Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(11): 26884–26904.
DOI: 10.3934/math.20251182
Received: 05 July 2025
Revised: 18 September 2025
Accepted: 09 October 2025
Published: 20 November 2025

Research article

Stochastic soliton dynamics in the perturbed Gerdjikov-Ivanov equation with multiplicative noise via the new Jacobi elliptic function expansion method

Nafissa T. Trouba[1,2], Huiying Xu[1,*], Reham M. A. Shohib[3,*], Mohamed E. M. Alngar[4], Mohammed El-Meligy[5], Xinzhong Zhu[1,6] and Adham E. Ragab[7]

1 School of Computer Science and Technology, Zhejiang Normal University, Jinhua, 321004, China

2 Zhejiang Institute of Photoelectronics, Jinhua, Zhejiang 321004, China

3 Basic Science Department, Faculty of Economics and Business Administration, Innovation University, Sharqia, Egypt

4 Mathematics Education Program, Faculty of Education and Arts, Sohar University, Sohar 311, Oman

5 Applied Science Research Center, Applied Science Private University, Amman, Jordan

6 College of Computer Science and Artificial Intelligence, Wenzhou University, Wenzhou 325035, China

7 Department of Industrial Engineering, College of Engineering, King Saud University, Riyadh 12372, Saudi Arabia

* **Correspondence:** Email: xhy@zjnu.edu.cn; r.shohib@yahoo.com.

Abstract: This study investigated the perturbed Gerdjikov-Ivanov equation (PGIE) affected by multiplicative noise in the Itô sense. The equation examined the influence of stochastic perturbations on its solitonic structures through an analytical and computational method. We found precise soliton solutions and examined their stability through the new Jacobi elliptic function expansion method. Subsequent to the mathematical study, graphical representations were executed. The results enhance the comprehensive understanding of nonlinear stochastic wave equations and their applications in optical fiber communications and many physical systems. Subsequently, the three-dimensional and two-dimensional model demonstrate the presence of bright and dark solitons, Jacobi-elliptic solutions, and periodic solutions for various values of σ.

Keywords: Gerdjikov-Ivanov equation; solitons; Itô sense
Mathematics Subject Classification: 35C07, 35Q40, 35Q55, 60H15, 81Q15

1. Introduction

The Gerdjikov-Ivanov (GI) equation is an important component of the derivative nonlinear Schrödinger (DNLS) family, extensively utilized to represent nonlinear wave propagation across several physical scenarios [1]. Abdelrahman et al. [2] investigated the impact of multiplicative noise on precise solutions of the nonlinear Schrödinger equation, demonstrating how stochastic perturbations can affect waveform fidelity. Albosaily et al. [3] examined a stochastic chiral nonlinear Schrödinger equation, uncovering multi-dimensional soliton dynamics influenced by noise. Khan [4] investigated sub-picosecond envelope solitons in the Triki-Biswas equation incorporating multiphoton absorption and filtering, and subsequently [5] examined extended anti-cubic nonlinearity with analogous stochastic effects. Stochastic perturbations are common in authentic nonlinear systems. Khan [6] investigated optical solitons characterized by quadratic-cubic nonlinear refractive indices in the presence of multiplicative noise, demonstrating significant sensitivity to amplitude and phase. Mohammed et al. [7] obtained precise solutions of the stochastic Ginzburg-Landau equation, linking dispersion and nonlinearity effects to noise-induced instabilities. Mohammed et al. [8] expanded this methodology to the Hirota-Maccari system, whilst Mohammed et al. [9] examined the stochastic novel coupled Konno-Oono equation, both emphasizing the influence of multiplicative noise on soliton profiles. Kudryashov [10] advanced theoretical insights into highly dispersive and perturbed Schrödinger-type systems by deriving optical solitons for the generalized eighth-order Schrödinger equation, while Kudryashov [11] examined solitary wave hierarchies featuring non-local nonlinearity. Kudryashov [12] also investigated highly dispersive solitary waves in perturbed nonlinear Schrödinger equations. Kudryashov et al. [13] conducted a comprehensive examination of the GI model, emphasizing bifurcations, precise solutions, and conservation laws, whereas Kudryashov and Nifontov [14] analyzed Hamiltonian structures in models characterized by unconstrained dispersion and polynomial nonlinearity. The GI equation has been examined using various transformation and analytical methods. Abbas [15] utilized the Darboux transformation to derive accurate solutions for the coupled GI system. Ahmed et al. [16] investigated a generalized stochastic GI equation with multiplicative white noise, yielding both bright and dark soliton solutions. Arshed [17] employed two analytical techniques on the perturbed GI problem, yielding various classes of soliton solutions. Chen and Zhen [18] examined the GI equation inside a complicated Hamiltonian framework, demonstrating its applicability in simulating optical systems. Hosseini et al. [19] examined the dynamics of optical solitons in perturbed generalized integrable systems, whereas Hu et al. [20] expanded the Darboux transformation to encompass a variable-coefficient nonlocal generalized integrable equation. Kaur and Wazwaz [21] found optical soliton solutions for the perturbed GI problem, whereas Khuri [22] provided novel traveling wave solutions. Liu and Li [23] conducted an examination of the dynamical behavior of the fractional perturbed GI equation. Ozisik [24] examined quintic GI models, whereas Ahmed et al. [25] assessed the impact of multiplicative noise on birefringent fiber GI models. Ultimately, Biswas and Milovic [26] investigated bright and dark solitons within extended nonlinear Schrödinger equations, offering insights pertinent to GI-type systems. In practical optical fiber applications, higher-order nonlinear effects are substantial, and systems described by the GI equation frequently encounter external noise. Incorporating stochastic effects into the GI framework enhances the depiction of physical systems influenced by unpredictability and random perturbations. Multiplicative white noise is significant because its amplitude correlates with wave strength, resulting in jitter, waveform distortion, or potential soliton annihilation in optical communication systems. Analytical methods include the Darboux transformation, simplest equation method, and enhanced extended tanh-function, and Kudryashov techniques have been employed to get exact solutions, each demonstrating the impact of stochastic perturbations on soliton properties. Overall, the GI equation, particularly in its stochastic form incorporating multiplicative noise in the Itô sense, provides a robust framework for analyzing the interactions among nonlinearity, dispersion, and randomness in wave propagation. The ongoing analytical and computational study of these systems enhances theoretical understanding and guides the engineering of resilient optical technologies that can preserve soliton integrity amid environmental variations.

This study presents a novel approach by simultaneously addressing multiple soliton families–bright, dark, singular, and periodic–within the stochastic GI equation influenced by multiplicative noise, utilizing a unified Jacobi elliptic function expansion method. In contrast to previous studies that focused on deterministic GI models or limited their analysis to a single soliton type, our methodology incorporates the noise term directly into the analytical solution process. This enables the derivation of precise stochastic solutions and explicit stability criteria, offering insights that are not readily achievable through existing methods like the Darboux transformation, extended tanh-function method, or Kudryashov method.

This paper's organization continues below. We provide the perturbed GI equation with multiplicative noise in the Itô sense and analyze its physical meaning in Subsection 1.1. Section 2 describes the new Jacobi elliptic function expansion technique used to generate accurate solutions, comparable to previous nonlinear wave research [15,27]. The approach is used to find numerous classes of soliton and periodic perturbed GI equation solutions in Section 3. Detailed stability analysis is performed in Section 4 to evaluate whether these solutions remain resilient in noisy situations. Section 5 shows multiplicative noise's influence on solution profiles in three dimensions. Section 6 links the findings to practical optical fiber and photonic systems for physical understanding. Section 7 concludes with a summary of the key results, practical consequences, and research prospects.

1.1. Main system

We investigate the PGIE with multiplicative noise in the Itô sense:

$$i\Psi_t + \alpha_1\Psi_{xx} + \alpha_2 |\Psi|^4 \Psi + i\alpha_3\Psi^2\Psi_x^* - i\left[\alpha_4\Psi_x + \alpha_5\left(|\Psi|^2\Psi\right)_x + \alpha_6\left(|\Psi|^2\right)_x\Psi\right] + \sigma\Psi\frac{dW(t)}{dt} = 0. \quad (1.1)$$

In this context, the function $\Psi = \Psi(x,t)$ denotes a complex function, where x and t correspond to distance and time in dimensionless form. It is important to recognize that the variable i is defined as the square root of negative one, or $\sqrt{-1}$. Variables $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$, and σ are defined as constants. Specifically, α_1 denotes chromatic dispersion (CD), while α_2 signifies self-phase modulation (SPM). Meanwhile, α_4 and α_5 provide the inter-modal dispersion (IMD) and the self-steepening (SS) term, respectively, while α_3 and α_6 represent the coefficients of nonlinear dispersion. In this instance, σ indicates the noise strength coefficient, and $W(t)$ denotes the standard Wiener process. The concept of "white noise" is represented by the equation $dW(t)/dt$. The stochastic process exhibits the subsequent characteristics: For $t \geq 0$, the function $W(t)$ remains continuous. (ii) The difference between $W(s)$ and $W(t)$, when t is higher than s, follows a normal distribution with a mean of zero and a variance equal to t minus s. This phenomenon is known as Brownian motion. The PGIE is presented in the paper in a modified form containing multiplicative noise in the Itô sense. This equation clarifies the dynamics of nonlinear wave occurrences with a focus on solitonic structures under random effects. The inclusion of multiplicative noise helps to address real-world problems affecting wave propagation, therefore raising the usefulness of the model in pragmatic situations. This equation is used in practical form in optical fiber communications to assess the propagation of light pulses, in which case noise may generate signal distortions.

It deepens understanding of how solitons preserve their form across extended distances in spite of environmental perturbations. In plasma physics, where it clarifies wave interactions in plasmas with stochasticity, the equation is really useful. It predicts in Bose-Einstein condensates how matter waves would behave under external influences. Applications of the equation are found in fluid dynamics, most especially in wave pattern analysis of turbulent flows. It clarifies nonlinear optics issues including beam propagation and self-focusing. It is used by researchers to design noise-free communication systems enhancing signal integrity. At last, the understanding of this equation aids in the development of strong photonic devices.

2. New Jacobi elliptic function expansion methodology and related descriptions

The perturbed Gerdjikov-Ivanov equation (PGIE) with multiplicative noise presents considerable analytical difficulties owing to the concurrent existence of nonlinearity, dispersion, and stochasticity. Conventional methods–such as the Darboux transformation, extended tanh-function techniques, and Kudryashov approaches–have proven effective in generating deterministic soliton solutions. Nevertheless, these methods frequently exhibit two principal limitations: (i) They generally produce a limited category of solutions (e.g., exclusively bright or dark solitons), and (ii) they are not readily amenable to the direct integration of stochastic effects. This encourages the implementation of a more adaptable approach that may encompass the variety of solution families while incorporating the effects of noise during the development phase. This encourages the implementation of a more adaptable approach that can encompass the variety of solution sets while incorporating the effects of noise throughout the development phase.

The Jacobi elliptic function expansion approach [27] is ideal for this. Periodic, hyperbolic, and trigonometric solutions are naturally linked by Jacobi elliptic functions. By adjusting the modulus parameter m, the approach seamlessly transitions between periodic ($m \to 0$) and solitary waves ($m \to 1$). This versatility allows the derivation of bright, dark, singular, and periodic soliton families from a single ansatz, avoiding the need for separate derivations. This method also allows direct incorporation of multiplicative noise in the Itô sense into the analytical solution, making stochastic perturbations inherent to wave dynamics rather than post-processing adjustments.

