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Research article

A method for solving the initial-boundary value problem for hyperbolic integro-differential systems with functional term

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Abstract: In this paper, we propose a method for solving initial-boundary value problems (IBVPs) for hyperbolic integro-differential systems (HIDSs) with functional terms. The equations under consideration involve derivatives with respect to a spatial variable with a generalized piecewise constant argument (GPCA), as well as integral operators acting on the time variable. For a qualitative analysis of such problems, we construct auxiliary problems for first-order integro-differential systems with distributed parameters and additional integral constraints. Using the parameterization of the Dzhumabaev method, the auxiliary problem is reduced to a family of parameterized integral problems. Under explicit invertibility conditions for certain matrices, we establish theorems on the existence and uniqueness of solutions for this family of problems. Furthermore, we develop a constructive iterative algorithm to obtain solutions to the auxiliary problem and prove its convergence. The results extend the theory of HIDEs and open up prospects for applications in models with discontinuities, impulse effects, and integral memory terms.

Keywords: hyperbolic integro-differential systems; initial-integral condition; parametrized integral-problems; distributed functions
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1. Introduction

Functional-partial differential equations (FPDEs) remain an actively developing area of research, largely due to their important role in modeling phenomena with aftereffects [1, 2], discontinuities [3, 4], and memory [5, 6]. Hyperbolic FPDEs (HFPDEs) with mixed derivatives attract particular attention because of their ability to describe wave dynamics in non-classical settings. Distinctive features of such systems include the presence of GPCA [2, 6, 7], retarded terms [8, 9], and nonlocal integral terms [10–12]. The appearance of these structures complicates the analysis and necessitates developing new mathematical methods.

Previous studies have mainly focused on problems in which systems with generalized arguments or integral terms are considered separately [11, 13].

The motivation for studying the initial-boundary value problem for a hyperbolic integro-differential system arises, first, from the numerous applications of such systems with various nonlocal conditions [6, 11, 12], and second, from the lack of effective constructive methods for solving these problems.

Hyperbolic integro-differential systems with a functional terms and integral conditions describe the propagation of waves and signals in media with delay [2, 4], memory, and nonlocal interactions, where the classical local Huygens-type equations are inadequate [5–7].

The novelty of the present work lies in the fact that, for the first time, an initial-integral problem for HIDSs is examined that simultaneously includes both a term with a generalized argument and an integral term. This formulation gives rise to new questions regarding the solvability of the problem, since standard approaches to HFPDEs cannot be applied directly.

Our approach is based on the parametrization method of Dzhumabaev, originally proposed in [14], which makes it possible to reduce the problem for an HIDS to an auxiliary problem for a system of first-order FPDEs with distributed parameters and additional integral constraints. The main goal of this paper is to establish rigorous solvability criteria for the auxiliary problem and, through their equivalence, for the initial–integral problem for HIDSs. This approach not only provides explicit, verifiable conditions for the existence and uniqueness of solutions, but also yields a constructive iterative algorithm whose convergence is rigorously justified.

The obtained results broaden the existing theory by integrating methods for solving HIDSs, FPDEs, and related nonlocal problems. The developed framework provides a solid foundation for further research on nonlinear problems, discontinuous systems, and models with impulse effects, thereby extending the applicability of HIDSs to the mathematical modeling of real processes in the natural sciences.

The structure of the paper is as follows. Section 1 provides an overview of the work and highlights the relevance and novelty of the obtained results. Section 2 presents the problem statement and the necessary preliminary material. Section 3 describes the reduction of the original problem to an equivalent auxiliary one involving parameterized integral problems for a system of first-order IDEs with integral constraints and distributed parameters. Section 4 examines these parameterized integral problems and establishes conditions for their unique solvability in terms of the coefficients and boundary matrices. Section 5 develops a constructive solution method and formulates criteria ensuring the unique solvability of the parameterized integral problems for the system of first-order IDEs with integral constraints and distributed parameters. Finally, based on the established equivalence between the auxiliary and original problems, we derive conditions guaranteeing the existence and uniqueness of a solution to the initial - integral problem in terms of the system matrices and the integral condition.

2. Statement of problem

Let us introduce a domain $\Lambda = [0, Y] \times [0, Z]$ and a function $\beta(y)$ which is a piecewise-constant generalized argument defined by $\beta(y) = \gamma_{i-1}$ for $y \in [y_{i-1}, y_i)$ with nodes $0 = y_0 < y_1 < ... < y_{L-1} <$

$y_L = Y$, and constants $\gamma_i \in (y_{i-1}, y_i)$.

Consider the initial-integral problem for a HIDS of the following type:

$$\frac{\partial^2 w}{\partial z \partial y} = A(y,z)\frac{\partial w(y,z)}{\partial z} + B(y,z)\frac{\partial w(y,z)}{\partial y} + C(y,z)w(y,z) + g(y,z) + D(y,z)\frac{\partial w(\beta(y),z)}{\partial z} + K(y,z)\int_0^Y M(\xi,z)\frac{\partial w(\xi,z)}{\partial z}d\xi, \qquad (y,z) \in \Lambda, \tag{2.1}$$

subject to the integral condition

$$P(z)\frac{\partial w(0,z)}{\partial z} + S(z)\int_0^Y M(\xi,z)\frac{\partial w(\xi,z)}{\partial z}d\xi = \varphi(z), \qquad z \in [0,Z], \tag{2.2}$$

and the initial condition

$$w(y,0) = \psi(y), \qquad y \in [0,Y]. \tag{2.3}$$

Here, $w(y,z) = (w_1(y,z), ..., w_n(y,z))'$ is the unknown vector function. The coefficient matrices $A(y,z)$, $B(y,z)$, $C(y,z)$, $D(y,z)$, $K(y,z)$, $M(y,z)$ and the vector function $g(y,z)$ are assumed continuous on Λ, $\frac{\partial w(\beta(y),z)}{\partial z}$ denotes the partial derivative with respect to z of the desired function $w(t,z)$ that involves a piecewise constant generalized argument with respect to the time variable.

Furthermore, $P(z)$ and $S(z)$ are continuous $m \times m$ matrices, while $\varphi(z)$ is a continuous vector function on $[0,Z]$. The initial data $\psi(y)$ is assumed continuously differentiable on $[0,Y]$.

A function $w(y,z) \in C(\Lambda, \mathbb{R}^n)$ is called a solution to problems (2.1)–(2.3) if the following conditions are satisfied:

(1) The partial derivatives $\frac{\partial w(y,z)}{\partial z}$ and $\frac{\partial w(y,z)}{\partial y}$ exist and are continuous on Λ.

(2) The mixed derivative $\frac{\partial^2 w(y,z)}{\partial z \partial y}$ exists on Λ, except possibly at the discontinuity points (y_{i-1}, z), where one-sided derivatives exist.

(3) Equation (2.1) holds on each subdomain $[y_{i-1}, y_i) \times [0,Z]$, and at the points (y_{i-1}, z) the right-hand mixed derivatives satisfy the equation.

(4) The integral condition (2.2) and the condition on characteristic (2.3) are satisfied.

The piecewise-constant generalized argument $w(\beta(y), z)$ in system (2.1) reflects mirror delays or the "memory" effect in the system. For example, wave propagation in media with memory, such as viscoelastic materials, plasma, or biological tissue; or thermoelasticity with delay, where the deformation at time t depends on the temperature corresponding to a previous moment in time [6, 12]; or controlled systems with periodic or piecewise-constant changes in parameters (e.g., switched or modulated media) [13, 15, 16].

Integral condition (2.2) reflects the direct influence of the entire domain, or part of it, on the behavior of the solution. Such conditions arise in the following contexts: wave equations with integral constraints, which model processes where energy or mass is distributed and controlled globally; elasticity and acoustics, where deformation or pressure are related through integral relations over the domain—for example, when modeling thin plates or membranes with distributed constraints [17–19].

