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Research article

Linear difference inequalities with constant coefficients with the sum equal to zero

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Abstract: Many difference equations of the form

$$x_{n+k} = f(x_{n+k-1}, \dots, x_n), \quad n \in \mathbb{N},$$

where $k \in \mathbb{N}$, model some phenomena in nature and society. The most interesting cases usually occur when the function f satisfies the condition $f(x, \dots, x) = x$ on its domain of definition. Because of this the difference inequalities $x_{n+k} \le f(x_{n+k-1}, \dots, x_n)$ and $x_{n+k} \ge f(x_{n+k-1}, \dots, x_n)$ are of some interest. If f is a smooth function, then it can be approximated by a linear function. Motivated by some concrete examples, here we mostly consider the sequences that satisfy the linear difference inequality

$$\sum_{j=1}^{k} a_j x_{n+l-j} \ge 0, \quad n \in \mathbb{N}_0,$$

where $k \in \mathbb{N}_2$, $l \in \mathbb{N}_0$, and the coefficients $a_j \in \mathbb{R}$, $j = \overline{2, k-1}$, $a_1, a_k \in \mathbb{R} \setminus \{0\}$, satisfy the condition $\sum_{j=1}^{k} a_j = 0$.

Keywords: linear difference inequalities; constant coefficients; difference equations; bounded sequences
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1. Introduction

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$, that is, the set of all positive whole numbers, $\mathbb{Z}$ be the set of all whole numbers, and $\mathbb{R}$ be the set of real numbers. Let $\mathbb{N}_k = \{n \in \mathbb{Z} : n \geq k\}$, where $k \in \mathbb{Z}$ is fixed. If $p, q \in \mathbb{Z}$ satisfy the condition $p \leq q$, then we use the notation $i = \overline{p, q}$ instead of writing the expression: $p \leq i \leq q$, $i \in \mathbb{Z}$. If $k, l \in \mathbb{N}$, then $\gcd(k, l)$ denotes the greatest common divisor of the numbers k and l.

Difference equations have been studied systematically since the time of de Moivre and Daniel Bernoulli, who started investigating their solvability and presented some methods for solving linear difference equations [5, 9]. The study was continued by Lagrange, Laplace, and some other mathematicians of the 18th century [16–18]. For some later presentations and studies on solvability, see, for example, [11, 13, 19]. For some recent results in the direction, see, for example, [14, 23, 34] and the papers quoted therein. The solvability theory of linear difference equations and systems of difference equations with constant coefficients was essentially finished up to the end of the 18th century and during the 19th century has been refined. In the recent papers [23, 34], solvability of the linear difference equations and systems decides the solvability of the nonlinear difference equations and systems considered therein. Linear and nonlinear difference equations and systems of difference equations occur in many areas of mathematics and science, for example, in computational mathematics [8], combinatorics [15, 20, 26], summations of series [1, 21, 22], theoretical biology and ecology [24], economics, etc.

For example, the following model

$$x_{n+1} = ax_n + \frac{bx_{n-1}}{1 + cx_{n-1} + dx_n}, \quad n \in \mathbb{N}, \tag{1.1}$$

where $\min\{x_0, x_1\} > 0$, $a \in (0, 1)$, $b \in (0, +\infty)$, and $c, d \in [0, +\infty)$, with $c + d > 0$, is the generalized Beddington-Holt stock recruitment model, and

$$x_{n+3} = ax_{n+2} + bx_n e^{-(cx_{n+2}+dx_n)}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

where $\min\{x_0, x_1, x_2\} > 0$, $a, b, c, d \geq 0$, and $c + d > 0$, is the flour beetle population model (see, e.g., [28]).

The most interesting case for the difference equations in (1.1) and (1.2) is when the sum of the main coefficients of the equations is equal to one, that is, when $a + b = 1$ in the equations.

Since the initial values are positive and the parameters are nonnegative, in these cases, from (1.1), we have

$$x_{n+1} \leq ax_n + bx_{n-1}, \tag{1.3}$$

for $n \in \mathbb{N}$, and from (1.2), we have

$$x_{n+3} \leq ax_{n+2} + bx_n, \quad n \in \mathbb{N}_0. \tag{1.4}$$

Note that the inequalities (1.3) and (1.4) are linear difference inequalities with the property that the sums of the coefficients on their right-hand sides are equal to one, or equivalently, with the property that the sums of all the coefficients are equal to zero. Such a situation occurs frequently.

Sometimes it is not possible to use only difference equations or systems for solving concrete theoretical or applicable problems, because of which together with them, are used some related difference inequalities, which are conveniently chosen to solve the problems. Such a situation can be found even in quite old literature (for instance, in [25]). For some later results where difference inequalities are used, see, e.g., [6, 7, 27] and the references therein. To show the global convergence of solutions to the max-type difference equations in [33], a difference inequality has been used. The difference equation

$$x_{n+1} \le ax_n + b, \quad n \in \mathbb{N},$$

has been frequently used in the literature.

Difference inequalities are frequently useful in getting some comparison results in the theory of difference equations and systems of difference equations (see, for example, [2, 3, 30]). In these papers such inequalities are used to obtain some 'frame' sequences by which some properties of solutions to the corresponding difference equations can be obtained. For example, in [30] was proved such a result by which were found the second members in the asymptotics of some positive solutions to some special cases of the generalized Beverton-Holt stock recruitment model [4], the flour beetle population model [28], and some mosquito population models. A generalization of the inclusion theorem in [30] was given in [31]. By using the inclusion theorem, the existence of a monotone solution to a rational difference equation converging to the equilibrium exponentially was proved, whereas in [32] the existence of nontrivial solutions of a class of difference equations of arbitrary order was proved. These examples show that difference inequalities frequently appear in many situations and play some important roles in studying solutions of difference equations and systems of difference equations, as well as some other types of mathematical objects.

Motivated by the above-mentioned investigations in the theory of difference equations and systems, models in theoretical biology and other branches of science, as well as some concrete linear and nonlinear difference inequalities, including the ones in (1.3) and (1.4), here we consider the sequences of real numbers that satisfy the following linear difference inequality:

$$\sum_{j=1}^{k} a_j x_{n+l-j} \ge 0, \tag{1.5}$$

for every $n \in \mathbb{N}_0$, and some $k \in \mathbb{N}_2$, $l \in \mathbb{N}_0$, where the coefficients $a_j \in \mathbb{R}$, $j = \overline{2, k-1}$, $a_1, a_k \in \mathbb{R} \setminus \{0\}$, satisfy the condition

$$\sum_{j=1}^{k} a_j = 0.$$

Some of the results in the paper we obtained a long time ago, but have never been published or presented so far. Some of the results in the paper could be a matter of folklore, but we have not managed to find specific references for them, which, if they exist, could be scattered in the literature, or could be some auxiliary results in dealing with some difference equations, iteration processes, and related topics. We include them here for a better presentation and for the benefit of the reader, who can get a better picture on the topic and its possible applications.