Formally, we start with a general nonlinear partial differential equation (NLPDE):

$$P(\Psi, \Psi_t, \Psi_x, \Psi_{tt}, \Psi_{xx}, ...) = 0, \tag{2.1}$$

where P is a polynomial in $\Psi(x, t)$ that involves partial derivatives, including nonlinear terms and highest-order derivatives. Assume that the NLPDE given above may be converted into an nonlinear ordinary differential equation (NLODE) by

$$\varrho(\chi, \chi', \chi'', ...) = 0, \tag{2.2}$$

through adopting the transformation

$$\Psi(x,t) = \chi(\eta)\, e^{i\left[-\kappa x+\omega t+\sigma W(t)-\sigma^2 t\right]}, \qquad \eta = \mu x - \beta t.$$

This method presents Eq (2.2) assuming the solution to be

$$\chi(\eta) = g_0 + \sum_{j=1}^{N} \left[\frac{z(\eta)}{1+z^2(\eta)}\right]^{j-1} \left\{g_j\left(\frac{z(\eta)}{1+z^2(\eta)}\right) + f_j\left(\frac{1-z^2(\eta)}{1+z^2(\eta)}\right)\right\}, \tag{2.3}$$

and $z(\eta)$ is the solution of the equation

$$z'^2(\eta) = a + b\, z^2(\eta) + c\, z^4(\eta). \tag{2.4}$$

If N is a positive integer ($(N \in Z^+)$, then a, b, c, g_0, g_j, and f_j (for $j = 1, 2, ..., N$) must all be constants, provided that $g_N \neq 0$ or $f_N \neq 0$. N represents the order of the pole in the general solution. Common recognition indicates that Eq (2.4) allows for solutions expressed as generalized Jacobian elliptic functions (JEFs).

Table 1 displays the functions $sn(\eta,m)$, $cn(\eta,m)$, $dn(\eta,m)$, and so on. These functions belong to the third sort of Jacobi elliptic functions, and the modulus of these functions is denoted by m, where $0 < m < 1$. As is well known, the JEFs become hyperbolic functions as m gets closer to 1^-: $sn(\eta,1) \to \tanh(\eta)$, $cn(\eta,1) \to \text{sech}(\eta)$, $dn(\eta,1) \to \text{sech}(\eta)$, $ns(\eta,1) \to \coth(\eta)$, $dc(\eta,1) \to 1$, $ds(\eta,1) \to \text{cosech}(\eta)$, $sc(\eta,1) \to \sinh(\eta)$, $sd(\eta,1) \to \sinh(\eta)$, $cs(\eta,1) \to \text{cosech}(\eta)$, and becomes the following trigonometric functions when $m \to 0^+$: $sn(\eta,0) \to \sin(\eta)$, $cn(\eta,0) \to \cos(\eta)$, $dn(\eta,0) \to 1$, $ns(\eta,0) \to \text{cosec}(\eta)$, $cs(\eta,0) \to \cot(\eta)$, $ds(\eta,0) \to \text{cosec}(\eta)$, $sc(\eta,0) \to \tan(\eta)$, $sd(\eta,0) \to \sin(\eta)$, $dc(\eta,0) \to \sec(\eta)$.

Table 1. Jacobi elliptic functions and the modulus of these functions.

No.	a	b	c	$z(\eta)$
(1)	1	$-\left(1+m^2\right)$	m^2	$\text{sn}(\eta,m)$ or $\text{cd}(\eta,m)$
(2)	$1-m^2$	$2m^2-1$	$-m^2$	$\text{cn}(\eta,m)$
(3)	m^2-1	$2-m^2$	-1	$\text{dn}(\eta,m)$
(4)	m^2	$-\left(1+m^2\right)$	1	$\text{ns}(\eta,m)$ or $\text{dc}(\eta,m)$
(5)	$-m^2$	$2m^2-1$	$1-m^2$	$\text{nc}(\eta,m)$
(6)	-1	$2-m^2$	m^2-1	$\text{nd}(\eta,m)$
(7)	1	$2m^2-1$	$-m^2\left(1-m^2\right)$	$\text{sd}(\eta,m)$
(8)	$-m^2\left(1-m^2\right)$	$2m^2-1$	1	$\text{ds}(\eta,m)$
(9)	$\frac{1-m^2}{4}$	$\frac{1+m^2}{2}$	$\frac{1-m^2}{4}$	$\text{nc}(\eta,m) \pm \text{sc}(\eta,m)$, $\frac{\text{cn}(\eta,m)}{1\pm\text{sn}(\eta,m)}$
(10)	$\frac{-\left(1-m^2\right)^2}{4}$	$\frac{1+m^2}{2}$	$\frac{-1}{4}$	$m\text{cn}(\eta,m) \pm \text{dn}(\eta,m)$
(11)	$\frac{1}{4}$	$\frac{1-2m^2}{2}$	$\frac{1}{4}$	$\frac{\text{sn}(\eta,m)}{1\pm\text{cn}(\eta,m)}$
(12)	$\frac{1}{4}$	$\frac{1+m^2}{2}$	$\frac{\left(1-m^2\right)^2}{4}$	$\frac{\text{sn}(\eta,m)}{\text{cn}(\eta,m)\pm\text{dn}(\eta,m)}$

Inserting functions (2.3) and (2.4) into Eq (2.2) and performing symbolic computations yields an algebraic system, the solution of which produces optical solitons of the NLPDE (2.1). The present Jacobi elliptic function expansion method is notable for its capacity to systematically produce various solution classes–bright, dark, singular, and periodic–within a unified analytical framework, while also directly incorporating multiplicative stochastic perturbations. In contrast to the Darboux transformation [15, 17] or the extended tanh-function method [21], which generally provide restricted solution families or necessitate distinct derivations for various wave types, the current approach generates all these forms from a single ansatz by utilizing the comprehensive set of Jacobi elliptic function identities. The method incorporates the stochastic Itô term directly into the solution construction, thereby removing the necessity for post-processing or approximation to address noise effects. The solutions and stability criteria derived in this study are not easily obtainable through current deterministic methods, especially concerning stochastic GI equations.

3. Perturbed Gerdjikov-Ivanov equation and its optical solitons

To address the PGIE, we perform a wave transformation described as

$$\Psi(x,t) = \chi(\eta)\, e^{i\left[-\kappa x+\omega t+\sigma W(t)-\sigma^2 t\right]}, \tag{3.1}$$

and

$$\eta = \mu x - \beta t. \tag{3.2}$$

In this context, κ, ω, μ, and β are non-zero constants to be determined, while the real-valued function $\chi(\eta)$ denotes the amplitude component. In this context, μ represents the directional ratio, which indicates the width along the spatial dimension. β denotes the soliton's velocity, κ signifies the frequency, and ω refers to the wave number. The following findings may be derived by putting Eqs (3.1) and (3.2) into Eq (1.1) and separating the real and imaginary components:

$$\Re : \alpha_1\mu^2\chi''(\eta) - \left(\omega - \sigma^2 + \alpha_1\kappa^2 + \alpha_4\kappa\right)\chi(\eta) - \kappa\left(\alpha_3 + \alpha_5\right)\chi^3(\eta) + \alpha_2\,\chi^5(\eta) = 0, \tag{3.3}$$

and

$$\Im : (\beta + 2\alpha_1\kappa\mu + \alpha_4\mu)\chi'(\eta) + (\alpha_3\mu - 3\alpha_5\mu - 2\alpha_6\mu)\chi^2(\eta)\chi'(\eta) = 0. \tag{3.4}$$

From Eq (3.4), we get the velocity of soliton

$$\beta = -\mu\left(\alpha_4 + 2\alpha_1\kappa\right), \tag{3.5}$$

and the nonlinear dispersion coefficient is α_3:

$$\alpha_3 = 3\alpha_5 + 2\alpha_6. \tag{3.6}$$

The balance number $N = \frac{1}{2}$ is derived by balancing the terms $\chi''(\eta)$ and $\chi^5(\eta)$ in Eq (3.3). Given that N is not an integer, we implement the following transformation:

$$F(\eta) = \chi^{\frac{1}{2}}(\eta), \tag{3.7}$$

where $\chi(\eta)$ is a new function of η. Currently, Eq (3.3) shifts to the new equation

$$\alpha_1\mu^2\left[F'^2(\eta) - 2F(\eta)F''(\eta)\right] + 4\left(\alpha_1\kappa^2 + \alpha_4\kappa + \omega - \sigma^2\right)F^2(\eta) + 8\kappa\left(2\alpha_5 + \alpha_6\right)F^3(\eta) - 4\alpha_2F^4(\eta) = 0. \tag{3.8}$$

In Eq (3.8), we balance $F(\eta)F''(\eta)$ with $F^4(\eta)$, leading to $N = 1$. Equation (3.8)'s formal solution is

$$F(\eta) = g_0 + g_1\left(\frac{z(\eta)}{1+z^2(\eta)}\right) + f_1\left(\frac{1-z^2(\eta)}{1+z^2(\eta)}\right). \tag{3.9}$$

Here, g_0, g_1, and f_1 are constants provided with $g_1 \neq 0$ or $f_1 \neq 0$. By merging (2.4) and (3.9) with Eq (3.8), we derive the resulting collection of algebraic equations:

$$\begin{aligned}
z^8(\eta):\ & -8\kappa(2\alpha_5+\alpha_6)\left(f_1^3-3g_0f_1^2-g_0^3+3g_0f_1\right)-4\left[6\alpha_2g_0^2-2c\mu^2\alpha_1-\kappa^2\alpha_1-\kappa\alpha_4-\left(\omega-\sigma^2\right)\right]f_1^2\\
& -8g_0\left[-2g_0^2\alpha_2+\left(c\mu^2+\kappa^2\right)\alpha_1+\kappa\alpha_4-\sigma^2+\omega\right]f_1-4\alpha_2(f_1-4g_0)f_1^3\\
& +4(\kappa^2\alpha_1-\alpha_2g_0^2+\kappa\alpha_4-\sigma^2+\omega)g_0^2+c\mu^2\alpha_1g_1^2=0,\\
z^7(\eta):\ & 2g_1(f_1-g_0)\\
& \times\left[\mu^2(b-6c)\alpha_1-4\kappa^2\alpha_1+12\kappa(2\alpha_5+\alpha_6)(f_1-g_0)+8\alpha_2(f_1-g_0)^2-4\kappa\alpha_4+4(\sigma^2-\omega)\right]=0,\\
z^6(\eta):\ & 16b\mu^2\alpha_1f_1^2-16b\mu^2\alpha_1f_1g_0-b\mu^2\alpha_1g_1^2-16c\mu^2\alpha_1f_1^2+16c\mu^2\alpha_1f_1g_0+10c\mu^2\alpha_1g_1^2-16\kappa^2\alpha_1f_1g_0\\
& +16\kappa^2\alpha_1g_0^2+32\kappa\alpha_6g_0^3+16\sigma^2f_1g_0-32\alpha_2f_1^3g_0+4\kappa^2\alpha_1g_1^2+32\kappa\alpha_5f_1^3+16\kappa\alpha_4g_0^2+4\kappa\alpha_4g_1^2\\
& -96\kappa\alpha_5f_1g_0^2-48\kappa\alpha_5f_1g_1^2+64\kappa\alpha_5g_0^3+48\kappa\alpha_5g_0g_1^2+16\kappa\alpha_6f_1^3-48\kappa\alpha_6f_1g_0^2-24\kappa\alpha_6f_1g_1^2\\
& -24\alpha_2f_1^2g_1^2+32\alpha_2f_1g_0^3+48\alpha_2f_1g_0g_1^2-16\alpha_2g_0^4-24\alpha_2g_0^2g_1^2-16\kappa\alpha_4f_1g_0+16\alpha_2f_1^4\\
& -16\sigma^2g_0^2-4\sigma^2g_1^2-16\omega f_1g_0+16\omega g_0^2+4\omega g_1^2+24\kappa\alpha_6g_0g_1^2=0,\\
z^5(\eta):\ & \alpha_1\left(2a\mu^2f_1-2a\mu^2g_0-11b\mu^2f_1+5b\mu^2g_0+16c\mu^2f_1+4c\mu^2g_0-4\kappa^2f_1+12\kappa^2g_0\right)\\
& +8\kappa\alpha_5\left(9g_0^2+g_1^2\right)-4\kappa\alpha_6\left(3f_1^2+6f_1g_0-9g_0^2-g_1^2\right)\\
& -8\alpha_2\left(3f_1^3-3f_1^2g_0-3f_1g_0^2+3g_0^3+g_0g_1^2-f_1g_1^2\right)-4\kappa+4\left(\sigma^2-\omega\right)(f_1-3g_0)\alpha_4(f_1+3g_0)\\
& -24\kappa\alpha_5f_1(2g_0+f_1)=0,\\
z^4(\eta):\ & 24a\mu^2\alpha_1f_1^2-24a\mu^2\alpha_1f_1g_0-3a\mu^2\alpha_1g_1^2-16b\mu^2\alpha_1f_1^2+10b\mu^2\alpha_1g_1^2+24c\mu^2\alpha_1f_1^2\\
& +24c\mu^2\alpha_1f_1g_0+8\kappa^2\alpha_1g_1^2-96\kappa\alpha_5f_1^2g_0+96\kappa\alpha_5g_0^3+96\kappa\alpha_5g_0g_1^2-48\kappa\alpha_6f_1^2g_0+48\kappa\alpha_6g_0^3\\
& +48\kappa\alpha_6g_0g_1^2-24\alpha_2f_1^4-48\alpha_2g_0^2g_1^2-4\alpha_2g_1^4-8\kappa\alpha_4f_1^2+24\kappa\alpha_4g_0^2+8\kappa\alpha_4g_1^2+8\sigma^2f_1^2\\
& -24\sigma^2g_0^2-8\sigma^2g_1^2-8\omega f_1^2+24\omega g_0^2-8\kappa^2\alpha_1f_1^2+24\kappa^2\alpha_1g_0^2+48\alpha_2f_1^2g_0^2+48\alpha_2f_1^2g_1^2\\
& -24\alpha_2g_0^4-3c\mu^2\alpha_1g_1^2+8\omega g_1^2=0,\\
z^3(\eta):\ & 16a\mu^2\alpha_1f_1-11b\mu^2\alpha_1f_1-5b\mu^2\alpha_1g_0+2c\mu^2\alpha_1f_1+2(c-2a)\alpha_1g_0\mu^2-4\kappa^2\alpha_1f_1\\
& -12\left(\kappa^2\alpha_1-\sigma^2\right)g_0-72\kappa\alpha_5g_0^2-4\kappa(2\alpha_5+\alpha_6)g_1^2+12\kappa\alpha_6f_1^2-24\kappa(\alpha_6+2\alpha_5)f_1g_0\\
& -36\kappa\alpha_6g_0^2-24\alpha_2f_1-24\alpha_2f_1^2g_0+24\alpha_2f_1g_0^2+8\alpha_2f_1g_1^2+24\alpha_2g_0^3+8\alpha_2g_0g_1^2\\
& -4(3\kappa\alpha_4+\omega f_1\omega)g_0+4\left(\sigma^2-\kappa\alpha_4\right)f_1+24\kappa\alpha_5f_1^2=0,\\
z^2(\eta):\ & -16a\mu^2\alpha_1f_1^2-16a\mu^2\alpha_1f_1g_0+10a\mu^2\alpha_1g_1^2+16b\mu^2\alpha_1f_1^2+16b\mu^2\alpha_1f_1g_0-b\mu^2\alpha_1g_1^2\\
& +16\kappa^2\alpha_1f_1g_0+16\kappa^2\alpha_1g_0^2+4\kappa^2\alpha_1g_1^2-32\kappa\alpha_5f_1^3+96\kappa\alpha_5f_1g_0^2+48\kappa\alpha_5f_1g_1^2+64\kappa\alpha_5g_0^3\\
& +48\kappa\alpha_5g_0g_1^2-16\kappa\alpha_6f_1^3+48\kappa\alpha_6f_1g_0^2+24\kappa\alpha_6f_1g_1^2+32\kappa\alpha_6g_0^3+24\kappa\alpha_6g_0g_1^2+16\alpha_2f_1^4\\
& +32\alpha_2f_1^3g_0-24\alpha_2f_1^2g_1^2-32\alpha_2f_1g_0^3-48\alpha_2f_1g_0g_1^2-24\alpha_2g_0^2g_1^2+16\kappa\alpha_4f_1g_0+16\kappa\alpha_4g_0^2\\
& +4\kappa\alpha_4g_1^2-4\left(\sigma^2-\omega\right)\left[4g_0^2+4f_1g_0+g_1^2\right]-16\alpha_2g_0^4=0,
\end{aligned}$$

$$\begin{aligned}
z(\eta) : &(6a-b)\mu^2\alpha_1 f_1 + 6a\mu^2\alpha_1 g_0 - b\mu^2\alpha_1 g_0 + 4\kappa^2\alpha_1 f_1 + 4\kappa^2\alpha_1 g_0 + 24\kappa\alpha_5 f_1^2 + 48\kappa\alpha_5 f_1 g_0 \\
&+ 24\kappa\alpha_5 g_0^2 + 24\kappa\alpha_6 f_1 g_0 + 12\kappa\alpha_6 g_0^2 - 8\alpha_2 f_1^3 - 24\alpha_2 f_1^2 g_0 - 24\alpha_2 f_1 g_0^2 - 8\alpha_2 g_0^3 \\
&+ 4\kappa\alpha_4 (f_1 + g_0) + 12\kappa\alpha_6 f_1^2 - 4\left(\sigma^2 - \omega\right)(f_1 + g_0) = 0, \\
z^0(\eta) : &8a\mu^2\alpha_1 f_1^2 + 8a\mu^2\alpha_1 f_1 g_0 + a\mu^2\alpha_1 g_1^2 + 4\kappa^2\alpha_1 f_1^2 + 8\kappa^2\alpha_1 f_1 g_0 + 4\kappa^2\alpha_1 g_0^2 + 16\kappa\alpha_5 f_1^3 + 48\kappa\alpha_5 f_1^2 g_0 \\
&+ 8\kappa\alpha_6 f_1^3 + 24\kappa\alpha_6 f_1^2 g_0 + 24\kappa\alpha_6 f_1 g_0^2 + 8\kappa\alpha_6 g_0^3 - 4\alpha_2 f_1^4 - 16\alpha_2 f_1^3 g_0 - 24\alpha_2 f_1^2 g_0^2 - 16\alpha_2 f_1 g_0^3 \\
&- 4\alpha_2 g_0^4 + 4\kappa\alpha_4 f_1^2 + 8\kappa\alpha_4 f_1 g_0 + 48\kappa\alpha_5 f_1 g_0^2 + 16\kappa\alpha_5 g_0^3 + 4\kappa\alpha_4 g_0^2 - 4\sigma^2 f_1^2 - 8\sigma^2 f_1 g_0 \\
&- 4\sigma^2 g_0^2 + 4\omega f_1^2 + 8\omega f_1 g_0 + 4\omega g_0^2 = 0.
\end{aligned} \tag{3.10}$$

Thus, Eq (3.10) hold the results:

$$\begin{aligned}
&\kappa = \frac{-\alpha_4 + \sqrt{-4\mu^2\alpha_1^2(3a-b) - 4\left(\omega-\sigma^2\right)\alpha_1 + \alpha_4^2}}{2\alpha_1},\ \alpha_6 = -2\alpha_5 + \frac{2(3a-2b+c)\sqrt{\frac{3\alpha_1\alpha_2}{(a-b+c)}}}{3\left(-\alpha_4 + \sqrt{-4\mu^2\alpha_1^2(3a-b) - 4\alpha_1\left(\omega-\sigma^2\right) + \alpha_4^2}\right)}, \\
&g_0 = \sqrt{\frac{3(a-b+c)\mu^2\alpha_1}{4\alpha_2}},\ g_1 = 0,\ f_1 = -\sqrt{\frac{3(a-b+c)\mu^2\alpha_1}{4\alpha_2}},
\end{aligned} \tag{3.11}$$

provided $(a-b+c)\,\alpha_1\alpha_2 > 0$ and $\left(-4\mu^2\alpha_1^2(3a-b) - 4\left(\omega-\sigma^2\right)\alpha_1 + \alpha_4^2\right) > 0$. From (3.7), (3.9), and (3.11), we have the solutions:

$$F(\eta) = \left\{\sqrt{\frac{3(a-b+c)\mu^2\alpha_1}{4\alpha_2}}\left[1 - \left(\frac{1 - z^2(\eta)}{1 + z^2(\eta)}\right)\right]\right\}^{\frac{1}{2}}. \tag{3.12}$$

The solution sets for Eq (1.1) are as follows:

Set 1. If we replace $a = 1$, $b = -\left(1+m^2\right)$, $c = m^2$ in (3.12) and utilize (1) from Table 1, Eq (1.1) has Jacobi elliptic solutions:

$$\Psi(x,t) = \left\{\sqrt{\frac{3\left(1+m^2\right)\mu^2\alpha_1}{2\alpha_2}}\left[1 - \left(\frac{1-\mathrm{sn}^2(\mu x-\beta t,m)}{1+\mathrm{sn}^2(\mu x-\beta t,m)}\right)\right]\right\}^{\frac{1}{2}} \mathrm{e}^{i\left[-\kappa x+\omega t+\sigma W(t)-\sigma^2 t\right]}, \tag{3.13}$$

and

$$\Psi(x,t) = \left\{\sqrt{\frac{3\left(1+m^2\right)\mu^2\alpha_1}{2\alpha_2}}\left[1 - \left(\frac{1-\mathrm{cd}^2(\mu x-\beta t,m)}{1+\mathrm{cd}^2(\mu x-\beta t,m)}\right)\right]\right\}^{\frac{1}{2}} \mathrm{e}^{i\left[-\kappa x+\omega t+\sigma W(t)-\sigma^2 t\right]}. \tag{3.14}$$

Specifically, as m approaches 1^-, we acquire the solitary soliton solutions:

$$\Psi(x,t) = \left\{\sqrt{\frac{3\mu^2\alpha_1}{\alpha_2}}\left[1 - \left(\frac{1-\tanh^2(\mu x-\beta t)}{1+\tanh^2(\mu x-\beta t)}\right)\right]\right\}^{\frac{1}{2}} \mathrm{e}^{i\left[-\kappa x+\omega t+\sigma W(t)-\sigma^2 t\right]}. \tag{3.15}$$