Together, these structures capture complex physical phenomena involving delay, memory, and nonlocal interactions, extending the applicability of hyperbolic integro-differential systems beyond classical local models.

This formulation gives rise to a new class of initial-integral problems for HIDSs that warrant detailed investigation. One effective approach to solving such problems is to reduce them to simpler parameterized integral problems for first-order IDSs systems.

The next section is devoted to solving the initial-integral problem for HIDSs (2.1)–(2.3), based on its reduction to a parameterized problem for first-order FIDEs.

3. Equivalent parametrized integral-problems for system IDEs with distributed functions

For a qualitative analysis of the problems (2.1)–(2.3), we apply the effective approach proposed in [20] and introduce the following new functions:

$$u(y,z) = \frac{\partial w(y,z)}{\partial z}, \qquad v(y,z) = \frac{\partial w(y,z)}{\partial y}, \qquad (y,z) \in \Lambda.$$

The introduction of these functions makes it possible to rewrite the problem under consideration as a system of first-order IDEs with distributed functions:

$$\begin{aligned}\frac{\partial u(y,z)}{\partial y} &= A(y,z)u(y,z) + D(y,z)u(\beta(y),z) + K(y,z)\int_0^Y M(\xi,z)u(\xi,z)d\xi \\ &\quad + g(y,z) + B(y,z)v(y,z) + C(y,z)w(y,z), \qquad (y,z) \in \Lambda,\end{aligned} \tag{3.1}$$

together with the integral condition

$$P(z)u(0,z) + S(z)\int_0^Y M(\xi,z)u(\xi,z)d\xi = \varphi(z), \qquad z \in [0,Z], \tag{3.2}$$

and the integral constraints

$$w(y,z) = \psi(y) + \int_0^z u(y,\eta)d\eta, \qquad v(y,z) = \dot{\psi}(y) + \int_0^z \frac{\partial u(y,\eta)}{\partial y}d\eta, \qquad (y,z) \in \Lambda. \tag{3.3}$$

Here, $\dot{\psi}(y)$ is the derivative of function $\psi(y)$.

The variable z is treated as a parameter that varies continuously over the interval $[0,Z]$ for the parameterized integral problems. Relations (3.3) serve as integral constraints for the functions $w(y,z)$ and $v(y,z)$, involving the function $u(y,z)$ and its partial derivative with respect to y.

A triple $\{u(y,z), w(y,z), v(y,z)\}$ is called a solution of the parameterized integral problems (3.1)–(3.3) if the following hold:

⋄ $u(y,z) \in C(\Lambda, \mathbb{R}^n)$ and is continuous in both variables.

⋄ The partial derivative $\frac{\partial u(y,z)}{\partial y}$ exists for all $(y,z) \in \Lambda$, except possibly at points (y_{i-1}, z), where one-sided limits exist.

⋄ The system (3.1) is satisfied in every subdomain $[y_{i-1}, y_i) \times [0,Z]$, with right-hand derivatives at $y = y_{i-1}$.

⋄ The integral condition (3.2) holds for all $y \in [0,Y]$.

⋄ The relations (3.3) are fulfilled, thereby coupling $u(y,z)$ with the corresponding functions $w(y,z)$ and $v(y,z)$.

In [20], using the properties of solutions to parameterized integral problems for systems of differential equations of the first order with integral constraints, conditions for the unique and well-posed solvability of a nonlocal problem for a system of hyperbolic equations of the second order with an integral condition were established. An effective method for solving the nonlocal problem for hyperbolic systems second order, based on approximate solutions to parameterized integral problems for first-order differential systems, was also proposed.

Parameterized integral problems for a system of first-order IDEs with distributed functions and constraints (3.1)–(3.3) represent a new type of problem. This problem requires a separate investigation and an analysis of its solvability conditions.

We now focus our attention on a special type of auxiliary problem:

$$\frac{\partial u(y,z)}{\partial y} = A(y,z)u(y,z) + D(y,z)u(\beta(y),z) + K(y,z)\int_0^Y M(\xi,z)u(\xi,z)d\xi + G(y,z), \quad (y,z) \in \Lambda, \quad (3.4)$$

with the accompanying condition

$$P(z)u(0,z) + S(z)\int_0^Y M(\xi,z)u(\xi,z)d\xi = \varphi(z), \qquad z \in [0,Z], \quad (3.5)$$

where $G(y,z) \in C(\Lambda,\mathbb{R}^n)$ is a given function.

If we take $G(y,z) = g(y,z) + B(y,z)v(y,z) + C(y,z)w(y,z)$, we obtain system (3.1) with integral condition (3.2). By studying system (3.4) with condition (3.5), we can describe the properties of the parameterized integral problems (3.1), (3.2) for fixed v and w.

A function $u(y,z) \in C(\Lambda,\mathbb{R}^n)$ is said to be a solution of problem (3.4), (3.5) if the following conditions are satisfied:

⋄ $\frac{\partial u(y,z)}{\partial y}$ exists for all $(y,z) \in \Lambda$, except possibly at the discontinuity points (y_{i-1},z), $i = \overline{1,L}$, for all $z \in [0,Z]$, where one-sided derivatives exist.

⋄ system of IDEs (3.4) is satisfied in each sub-domain $[y_{i-1},y_i) \times [0,Z]$.

⋄ The integral condition (3.5) holds for all $z \in [0,Z]$.

The problem statement (3.4), (3.5) defines a new class of parameterized integral problems for systems of first-order IDEs. This class requires careful and independent qualitative analysis, since the combination of the parameter y, integral operators, and GPCA gives rise to challenging analytical questions. Moreover, such problems naturally arise in applications ranging from control theory to the mathematical modeling of neural networks [7].

4. Solvability conditions for the parameterized integral-problems

Conditions are identified that ensure the existence and uniqueness of the solution of parameterized integral problems (3.4) and (3.5). The approach is based on a modification of the parametrization Dzhumabaev method [14] and solving problems in each subdomain $\Lambda_i = [y_{i-1},y_i) \times [0,Z]$, in combination with continuity conditions on the boundary lines of the intervals $[y_{i-1},y_i)$, $i = 1,2,...,L$.

Now, we consider the parameterized integral-problems for the system of first-order IDEs (3.4) and (3.5).

Divide the region Λ along the lines $y = y_i$: $\Lambda_i = [y_{i-1}, y_i) \times [0, Z],\ i = 1, 2, ..., L.$

Let $u_i(y, z)$ denote the restriction of the desired function to the subdomain

$$\Lambda_i : u(y, z) = u_i(y, z), (y, z) \in \Lambda_i,\ i = 1, 2, ..., L.$$

The set of functions $\{u_i(y, z)\},\ i = 1, 2, ..., L$, satisfies the system IDEs

$$\frac{\partial u_i(y, z)}{\partial y} = A(y, z)u_i(y, z) + D(y, z)u_i(\gamma_{i-1}, z) + K(y, z)\sum_{i=1}^{L}\int_{y_{i-1}}^{y_i} M(\xi, z)u_i(\xi, z)d\xi + G(y, z), \tag{4.1}$$
$$(y, z) \in \Lambda_i, \qquad i = 1, 2, ..., L,$$

a continuity conditions

$$\lim_{y \to y_r - 0} u_r(y, z) = u_{r+1}(y_r, z), \qquad r = \overline{1, L-1}, \qquad z \in [0, Z], \tag{4.2}$$

an integral relations

$$P(z)u_1(0, z) + S(z)\sum_{i=1}^{L}\int_{y_{i-1}}^{y_i} M(\xi, z)u_i(\xi, z)d\xi = \varphi(z), \qquad z \in [0, Z]. \tag{4.3}$$

In system (4.1), it is assumed that $\beta(y) = \gamma_{i-1}$ if $y \in [y_{i-1}, y_i)$, $i = 1, ..., L.$

The following functional parameters are introduced:

$$\theta(z) = u_1(0, z), \qquad \lambda(z) = \sum_{i=1}^{L}\int_{y_{i-1}}^{y_i} M(\xi, z)u_i(\xi, z)d\xi, \qquad z \in [0, Z].$$