2. Main results

In this section we state and prove the main results in this paper. Our first auxiliary result considers a difference inequality that has nonconstant coefficients and is the only result dealing with such coefficients. One of the reasons for this is based on our idea to simplify the settings and use solvability of the linear difference equations with constant coefficients at some points.

Lemma 1. *Assume that a sequence $(x_n)_{n\in\mathbb{Z}}$ of real numbers satisfies the difference inequality*

$$x_n \le a_n^{(-k)} x_{n-k} + \cdots + a_n^{(-1)} x_{n-1} + a_n^{(1)} x_{n+1} + \cdots + a_n^{(l)} x_{n+l}, \tag{2.1}$$

for $n \in \mathbb{Z}$, where $k, l \in \mathbb{N}$, $a_n^{(j)} \ge 0$, $j \in \{-k+1, \dots, -2, 2, \dots, l-1\} \setminus \{0\}$, $\min\{a_n^{(-k)}, a_n^{(-1)}, a_n^{(1)}, a_n^{(l)}\} > 0$, $n \in \mathbb{Z}$, and

$$\sum_{j=-k, j\ne 0}^{l} a_n^{(j)} \le 1, \quad n \in \mathbb{Z}. \tag{2.2}$$

Then, the sequence is constant, or it cannot achieve the maximum.

Proof. Assume that $(x_n)_{n\in\mathbb{Z}}$ is a nonconstant sequence satisfying (2.1) that achieves the maximum M, say, at x_r. Then from (2.1) we have

$$\begin{aligned} M = & x_r \le a_r^{(-k)} x_{r-k} + \cdots + a_r^{(-1)} x_{r-1} + a_r^{(1)} x_{r+1} + \cdots + a_r^{(l)} x_{r+l} \\ \le & M \sum_{j=-k, j\ne 0}^{l} a_r^{(j)} \le M. \end{aligned}$$

From this and since $\min\{a_r^{(-1)}, a_r^{(1)}\} > 0$, we have

$$x_{r-1} = x_{r+1} = M.$$

Using the same procedure to the terms x_{r-1} and x_{r+1} we get

$$x_{r-2} = x_{r+2} = M.$$

A simple inductive argument shows that

$$x_{r-m} = x_{r+m} = M,$$

for every $m \in \mathbb{N}$, or equivalently $x_n = x_r$, for $n \in \mathbb{Z}$, that is, x_n is constant, which is a contradiction. □

2.1. *Difference inequality* (1.5) *in the case* $k = 3$

Here we consider the difference inequality (1.5) in the case $k = 3$. First, we consider the sequences on a finite discrete interval. The following result, which is a matter of folklore, can be proved similar to Lemma 1. We give a different proof for the benefit of the reader.

Proposition 1. *Assume that a sequence* $(x_n)_{n=\overline{0,N}}$, $N \in \mathbb{N}$, *of real numbers satisfies the difference inequality*

$$x_n \leq (1-\alpha)x_{n-1} + \alpha x_{n+1}, \tag{2.3}$$

for $n = \overline{1, N-1}$, *and some* $\alpha \in (0,1)$, *and*

$$x_0 = x_N. \tag{2.4}$$

Then

$$x_n \leq x_0, \quad n = \overline{0, N}. \tag{2.5}$$

Proof. Assume that (2.5) is not true. Then, there is $r \in \{1, \dots, N-1\}$ such that

$$x_r > x_0, \tag{2.6}$$

$$x_r > x_{r-1} \quad \text{and} \quad x_0 \geq x_j, \quad j = \overline{0, r-1}. \tag{2.7}$$

From (2.3) we have

$$x_{n+1} - x_n \geq \frac{1-\alpha}{\alpha}(x_n - x_{n-1}), \tag{2.8}$$

for $n = \overline{1, N-1}$.

If we apply (2.8) for $n = \overline{r,k}$, where $k \leq N-1$, and (2.7), we easily get

$$x_{k+1} - x_k \geq \left(\frac{1-\alpha}{\alpha}\right)^{k-r+1}(x_r - x_{r-1}) > 0. \tag{2.9}$$

This, together with (2.6) implies

$$x_N > x_{N-1} > \cdots > x_r > x_0,$$

from which, along with the assumption (2.4), we get a contradiction. □

Corollary 1. *Assume that a sequence* $(x_n)_{n=\overline{0,N}}$, $N \in \mathbb{N}$, *of real numbers satisfies the difference inequality in* (2.3) *for* $n = \overline{1, N-1}$, *and some* $\alpha \in (0,1)$, *that*

$$x_n \geq m, \quad \textit{for} \quad n = \overline{0, N}, \tag{2.10}$$

for some $m \in \mathbb{R}$, *and*

$$x_0 = x_N = m. \tag{2.11}$$

Then

$$x_n = m, \quad n = \overline{0, N}. \tag{2.12}$$

Proof. From Proposition 1 we get

$$x_n \le x_0 = m, \quad n = \overline{0, N},$$

from which, together with the assumption in (2.10), we get (2.12). □

Remark 1. If a sequence $(x_n)_{n=\overline{0,N}}$, $N \in \mathbb{N}$, of real numbers satisfies the difference inequality

$$x_n \le \alpha x_{n+1} + (1-\alpha)x_{n+2}, \tag{2.13}$$

for $n = \overline{0, N-2}$, as well as the conditions (2.10) and (2.11), then (2.12) need not hold. Indeed, the following sequence satisfy all the conditions

$$x_n = \begin{cases} m, & n \ne N-1, \\ m+c, & n = N-1. \end{cases}$$

for some $c > 0$, but is not constant.

Remark 2. If a sequence $(x_n)_{n=\overline{0,N}}$, $N \in \mathbb{N}$, of real numbers satisfies the difference inequality

$$x_{n+2} \le \alpha x_{n+1} + (1-\alpha)x_n, \tag{2.14}$$

for $n = \overline{0, N-2}$, as well as the conditions (2.10) and (2.11), then (2.12) need not hold. Indeed, the following sequence satisfy all the conditions

$$x_n = \begin{cases} m, & n \ne 1, \\ m+c, & n = 1. \end{cases}$$

for some $c > 0$, but is not constant.