When m nears 0^+, periodic solutions become available:

$$\Psi(x,t) = \left\{\sqrt{\frac{3\mu^2\alpha_1}{2\alpha_2}}\left[1 - \left(\frac{\cot^2(\mu x-\beta t)}{1+\csc^2(\mu x-\beta t)}\right)\right]\right\}^{\frac{1}{2}} \mathrm{e}^{i\left[-\kappa x+\omega t+\sigma W(t)-\sigma^2 t\right]}, \tag{3.16}$$

and

$$\Psi(x,t)=\left\{\sqrt{\frac{3\mu^2\alpha_1}{2\alpha_2}}\left[1-\left(\frac{\tan^2(\mu x-\beta t)}{1+\sec^2(\mu x-\beta t)}\right)\right]\right\}^{\frac{1}{2}}\mathrm{e}^{i[-\kappa x+\omega t+\sigma W(t)-\sigma^2 t]},\tag{3.17}$$

provided $\alpha_1\alpha_2>0$.
Set 2. Jacobi elliptic solutions for Eq (1.1) are obtained by using (2) from Table 1 and substituting $a=1-m^2$, $b=2m^2-1$, $c=-m^2$ in (3.12):

$$\Psi(x,t)=\left\{\sqrt{\tfrac{3\left(1-2m^2\right)\mu^2\alpha_1}{2\alpha_2}}\left[1-\left(\tfrac{1-\mathrm{cn}^2(\mu x-\beta t,m)}{1+\mathrm{cn}^2(\mu x-\beta t,m)}\right)\right]\right\}^{\frac{1}{2}}\mathrm{e}^{i[-\kappa x+\omega t+\sigma W(t)-\sigma^2 t]},\tag{3.18}$$

provided $\left(1-2m^2\right)\alpha_1\alpha_2>0$. Specifically, bright soliton solutions exist as m approaches 1^-:

$$\Psi(x,t)=\left\{\sqrt{-\frac{3\mu^2\alpha_1}{2\alpha_2}}\left[\frac{2}{1+\cosh^2(\mu x-\beta t)}\right]\right\}^{\frac{1}{2}}\mathrm{e}^{i[-\kappa x+\omega t+\sigma W(t)-\sigma^2 t]},\tag{3.19}$$

provided $\alpha_1\alpha_2<0$ and while if $m\to 0^+$, then we have the same periodic solution (3.17).
Set 3. Equation (1.1) has Jacobi elliptic solutions if we apply (3) from Table 1 and replace $a=-\left(1-m^2\right)$, $b=2-m^2$, and $c=-1$ in (3.12):

$$\Psi(x,t)=\left\{\sqrt{\tfrac{3\left(m^2-2\right)\mu^2\alpha_1}{2\alpha_2}}\left[1-\left(\tfrac{1-\mathrm{dn}^2(\mu x-\beta t,m)}{1+\mathrm{dn}^2(\mu x-\beta t,m)}\right)\right]\right\}^{\frac{1}{2}}\mathrm{e}^{i[-\kappa x+\omega t+\sigma W(t)-\sigma^2 t]},\tag{3.20}$$

provided $\left(m^2-2\right)<0$ and $\alpha_1\alpha_2<0$. We acquire the same bright soliton solution (3.17) particularly when $m\to 1^-$.
Set 4. Equation (1.1) has Jacobi elliptic solutions if $a=m^2$, $b=-\left(1+m^2\right)$, $c=1$ in (3.12) and we apply (4) from Table 1:

$$\Psi(x,t)=\left\{\sqrt{\tfrac{3\left(m^2+1\right)\mu^2\alpha_1}{2\alpha_2}}\left[1-\left(\tfrac{1-\mathrm{ns}^2(\mu x-\beta t,m)}{1+\mathrm{ns}^2(\mu x-\beta t,m)}\right)\right]\right\}^{\frac{1}{2}}\mathrm{e}^{i[-\kappa x+\omega t+\sigma W(t)-\sigma^2 t]},\tag{3.21}$$

and

$$\Psi(x,t)=\left\{\sqrt{\tfrac{3\left(m^2+1\right)\mu^2\alpha_1}{2\alpha_2}}\left[1-\left(\tfrac{1-\mathrm{dc}^2(\mu x-\beta t,m)}{1+\mathrm{dc}^2(\mu x-\beta t,m)}\right)\right]\right\}^{\frac{1}{2}}\mathrm{e}^{i[-\kappa x+\omega t+\sigma W(t)-\sigma^2 t]}.\tag{3.22}$$

Particularly, we have the singular soliton solutions if $m\to 1^-$:

$$\Psi(x,t)=\left\{\sqrt{\tfrac{3\mu^2\alpha_1}{\alpha_2}}\left[1-\left(\tfrac{1-\coth^2(\mu x-\beta t)}{1+\coth^2(\mu x-\beta t)}\right)\right]\right\}^{\frac{1}{2}}\mathrm{e}^{i[-\kappa x+\omega t+\sigma W(t)-\sigma^2 t]},\tag{3.23}$$

while the periodic solutions exist as m approaches 0^+:

$$\Psi(x,t)=\left\{\sqrt{\tfrac{3\mu^2\alpha_1}{2\alpha_2}}\left[1-\left(\tfrac{1-\csc^2(\mu x-\beta t)}{1+\csc^2(\mu x-\beta t)}\right)\right]\right\}^{\frac{1}{2}}\mathrm{e}^{i[-\kappa x+\omega t+\sigma W(t)-\sigma^2 t]},\tag{3.24}$$

and

$$\Psi(x,t)=\left\{\sqrt{\frac{3\mu^2\alpha_1}{2\alpha_2}}\left[1-\left(\frac{1-\sec^2(\mu x-\beta t)}{1+\sec^2(\mu x-\beta t)}\right)\right]\right\}^{\frac{1}{2}}\ \mathrm{e}^{i\left[-\kappa x+\omega t+\sigma W(t)-\sigma^2 t\right]},\tag{3.25}$$

provided $\alpha_1\alpha_2>0$.

Set 5. Applying (5) from Table 1 and replacing $a=-m^2$, $b=2m^2-1$, and $c=1-m^2$ in (3.12) yields Jacobi elliptic solutions to Eq (1.1):

$$\Psi(x,t)=\left\{\sqrt{\frac{3\left(1-2m^2\right)\mu^2\alpha_1}{2\alpha_2}}\left[1-\left(\frac{1-\mathrm{nc}^2(\mu x-\beta t,m)}{1+\mathrm{nc}^2(\mu x-\beta t,m)}\right)\right]\right\}^{\frac{1}{2}}\ \mathrm{e}^{i\left[-\kappa x+\omega t+\sigma W(t)-\sigma^2 t\right]},\tag{3.26}$$

provided $\left(1-2m^2\right)\alpha_1\alpha_2>0$. Specifically, the solitary soliton solutions exist as m approaches 1^-:

$$\Psi(x,t)=\left\{\sqrt{-\frac{3\mu^2\alpha_1}{2\alpha_2}}\left[1+\frac{\tanh^2(\mu x-\beta t)}{1+\mathrm{sech}^2(\mu x-\beta t)}\right]\right\}^{\frac{1}{2}}\ \mathrm{e}^{i\left[-\kappa x+\omega t+\sigma W(t)-\sigma^2 t\right]},\tag{3.27}$$

provided $\alpha_1\alpha_2<0$ and the periodic solution exists if and only if m approaches 0^+:

$$\Psi(x,t)=\left\{\sqrt{\frac{3\mu^2\alpha_1}{2\alpha_2}}\left[1+\left(\frac{\tan^2(\mu x-\beta t)}{1+\sec^2(\mu x-\beta t)}\right)\right]\right\}^{\frac{1}{2}}\ \mathrm{e}^{i\left[-\kappa x+\omega t+\sigma W(t)-\sigma^2 t\right]},\tag{3.28}$$

provided $\alpha_1\alpha_2>0$.

Set 6. Using (6) from Table 1 and substituting $a=-1$, $b=2-m^2$, and $c=m^2-1$ into (3.12), we get the Jacobi elliptic solutions to Eq (1.1):

$$\Psi(x,t)=\left\{\sqrt{\frac{3\left(m^2-2\right)\mu^2\alpha_1}{2\alpha_2}}\left[1-\left(\frac{1-\mathrm{nd}^2(\mu x-\beta t,m)}{1+\mathrm{nd}^2(\mu x-\beta t,m)}\right)\right]\right\}^{\frac{1}{2}}\ \mathrm{e}^{i\left[-\kappa x+\omega t+\sigma W(t)-\sigma^2 t\right]},\tag{3.29}$$

provided $\left(m^2-2\right)<0$ and $\alpha_1\alpha_2<0$. Specifically, the singular soliton solution (3.27) remains unchanged if m approaches 1^-.

Set 7. By exchanging $a=1$, $b=2m^2-1$, and $c=-m^2\left(1-m^2\right)$ in Eq (3.12) and applying (7) from Table 1, Jacobi elliptic solutions are obtained:

$$\Psi(x,t)=\left\{\sqrt{\frac{3(1-m^2)(2-m^2)\mu^2\alpha_1}{4\alpha_2}}\left[1-\left(\frac{1-\mathrm{sd}^2(\mu x-\beta t,m)}{1+\mathrm{sd}^2(\mu x-\beta t,m)}\right)\right]\right\}^{\frac{1}{2}}\ \mathrm{e}^{i\left[-\kappa x+\omega t+\sigma W(t)-\sigma^2 t\right]},\tag{3.30}$$

provided $(1-m^2)\alpha_1\alpha_2>0$. For instance, if $m\to 0^+$, we obtain the same periodic solution (3.16).

Set 8. To get Jacobi elliptic solutions for Eq (1.1), we may use (8) from Table 1 to substitute $a=-m^2\left(1-m^2\right)$, $b=2m^2-1$, and $c=1$ in Eq (3.12):

$$\Psi(x,t)=\left\{\sqrt{\frac{3(1-m^2)(2-m^2)\mu^2\alpha_1}{4\alpha_2}}\left[1-\left(\frac{1-\mathrm{ds}^2(\mu x-\beta t,m)}{1+\mathrm{ds}^2(\mu x-\beta t,m)}\right)\right]\right\}^{\frac{1}{2}}\ \mathrm{e}^{i\left[-\kappa x+\omega t+\sigma W(t)-\sigma^2 t\right]},\tag{3.31}$$

provided $(1-m^2)\alpha_1\alpha_2>0$. For example, if $m\to 0^+$, we obtain the periodic solution:

$$\Psi(x,t)=\left\{\sqrt{\frac{3\mu^2\alpha_1}{2\alpha_2}}\left[1+\left(\frac{\cot^2(\mu x-\beta t)}{1+\csc^2(\mu x-\beta t)}\right)\right]\right\}^{\frac{1}{2}}\ \mathrm{e}^{i\left[-\kappa x+\omega t+\sigma W(t)-\sigma^2 t\right]}.\tag{3.32}$$

Set 9. By substituting $a=\frac{1-m^2}{4}$, $b=\frac{1+m^2}{2}$, $c=\frac{1-m^2}{4}$ in (3.12) and using (9) from Table 1, Eq (1.1) has Jacobi elliptic solutions:

$$\Psi(x,t)=\left\{\sqrt{-\frac{3m^2\mu^2\alpha_1}{4\alpha_2}}\left[1-\left(\frac{1-[\mathrm{nc}(\mu x-\beta t,m)\pm\mathrm{sc}(\mu x-\beta t,m)]^2}{1+[\mathrm{nc}(\mu x-\beta t,m)\pm\mathrm{sc}(\mu x-\beta t,m)]^2}\right)\right]\right\}^{\frac{1}{2}}\ e^{i[-\kappa x+\omega t+\sigma W(t)-\sigma^2 t]}, \tag{3.33}$$

and

$$\Psi(x,t)=\left\{\sqrt{-\frac{3m^2\mu^2\alpha_1}{4\alpha_2}}\left[1-\left(\frac{[1\pm\mathrm{sn}(\mu x-\beta t,m)]^2-\mathrm{cn}^2(\mu x-\beta t,m)}{\mathrm{cn}^2(\mu x-\beta t,m)+[1\pm\mathrm{sn}(\mu x-\beta t,m)]^2}\right)\right]\right\}^{\frac{1}{2}}\ e^{i[-\kappa x+\omega t+\sigma W(t)-\sigma^2 t]}. \tag{3.34}$$

The dark soliton solutions exist if $m\to 1^-$:

$$\Psi(x,t)=\left\{\sqrt{-\frac{3\mu^2\alpha_1}{4\alpha_2}}\left[1\pm\tanh(\mu x-\beta t)\right]\right\}^{\frac{1}{2}}\ e^{i[-\kappa x+\omega t+\sigma W(t)-\sigma^2 t]}, \tag{3.35}$$

as well as bright-dark soliton solutions:

$$\Psi(x,t)=\left\{\sqrt{-\frac{3\mu^2\alpha_1}{4\alpha_2}}\left[\frac{\operatorname{sech}^2(\mu x-\beta t)}{1\pm\tanh(\mu x-\beta t)}\right]\right\}^{\frac{1}{2}}\ e^{i[-\kappa x+\omega t+\sigma W(t)-\sigma^2 t]}, \tag{3.36}$$

provided $\alpha_1\alpha_2<0$.