We also introduce new unknown functions $\widetilde{u}_i(y, z)$ on the subdomains $\Lambda_i = [y_{i-1}, y_i) \times [0, Z]$, $i = 1, 2, ..., L$. Assume that the functions $\widetilde{u}_i(y, z)$ are continuous on Λ_i, and have continuous partial derivatives with respect to z z for all $(y, z) \in \Lambda_i,\ i = 1, 2, ..., L.$

In the parameterized multipoint integral problems (4.1)–(4.3), the functions $u_i(y, z)$ are expressed in the following form:

$$u_i(y, z) = \widetilde{u}_i(y, z) + \theta(z) + \frac{y - y_{i-1}}{y_i - y_{i-1}}\lambda(z), \qquad (y, z) \in \Lambda_i, \qquad i = 1, 2, ..., L.$$

For the system (4.1), taking into account

$$\lambda(z) = \sum_{i=1}^{L}\int_{y_{i-1}}^{y_i} M(\xi, z)u_i(\xi, z)d\xi,$$

we have

$$\frac{\partial \widetilde{u}_i(y, z)}{\partial y} = A(y, z)\Big[\widetilde{u}_i(y, z) + \theta(z) + \frac{y - y_{i-1}}{y_i - y_{i-1}}\lambda(z)\Big] + D(y, z)\Big[\widetilde{u}_i(\gamma_{i-1}, z) + \theta(z) + \frac{\gamma_{i-1} - y_{i-1}}{y_i - y_{i-1}}\lambda(z)\Big]$$
$$+ G(y, z) + K(y, z)\lambda(z), \qquad (y, z) \in \Lambda_i, \quad i = 1, 2, ..., L.$$

Then, we obtain a parameterized initial problems for a system of first-order IDEs

$$\frac{\partial \widetilde{u}_i(y,z)}{\partial y} = A(y,z)\widetilde{u}_i(y,z) + D(y,z)\widetilde{u}_i(\gamma_{i-1},z) + G(y,z) + [A(y,z)+D(y,z)]\theta(z) + \Big[A(y,z)\frac{y-y_{i-1}}{y_i-y_{i-1}} + D(y,z)\frac{\gamma_{i-1}-y_{i-1}}{y_i-y_{i-1}} + K(y,z)\Big]\lambda(z),\ (y,z)\in\Lambda_i,\ i=1,2,...,L, \tag{4.4}$$

with initial conditions

$$\widetilde{u}_1(0,z)=0, \qquad z\in[0,Z], \tag{4.5}$$

$$\widetilde{u}_{r+1}(y_r,z) = \lim_{y\to y_r-0}\widetilde{u}_r(y,z) + \lambda(z), \qquad r=1,2,...,L-1, \qquad z\in[0,Z], \tag{4.6}$$

and relations

$$\Big[I - \sum_{i=1}^{L}\int_{y_{i-1}}^{y_i} M(\xi,z)\frac{\xi-y_{i-1}}{y_i-y_{i-1}}d\xi\Big]\lambda(z) - \sum_{i=1}^{L}\int_{y_{i-1}}^{y_i} M(\xi,z)d\xi\theta(z) = \sum_{i=1}^{L}\int_{y_{i-1}}^{y_i} M(\xi,z)\widetilde{u}_i(\xi,z)d\xi, \tag{4.7}$$

$$P(z)\theta(z) + S(z)\lambda(z) = \varphi(z), \qquad z\in[0,Z], \tag{4.8}$$

here I is an identity matrix of dimension n.

One of the advantages of the obtained problem is the presence of the initial condition (4.5) for the unknown function $\widetilde{u}_1(y,z)$ at $y=0$.

Relation (4.7) follows from

$$\lambda(z) = \sum_{i=1}^{L}\int_{y_{i-1}}^{y_i} M(\xi,z)u_i(\xi,z)d\xi = \sum_{i=1}^{L}\int_{y_{i-1}}^{y_i} M(\xi,z)\Big[\widetilde{u}_i(\xi,z) + \theta(z) + \frac{\xi-y_{i-1}}{y_i-y_{i-1}}\lambda(z)\Big]d\xi.$$

By the solution of the parameterized initial problems for the first-order system of IDEs (4.4)–(4.8), we mean a family of functions $(\{\widetilde{u}_i(y,z)\},\theta(z),\lambda(z))$, with elements $\widetilde{u}_i(y,z)\in C(\Lambda_i,\mathbb{R}^n)$, $\theta(z),\lambda(z)\in C([0,Z],\mathbb{R}^n)$, provided that:

◇ the partial derivatives with respect to the first variable $\frac{\partial \widetilde{u}_i(y,z)}{\partial y}$ exist at every point $(y,z)\in\Lambda_i$ except at the points (y_{i-1},z), $i=1,2,...,L$, for all $z\in[0,Z]$, where one-sided partial derivatives exist;

◇ the first order system IDEs (4.4) holds for $\widetilde{u}_i(y,z)$ in each subdomain $[y_{i-1},y_i)\times[0,Z]$, $i=1,2,...,L$, and right-hand partial derivatives with respect to y of $\widetilde{u}_i(y,z)$ exist at the points (y_{i-1},z), $i=1,2,...,L$, $z\in[0,Z]$;

◇ the initial conditions (4.5), (4.6) are satisfied by $\widetilde{u}_i(y,z)$ for all $z\in[0,Z]$, $i=1,2,...,L$;

◇ the integral relation (4.7) and the functional relation (4.8) hold for $\widetilde{u}_i(y,z)$, $\theta(z)$, and $\lambda(z)$ for all $z\in[0,Z]$.

Let $W_i(y,z)$ be the fundamental matrix of solutions to the differential system

$$\frac{\partial u_i(y,z)}{\partial y} = A(y,z)u_i(y,z), \qquad (y,z)\in\Lambda_i, \qquad i=1,2,...,L.$$

We also introduce the following notation:

$$U_i(\widetilde{L},y,z) = W_i(y,z)\int_{y_{i-1}}^{y} W_i^{-1}(\xi,z)\widetilde{L}(\xi,z)d\xi, \qquad z\in[0,Z], \qquad i=1,2,...,L. \tag{4.9}$$

Here $\widetilde{L}$ is a square matrix of dimension n. Sometimes $\widetilde{L}$ will denote a vector of dimension n. Let

$$K_i(y,z) = A(y,z)\frac{y-y_{i-1}}{y_i-y_{i-1}} + D(y,z)\frac{\gamma_{i-1}-y_{i-1}}{y_i-y_{i-1}} + K(y,z), \quad (y,z)\in\Lambda_i, \quad i=1,2,...,L.$$

Assume that the matrices $V_i(z) = I - U_i(D,\gamma_{i-1},z)$ are nonsingular for all $z\in[0,Z]$, $i=1,2,...,L$. Let

$$A_1(y,z) = U_1(D,y,z)[V_1(z)]^{-1}U_1(A+D,\gamma_0,z) + U_1(A+D,y,z), \qquad (y,z)\in\Lambda_1,$$
$$B_1(y,z) = U_1(D,y,z)[V_1(z)]^{-1}U_1(K_1,\gamma_0,z) + U_1(K_1,y,z), \qquad (y,z)\in\Lambda_1,$$
$$G_1(y,z) = U_1(D,y,z)[V_1(z)]^{-1}U_1(G,\gamma_0,z) + U_1(G,y,z), \qquad (y,z)\in\Lambda_1,$$
$$A_i(y,z) = W_i(y,z)W_i^{-1}(y_{i-1},z)A_{i-1}(y_{i-1},z) + U_i(A+D,y,z)+$$
$$+U_i(D,y,z)[V_i(z)]^{-1}[W_i(\gamma_{i-1},z)W_i^{-1}(y_{i-1},z)A_{i-1}(y_{i-1},z) + U_i(A+D,\gamma_{i-1},z)],$$
$$B_i(y,z) = W_i(y,z)W_i^{-1}(y_{i-1},z)\{B_{i-1}(y_{i-1},z)+I\} + U_i(K_i,y,z)+$$
$$+U_i(D,y,z)[V_i(z)]^{-1}[W_i(\gamma_{p-1},z)W_i^{-1}(y_{i-1},z)\{B_{i-1}(y_{i-1},z)+I\} + U_i(K_i,\gamma_{i-1},z)],$$
$$G_i(y,z) = W_i(y,z)W_i^{-1}(y_{i-1},z)G_{i-1}(y_{i-1},z) + U_i(G,y,z)+$$
$$+U_i(D,y,z)[V_i(z)]^{-1}[W_i(\gamma_{i-1},z)W_i^{-1}(y_{i-1},z)G_{i-1}(y_{i-1},z) + U_i(G,\gamma_{i-1},z)],$$
$$(y,z)\in\Lambda_i, \qquad i=2,3,...,L.$$