Remark 3. The case $\alpha = 1/2$ and $m = 0$ can be found in [36, p. 23].

Theorem 1. *Assume that a sequence* $(x_n)_{n=\overline{0,N+1}}$, $N \in \mathbb{N}$, *of real numbers satisfies the difference inequality* (2.3) *for* $n = \overline{1, N}$*, and some* $\alpha \in (0, 1)$, *and satisfies the conditions*

$$x_0 = x_N \quad \text{and} \quad x_1 = x_{N+1}. \tag{2.15}$$

Then

$$x_n = x_0, \quad n = \overline{0, N+1}. \tag{2.16}$$

Proof. Summing up the inequalities

$$(1-\alpha)x_{n-1} + \alpha x_{n+1} - x_n \ge 0,$$

for $n = \overline{1, N}$, and using the conditions in (2.15), we obtain

$$(1-\alpha)S_{N-1} + \alpha S_{N-1} - S_{N-1} \ge 0, \tag{2.17}$$

where

$$S_m = \sum_{n=0}^{m} x_n.$$

Since the left-hand side of inequality (2.17) is equal to zero, we have that all the summands are equal to zero, that is,

$$(1-\alpha)x_{n-1} + \alpha x_{n+1} - x_n = 0, \tag{2.18}$$

for $n = \overline{1, N}$. The roots of the characteristic polynomial

$$P_2(\lambda) = \alpha\lambda^2 - \lambda + 1 - \alpha = (\lambda - 1)(\alpha\lambda - (1-\alpha))$$

associated to the linear difference equation in (2.18) are equal to

$$\lambda_1 = 1 \quad \text{and} \quad \lambda_2 = \frac{1-\alpha}{\alpha}.$$

If $\alpha \neq 1/2$, then we have that the general solution to the equation is given by

$$x_n = c_1 + c_2\left(\frac{1-\alpha}{\alpha}\right)^n. \tag{2.19}$$

Since it must be

$$c_1 + c_2 = x_0 \quad \text{and} \quad c_1 + c_2\left(\frac{1-\alpha}{\alpha}\right) = x_1$$

by some calculation we obtain

$$c_1 = \frac{(1-\alpha)x_0 - \alpha x_1}{1-2\alpha} \quad \text{and} \quad c_2 = \frac{(x_1 - x_0)\alpha}{1-2\alpha}. \tag{2.20}$$

Using (2.20) in (2.19), we get

$$x_n = \frac{(1-\alpha)x_0 - \alpha x_1}{1-2\alpha} + \frac{(x_1 - x_0)\alpha}{1-2\alpha}\left(\frac{1-\alpha}{\alpha}\right)^n. \tag{2.21}$$

From (2.15) and (2.21), we get

$$x_0 = \frac{(1-\alpha)x_0 - \alpha x_1}{1-2\alpha} + \frac{(x_1 - x_0)\alpha}{1-2\alpha}\left(\frac{1-\alpha}{\alpha}\right)^N,$$

from which it follows that

$$x_1 - x_0 = (x_1 - x_0)\left(\frac{1-\alpha}{\alpha}\right)^N. \tag{2.22}$$

Since $\alpha \neq 1/2$ and $N \geq 1$, we have $\left(\frac{1-\alpha}{\alpha}\right)^N \neq 1$, from which, together with (2.22), it follows that

$$x_0 = x_1. \tag{2.23}$$

Employing (2.23) in (2.18), we get $x_2 = x_0$. A simple inductive argument shows that (2.16) holds in this case.

If $\alpha = 1/2$, then the general solution to equation (2.18) is given by

$$x_n = c_1 + c_2 n. \tag{2.24}$$

Since it must be

$$c_1 = x_0 \quad \text{and} \quad c_1 + c_2 = x_1,$$

we obtain

$$c_1 = x_0 \quad \text{and} \quad c_2 = x_1 - x_0. \tag{2.25}$$

Using (2.25) in (2.24), we get

$$x_n = x_0 + (x_1 - x_0)n. \tag{2.26}$$

From (2.15) and (2.26) we get

$$x_0 = x_0 + (x_1 - x_0)N,$$

from which and since $N \geq 1$, it follows that

$$x_0 = x_1. \tag{2.27}$$

Employing (2.27) in (2.18), we get $x_2 = x_0$. A simple inductive argument shows that (2.16) also holds in this case. □

Remark 4. For $\alpha = 1/4$ and $N = 101$, Problem 258 in [29] with shifted indices forward by one is obtained.

2.2. *Difference inequality* (1.5) *in the case* $k = 4$

Here we consider the difference inequality (1.5) in the case $k = 4$.

Theorem 2. *Let*

$$P_3(\lambda) = \lambda^3 + \alpha\lambda^2 + \beta\lambda + \gamma, \tag{2.28}$$

where $\alpha, \beta \in \mathbb{R}$, $\gamma \in \mathbb{R} \setminus \{0\}$,

$$\alpha + \beta + \gamma + 1 = 0, \tag{2.29}$$

and none of the roots of $P_3(\lambda)$ *satisfies the condition* $\lambda^k = 1$ *for some* $k \in \mathbb{N}_2$.

Assume that a sequence $(x_n)_{n=\overline{0,N+3}}$, $N \in \mathbb{N}$, *of real numbers satisfies the difference inequality*

$$x_{n+3} + \alpha x_{n+2} + \beta x_{n+1} + \gamma x_n \geq 0, \tag{2.30}$$

for $n = \overline{0, N-1}$, *with the conditions*

$$x_0 = x_N, \quad x_1 = x_{N+1} \quad \text{and} \quad x_2 = x_{N+2}. \tag{2.31}$$

Then

$$x_n = x_0, \quad n = \overline{0, N+2}. \tag{2.32}$$

Proof. Summing up the inequalities

$$x_{n+3} + \alpha x_{n+2} + \beta x_{n+1} + \gamma x_n \geq 0,$$

for $n = \overline{0, N-1}$, and using the conditions in (2.31) we obtain

$$S_{N-1} + \alpha S_{N-1} + \beta S_{N-1} + \gamma S_{N-1} \geq 0, \tag{2.33}$$

where

$$S_m = \sum_{n=0}^{m} x_n.$$

Since the left-hand side of inequality (2.33) is equal zero, we have that all the summands are equal to zero, that is,

$$x_{n+3} + \alpha x_{n+2} + \beta x_{n+1} + \gamma x_n = 0, \tag{2.34}$$

for $n = \overline{0, N-1}$.