Set 10. Equation (1.1) has Jacobi elliptic solutions if we utilize (10) from Table 1 and exchange $a=-\frac{(1-m^2)^2}{4}$, $b=\frac{1+m^2}{2}$, $c=-\frac{1}{4}$ in (3.12):

$$\Psi(x,t)=\left\{\sqrt{-\frac{3(m^4+4)\mu^2\alpha_1}{16\alpha_2}}\left[1-\left(\frac{1-[m\,\mathrm{cn}(\mu x-\beta t,m)\pm\mathrm{dn}(\mu x-\beta t,m)]^2}{1+[m\,\mathrm{cn}(\mu x-\beta t,m)\pm\mathrm{dn}(\mu x-\beta t,m)]^2}\right)\right]\right\}^{\frac{1}{2}}\ e^{i[-\kappa x+\omega t+\sigma W(t)-\sigma^2 t]}. \tag{3.37}$$

Specifically, bright soliton solutions correspond to $m\to 1^-$:

$$\Psi(x,t)=\left\{\sqrt{-\frac{15\mu^2\alpha_1}{\alpha_2}}\left[\frac{2}{4+\cosh^2(\mu x-\beta t)}\right]\right\}^{\frac{1}{2}}\ e^{i[-\kappa x+\omega t+\sigma W(t)-\sigma^2 t]}, \tag{3.38}$$

provided $\alpha_1\alpha_2<0$.

Set 11. The Jacobi elliptic solutions to Eq (1.1) may be obtained by substituting $a=\frac{1}{4}$, $b=\frac{1-2m^2}{2}$, $c=\frac{1}{4}$ in Eq (3.12) and using (11) from Table 1:

$$\Psi(x,t)=\left\{\sqrt{\frac{3m^2\mu^2\alpha_1}{4\alpha_2}}\left[1-\left(\frac{[1\pm\mathrm{cn}(\mu x-\beta t,m)]^2-\mathrm{sn}^2(\mu x-\beta t,m)}{[1\pm\mathrm{cn}(\mu x-\beta t,m)]^2+\mathrm{sn}^2(\mu x-\beta t,m)}\right)\right]\right\}^{\frac{1}{2}}\ e^{i[-\kappa x+\omega t+\sigma W(t)-\sigma^2 t]}. \tag{3.39}$$

Specifically, the singular soliton solutions exist as m approaches 1^-:

$$\Psi(x,t)=\left\{\sqrt{\frac{3\mu^2\alpha_1}{\alpha_2}}\left[\frac{1}{1+[\coth(\mu x-\beta t)\pm\operatorname{csch}(\mu x-\beta t)]^2}\right]\right\}^{\frac{1}{2}}\ e^{i[-\kappa x+\omega t+\sigma W(t)-\sigma^2 t]}, \tag{3.40}$$

provided $\alpha_1\alpha_2>0$.

Set 12. Entering $a=\frac{1}{4}$, $b=\frac{1+m^2}{2}$, $c=\frac{(1-m^2)^2}{4}$ in (3.12) and using (12) from Table 1 yields Jacobi elliptic solutions for Eq (1.1):

$$\Psi(x,t)=\left\{\sqrt{\frac{3m^2(m^2-4)\mu^2\alpha_1}{4\alpha_2}}\left[\frac{\mathrm{sn}^2(\mu x-\beta t,m)}{\mathrm{sn}^2(\mu x-\beta t,m)+[\mathrm{cn}(\mu x-\beta t,m)\pm\mathrm{dn}(\mu x-\beta t,m)]^2}\right]\right\}^{\frac{1}{2}}\ e^{i[-\kappa x+\omega t+\sigma W(t)-\sigma^2 t]}, \tag{3.41}$$

provided $\alpha_1\alpha_2 < 0$ and $\left(m^2-4\right) < 0$. If $m \to 1^-$, we have solitary solutions:

$$\Psi(x,t) = \left\{\sqrt{-\frac{9\mu^2\alpha_1}{4\alpha_2}}\left[\frac{\tanh^2(\mu x-\beta t)}{\tanh^2(\mu x-\beta t)+4\operatorname{sech}^2(\mu x-\beta t)}\right]\right\}^{\frac{1}{2}} e^{i\left[-\kappa x+\omega t+\sigma W(t)-\sigma^2 t\right]}, \tag{3.42}$$

provided $\alpha_1\alpha_2 < 0$.

4. Stability analysis

In this section, we will analyze the stability of Eq (1.1). To find the steady-state solutions, assume $F(\eta)$ is constant [28,29], and let

$$F(\eta) = F_g. \tag{4.1}$$

At the steady state,

$$\frac{d}{d\eta}F(\eta) = 0 \text{ and } \frac{d^2}{d\eta^2}F(\eta) = 0, \tag{4.2}$$

so, Eq (3.8) simplifies to

$$\left(\alpha_1\kappa^2+\alpha_4\kappa+\omega-\sigma^2\right)F_g^2 + 2\kappa\left(2\alpha_5+\alpha_6\right)F_g^3 - \alpha_2F_g^4 = 0. \tag{4.3}$$

This gives two possible solutions:

$$F_g = 0 \quad \text{or} \quad F_g = \frac{\kappa(2\alpha_5+\alpha_6)\mp\sqrt{\kappa^2(2\alpha_5+\alpha_6)^2+\alpha_2\left(\alpha_1\kappa^2+\alpha_4\kappa+\omega-\sigma^2\right)}}{\alpha_2}, \tag{4.4}$$

provided $\left[\kappa^2\left(2\alpha_5+\alpha_6\right)^2+\alpha_2\left(\alpha_1\kappa^2+\alpha_4\kappa+\omega-\sigma^2\right)\right] > 0$.

The possible equilibrium locations where the system may maintain stability are represented by these solutions. Let us look at a little perturbation around the steady state F_g to examine the stability of these steady-state solutions. Consider that

$$F(\eta) = F_g + \epsilon\overline{F}(\eta). \tag{4.5}$$

When the perturbation is $\overline{F}(\eta)$ and ϵ is a minor parameter parameter, entering (4.5) into Eq (3.8), one obtains

$$\begin{aligned}&\alpha_1\mu^2\left(\frac{d}{d\eta}\left[F_g+\epsilon\overline{F}(\eta)\right]\right)^2 - 2\alpha_1\mu^2\left[F_g+\epsilon\overline{F}(\eta)\right]\frac{d^2}{d\eta^2}\left[F_g+\epsilon\overline{F}(\eta)\right] + 4\left(\alpha_1\kappa^2+\alpha_4\kappa+\omega-\sigma^2\right)\left[F_g+\epsilon\overline{F}(\eta)\right]^2\\&+8\kappa\left(2\alpha_5+\alpha_6\right)\left[F_g+\epsilon\overline{F}(\eta)\right]^3 - 4\alpha_2\left[F_g+\epsilon\overline{F}(\eta)\right]^4 = 0.\end{aligned} \tag{4.6}$$

Here,

$$\left.\begin{aligned}\left(\tfrac{d}{d\eta}\left[F_g+\epsilon\overline{F}(\eta)\right]\right)^2 &= \left(\tfrac{d}{d\eta}\epsilon\overline{F}(\eta)\right)^2 = \epsilon^2\left(\tfrac{d}{d\eta}\overline{F}(\eta)\right)^2,\\ \left[F_g+\epsilon\overline{F}(\eta)\right]\tfrac{d^2}{d\eta^2}\left[F_g+\epsilon\overline{F}(\eta)\right] &= \epsilon F_g\tfrac{d^2}{d\eta^2}\overline{F}(\eta) + \epsilon^2\overline{F}(\eta)\tfrac{d^2}{d\eta^2}\overline{F}(\eta),\\ \left[F_g+\epsilon\overline{F}(\eta)\right]^2 &= F_g^2 + 2\epsilon F_g\overline{F}(\eta) + \epsilon^2\overline{F}^2(\eta),\\ \left[F_g+\epsilon\overline{F}(\eta)\right]^3 &= F_g^3 + 3\epsilon F_g^2\overline{F}(\eta) + 3\epsilon^2F_g\overline{F}^2(\eta) + \epsilon^3\overline{F}^3(\eta),\\ \left[F_g+\epsilon\overline{F}(\eta)\right]^4 &= F_g^4 + 4\epsilon F_g^3\overline{F}(\eta) + 6\epsilon^2F_g^2\overline{F}^2(\eta) + 4\epsilon^3F_g\overline{F}^3(\eta) + \epsilon^4\overline{F}^4(\eta).\end{aligned}\right\} \tag{4.7}$$

We linearize the equation by retaining just the first-order terms in ϵ since it is small. One derivation is

$$\alpha_1\mu^2 F_g \frac{d^2}{d\eta^2}\overline{F}(\eta) - 4\left(\alpha_1\kappa^2 + \alpha_4\kappa + \omega - \sigma^2\right)F_g\overline{F}(\eta) - 12\kappa\left(2\alpha_5 + \alpha_6\right)F_g^2\overline{F}(\eta) + 8\alpha_2 F_g^3\overline{F}(\eta) = 0. \quad (4.8)$$

The stability is established by assuming

$$\overline{F}(\eta) = e^{\lambda\eta}. \quad (4.9)$$

A constant is denoted by λ. A characteristic equation may be obtained by substituting Eq (4.9) into the linearized equation (4.8):

$$\alpha_1\mu^2\lambda^2 F_g - 4\left[\left(\alpha_1\kappa^2 + \alpha_4\kappa + \omega - \sigma^2\right)F_g + 3\kappa\left(2\alpha_5 + \alpha_6\right)F_g^2 - 2\alpha_2 F_g^3\right] = 0. \quad (4.10)$$

We shall next examine the two equilibrium points:

Case 1: To determine the characteristic equation, enter $F_g = 0$ into Eq (4.10). The characteristic equation of the system does not provide any useful information on stability at $F_g = 0$, indicating neutral stability. Small perturbations neither increase nor decrease in the absence of a linear restoring force, suggesting that $F_g = 0$ is a weakly stable equilibrium point. The system's specific behavior may be influenced by outside variables or nonlinear influences.