We construct a $2n\times 2n$ matrix $Q_*(z)$:

$$Q_*(z) = \begin{bmatrix} -\sum_{i=1}^{L}\int_{y_{i-1}}^{y_i} M(\xi,z)[I+A_i(\xi,z)]d\xi, & I-\sum_{i=1}^{L}\int_{y_{i-1}}^{y_i} M(\xi,z)\Big\{\frac{\xi-y_{i-1}}{y_i-y_{i-1}} - B_i(\xi,z)\Big\}d\xi \\ P(z) & S(z) \end{bmatrix}. \tag{4.10}$$

Theorem 1. *Assume that*

a) the $n\times n$ matrices $V_i(z) = I - U_i(D,\gamma_{i-1},z)$ are nonsingular for all $z\in[0,Z]$, where

$$U_i(D,\gamma_{i-1},z) = W_i(\gamma_{i-1},z)\int_{y_{i-1}}^{\gamma_{i-1}} W_i^{-1}(\xi,z)D(\xi,z)d\xi, \qquad z\in[0,Z],$$

and $W_i(y,z)$ is the fundamental matrix of solutions to the system of differential equations

$$\frac{\partial u_i(y,z)}{\partial y} = A(y,z)u_i(y,z), \qquad (y,z)\in\Lambda_i, \qquad i=1,2,...,L;$$

b) the $2n\times 2n$ matrix $Q_*(z)$, given by (4.10), is nonsingular for all $z\in[0,Z]$.

Then, the parameterized initial value problem for the IDE systems (4.4)–(4.8) has a unique solution $(\{\widetilde{u}_i^*(y,z)\},\theta^*(z),\lambda^*(z))$, where the functions $\widetilde{u}_i^*(y,z)$ admit the representations

$$\widetilde{u}_i^*(y,z) = A_i(y,z)\theta(z) + B_i(y,z)\lambda(z) + G_i(y,z), \qquad (y,z)\in\Lambda_i, \qquad i=1,2,...,L; \tag{4.11}$$

and the functional parameters $\theta^*(z)$ and $\lambda^*(z)$ are the solution to system

$$Q_*(z)\begin{pmatrix}\theta(z)\\ \lambda(z)\end{pmatrix} = \begin{pmatrix}\sum_{i=1}^{L}\int_{y_{i-1}}^{y_i} M(\xi,z)G_i(\xi,z)d\xi \\ \varphi(z)\end{pmatrix}, \qquad z\in[0,Z]. \tag{4.12}$$

Proof. For fixed values of the parameters $\theta(z), \lambda(z) \in C([0, Z], \mathbb{R}^n)$, we construct solutions to the parameterized initial value problems for the system IDEs (4.4)–(4.6).

We begin by constructing a solution in the subdomain Λ_1. Using the representation (4.9), the solutions to the parameterized initial problems for the system IDEs (4.4), and (4.5) can be written as

$$\widetilde{u}_1(y, z) = U_1(D, y, z)\widetilde{u}_1(\gamma_0, z) + U_1(A + D, y, z)\theta(z) + U_1(\mathcal{K}_1, y, z)\lambda(z) + U_1(G, y, z), \tag{4.13}$$

where $(y, z) \in \Lambda_1$.

At $y = \gamma_0$, from (4.13) we obtain:

$$\widetilde{u}_1(\gamma_0, z) = U_1(D, \gamma_0, z)\widetilde{u}_1(\gamma_0, z) + U_1(A + D, \gamma_0, z)\theta(z) + U_1(\mathcal{K}_1, \gamma_0, z)\lambda(z) + U_1(G, \gamma_0, z).$$

From this, it follows that

$$[I - U_1(D, \gamma_0, z)]\widetilde{u}_1(\gamma_0, z) = U_1(A + D, \gamma_0, z)\theta(z) + U_1(\mathcal{K}_1, \gamma_0, z)\lambda(z) + U_1(G, \gamma_0, z), \ z \in [0, Z]. \tag{4.14}$$

According to condition a) of the Theorem, the matrix $V_1(z) = I - U_1(D, \gamma_0, z)$ is nonsingular for all $z \in [0, Z]$.

Then the system (4.14) admits a unique solution, which can be written as:

$$\widetilde{u}_1(\gamma_0, z) = [V_1(z)]^{-1}U_1(A + D, \gamma_0, z)\theta(z) + [V_1(z)]^{-1}U_1(K_1, \gamma_0, z)\lambda(z) + [V_1(z)]^{-1}U_1(G, \gamma_0, z), \tag{4.15}$$

for all $z \in [0, Z]$.

Substituting the representation (4.15) for $\widetilde{u}_1(\gamma_0, z)$ into (4.13), we obtain an explicit form of the solution to problems (4.4) and (4.5):

$$\widetilde{u}_1(y, z) = A_1(y, z)\theta(z) + B_1(y, z)\lambda(z) + G_1(y, z), \qquad (y, z) \in \Lambda_1, \tag{4.16}$$

where $A_1(y, z)$, $B_1(y, z)$, and $G_1(y, z)$ are given above.

Next, we examine system (4.4) within the subdomain Λ_2:

$$\begin{aligned} \frac{\partial \widetilde{u}_2(y, z)}{\partial y} &= A(y, z)\widetilde{u}_2(y, z) + D(y, z)\widetilde{u}_2(\gamma_1, z) + [A(y, z) + D(y, z)]\theta(z) \\ &+ \mathcal{K}_2(y, z)\lambda(z) + G(y, z), \qquad (y, z) \in \Lambda_2. \end{aligned} \tag{4.17}$$

From the representation (4.16) we find the limit value at $y \to y_1 - 0$ for $\widetilde{u}_1(y, z)$:

$$\lim_{y \to y_1 - 0} \widetilde{u}_1(y, z) = A_1(y_1, z)\theta(z) + B_1(y_1, z)\lambda(z) + G_1(y_1, z), \qquad z \in [0, Z]. \tag{4.18}$$

Using the initial conditions (4.6) at $r = 1$, and taking into account the limit value (4.18), we can determine the initial condition along the line $y = y_1$:

$$\widetilde{u}_2(y_1, z) = A_1(y_1, z)\theta(z) + \{B_1(y_1, z) + I\}\lambda(z) + G_1(y_1, z), \qquad z \in [0, Z]. \tag{4.19}$$

Then the solutions to the parameterized initial problems for the system IDEs (4.17), and (4.19) can be written as:

$$\begin{aligned} \widetilde{u}_2(y, z) &= W_2(y, z)W_2^{-1}(y_1, z)[A_1(y_1, z)\theta(z) + \{B_1(y_1, z) + I\}\lambda(z) + G_1(y_1, z)] \\ &+ U_2(D, y, z)\widetilde{u}_2(\gamma_1, z) + U_2(A + D, y, z)\theta(z) + U_2(\mathcal{K}_2, y, z)\lambda(z) + U_2(G, y, z), \ (y, z) \in \Lambda_2. \end{aligned} \tag{4.20}$$

At $y = \gamma_1$, representation (4.20) yields

$$\begin{aligned}[I - U_2(D, \gamma_1, z)]\widetilde{u}_2(\gamma_1, z) = {} & [W_2(\gamma_1, z)W_2^{-1}(y_1, z)A_1(y_1, z) + U_2(A + D, \gamma_1, z)]\theta(z) \\ & + [W_2(\gamma_1, z)W_2^{-1}(y_1, z)\{B_1(y_1, z) + I\} + \mathcal{U}_2(K_2, \gamma_1, z)]\lambda(z) \\ & + W_2(\gamma_1, z)W_2^{-1}(y_1, z)G_1(y_1, z) + U_2(G, \gamma_1, z),\end{aligned} \tag{4.21}$$

for $z \in [0, Z]$.