Now note that the polynomial (2.28) is the characteristic polynomial associated with the linear difference equation in (2.34). The condition (2.29) implies that one of its roots is equal to one, say, λ_1. Bearing in mind this fact, there are five cases to be considered. Before considering the cases note that since

$$0 \neq \gamma = -\lambda_1 \lambda_2 \lambda_3,$$

we have that none of the roots is equal to zero.

Case $\lambda_2 = \lambda_3 = 1$. In this case the general solution to Eq (2.34) is given by

$$x_n = c_1 + c_2 n + c_3 n^2. \tag{2.35}$$

From this and the conditions in (2.31), we have

$$\begin{aligned} c_1 =& c_1 + c_2 N + c_3 N^2, \\ c_1 + c_2 + c_3 =& c_1 + c_2(N+1) + c_3(N+1)^2, \end{aligned}$$

from which it follows that

$$\begin{aligned} 0 =& N(c_2 + N c_3), \\ 0 =& N(c_2 + (N+2) c_3). \end{aligned}$$

Hence,

$$c_2 + N c_3 = 0 \quad \text{and} \quad c_2 + (N+2) c_3 = 0,$$

and consequently

$$c_2 = c_3 = 0. \tag{2.36}$$

Using (2.36) in (2.35), and since it is obviously $c_1 = x_0$, we get (2.32).

Case $\lambda_2 = 1$, $\lambda_3 \neq 1$. In this case the general solution to Eq (2.34) is given by

$$x_n = c_1 + c_2 n + c_3 \lambda_3^n. \tag{2.37}$$

From this and the conditions in (2.31), we have

$$\begin{aligned} c_1 + c_3 =& c_1 + c_2 N + c_3 \lambda_3^N, \\ c_1 + c_2 + c_3\lambda_3 =& c_1 + c_2(N+1) + c_3\lambda_3^{N+1}, \end{aligned}$$

from which it follows that

$$\begin{aligned} 0 =& c_2 N + c_3(\lambda_3^N - 1), \\ 0 =& c_2 N + c_3\lambda_3(\lambda_3^N - 1). \end{aligned}$$

Hence

$$c_3(\lambda_3^N - 1)(\lambda_3 - 1) = 0.$$

From this and since $\lambda_3^N \neq 1$ for each $N \in \mathbb{N}$, we get $c_3 = 0$, and consequently $c_2 = 0$. Using these facts in (2.37), and since it obviously holds $c_1 = x_0$, we obtain (2.32) in this case.
Case $\lambda_2 \neq 1$, $\lambda_3 = 1$. This case is dual to the case $\lambda_2 = 1$, $\lambda_3 \neq 1$ (only the letters λ_2 and λ_3 should be interchanged to get the previous case). Hence, it is omitted.
Case $\lambda_2 = \lambda_3 \neq 1$. In this case the general solution to equation (2.34) is given by

$$x_n = c_1 + c_2\lambda_2^n + c_3 n\lambda_2^n. \tag{2.38}$$

From this and the conditions in (2.31), we have

$$\begin{aligned} c_1 + c_2 =& c_1 + c_2\lambda_2^N + c_3 N\lambda_2^N, \\ c_1 + c_2\lambda_2 + c_3\lambda_2 =& c_1 + c_2\lambda_2^{N+1} + c_3(N+1)\lambda_2^{N+1}, \end{aligned}$$

from which it follows that

$$\begin{aligned} 0 =& c_2(\lambda_2^N - 1) + c_3 N\lambda_2^N, \\ 0 =& \lambda_2(c_2(\lambda_2^N - 1) + c_3((N+1)\lambda_2^N - 1)). \end{aligned}$$

Hence

$$c_2(\lambda_2^N - 1) + c_3 N\lambda_2^N = 0 \quad \text{and} \quad c_2(\lambda_2^N - 1) + c_3((N+1)\lambda_2^N - 1) = 0,$$

and consequently

$$c_3(\lambda_2^N - 1) = 0.$$

From this and since $\lambda_2^N \neq 1$, we get $c_3 = 0$, from which, along with the assumption $\lambda_2^N \neq 1$, we have that $c_2 = 0$. Using these facts in (2.38), and since it is obviously $c_1 = x_0$, we get (2.32) in this case.
Case $1 \neq \lambda_2 \neq \lambda_3 \neq 1$. In this case the general solution to Eq (2.34) is given by

$$x_n = c_1 + c_2\lambda_2^n + c_3\lambda_3^n. \tag{2.39}$$

From this and the conditions in (2.31), we have

$$\begin{aligned} c_1 + c_2 + c_3 =& c_1 + c_2\lambda_2^N + c_3\lambda_3^N, \\ c_1 + c_2\lambda_2 + c_3\lambda_3 =& c_1 + c_2\lambda_2^{N+1} + c_3\lambda_3^{N+1}, \end{aligned}$$

from which it follows that

$$\begin{aligned} 0 =& c_2(\lambda_2^N - 1) + c_3(\lambda_3^N - 1), \\ 0 =& c_2\lambda_2(\lambda_2^N - 1) + c_3\lambda_3(\lambda_3^N - 1). \end{aligned}$$

Hence

$$c_3(\lambda_3^N - 1)(\lambda_3 - \lambda_2) = 0.$$

From this and since $\lambda_2 \neq \lambda_3$ and $\lambda_3^N \neq 1$, we get $c_3 = 0$, from which, together with the assumption $\lambda_2^N \neq 1$, we get $c_2 = 0$. Using these facts in (2.39), and since it is obviously $c_1 = x_0$, we get (2.32) in this case. □

Remark 5. If the polynomial (2.28) has a root λ_2 such that $\lambda_2^N = 1$, then Theorem 2 need not hold. Namely, assume that λ_2 is such a root of the corresponding polynomial P_3, then for $N = 2$, $\lambda_2 = -1$, the sequence

$$x_n := (-1)^n$$

will be a solution to the difference Eq (2.34) satisfying the conditions in (2.31) for $N = 2$, which is obviously not constant.

If $N \in \mathbb{N}_3$, then if $\lambda_2^N = 1$ and $\lambda_2 \in \mathbb{C} \setminus \mathbb{R}$ we have that $\lambda_3 = \overline{\lambda}_2$ is also a root of the polynomial P_3, and the sequence

$$x_n := c\lambda_2^n + \bar{c}\overline{\lambda}_2^n,$$

where $c \in \mathbb{C} \setminus \{0\}$, will be a real-valued solution to Eq (2.34) satisfying the conditions in (2.31) for this N, which is obviously not constant.