Case 2: When the non-zero steady state $F_g = \frac{\kappa(2\alpha_5+\alpha_6)\mp\sqrt{\kappa^2(2\alpha_5+\alpha_6)^2+\alpha_2\left(\alpha_1\kappa^2+\alpha_4\kappa+\omega-\sigma^2\right)}}{\alpha_2}$ is substituted into the linearized equation (4.10), we have the characteristic equation with the following coefficients:

$$\lambda^2 + \frac{4\alpha_2\left(\alpha_1\kappa^2 + \alpha_4\kappa + \omega - \sigma^2\right) + 4\kappa^2\left(2\alpha_5 + \alpha_6\right)^2 \pm 4\kappa\left(2\alpha_5 + \alpha_6\right)\sqrt{\kappa^2\left(2\alpha_5 + \alpha_6\right)^2 + \alpha_2\left(\alpha_1\kappa^2 + \alpha_4\kappa + \omega - \sigma^2\right)}}{\alpha_1\alpha_2\mu^2} = 0. \quad (4.11)$$

The roots of the characteristic equation λ_1 and λ_2 are given by

$$\lambda = 2\sqrt{\frac{-\alpha_2\left(\alpha_1\kappa^2 + \alpha_4\kappa + \omega - \sigma^2\right) \mp \kappa\left(2\alpha_5 + \alpha_6\right)\sqrt{\kappa^2\left(2\alpha_5 + \alpha_6\right)^2 + \alpha_2\left(\alpha_1\kappa^2 + \alpha_4\kappa + \omega - \sigma^2\right)} - \kappa^2\left(2\alpha_5 + \alpha_6\right)^2}{\alpha_1\alpha_2\mu^2}}. \quad (4.12)$$

The steady state's stability is thus dependent on the actual components of these roots: (1) For stability in the system: Both roots must be negative, and the following requirements must be fulfilled:

$$\begin{aligned}&-\alpha_2\left(\alpha_1\kappa^2 + \alpha_4\kappa + \omega - \sigma^2\right) \mp \kappa\left(2\alpha_5 + \alpha_6\right)\sqrt{\kappa^2\left(2\alpha_5 + \alpha_6\right)^2 + \alpha_2\left(\alpha_1\kappa^2 + \alpha_4\kappa + \omega - \sigma^2\right)}\\ &-\kappa^2\left(2\alpha_5 + \alpha_6\right)^2 > 0.\end{aligned} \quad (4.13)$$

This inequality provides a condition on the parameters $\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6, \kappa, \mu, \omega$, and σ for the equilibrium point $F_g = \frac{\kappa(2\alpha_5+\alpha_6)\mp\sqrt{\kappa^2(2\alpha_5+\alpha_6)^2+\alpha_2\left(\alpha_1\kappa^2+\alpha_4\kappa+\omega-\sigma^2\right)}}{\alpha_2}$ to be stable.

(2) If there is instability in the system: In the event that one root is positive or the subsequent requirement is met:

$$\begin{aligned}&-\alpha_2\left(\alpha_1\kappa^2 + \alpha_4\kappa + \omega - \sigma^2\right) \mp \kappa\left(2\alpha_5 + \alpha_6\right)\sqrt{\kappa^2\left(2\alpha_5 + \alpha_6\right)^2 + \alpha_2\left(\alpha_1\kappa^2 + \alpha_4\kappa + \omega - \sigma^2\right)}\\ &-\kappa^2\left(2\alpha_5 + \alpha_6\right)^2 < 0.\end{aligned} \quad (4.14)$$

(3) A little instability in the system: If the following circumstance is met:

$$\begin{aligned}&\alpha_2\left(\alpha_1\kappa^2 + \alpha_4\kappa + \omega - \sigma^2\right) \pm \kappa\left(2\alpha_5 + \alpha_6\right)\sqrt{\kappa^2\left(2\alpha_5 + \alpha_6\right)^2 + \alpha_2\left(\alpha_1\kappa^2 + \alpha_4\kappa + \omega - \sigma^2\right)}\\ &+\kappa^2\left(2\alpha_5 + \alpha_6\right)^2 = 0,\end{aligned} \quad (4.15)$$

resulting in actual roots that are repeated. The study demonstrates that the simple equilibrium point represented by the naive steady-state solution $F_g = 0$ is dependent on certain system factors for stability. Alternatively, the non-trivial steady-state solution $F_g = \frac{\kappa(2\alpha_5+\alpha_6)\mp\sqrt{\kappa^2(2\alpha_5+\alpha_6)^2+\alpha_2\left(\alpha_1\kappa^2+\alpha_4\kappa+\omega-\sigma^2\right)}}{\alpha_2}$ calls for a more nuanced analysis since the stability is affected by a number of variables, including the coefficients $\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6, \kappa, \mu, \omega$, and σ. Predicting the system's behavior under minor perturbations requires an understanding of these relationships.

5. Graphical representations

We offer numerical simulations of various solution types of the perturbed Gerdjikov-Ivanov equation to demonstrate the analytical findings and emphasize the impact of multiplicative noise. To enhance understanding, we selected representative parameter sets and illustrated both three-dimensional and two-dimensional perspectives to highlight the spatial-temporal dynamics and the influence of randomness.

Jacobi-elliptic wave pattern (Eq (3.14)): Figure 1 depicts an oscillating sequence of pulses characterized by Jacobi elliptic functions, rendered for $t = 5$ with parameters $m = \sqrt{3}/2$, $\alpha_1 = 0.1$, $\alpha_2 = 0.2$, $\alpha_4 = 0.2$, $\mu = 0.1$, $\omega = 0.01$, $\kappa = -1 + \sqrt{0.8525 + 10\sigma^2}$, and $\beta = -0.02(1+\kappa)$. The stochastic term $W(t) = \cosh(2t)$ changes the amplitude. The two-dimensional projection distinctly illustrates periodic humps, whose shape and height exhibit minor fluctuations attributable to noise, simulating noisy optical fiber pulse trains.

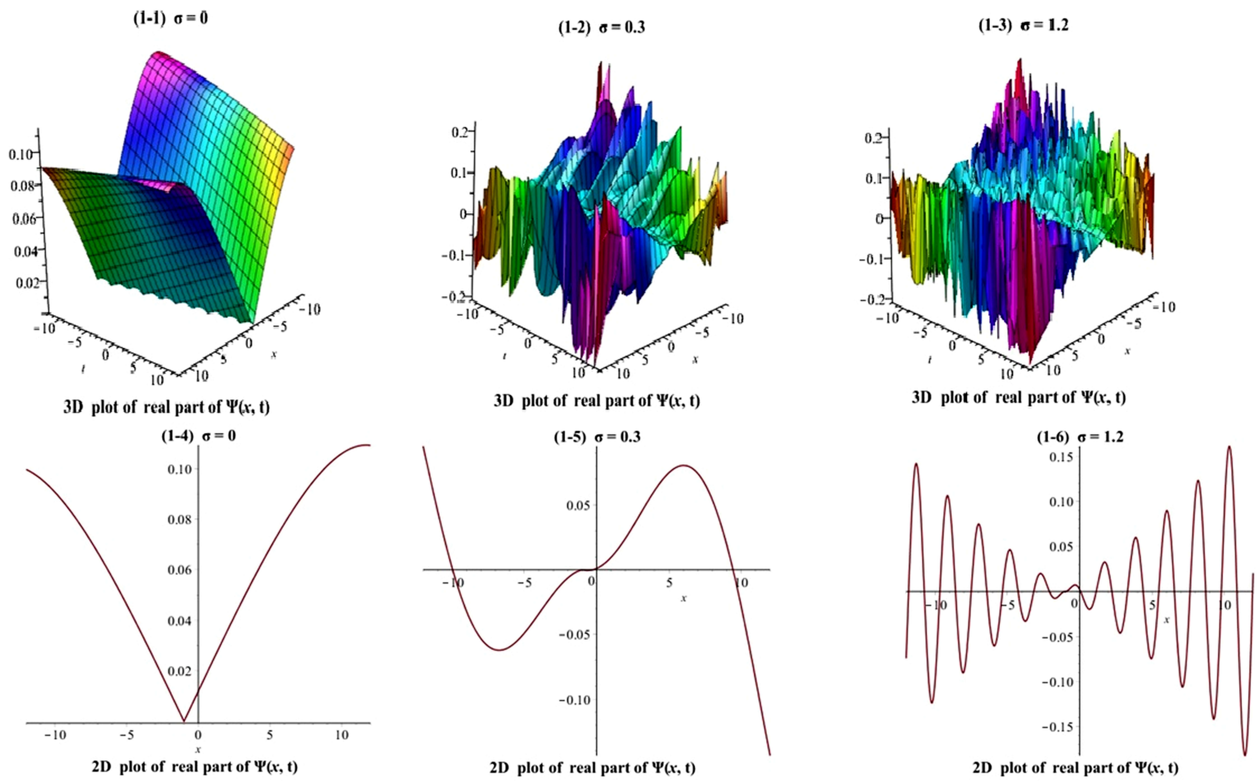

Figure 1. Plot of Jacobi-elliptic wave pattern Eq (3.14).

Bright soliton (Eq (3.19)): Figure 2 illustrates a solitary localized pulse propagating without dispersion at $t = 4$ with $\alpha_1 = 0.1$, $\alpha_2 = -0.3$, $\alpha_4 = 0.2$, $\mu = 0.1$, $\omega = 0.03$, $\kappa = -1 + \sqrt{0.71 + 10\sigma^2}$, $\beta = -0.02(1 + \kappa)$, as well $W(t) = \cos(t)$, despite random perturbations. The two-dimensional slice illustrates that the soliton peak retains its sharpness, exhibiting minimal amplitude fluctuations caused by noise, underscoring the resilience of bright solitons in optical systems.