According to condition a) of the Theorem, the matrix $V_2(z) = I - U_2(D, \gamma_1, z)$ is nonsingular for all $z \in [0, Z]$.

Then the system (4.21) admits a unique solution, represented as:

$$\begin{aligned}\widetilde{u}_2(\gamma_1, z) = {} & [V_2(z)]^{-1}[W_2(\gamma_1, z)W_2^{-1}(y_1, z)A_1(y_1, z) + U_2(A + D, \gamma_1, z)]\theta(z) \\ & + [V_2(z)]^{-1}[W_2(\gamma_1, z)W_2^{-1}(y_1, z)\{B_1(y_1, z) + I\} + U_2(K_2, \gamma_1, z)]\lambda(z) \\ & + [V_2(z)]^{-1}[W_2(\gamma_1, z)W_2^{-1}(y_1, z)G_1(y_1, z) + U_2(G, \gamma_1, z)], \quad z \in [0, Z].\end{aligned} \tag{4.22}$$

Substituting expression (4.22) for $\widetilde{u}_2(\gamma_1, z)$ into representation (4.20), we obtain an explicit formula for the solution to problems (4.17) and (4.19):

$$\widetilde{u}_2(y, z) = A_2(y, z)\theta(z) + B_2(y, z)\lambda(z) + G_2(y, z), \qquad (y, z) \in \Lambda_2, \tag{4.23}$$

where $A_2(y, z)$, $B_2(y, z)$, and $G_2(y, z)$ are given above.

Continuing in this way, we consider system (4.4) on the subdomain Λ_p:

$$\begin{aligned}\frac{\partial \widetilde{u}_p(y, z)}{\partial y} = {} & A(y, z)\widetilde{u}_p(y, z) + D(y, z)\widetilde{u}_2(\gamma_{p-1}, z) + [A(y, z) + D(y, z)]\theta(z) \\ & + K_p(y, z)\lambda(z) + G(y, z), \qquad (y, z) \in \Lambda_p,\end{aligned} \tag{4.24}$$

and the initial condition is given by:

$$\widetilde{u}_p(y_{p-1}, z) = A_{p-1}(y_{p-1}, z)\theta(z) + \{B_{p-1}(y_{p-1}, z) + I\}\lambda(z) + G_{p-1}(y_{p-1}, z), \qquad z \in [0, Z], \tag{4.25}$$

where p denotes an arbitrary integer in the range $2 \le p \le L$.

The solutions to the parameterized initial problems for the system IDEs (4.24) and (4.25) can be written as:

$$\begin{aligned}\widetilde{u}_p(y, z) = {} & W_p(y, z)W_p^{-1}(y_{p-1}, z)[A_{p-1}(y_{p-1}, z)\theta(z) + \{B_{p-1}(y_{p-1}, z) + I\}\lambda(z) + G_{p-1}(y_{p-1}, z)] \\ & + U_p(D, y, z)\widetilde{u}_p(\gamma_{p-1}, z) + U_p(A + D, y, z)\theta(z) + U_p(K_p, y, z)\lambda(z) + U_p(G, y, z), \quad (y, z) \in \Lambda_p.\end{aligned} \tag{4.26}$$

At $y = \gamma_{p-1}$, representation (4.26) yields

$$\begin{aligned}[I - U_p(D, \gamma_{p-1}, z)]\widetilde{u}_p(\gamma_{p-1}, z) = {} & [W_p(\gamma_{p-1}, z)W_p^{-1}(y_{p-1}, z)A_{p-1}(y_{p-1}, z) + U_p(A + D, \gamma_{p-1}, z)]\theta(z) \\ & + [W_p(\gamma_{p-1}, z)W_p^{-1}(y_{p-1}, z)\{B_{p-1}(y_{p-1}, z) + I\} + U_p(K_p, \gamma_{p-1}, z)]\lambda(z) \\ & + W_p(\gamma_{p-1}, z)W_p^{-1}(y_{p-1}, z)G_{p-1}(y_{p-1}, z) + U_p(G, \gamma_{p-1}, z), \quad z \in [0, Z].\end{aligned} \tag{4.27}$$

Assume that the matrix $V_p(z) = I - U_p(D, \gamma_{p-1}, z)$ is nonsingular for all $z \in [0, Z]$.

Then the system (4.27) admits a unique solution, represented as:

$$\begin{aligned}\widetilde{u}_p(\gamma_{p-1},z) &= [V_p(z)]^{-1}[W_p(\gamma_{p-1},z)W_p^{-1}(y_{p-1},z)A_{p-1}(y_{p-1},z) + U_p(A+D,\gamma_{p-1},z)]\theta(z) \\ &+ [V_p(z)]^{-1}[W_p(\gamma_{p-1},z)W_p^{-1}(y_{p-1},z)\{B_{p-1}(y_{p-1},z)+I\} + U_p(K_p,\gamma_{p-1},z)]\lambda(z) \\ &+ [V_p(z)]^{-1}W_p(\gamma_{p-1},z)W_p^{-1}(y_{p-1},z)G_{p-1}(y_{p-1},z) + U_p(G,\gamma_{p-1},z).\end{aligned} \tag{4.28}$$

Substituting expression (4.28) for $\widetilde{u}_p(\gamma_{p-1},z)$ into (4.26), we obtain an explicit representation of the solution to problems (4.24) and (4.25):

$$\widetilde{u}_p(y,z) = A_p(y,z)\theta(z) + B_p(y,z)\lambda(z) + G_p(y,z), \qquad (y,z)\in\Lambda_p, \tag{4.29}$$

where $A_p(y,z)$, $B_p(y,z)$, and $G_p(y,z)$ are given above.

Thus, we have obtained an explicit solution to the parameterized initial problems for the system IDEs (4.4)–(4.6), given by formula (4.13) for $i = 1$ and by the recurrence relations (4.29) for $i = p$, where $p = 2, 3, ..., L$.

According to condition a) of the Theorem, the matrices $V_p(z) = I - U_p(D,\gamma_{p-1},z)$ of dimension n are nonsingular for all $z \in [0,Z]$ and for $p = 1, 2, ..., L$.

Let us note that there also exists a finite left-hand limit $\lim\limits_{y\to Y-0}\widetilde{u}_L(y,z)$ for all $z\in[0,Z]$.

Now, in relation (4.7), instead of $\widetilde{u}_i(\xi,z)$ in the integral sum, we substitute formulas (4.16) and (4.29) at $y=\xi$. As a result, this yields

$$\begin{aligned}&-\sum_{i=1}^{L}\int_{y_{i-1}}^{y_i} M(\xi,z)[I+A_i(\xi,z)]d\xi\theta(z) + \Big[I - \sum_{i=1}^{L}\int_{y_{i-1}}^{y_i} M(\xi,z)\Big\{\frac{\xi-y_{i-1}}{y_i-y_{i-1}} - B_i(\xi,z)\Big\}d\xi\Big]\lambda(z) \\ &= \sum_{i=1}^{L}\int_{y_{i-1}}^{y_i} M(\xi,z)G_i(\xi,z)d\xi,\end{aligned} \tag{4.30}$$

$$P(z)\theta(z) + S(z)\lambda(z) = \varphi(z), \qquad z\in[0,Z], \tag{4.31}$$

From the coefficients of the left-hand sides of systems (4.30) and (4.31), we construct a $2n\times 2n$ matrix $Q_*(z)$ in the form (4.10).

