4}(4-c^2)^3 + \frac{477}{4}c^2(4-c^2)^2 + 162(4-c^2)^2 \\ &\quad + \frac{49}{4}c^6 + 162c^2(4-c^2) + \frac{285}{4}c^4(4-c^2). \end{aligned}$$

Using $\frac{\partial k_8}{\partial c} = 0$, we determined that $c_0 = 2$, its maximum value of the function

$$f(c,1,0) = 946.$$

When $\alpha = 0$ and $\beta = 1$ in (7.3) , we get

$$\begin{aligned} k_9(c) &= f(c,0,1) = 144(4-c^2)^2 + 81c^2 + \frac{81}{2}c^3 + \frac{49}{4}c^6 \\ &\quad + 246c^3(4-c^2) + 162c^2(4-c^2). \end{aligned}$$

The function reaches its maximum value at $c = 1.471$.

$$f(c,0,1) = 2982.33.$$

Put $c = 0$ and $\beta = 0$ in (7.3), and we get

$$k_{10}(\alpha) = f(0,\alpha,0) = 162\alpha + 6480\alpha^3 + 2592\alpha(1-\alpha^2).$$

The function reaches a maximum value of approximately $\alpha \approx 0.9999$. In other words, this is the highest value that it attains:

$$f(0,\alpha,0) \approx 6641.91.$$

When we put $c = 0$ and $\alpha = 0$ in (7.3), we have

$$k_{11}(\beta) = f(0,0,\beta) = 2304\beta^2.$$

Using $\frac{\partial k_{11}}{\partial \beta} > 0$, while k_{11} represents the maximum value at $\beta = 1$, we obtain the following:

$$f(c,\alpha,\beta) = 2304.$$

$$\left|H_{3,1}(f)\right| \le \frac{1}{20736}\left(f(c,\alpha,\beta)\right) \le \frac{1}{9}.$$

□

8. Conclusions

There is considerable literature on Hankel determinants in geometric function theory. However, finding the sharp bound for the third-order Hankel determinant remains challenging. This study the nephroid in various domains. We establish bounds on the logarithmic coefficients of these starlike functions, along with the third-order Hankel determinant and various coefficient estimates. The method presented in this paper can be applied to establish the sharp upper bound of the initial coefficients for various classes of analytic functions. The nephroid function is applied in investigating the characteristics of starlike and univalent functions, particularly within regions enclosed by nephroid-shaped. We investigate the growth, distortion, and covering characteristics of analytic functions by mapping the unit disk onto a nephroid-shaped domain. The third Hankel determinant finds applications in the study of analytic functions, as well as in problems related to stability and function approximation. The third-order Hankel determinant is especially useful for examining the nonlinear characteristics of subclasses such as starlike, convex, and close-to-convex functions. In future work, we plan to investigate the sharpness of these results. This study aims to evaluate the fourth Hankel determinants for similar classes of analytic functions that will be investigated in further research.

Author contributions

Wahid Ullah: Conceptualization, investigation; Sarfraz Malik: Conceptualization, formal analysis, visualization, supervision; Daniel Breaz: Resources, data curation, funding acquisition; Luminita-Ioana Cotirla: Validation, project administration, funding acquisition. All authors have read and agreed to the published version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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