Remark 6. Difference inequality (1.5) in the case $k \geq 5$ can be considered similarly, but the calculations are more complex and there are more cases that should be dealt with separately.

2.3. The difference equation corresponding to (2.3)

Here we consider the following difference equation

$$\alpha y_{n+1} - y_n + (1 - \alpha)y_{n-1} = 0, \tag{2.40}$$

where $\alpha \in (0, 1)$, which corresponds to the difference inequality in (2.3).

The roots of the characteristic polynomial $p_2(\lambda) = \alpha\lambda^2 - \lambda + 1 - \alpha$, associated to Eq (2.40), are $\lambda_1 = 1$ and $\lambda_2 = \frac{1-\alpha}{\alpha}$, from which it follows that the solution with the initial values y_0 and y_1 is given by

$$y_n = \frac{(\alpha - 1)y_0 + \alpha y_1}{2\alpha - 1} + \frac{\alpha(y_0 - y_1)}{2\alpha - 1}\left(\frac{1 - \alpha}{\alpha}\right)^n, \tag{2.41}$$

for $n \in \mathbb{Z}$, when $\alpha \neq 1/2$, whereas

$$y_n = y_0 + n(y_1 - y_0), \tag{2.42}$$

for $n \in \mathbb{Z}$, when $\alpha = 1/2$.

If $y_0 = y_1$, then from (2.41) and (2.42) we obtain

$$y_n = y_0, \quad n \in \mathbb{Z}. \tag{2.43}$$

Hence, from now on we assume that $y_0 \neq y_1$.

If we write (2.40) in the form

$$\alpha(y_{n+1} - y_n) = (1 - \alpha)(y_n - y_{n-1}), \quad n \in \mathbb{Z},$$

we see that

$$\operatorname{sign}(y_{n+1} - y_n) = \operatorname{sign}(y_n - y_{n-1}), \quad n \in \mathbb{Z}. \tag{2.44}$$

Assume that $\alpha \in (1/2, 1)$. Then $\frac{1-\alpha}{\alpha} \in (0, 1)$. Hence

$$\lim_{n\to+\infty} y_n = \frac{(\alpha - 1)y_0 + \alpha y_1}{2\alpha - 1}. \tag{2.45}$$

If $y_1 > y_0$, then from (2.44) we see that the sequence y_n increasingly converges to the limit in (2.45) (moreover, the sequence is increasing on $\mathbb{Z}$), whereas if $y_1 < y_0$, then the sequence y_n decreasingly converges to the limit (moreover, the sequence is decreasing on $\mathbb{Z}$).

On the other hand, we have

$$\lim_{n\to-\infty} y_n = -\infty, \tag{2.46}$$

if $y_1 > y_0$, whereas

$$\lim_{n\to-\infty} y_n = +\infty, \tag{2.47}$$

if $y_1 < y_0$.

Assume that $\alpha \in (0, 1/2)$. Then $\frac{1-\alpha}{\alpha} > 1$. Hence

$$\lim_{n\to-\infty} y_n = \frac{(1 - \alpha)y_0 - \alpha y_1}{1 - 2\alpha}. \tag{2.48}$$

If $y_1 > y_0$, then from (2.44) we see that the sequence y_n decreasingly converges to the limit in (2.48) (moreover, the sequence is decreasing on $-\mathbb{Z}$), whereas if $y_1 < y_0$, then the sequence y_n increasingly converges to the limit (moreover, the sequence is increasing on $-\mathbb{Z}$).

On the other hand, we have

$$\lim_{n\to+\infty} y_n = +\infty, \tag{2.49}$$

if $y_1 > y_0$, whereas

$$\lim_{n\to+\infty} y_n = -\infty, \tag{2.50}$$

if $y_1 < y_0$.

2.4. On the case $\alpha = 1/2$

Here we present some results on the sequences satisfying the inequality (2.3) for $\alpha = 1/2$. The following result should be a matter of folklore. We include it for completeness.

Theorem 3. *Assume that a sequence $(x_n)_{n\in\mathbb{Z}}$ of real numbers satisfies the difference inequality*

$$x_n \le \frac{x_{n+1} + x_{n-1}}{2}, \tag{2.51}$$

for $n \in \mathbb{Z}$. If the sequence is non-constant, then it must be unbounded.

Proof. From (2.51), we have

$$x_{n+1} - x_n \ge x_n - x_{n-1}, \quad n \in \mathbb{Z}. \tag{2.52}$$

If there is $r \in \mathbb{Z}$, such that

$$x_r > x_{r-1}, \tag{2.53}$$

then from (2.52) we get

$$x_n - x_{n-1} \ge x_{n-1} - x_{n-2} \ge \cdots \ge x_{r+1} - x_r \ge x_r - x_{r-1}.$$

From this and since

$$x_n - x_r = \sum_{j=r+1}^{n} (x_j - x_{j-1}),$$

we have

$$x_n \ge x_r + (n-r)(x_r - x_{r-1}), \tag{2.54}$$

for $n \ge r$.

Letting $n \to +\infty$ in (2.54) and using (2.53), we have

$$\lim_{n\to+\infty} x_n = +\infty,$$

which means that the sequence x_n is unbounded in this case.

Otherwise, it must be

$$x_r \le x_{r-1}, \tag{2.55}$$

for every $r \in \mathbb{Z}$.

If

$$x_r = x_{r-1},$$

for every $r \in \mathbb{Z}$, then the sequence $(x_n)_{n\in\mathbb{Z}}$ is constant.

If there is $r \in \mathbb{Z}$, such that

$$x_r < x_{r-1}. \tag{2.56}$$

From (2.52) we get

$$x_r - x_{r-1} \geq x_{r-1} - x_{r-2} \geq \cdots \geq x_{r-k+1} - x_{r-k}.$$

From this and since

$$x_r - x_{r-k} = \sum_{j=r-k+1}^{r} (x_j - x_{j-1}),$$

we have

$$x_{r-k} \geq x_r + k(x_{r-1} - x_r), \tag{2.57}$$

for $k \in \mathbb{N}$.

Letting $k \to +\infty$ in (2.57) and using (2.56), we have

$$\lim_{k\to+\infty} x_{r-k} = +\infty,$$

which means that the sequence x_n is unbounded in this case. □

Corollary 2. *Assume that a bounded sequence $(x_n)_{n\in\mathbb{Z}}$ of real numbers satisfies inequality* (2.51) *for $n \in \mathbb{Z}$. Then, it is constant.*

If $\alpha \in (0, 1) \setminus \{1/2\}$, then the situation is different. Namely, the following result holds.