Then equations (4.30) and (4.31) can be written in the form (4.12).

According to condition b) of the Theorem, the matrix $Q_*(z)$ of size $2n\times 2n$, is nonsingular for all $z\in[0,Z]$. This means that its determinant is nonzero:

$$\Big|-\sum_{i=1}^{L}\int_{y_{i-1}}^{y_i} M(\xi,z)[I+A_i(\xi,z)]d\xi S(z) - P(z)\Big[I - \sum_{i=1}^{L}\int_{y_{i-1}}^{y_i} M(\xi,z)\Big\{\frac{\xi-y_{i-1}}{y_i-y_{i-1}} - B_i(\xi,z)\Big\}d\xi\Big]\Big| \neq 0$$

for all $z\in[0,Z]$.

It follows that system (4.12) has a unique solution $\begin{pmatrix}\theta^*(z)\\ \lambda^*(z)\end{pmatrix}$ in the form:

$$\begin{pmatrix}\theta^*(z)\\ \lambda^*(z)\end{pmatrix} = [Q_*(z)]^{-1}\begin{pmatrix}\sum\limits_{i=1}^{L}\int\limits_{y_{i-1}}^{y_i} M(\xi,z)G_i(\xi,z)d\xi \\ \varphi(z)\end{pmatrix}.$$

Hence, Theorem 1 is proved. □

From the equivalence of problems (3.4), (3.5), and (4.4)–(4.8), the following statement follows:

Theorem 2. *Let conditions a) and b) of Theorem 1 be satisfied.*

Then the parameterized integral-problems for the system IDEs (3.4), and (3.5) have a unique solution $u^*(y,z)$, determined by the equalities:

$$u^*(y,z) = \widetilde{u}_i^*(y,z) + \theta^*(z) + \frac{y - y_{i-1}}{y_i - y_{i-1}}\lambda^*(z), \qquad (y,z) \in \Lambda_i, \qquad i = 1,2,...,L, \tag{4.32}$$

$$u^*(Y,z) = \lim_{y \to Y-0} \widetilde{u}_L^*(y,z) + \theta^*(z) + \frac{Y - y_{L-1}}{Y - y_{L-1}}\lambda^*(z), \qquad z \in [0,Z], \tag{4.33}$$

where the functions $\widetilde{u}_i^*(y,z)$ admit the representations (4.11), and the functional parameters $\theta^*(z)$ and $\lambda^*(z)$ are the solution to system (4.12).

Recent studies have explored several related directions in the theory of integro-differential and integrable systems. For instance, advanced collocation-based numerical methods have been developed for highly oscillatory algebraic singular Volterra integral equations [21], and multi-component integrable equations with infinitely many symmetries and conservation laws have been investigated within the framework of Hamiltonian systems [22]. The parametrization method proposed in this paper could, in principle, be extended to such classes of problems, particularly to nonlinear systems with oscillatory kernels, memory effects, or Hamiltonian structures. These perspectives open new possibilities for applying the developed approach to a wider range of models describing nonlinear wave propagation and related phenomena.

5. Constructive method for solving the equivalent problem and convergence of the algorithm

Let us now return to the study of the parameterized integral-problems for the first-order system IDEs with distributed functions (3.1)–(3.3).

We define the triple system $\{u^*(y,z), w^*(y,z), v^*(y,z)\}$ as the limit of the sequence of triple systems $\{u^{(k)}(y,z), w^{(k)}(y,z), v^{(k)}(y,z)\}$, $k = 0,1,2,\ldots$.

An iterative approach for constructing solutions to the parameterized integral problems for first-order system IDEs with distributed functions (3.1)–(3.3) is proposed.

The approximate solutions will be obtained using the following algorithm:

Stage 0.

(1) Let $w(y,z) = \psi(y)$, $v(y,z) = \dot{\psi}(y)$ in the right-hand side of system (3.1). The initial approximation $u^{(0)}(y,z)$ is obtained as the solution of the parameterized integral-problems for the first-order system IDEs

$$\frac{\partial u(y,z)}{\partial y} = A(y,z)u(y,z) + D(y,z)u(\beta(y),z) + K(y,z)\int_0^Y M(\xi,z)u(\xi,z)d\xi \\ + g(y,z) + B(y,z)\dot{\psi}(y) + C(y,z)\psi(y), \qquad (y,z) \in \Lambda, \tag{5.1}$$

$$P(z)u(0,z) + S(z)\int_0^Y M(\xi,z)u(\xi,z)d\xi = \varphi(z), \qquad z \in [0,Z]. \tag{5.2}$$

(2) Substituting the obtained function $u^{(0)}(y,z)$ and its partial derivative $\frac{\partial u^{(0)}(y,z)}{\partial y}$ into the integral constraints (3.3), we determine the initial approximations $w^{(0)}(y,z)$ and $v^{(0)}(y,z)$:

$$w^{(0)}(y,z) = \psi(y) + \int_0^z u^{(0)}(y,\eta)d\eta, \qquad v^{(0)}(y,z) = \dot{\psi}(y) + \int_0^z \frac{\partial u^{(0)}(y,\eta)}{\partial y}d\eta, \qquad (y,z) \in \Lambda. \tag{5.3}$$

Continuing in this way, at the k-th stage we construct a sequence of triple systems $\{u^{(k)}(y,z), w^{(k)}(y,z), v^{(k)}(y,z)\}$

Stage $(k+1)$.

(1) Let $w(y,z) = w^{(k)}(y,z)$, $v(y,z) = v^{(k)}(y,z)$ in the right-hand side of system (3.1). The $(k+1)$-st approximation $u^{(k+1)}(y,z)$ is obtained as the solution of the parameterized integral-problems for the system of first-order IDEs

$$\begin{aligned} \frac{\partial u(y,z)}{\partial y} &= A(y,z)u(y,z) + D(y,z)u(\beta(y),z) + K(y,z)\int_0^Y M(\xi,z)u(\xi,z)d\xi \\ &\quad + g(y,z) + B(y,z)v^{(k)}(y,z) + C(y,z)w^{(k)}(y,z), \qquad (y,z) \in \Lambda, \end{aligned} \tag{5.4}$$

$$P(z)u(0,z) + S(z)\int_0^Y M(\xi,z)u(\xi,z)d\xi = \varphi(z), \qquad z \in [0,Z]. \tag{5.5}$$

(2) By substituting the obtained function $u^{(k+1)}(y,z)$ and its partial derivative $\frac{\partial u^{(k+1)}(y,z)}{\partial y}$ into the integral constraints (3.3), we determine the $(k+1)$-st approximations $w^{(k+1)}(y,z)$ and $v^{(k+1)}(y,z)$:

$$w^{(k+1)}(y,z) = \psi(y) + \int_0^z u^{(k+1)}(y,\eta)d\eta, \qquad v^{(k+1)}(y,z) = \dot{\psi}(y) + \int_0^z \frac{\partial u^{(k+1)}(y,\eta)}{\partial y}d\eta, \tag{5.6}$$

$(y,z) \in \Lambda, \quad k = 0,1,2,....$

The procedure for obtaining the solution to the parameterized integral problems (3.1)–(3.3) for first-order system IDEs with distributed functions consists of two phases. At the first phase, the parameterized integral problems are solved for the first-order system IDEs with fixed distributed functions $w(y,z)$ and $v(y,z)$. At the second phase, the distributed functions are determined from the integral constraints.

The conditions for the applicability and implementation of the proposed method for solving the parameterized integral problems for first-order systems IDEs with distributed functions (3.1)–(3.3) are given in the following statement.