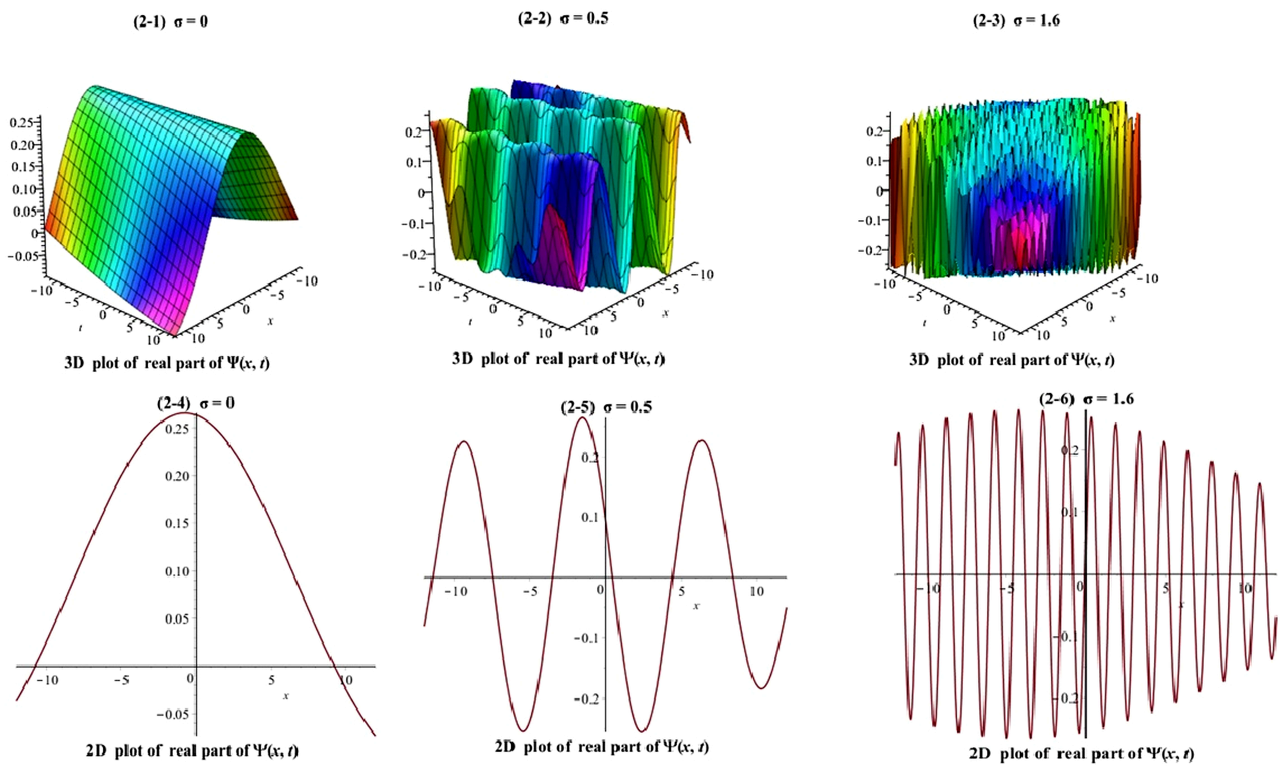

Figure 2. Plot of bright soliton Eq (3.19).

Periodic solution (Eq (3.28)): For $t = 2$, $\alpha_1 = 0.1$, $\alpha_2 = 0.2$, $\alpha_4 = 0.2$, $\mu = 0.1$, $\omega = 0.01$, $\kappa = -1 + \sqrt{0.8525 + 10\sigma^2}$, $\beta = -0.02(1 + \kappa)$, as well as $W(t) = \cosh(2t)$, the solution creates a sinusoidal wave pattern that repeats in space and time. Figure 3 depicts a clean sinusoid affected by modest random modulations, demonstrating how stochasticity impacts continuous wave trains and mode-locked systems.

Dark soliton (Eq (3.35)): With parameters set at $t = 1$ with $\alpha_1 = 0.5$, $\alpha_2 = -0.5$, $\alpha_4 = 0.1$, $\mu = 0.2$, $\omega = -0.01$, $\kappa = -0.1 + \sqrt{0.05 + 2\sigma^2}$, $\beta = -0.2(0.1 + \kappa)$, as well as $W(t) = 3t^2$, Figure 4 illustrates a dip within a predominantly continuous wave background. The two-dimensional profile illustrates the localized intensity reduction and its susceptibility to noise, which has the potential to alter or distort the notch position.

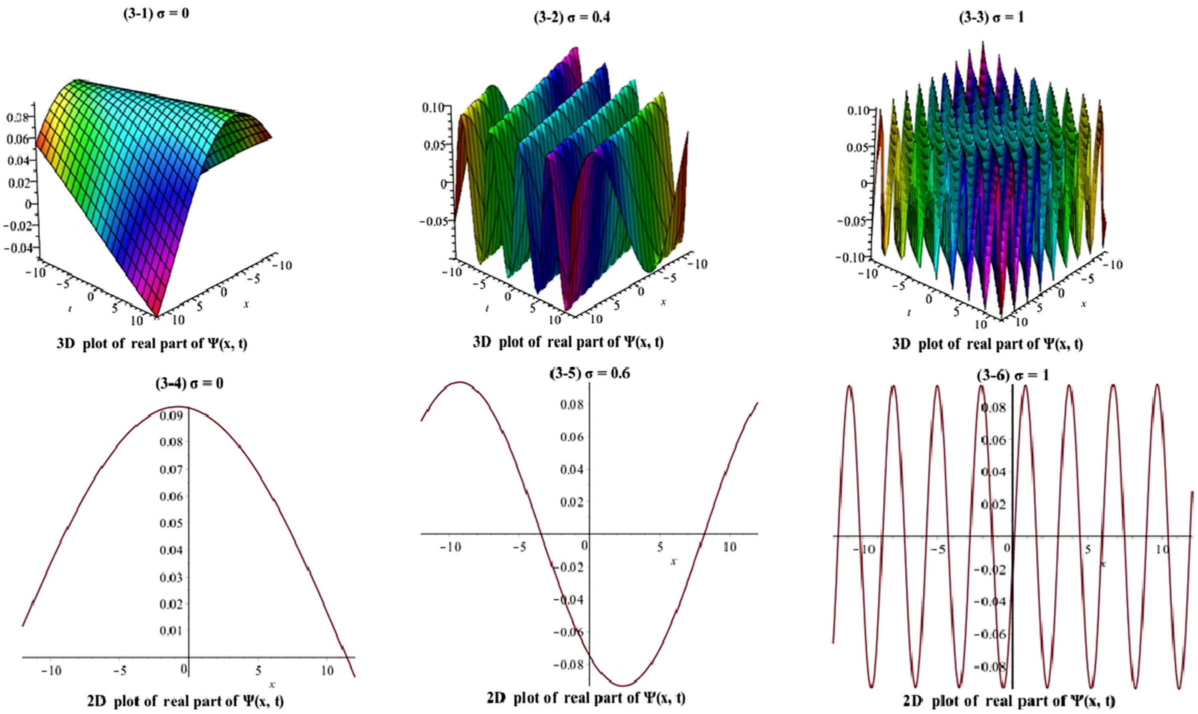

Figure 3. Plot of periodic solution Eq (3.28).

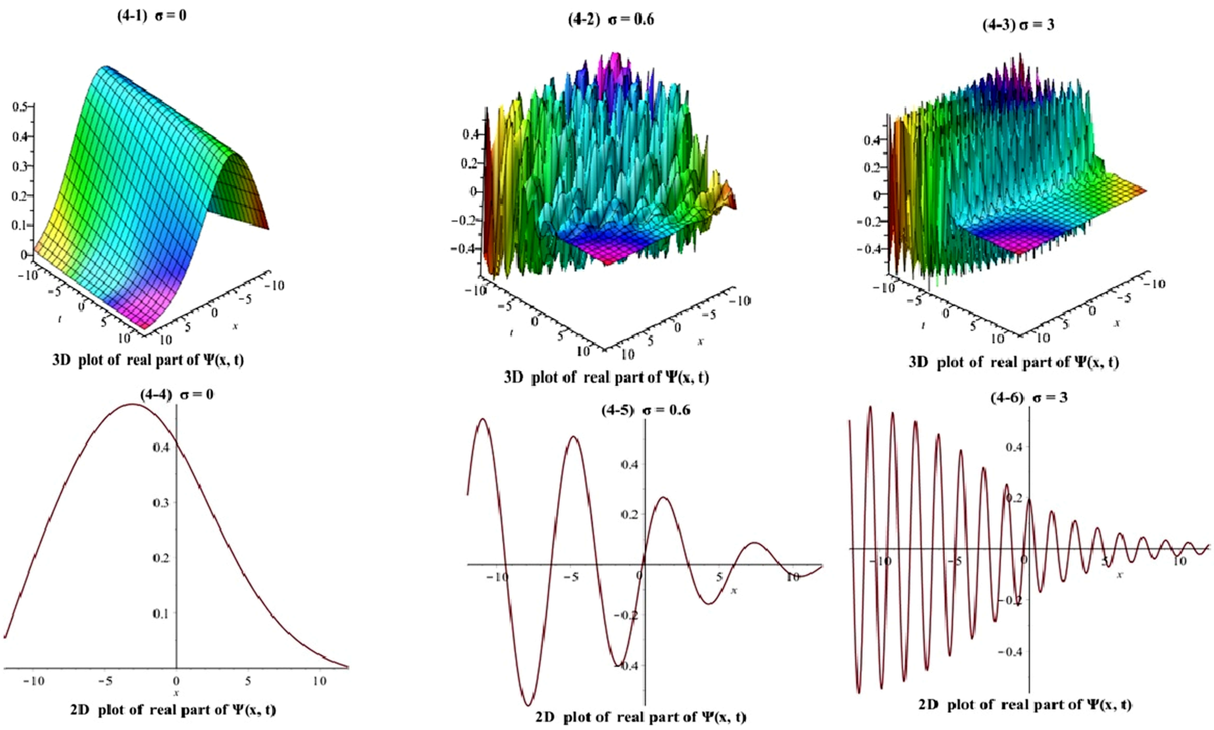

Figure 4. Plot of dark soliton Eq (3.35).

The 3D views show the shape and development of the soliton as a whole, while the 2D projections make the effect of random factors clear. When there is noise, bright solitons stay mostly fixed, but dark solitons are more likely to get distorted. The stochastic term causes changes in the magnitude and phase of the solutions that are periodic and Jacobi-elliptic. These pictures show that multiplicative noise causes fluctuations that can be measured and managed. The images also show random effects on the solutions, which is what the reviewer asked for.

6. Physical interpretation

The perturbed Gerdjikov-Ivanov equation with multiplicative noise encapsulates a complex interaction among nonlinearity, dispersion, and stochasticity. The solutions obtained by the Jacobi elliptic function expansion method yield both precise mathematical representations and significant scientific insights into wave propagation in realistic settings.

The incorporation of multiplicative noise inside soliton solutions underscores the influence of medium variations on the amplitude, phase, and stability of localized waves. In optical fiber communication systems, this pertains to stochastic fluctuations in the refractive index or amplifier noise, which may result in jitter, waveform distortion, or soliton annihilation. The solutions illustrate that bright solitons maintain energy localization under moderate noise, whereas dark and isolated solitons have heightened sensitivity, reflecting varying degrees of robustness among soliton families.

In practical terms, bright, dark, singular, periodic, and mixed soliton structures describe transmission modes in nonlinear optical systems. High-bit-rate optical channels benefit from bright solitons' energy confinement, whereas background-intensity-supported propagation benefits from dark ones. In mode-locked lasers and photonic lattices, periodic and Jacobi elliptic-type solutions describe wave trains and pulse trains.

Explicit stability analysis also offers physical criteria for solitons to survive stochastic disturbances. The stability criteria can be used to develop noise-resilient photonic devices. To reduce instability thresholds and improve robustness in noisy situations, engineers can optimize system parameters by tuning dispersion (α_1) and self-phase modulation (α_2).

In addition to optics, noise-driven solitons affect wave-particle interactions in plasma physics and stochastic external fields dictate matter-wave dynamics in Bose-Einstein condensates. Fluid system solutions describe stochastic surface wave patterns, making the research interdisciplinary. Based on our physical understanding, different soliton families exhibit varying responses to noise, revealing their robustness. Stability conditions help engineers build stable optical communication systems. Solution variety improves comprehension of real-world nonlinear stochastic systems in optics, plasma, and condensed matter physics. This approach goes beyond mathematical derivations and sheds light on the physical meaning and applications of the solutions.

7. Discussion and conclusions

This research examined the perturbed GI equation in the Itô framework, integrating multiplicative noise to simulate genuine environmental disturbances. Utilizing the new Jacobi elliptic function expansion approach, we obtained several categories of precise solutions, including bright, dark, singular, and periodic solitons, and analyzed their behavior under stochastic perturbations. The stability study indicated that soliton persistence is fundamentally influenced by the interaction of dispersion, nonlinearity, and noise intensity, with analytical criteria defined for both stable and unstable settings.