Theorem 4. *Assume that $\alpha \in (0, 1) \setminus \{1/2\}$. Then, there are sequences of real numbers satisfying inequality* (2.3) *for every $n \in \mathbb{Z}$, which are bounded.*

Proof. Assume that $\alpha \in (0, 1/2)$. Then, the sequence defined in (2.41) satisfies (2.48) and if $y_0 > y_1$, then the sequence y_n increasingly converges to the limit therein as $n \to -\infty$, that is, we have

$$\lim_{n\to+\infty} y_{-n} = \frac{(1-\alpha)y_0 - \alpha y_1}{1-2\alpha} > \cdots > y_{-n} > y_{-n+1} > \cdots > y_{-1} > y_0 > y_1. \tag{2.58}$$

Note that since $y_0 > y_1$ and $\alpha \in (0, 1/2)$, we have

$$y_0 > \frac{\alpha}{1-\alpha} y_1.$$

Since for the sequence the relation (2.50) holds, we should change the sequence on the set $n \geq 2$.

Further, assume that $y_1 > 0$. Since it must be

$$\alpha y_2 + (1-\alpha)y_0 \geq y_1, \tag{2.59}$$

we have to choose

$$y_2 \in \Big(\max\Big\{0, y_1 - \frac{1-\alpha}{\alpha}(y_0 - y_1)\Big\}, y_1 \Big),$$

so that holds (2.59) and $y_1 > y_2 > 0$.

Assume that y_n was chosen such that $y_1 > y_2 > \cdots > y_n > 0$. Then, since it must be

$$\alpha y_{n+1} + (1-\alpha)y_{n-1} \geq y_n, \tag{2.60}$$

we have to choose

$$y_{n+1} \in \Big(\max\Big\{0, y_n - \frac{1-\alpha}{\alpha}(y_{n-1} - y_n)\Big\}, y_n\Big),$$

so that it holds (2.60) and $y_n > y_{n+1} > 0$.

Assume that $\alpha \in (1/2, 1)$. Namely, we can take the sequence defined in (2.41) satisfying (2.45) and if $y_0 < y_1$, then the sequence y_n increasingly converges to the limit therein as $n \to +\infty$, that is, we have

$$y_0 < y_1 < \cdots < y_n < y_{n+1} < \cdots < \frac{\alpha y_1 - (1-\alpha)y_0}{2\alpha - 1} = \lim_{n\to+\infty} y_n. \tag{2.61}$$

Note that since $y_0 < y_1$ and $\alpha \in (1/2, 1)$, we have $y_0 < \frac{\alpha}{1-\alpha} y_1$.

Since for the sequence the relation (2.46) holds, we should change the sequence on the set $n \le -1$.

Further, assume that $y_0 > 0$. Since it must be

$$\alpha y_1 + (1-\alpha)y_{-1} \ge y_0, \tag{2.62}$$

we have to choose

$$y_{-1} \in \Big(\max\Big\{0, y_0 - \frac{\alpha}{1-\alpha}(y_1 - y_0)\Big\}, y_0\Big),$$

so that holds (2.62) and $0 < y_{-1} < y_0$.

Assume that y_{-n} was chosen such that $y_0 > y_{-1} > \cdots > y_{-n} > 0$. Then, since it must be

$$\alpha y_{-n+1} + (1-\alpha)y_{-n-1} \ge y_{-n}, \tag{2.63}$$

we have to choose

$$y_{-n-1} \in \Big(\max\Big\{0, y_{-n} - \frac{\alpha}{1-\alpha}(y_{-(n-1)} - y_{-n})\Big\}, y_{-n}\Big),$$

so that it holds (2.63) and $y_{-n} > y_{-n-1} > 0$. □

Remark 7. Inequality (2.3) looks like a convexity relation, so a natural question is how it is possible that (2.58) holds. The answer is that the sequence for $n \le 0$, is, in fact, concave. Indeed, since $\alpha \in (0, 1/2)$, and (2.58) holds, we have

$$y_n = \alpha y_{n+1} + (1-\alpha)y_{n-1} > \frac{y_{n+1} + y_{n-1}}{2},$$

for $n \le 0$.

The following result could be a matter of folklore. We give a proof for the completeness and since it has some interesting details.

Proposition 2. *Let $(x_n)_{n\in\mathbb{Z}}$ be a sequence of real numbers such that*

$$x_n \ge \frac{x_{n-1} + x_{n+1}}{2}, \quad n \in \mathbb{Z}. \tag{2.64}$$

Then,

$$x_n \ge \frac{x_{n-2^k} + \cdots + x_{n-1} + x_{n+1} + \cdots + x_{n+2^k}}{2^{k+1}}, \tag{2.65}$$

for every $n \in \mathbb{Z}$ and $k \in \mathbb{N}_0$.

Proof. From (2.64) we see that (2.65) holds for $k = 0$. Further, from (2.64) we also have

$$x_{n+2} + x_n \leq 2x_{n+1} \quad \text{and} \quad x_n + x_{n-2} \leq 2x_{n-1},$$

for $n \in \mathbb{Z}$, from which, along with (2.64), it follows that

$$x_{n+2} + 2x_n + x_{n-2} \leq 2(x_{n+1} + x_{n-1}) \leq 4x_n,$$

for $n \in \mathbb{Z}$, and consequently

$$x_{n+2} + x_{n-2} \leq 2x_n, \tag{2.66}$$

for $n \in \mathbb{Z}$.

From (2.64) and (2.66), we obtain

$$\frac{x_{n-2} + x_{n-1} + x_{n+1} + x_{n+2}}{4} \leq x_n,$$

for $n \in \mathbb{Z}$, which is inequality (2.65) for $k = 1$.

Assume that we have proved the inequality

$$x_n \geq \frac{x_{n-2^{k-1}} + x_{n+2^{k-1}}}{2}, \tag{2.67}$$

and

$$x_n \geq \frac{x_{n-2^{k-1}} + \cdots + x_{n-1} + x_{n+1} + \cdots + x_{n+2^{k-1}}}{2^k}, \tag{2.68}$$

for every $n \in \mathbb{Z}$ and a fixed $k \in \mathbb{N}$.