Theorem 3. *Suppose that the conditions a) and b) of Theorem 1 hold.*

Then the parameterized integral-problems for the first-order system IDEs with distributed functions and integral constraints (3.1)–(3.3) admits a unique solution $\{u^*(y,z), w^*(y,z), v^*(y,z)\}$, *where the function* $u^*(y,z)$ *is the solution of the parameterized integral-problems for the first-order* *system IDEs (3.4) and (3.5) with the function* $G(y,z) = g(y,z) + B(y,z)v^*(y,z) + C(y,z)w^*(y,z)$, *while the distributed functions* $w^*(y,z)$ *and* $v^*(y,z)$ *satisfy the integral constraints*

$$w^*(y,z) = \psi(y) + \int_0^z u^*(y,\eta)d\eta, \qquad v^*(y,z) = \dot{\psi}(y) + \int_0^z \frac{\partial u^*(y,\eta)}{\partial y} d\eta$$

for all $(y,z) \in \Lambda$.

Proof. By condition, the assumptions of Theorem 1 are fulfilled. This implies that the parameterized integral-problems for the first-order system IDEs (3.4), and (3.5) are uniquely solvable for $G(y,z) \in C(\Lambda, \mathbb{R}^n)$ è $\varphi(z) \in C([0,Z], \mathbb{R}^n)$.

Let

$$\alpha(z) = \max_{y\in[0,Y]} \|A(y,z)\|, \qquad \delta(z) = \max_{y\in[0,Y]} \|D(y,z)\|, \qquad \beta(z) = \max_{y\in[0,Y]} \|B(y,z)\|,$$
$$\sigma(z) = \max_{y\in[0,Y]} \|C(y,z)\|, \qquad \kappa(z) = \max_{y\in[0,Y]} \|K(y,z)\|, \qquad \zeta(z) = \max_{y\in[0,Y]} \|M(y,z)\|.$$

According to Stage 0, we solve the parameterized integral problems (5.1) and (5.2) and obtain a unique solution $u^{(0)}(y,z)$, together with its partial derivative $\frac{\partial u^{(0)}(y,z)}{\partial y}$, for which the following estimates hold:

$$\max_{y\in[0,Y]} \|u^{(0)}(y,z)\| \le C_1(z) \max\Big(\max_{y\in[0,Y]} \|\psi(y)\|, \max_{y\in[0,Y]} \|\dot{\psi}(y)\|, \max_{y\in[0,Y]} \|g(y,z)\|, \|\varphi(z)\|\Big), \tag{5.7}$$

$$\begin{aligned}\max_{y\in[0,Y]} \Big\|\frac{\partial u^{(0)}(y,z)}{\partial y}\Big\| \le &\Big\{[\alpha(z) + \delta(z) + \kappa(z)Y\zeta(z)]C_1(z) + 1 + \beta(z) + \sigma(z)\Big\} \\ &\times \max\Big(\max_{y\in[0,Y]} \|\psi(y)\|, \max_{y\in[0,Y]} \|\dot{\psi}(y)\|, \max_{y\in[0,Y]} \|g(y,z)\|, \|\varphi(z)\|\Big).\end{aligned} \tag{5.8}$$

Here, the function $C_1(z)$ is determined by the given data of the problem using the representations (4.32) and (4.33) and does not depend on $g(y,z)$ and $\varphi(z)$.

From the integral relations (5.1) we obtain an inequality for the functions $w^{(0)}(y,z)$ and $v^{(0)}(y,z)$:

$$\begin{aligned}&\max\Big(\max_{y\in[0,Y]} \|w^{(0)}(y,z)\|, \max_{y\in[0,Y]} \|v^{(0)}(y,z)\|\Big) \\ &\le \max\Big(\max_{y\in[0,Y]} \|\psi(y)\|, \max_{y\in[0,Y]} \|\dot{\psi}(y)\|\Big) + \int_0^z \max\Big(\max_{y\in[0,Y]} \|u^{(0)}(y,\eta)\|, \max_{y\in[0,Y]} \Big\|\frac{\partial u^{(0)}(y,\eta)}{\partial y}\Big\|\Big)d\eta.\end{aligned} \tag{5.9}$$

Continuing the process, at the $(k+1)$-st stage, from (5.2)–(5.4) we establish the next estimates:

$$\max_{y\in[0,Y]} \|u^{(k+1)}(y,z)\| \le C_1(z) \max\Big(\max_{y\in[0,Y]} \|w^{(k)}(y,z)\|, \max_{y\in[0,Y]} \|v^{(k)}(y,z)\|, \max_{y\in[0,Y]} \|g(y,z)\|, \|\varphi(z)\|\Big), \tag{5.10}$$

$$\max_{y\in[0,Y]} \Big\|\frac{\partial u^{(k+1)}(y,z)}{\partial y}\Big\| \le C_2(z) \max\Big(\max_{y\in[0,Y]} \|w^{(k)}(y,z)\|, \max_{y\in[0,Y]} \|v^{(k)}(y,z)\|, \max_{y\in[0,Y]} \|g(y,z)\|, \|\varphi(z)\|\Big), \tag{5.11}$$

$$\begin{aligned}\max\Big(\max_{y\in[0,Y]} \|w^{(k+1)}(y,z)\|, \max_{y\in[0,Y]} \|v^{(k+1)}(y,z)\|\Big) \le \max\Big(\max_{y\in[0,Y]} \|\psi(y)\|, \max_{y\in[0,Y]} \|\dot{\psi}(y)\|\Big) \\ + \int_0^z \max\Big(\max_{y\in[0,Y]} \|u^{(k+1)}(y,\eta)\|, \max_{y\in[0,Y]} \Big\|\frac{\partial u^{(k+1)}(y,\eta)}{\partial y}\Big\|\Big)d\eta.\end{aligned} \tag{5.12}$$

where $C_2(z) = [\alpha(z) + \delta(z) + \kappa(z)Y\zeta(z)]C_1(z) + 1 + \beta(z) + \sigma(z)$, $k = 0, 1, 2, \dots$.

Now, let us form the differences

$$\Delta u^{(k+1)}(y,z) = u^{(k+1)}(y,z) - u^{(k)}(y,z), \quad \Delta\frac{\partial u^{(k+1)}(y,z)}{\partial y} = \frac{\partial u^{(k+1)}(y,z)}{\partial y} - \frac{\partial u^{(k)}(y,z)}{\partial y},$$

$$\Delta w^{(k+1)}(y,z) = w^{(k+1)}(y,z) - w^{(k)}(y,z), \quad \Delta v^{(k+1)}(y,z) = v^{(k+1)}(y,z) - v^{(k)}(y,z),\ k = 0, 1, 2, \dots.$$

Then, similarly to estimates (5.7)–(5.12), the following inequalities are established:

$$\begin{aligned}&\max\Big(\max_{y\in[0,Y]}\|\Delta u^{(k+1)}(y,z)\|, \max_{y\in[0,Y]}\Big\|\Delta\frac{\partial u^{(k+1)}(y,z)}{\partial y}\Big\|\Big)\\ &\le \max\{C_1(z), C_2(z)\}\max\Big(\max_{y\in[0,Y]}\|\Delta w^{(k)}(y,z)\|, \max_{y\in[0,Y]}\|\Delta v^{(k)}(y,z)\|\Big),\end{aligned} \tag{5.13}$$

$$\begin{aligned}&\max\Big(\max_{y\in[0,Y]}\|\Delta w^{(k+1)}(y,z)\|, \max_{y\in[0,Y]}\|\Delta v^{(k+1)}(y,z)\|\Big)\\ &\le \int_0^z \max\Big(\max_{y\in[0,Y]}\|\Delta u^{(k+1)}(y,\eta)\|, \max_{y\in[0,Y]}\Big\|\Delta\frac{\partial u^{(k+1)}(y,\eta)}{\partial y}\Big\|\Big)d\eta.\end{aligned} \tag{5.14}$$

In view of estimate (5.13), the relation (5.14) yields the fundamental inequality

$$\begin{aligned}&\max\Big(\max_{y\in[0,Y]}\|\Delta w^{(k+1)}(y,z)\|, \max_{y\in[0,Y]}\|\Delta v^{(k+1)}(y,z)\|\Big)\\ &\le \int_0^z \max\{C_1(\eta), C_2(\eta)\}\max\Big(\max_{y\in[0,Y]}\|\Delta w^{(k)}(y,\eta)\|, \max_{y\in[0,Y]}\|\Delta v^{(k)}(y,\eta)\|\Big)d\eta,\end{aligned} \tag{5.15}$$

$k = 0, 1, 2, \dots.$

From the assumptions on the coefficients of system (2.1), it follows that the functions $C_1(z)$ and $C_2(z)$ are continuous on $[0, Z]$.