Graphical simulations (Figures 1–4) demonstrated that multiplicative noise may cause phase jitter, amplitude modulation, and shape distortion in soliton profiles, with the extent of these effects varying based on the individual solution type and parameter configuration. The dark soliton instance exhibited significant sensitivity to increased noise terms, consistent with theoretical stability thresholds.

Recent investigations [30,31] have found noise-modulated soliton dynamics in optical fibers and other dispersive systems. The stochastic effects represented in our modified GI equation strongly align with the experimental noise sources in these studies. Consequently, the soliton solutions presented herein–including bright, dark, and periodic forms–may be actualized and quantified via analogous experimental setups, offering a means for direct corroboration of our theoretical forecasts.

Recent studies, including those by Ahmed et al. [16] and Hosseini et al. [19], have investigated the GI equation under perturbations, primarily concentrating on bright soliton solutions while neglecting a comprehensive classification of solution types. Kudryashov et al. [13] examined the deterministic GI model and derived analytical structures devoid of stochastic influences. This study integrates multiplicative noise in the Itô framework and systematically employs the Jacobi elliptic function expansion method to derive a variety of solutions, encompassing bright, dark, singular, and periodic solitons. Moreover, while earlier studies have frequently examined stability in a qualitative manner, our research establishes clear analytical stability conditions and supports them with graphical evidence across various solution families. This comprehensive framework, which incorporates noise, offers an expanded viewpoint that surpasses existing literature and enhances the connection between theoretical predictions and possible experimental validation.

Future research may investigate adaptive control methods and noise-mitigation approaches to maintain soliton integrity in noisy dispersive media, and may also expand the paradigm to higher-dimensional or nonlocal GI-type equations. The findings presented herein enhance the theoretical comprehension of stochastic nonlinear wave equations and inform the practical development of resilient optical and photonic systems that provide steady signal transmission amidst real-world environmental variations.

Author contributions

Nafissa T. Trouba: Writing–original draft, Formal analysis; Huiying Xu: Conceptualization, Funding acquisition; Reham M. A. Shohib: Methodology, Writing–original draft, Writing–review & editing; Mohamed E. M. Alngar: Formal analysis, Methodology; Mohamed Abdelhamid: Supervision, Validation; Xinzhong Zhu: Conceptualization, Language editing; Adham E. Ragab: Writing–review & editing, Supervision. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors present their appreciation to King Saud University for funding this research through the Ongoing Research Funding program (ORF-2025-711), King Saud University, Riyadh, Saudi Arabia. This work was supported by the National Natural Science Foundation of China (62376252); Zhejiang Province Province-Land Synergy Program (2025SDXT004-3), and Zhejiang Province Leading Geese Plan (2025C02025, 2025C01056).

Conflict of interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

1. M. A. E. Abdelrahman, W. W. Mohammed, M. Alesemi, S. Albosaily, The effect of multiplicative noise on the exact solutions of nonlinear Schrödinger equation, *AIMS Math.*, **6** (2021), 2970–2980. https://doi.org/10.3934/math.2021180
2. S. Albosaily, W. W. Mohammed, M. A. Aiyashi, M. A. E. Abdelrahman, Exact solutions of the (2+1)-dimensional stochastic chiral nonlinear Schrödinger equation, *Symmetry*, **12** (2020), 1–12. https://doi.org/10.3390/sym12111874
3. S. Khan, Stochastic perturbation of sub-pico second envelope solitons for Triki-Biswas equation with multi-photon absorption and bandpass filters, *Optik*, **183** (2019), 174–178. https://doi.org/10.1016/j.ijleo.2019.02.065
4. S. Khan, Stochastic perturbation of optical solitons having generalized anti-cubic nonlinearity with bandpass filters and multi-photon absorption, *Optik*, **200** (2020), 163405. https://doi.org/10.1016/j.ijleo.2019.163405
5. S. Khan, Stochastic perturbation of optical solitons with quadratic-cubic nonlinear refractive index, *Optik*, **212** (2020), 164706. https://doi.org/10.1016/j.ijleo.2020.164706
6. W. W. Mohammed, H. Ahmad, A. E. Hamza, E. S. Aly, M. El-Morshedy, E. M. Elabbasy, The exact solutions of the stochastic Ginzburg-Landau equation, *Results Phys.*, **23** (2021), 103988. https://doi.org/10.1016/j.rinp.2021.103988
7. W. W. Mohammed, H. Ahmad, H. Boulares, F. Kheli, M. El-Morshedy, Exact solutions of Hirota-Maccari system forced by multiplicative noise in the Itô sense, *J. Low Freq. Noise Vibration Active Control*, **41** (2022), 74–84. https://doi.org/10.1177/14613484211028100
8. N. T. Trouba, M. E. M. Alngar, H. A. Mahmoud, R. M. A. Shohib, Optical soliton solutions of the stochastic resonant nonlinear Schrödinger equation with spatio temporal and inter-modal dispersion under generalized Kudryashov's law non-linearity, *Ain Shams Eng. J.*, **15** (2024), 103117. https://doi.org/10.1016/j.asej.2024.103117
9. W. W. Mohammed, N. Iqbal, A. Ali, M. El-Morshedy, Exact solutions of the stochastic new coupled Konno-Oono equation, *Results Phys.*, **21** (2021), 103830. https://doi.org/10.1016/j.rinp.2021.103830

10. W. W. Mohammed, M. El-Morshedy, The influence of multiplicative noise on the stochastic exact solutions of the Nizhnik-Novikov-Veselov system, *Math. Comput. Simul.*, **190** (2021), 192–202. https://doi.org/10.1016/j.matcom.2021.05.022
11. N. A. Kudryashov, Highly dispersive optical solitons of the generalized nonlinear eighth-order Schrödinger equation, *Optik*, **206** (2020), 164335. https://doi.org/10.1016/j.ijleo.2020.164335
12. N. A. Kudryashov, Solitary wave solutions of hierarchy with non-local nonlinearity, *Appl. Math. Lett.*, **103** (2020), 106155. https://doi.org/10.1016/j.aml.2019.106155
13. N. A. Kudryashov, Highly dispersive solitary wave solutions of perturbed nonlinear Schrödinger equations, *Appl. Math. Comput.*, **371** (2020), 124972. https://doi.org/10.1016/j.amc.2019.124972
14. N. A. Kudryashov, S. F. Lavrova, D. R. Nifontov, Bifurcations of phase portraits, exact solutions and conservation laws of the generalized Gerdjikov-Ivanov model, *Mathematics*, **11** (2023), 1–20. https://doi.org/10.3390/math11234760
15. N. A. Kudryashov, D. R. Nifontov, Conservation laws and Hamiltonians of the mathematical model with unrestricted dispersion and polynomial nonlinearity, *Chaos Solitons Fract.*, **175** (2023), 114076. https://doi.org/10.1016/j.chaos.2023.114076
16. Z. Abbas, Coupled Gerdjikov-Ivanov system and its exact solutions through Darboux transformation, *Punjab Univ. J. Math.*, **54** (2025), 163–180.
17. M. S. Ahmed, A. H. Arnous, Y. Yildirim, Optical solitons of the generalized stochastic Gerdjikov-Ivanov equation in the presence of multiplicative white noise, *Ukr. J. Phys. Opt.*, **25** (2024), S1111–S1130.
18. S. Arshed, Two reliable techniques for the soliton solutions of perturbed Gerdjikov-Ivanov equation, *Optik*, **164** (2018), 93–99. https://doi.org/10.1016/j.ijleo.2018.02.119
19. J. B. Chen, Y. P. Zhen, The complex Hamiltonian system in the Gerdjikov-Ivanov equation and its applications, *Anal. Math. Phys.*, **12** (2022), 100. https://doi.org/10.1007/s13324-022-00704-7
20. K. Hosseini, M. Mirzazadeh, M. Ilie, S. Radmehr, Dynamics of optical solitons in the perturbed Gerdjikov-Ivanov equation, *Optik*, **206** (2020), 164350. https://doi.org/10.1016/j.ijleo.2020.164350
21. Y. R. Hu, F. Zhang, X. P. Xin, H. Z. Liu, Darboux transformation and exact solutions of the variable-coefficient nonlocal Gerdjikov-Ivanov equation, *Theoret. Math. Phys.*, **211** (2022), 460–472. https://doi.org/10.1134/S004057792204002X
22. L. Kaur, A. M. Wazwaz, Optical solitons for perturbed Gerdjikov-Ivanov equation, *Optik*, **174** (2018), 447–451. https://doi.org/10.1016/j.ijleo.2018.08.072
23. S. A. Khuri, New optical solitons and traveling wave solutions for the Gerdjikov-Ivanov equation, *Optik*, **268** (2022), 169784. https://doi.org/10.1016/j.ijleo.2022.169784
24. C. Y. Liu, Z. Li, The dynamical behavior analysis of the fractional perturbed Gerdjikov-Ivanov equation, *Results Phys.*, **59** (2024), 107537. https://doi.org/10.1016/j.rinp.2024.107537
25. M. Ozisik, Optical soliton perturbation of the quintic Gerdjikov-Ivanov model, *New Trends Math. Sci.*, **11** (2023), 7–22. https://doi.org/10.20852/ntmsci.2023.505
26. A. Biswas, D. Milovic, Bright and dark solitons of the generalized nonlinear Schrödinger's equation, *Commun. Nonlinear Sci. Numer. Simul.*, **15** (2010), 1473–1484. https://doi.org/10.1016/j.cnsns.2009.06.017

27. E. M. E. Zayed, R. M. A. Shohib, A. Biswas, Y. Yildirim, F. Mallawi, M. R. Belic, Chirped and chirp-free solitons in optical fiber Bragg gratings with dispersive reflectivity having parabolic law nonlinearity by Jacobi's elliptic function, *Results Phys.*, **15** (2019), 102784. https://doi.org/10.1016/j.rinp.2019.102784
28. M. Alosaimi, M. A. S. Al-Malki, K. A. Gepreel, Optical soliton solutions in optical metamaterials with full nonlinearity, *J. Nonlinear Opt. Phys. Mater.*, **34** (2025), 2450021. https://doi.org/10.1142/S0218863524500218
29. N. T. Trouba, H. Y. Xu, M. E. M. Alngar, R. M. A. Shohib, H. A. Mahmoud, X. Z. Zhu, Soliton solutions and stability analysis of the stochastic nonlinear reaction-diffusion equation with multiplicative white noise in soliton dynamics and optical physics, *AIMS Math.*, **10** (2025), 1859–1881. https://doi.org/10.3934/math.2025086
30. Z. T. Ju, Z. Z. Si, L. F. Ren, H. Y. Feng, X. Yan, J. H. Zhang, et al., Energy disturbance-induced collision between soliton molecules and switch dynamics in fiber lasers: periodic collision, oscillation, and annihilation, *Opt. Lett.*, **50** (2025), 1321–1324. https://doi.org/10.1364/OL.549637
31. Y. H. Jia, Z. Z. Si, Z. T. Ju, H. Y. Feng, J. H. Zhang, X. Yan, et al., Convolutional-recurrent neural network for the prediction of formation and switching dynamics for multicolor solitons, *Sci. China Phys. Mech. Astron.*, **68** (2025), 284211. https://doi.org/10.1007/s11433-025-2679-8

AIMS Press