From (2.67) we have

$$x_{n-2^k} + x_n \leq 2x_{n-2^{k-1}} \quad \text{and} \quad x_n + x_{n+2^k} \leq 2x_{n+2^{k-1}},$$

for every $n \in \mathbb{Z}$, from which, along with (2.67), it follows that

$$x_{n-2^k} + 2x_n + x_{n+2^k} \leq 2(x_{n-2^{k-1}} + x_{n+2^{k-1}}) \leq 4x_n,$$

for $n \in \mathbb{Z}$, and consequently

$$x_{n-2^k} + x_{n+2^k} \leq 2x_n, \tag{2.69}$$

for $n \in \mathbb{Z}$. Hence, the inductive argument shows that (2.69) holds for every $n \in \mathbb{Z}$ and $k \in \mathbb{N}_0$.

From (2.68) we have

$$x_{n-2^k} + \cdots + x_{n-2^{k-1}-1} + x_{n-2^{k-1}+1} + \cdots + x_{n-1} + x_n \leq 2^k x_{n-2^{k-1}}$$

and

$$x_{n+2^k} + \cdots + x_{n+2^{k-1}+1} + x_{n+2^{k-1}-1} + \cdots + x_{n+1} + x_n \leq 2^k x_{n+2^{k-1}}.$$

Summing up these inequalities and using (2.67), we get

$$\begin{aligned}&x_{n-2^k}+\cdots+x_{n-2^{k-1}-1}+x_{n-2^{k-1}+1}+\cdots+x_{n-1}+x_n\\&+x_{n+2^k}+\cdots+x_{n+2^{k-1}+1}+x_{n+2^{k-1}-1}+\cdots+x_{n+1}+x_n\\\le&2^k(x_{n-2^{k-1}}+x_{n+2^{k-1}})\le 2^{k+1}x_n,\end{aligned}$$

for $n\in\mathbb{Z}$, from which it follows that

$$\begin{aligned}&x_{n-2^k}+\cdots+x_{n-2^{k-1}-1}+x_{n-2^{k-1}+1}+\cdots+x_{n-1}\\&+x_{n+2^k}+\cdots+x_{n+2^{k-1}+1}+x_{n+2^{k-1}-1}+\cdots+x_{n+1}\\\le&(2^{k+1}-2)x_n,\end{aligned}$$

for $n\in\mathbb{Z}$, which is equivalent to

$$\begin{aligned}&x_{n-2^k}+\cdots+x_{n-2^{k-1}-1}+x_{n-2^{k-1}}+x_{n-2^{k-1}+1}+\cdots+x_{n-1}\\&+x_{n+2^k}+\cdots+x_{n+2^{k-1}+1}+x_{n+2^{k-1}}+x_{n+2^{k-1}-1}+\cdots+x_{n+1}\\\le&(2^{k+1}-2)x_n+x_{n-2^{k-1}}+x_{n+2^{k-1}},\end{aligned}$$

for $n\in\mathbb{Z}$, and consequently by using again inequality (2.67), we get

$$\begin{aligned}&x_{n-2^k}+\cdots+x_{n-1}+x_{n+1}+\cdots+x_{n+2^k}\\\le&(2^{k+1}-2)x_n+x_{n-2^{k-1}}+x_{n+2^{k-1}}\le(2^{k+1}-2)x_n+2x_n=2^{k+1}x_n,\end{aligned}$$

for $n\in\mathbb{Z}$, finishing the inductive proof of inequality (2.65). □

By using the change of variables

$$y_n:=-x_n \tag{2.70}$$

in Proposition 2 we obtain the following corollary.

Corollary 3. *Let* $(x_n)_{n\in\mathbb{Z}}$ *be a sequence of real numbers such that*

$$x_n\le\frac{x_{n-1}+x_{n+1}}{2},\quad n\in\mathbb{Z}.$$

Then,

$$x_n\le\frac{x_{n-2^k}+\cdots+x_{n-1}+x_{n+1}+\cdots+x_{n+2^k}}{2^{k+1}},$$

for every $n\in\mathbb{Z}$ *and* $k\in\mathbb{N}_0$.

From the proof of Proposition 2 we see that the following results also hold.

Corollary 4. *Let* $(x_n)_{n\in\mathbb{N}_0}$ *be a sequence of real numbers such that*

$$x_n\ge\frac{x_{n-1}+x_{n+1}}{2},\quad n\in\mathbb{N},$$

and $k \in \mathbb{N}_2$. *Then,*

$$x_n \geq \frac{x_{n-2^k} + \cdots + x_{n-1} + x_{n+1} + \cdots + x_{n+2^k}}{2^{k+1}},$$

for every $n \geq 2^k$.

Corollary 5. *Let* $(x_n)_{n\in\mathbb{Z}\setminus\mathbb{N}}$ *be a sequence of real numbers such that*

$$x_n \geq \frac{x_{n-1} + x_{n+1}}{2}, \quad n \in \mathbb{Z} \setminus \mathbb{N}_0,$$

and $k \in \mathbb{N}_2$. *Then,*

$$x_n \geq \frac{x_{n-2^k} + \cdots + x_{n-1} + x_{n+1} + \cdots + x_{n+2^k}}{2^{k+1}},$$

for every $n \leq -2^k$.

By using the change of variables (2.70), from Corollaries 4 and 5 we obtain the following results, respectively.

Corollary 6. *Let* $(x_n)_{n\in\mathbb{N}_0}$ *be a sequence of real numbers such that*

$$x_n \leq \frac{x_{n-1} + x_{n+1}}{2}, \quad n \in \mathbb{N},$$

and $k \in \mathbb{N}_2$. *Then,*

$$x_n \leq \frac{x_{n-2^k} + \cdots + x_{n-1} + x_{n+1} + \cdots + x_{n+2^k}}{2^{k+1}},$$

for every $n \geq 2^k$.

Corollary 7. *Let* $(x_n)_{n\in\mathbb{Z}\setminus\mathbb{N}}$ *be a sequence of real numbers such that*

$$x_n \leq \frac{x_{n-1} + x_{n+1}}{2}, \quad n \in \mathbb{Z} \setminus \mathbb{N}_0,$$

and $k \in \mathbb{N}_2$. *Then,*

$$x_n \leq \frac{x_{n-2^k} + \cdots + x_{n-1} + x_{n+1} + \cdots + x_{n+2^k}}{2^{k+1}},$$

for every $n \leq -2^k$.

2.5. A result related to Lemma 1 and a counterexample

The following result is a nontrivial improvement of Lemma 1 for a special case of inequality (2.1). However, it also considers the case when the coefficients $a_n^{(-1)}$ and $a_n^{(1)}$ are equal to zero.