Define for $z \in [0, Z]$

$$D_k(z) = \max\Big(\max_{y\in[0,Y]}\|\Delta w^{(k)}(y,z)\|, \max_{y\in[0,Y]}\|\Delta v^{(k)}(y,z)\|\Big),$$

and $\omega(z) = \max\{C_1(\eta), C_2(\eta)\}$, so inequality (5.15) becomes

$$D_{k+1}(z) \le \int_0^z \omega(\eta)D_k(\eta)d\eta, \qquad k = 0, 1, 2, \dots \tag{5.16}$$

For $D_0(z)$, taking into account the estimates (5.7)–(5.9), we obtain

$$D_0(z) \le \Big(1 + \int_0^Z \omega(\eta)d\eta\Big)\max\Big(\max_{y\in[0,Y]}\|\psi(y)\|, \max_{y\in[0,Y]}\|\dot\psi(y)\|, \max_{(y,z)\in\Lambda}\|g(y,z)\|, \max_{z\in[0,Z]}\|\varphi(z)\|\Big) = \bar D_0.$$

Then

$$D_{k+1}(z) \le \bar D_0\frac{1}{(k+1)!}\Big(\int_0^z \omega(\eta)d\eta\Big)^{k+1}, \qquad k = 0, 1, 2, \dots \tag{5.17}$$

For each fixed $z \in [0, Z]$,

$$D_k(z) \leq \bar{D}_0 \frac{1}{k!}\Big(\int_0^z \omega(\eta)d\eta\Big)^k \leq \bar{D}_0 \frac{1}{k!}\Big(\int_0^Z \omega(\eta)d\eta\Big)^k \longrightarrow 0, \qquad k \to \infty, \tag{5.18}$$

since $k!$ grows faster than any power. Then, the convergence is uniform in z, and D_k bounds maxima over y uniformly on the whole domain $\Lambda = [0, Y] \times [0, Z]$.

Therefore, from inequality (5.17), it follows that the sequences $\{w^{(k)}(y, z)\}$ and $\{v^{(k)}(y, z)\}$ in the domain Λ converge uniformly, as $k \to \infty$, to the functions $w^*(y, z)$ and $v^*(y, z)$, respectively.

From inequality (5.13) we have

$$\begin{aligned} &\max\Big(\max_{y\in[0,Y]} \|\Delta u^{(k+1)}(y, z)\|, \max_{y\in[0,Y]}\Big\|\Delta \frac{\partial u^{(k+1)}(y, z)}{\partial y}\Big\|\Big) \\ &\leq \max\{C_1(z), C_2(z)\} \max\Big(\max_{y\in[0,Y]} \|\Delta w^{(k)}(y, z)\|, \max_{y\in[0,Y]} \|\Delta v^{(k)}(y, z)\|\Big) \leq \bar{D}_0 \frac{1}{(k+1)!}\Big(\int_0^z \omega(\eta)d\eta\Big)^{k+1}, \end{aligned} \tag{5.19}$$

From (5.19) it follows that the sequences $\{u^{(k)}(y, z)\}$ and $\{\frac{\partial u^{(k)}(y,z)}{\partial y}\}$ in the domain Λ converge uniformly to the functions $u^*(y, z)$ and $\frac{\partial u^*(y,z)}{\partial y}$, respectively. The functions $u^*(y, z)$ and $\frac{\partial u^*(y,z)}{\partial y}$ satisfy the inequalities

$$\begin{aligned} &\max_{y\in[0,Y]} \|u^*(y, z)\| \\ &\leq C_1(z)\Big(\bar{D}_0 \exp\Big\{\int_0^Z \omega(\eta)d\eta\Big\} + \max\Big(\max_{y\in[0,Y]} \|\psi(y)\|, \max_{y\in[0,Y]} \|\dot{\psi}(y)\|, \max_{(y,z)\in\Lambda} \|g(y, z)\|, \|\varphi(z)\|\Big)\Big). \end{aligned} \tag{5.20}$$

$$\begin{aligned} &\max_{y\in[0,Y]}\Big\|\frac{\partial u^*(y, z)}{\partial y}\Big\| \\ &\leq C_2(z)\Big(\bar{D}_0 \exp\Big\{\int_0^Z \omega(\eta)d\eta\Big\} + \max\Big(\max_{y\in[0,Y]} \|\psi(y)\|, \max_{y\in[0,Y]} \|\dot{\psi}(y)\|, \max_{(y,z)\in\Lambda} \|g(y, z)\|, \|\varphi(z)\|\Big)\Big). \end{aligned} \tag{5.21}$$

Thus, the triple system $\{u^*(y, z), w^*(y, z), v^*(y, z)\}$ is a solution to the parameterized integral problems for the first-order system IDEs with distributed functions (3.1)–(3.3).

The uniqueness of the solution is proved by contradiction, using inequalities (5.13)–(5.15). Hence, Theorem 3 is proved. □

The equivalence between the original initial-integral problems (2.1)–(2.3) and the parameterized integral problems for the first-order system IDEs with distributed functions (3.1)–(3.3) implies the validity of the statement.

Based on the equivalence of the initial-integral problems (2.1)–(2.3) and the parameterized integral-problems for the system of first-order IDEs with distributed functions (3.1)–(3.3), the validity of the statement follows.

Theorem 4. *Suppose that the conditions a) and b) of Theorem 1 hold.*

Then the initial-integral problem for the HIDS (2.1)–(2.3) admits a unique solution $w^*(y,z)$*, where the function* $w^*(y,z)$*, together with its partial derivatives*

$$\frac{\partial w^*(y,z)}{\partial z} = u^*(y,z), \frac{\partial w^*(y,z)}{\partial y} = v^*(y,z),$$

constitutes a solution of the parameterized integral-problems for the first-order system IDEs with distributed functions (3.1)–(3.3) for all $(y,z) \in \Lambda$.

6. Conclusions

In this paper, we propose a new approach to solving the initial-integral problem for HIDS (2.1)–(2.3). Solvability criteria are derived in terms of matrix conditions that are explicitly constructed from the coefficients of the system together with the integral constraints. In addition, we develop a constructive scheme for computing approximate solutions and prove that these approximations converge to the exact solution of the corresponding parameterized integral problems for the first-order system IDEs with distributed functions (3.1)–(3.3). Within the framework of this approach, one component of the triple of solutions to problems (3.1)–(3.3) coincides with the solution of the initial–integral problem for HIDS (2.1)–(2.3), thereby establishing the equivalence of the two formulations and paving the way for numerical implementation.

Moreover, the developed approach can be naturally extended to a wider range of problems [3, 7], including initial-integral problems for HIDS with discontinuities [23,24] and impulse effects [25–27], HIDS with delays [8,12,15] and parameters [9,28], nonlinear HIDS [16,29], as well as applications to mathematical models in the biomedical sciences and engineering informatics [30].

Author contributions

Anar T. Assanova: Conceptualization, Methodology, Writing-original draft, Writing-review & editing; Sailaubay S. Zhumatov: Methodology, Writing-original draft, Writing-review & editing. Both authors have been working together on the mathematical development of the manuscript. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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