Theorem 5. *Assume that a sequence* $(x_n)_{n\in\mathbb{Z}}$ *of real numbers satisfies the difference inequality*

$$x_n \leq (1-\alpha)x_{n-l} + \alpha x_{n+k}, \tag{2.71}$$

for every $n \in \mathbb{Z}$, *some* $\alpha \in (0,1)$, $k, l \in \mathbb{N}$ *such that*

$$\gcd(k,l) = 1. \tag{2.72}$$

Then, the sequence is constant, or it cannot achieve the maximum.

Proof. Assume that a nonconstant sequence $(x_n)_{n\in\mathbb{Z}}$ of real numbers satisfying difference inequality (2.71) achieves the maximum. Let

$$x_{n_0} = \max_{n\in\mathbb{Z}} x_n.$$

Since (2.72) holds, then there are $p, q \in \mathbb{N}$ such that

$$pk - ql = 1. \tag{2.73}$$

Indeed, we know that there are $m_1, m_2 \in \mathbb{Z}$, such that

$$m_1 k + m_2 l = 1$$

(see, e.g., [10]). Since $k, l \in \mathbb{N}$, it is clear that m_1 and m_2 are two integers of the opposite sign. So, we have $m_1 k - (-m_2) l = 1$, where m_1 and $-m_2$ are two integers of the same sign. If $m_1, -m_2 \in \mathbb{N}$, we found such numbers. If m_1 and $-m_2$ are two negative integers, then note that they are a solution to the Diophantine equation

$$xk - yl = 1. \tag{2.74}$$

It is well known that all the solutions to equation (2.74) are given by

$$x = m_1 + tl, \quad y = -m_2 + tk, \tag{2.75}$$

where $t \in \mathbb{Z}$ (see, e.g., [12]). For sufficiently large $t \in \mathbb{N}$, the solutions in (2.75) are positive, from which the claim follows in this case.

Now note that

$$\{n_0 + m(pk - ql) : m \in \mathbb{Z}\} = \mathbb{Z}.$$

From (2.71) and the definition of x_{n_0}, we have

$$x_{n_0} \le (1-\alpha)x_{n_0-l} + \alpha x_{n_0+k} \le x_{n_0},$$

from which it follows that

$$x_{n_0} = x_{n_0-l} = x_{n_0+k}.$$

Using this procedure p times, among other equalities, we get

$$x_{n_0} = x_{n_0+pk},$$

and then using it q times to x_{n_0+pk}, among other equalities, we get

$$x_{n_0} = x_{n_0+pk} = x_{n_0+pk-ql},$$

that is, we obtain

$$x_{n_0} = x_{n_0+1}.$$

A simple inductive argument shows that

$$x_{n_0} = x_{n+1}, \tag{2.76}$$

for every $n \geq n_0$.

From (2.71) and (2.76), since $l \in \mathbb{N}$, and the definition of x_{n_0}, we have

$$x_{n_0} = x_{n_0+l-1} \leq (1-\alpha)x_{n_0-1} + \alpha x_{n_0+l+k+1} \leq x_{n_0},$$

from which it follows that

$$x_{n_0-1} = x_{n_0}.$$

A simple inductive argument shows that

$$x_{n_0} = x_{n-1}, \tag{2.77}$$

for every $n \leq n_0$.

Hence,

$$x_{n_0} = x_n, \quad n \in \mathbb{Z},$$

that is, it is a constant sequence, which is a contradiction. □

Now we consider the case when the parameters k and l in Theorem 5 satisfy the condition $\gcd(k,l) > 1$. In this case Theorem 5 does not hold. Let

$$r := \gcd(k,l),$$

then the inequality (2.71) can be written in the form

$$x_n \leq (1-\alpha)x_{n-l_1 r} + \alpha x_{n+k_1 r}, \tag{2.78}$$

for $n \in \mathbb{Z}$, where

$$l_1 = \frac{l}{r} \quad \text{and} \quad k_1 = \frac{k}{r}.$$

Note that

$$\gcd(k_1, l_1) = 1.$$

Since each $n \in \mathbb{Z}$ can be written in the form $n = mr + q$, where $m \in \mathbb{Z}$ and $q \in \{0, 1, \ldots, r-1\}$, inequality (2.78) can be written in the form

$$x_{mr+q} \leq (1-\alpha)x_{(m-l_1)r+q} + \alpha x_{(m+k_1)r+q}, \tag{2.79}$$

where $m \in \mathbb{Z}$ and $q \in \{0, 1, \ldots, r-1\}$.

From (2.79), we see that the following r sequences

$$y_m^{(q)} := x_{mr+q}, \quad m \in \mathbb{Z}, \tag{2.80}$$

satisfy the difference inequality

$$y_m \leq (1-\alpha)y_{m-l_1} + \alpha y_{m+k_1}, \tag{2.81}$$

for $m \in \mathbb{Z}$.

By Theorem 5 each of these r sequences in (2.80) is constant, or it cannot achieve the maximum.

Now note that the sets of indices $S_q = \{mr + q : m \in \mathbb{Z}\}$ are such that

$$S_q \cap S_p = \emptyset,$$

when $q \neq p$, and that

$$\bigcup_{j=0}^{r-1} S_j = \mathbb{Z}.$$

It is easy to see that the sequence

$$y_m = c, \quad m \in \mathbb{Z}, \tag{2.82}$$

satisfy inequality (2.81) for each $c \in \mathbb{R}$.

So, if we choose the sequences in (2.80) to be constant, that is, equal to a constant c_q, $q = \overline{0, r-1}$, respectively, and such that at least two of the constants are different, that is, if

$$y_m^{(q)} = c_q \quad \text{and} \quad y_m^{(p)} = c_p,$$

for every $m \in \mathbb{Z}$ and some $q, p \in \{0, 1, \dots, r-1\}$ such that $q \neq p$ and $c_q \neq c_p$, then the corresponding sequence $(x_m)_{m\in\mathbb{Z}}$ obtained by the r relations in (2.80) will satisfy inequality (2.78). However, the sequence is not constant and

$$\max_{n\in\mathbb{Z}} x_n = \max_{j=\overline{0,r-1}} c_j,$$

that is, the sequence achieves the maximum.

Remark 8. If we use the terminology that refers to some difference equations in [35], for the difference inequality (2.78) when $r > 1$, we can say that it is a difference inequality with interlacing indices.

3. Conclusions

Some classes of linear difference inequalities, mostly with constant coefficients, are considered in this paper. Many results, ideas, tricks, and remarks are presented. The inequalities could be some motivations for further investigations in this direction, as well as some motivations for their applications in any of the areas that use difference equations or iteration processes. One of the natural directions for further investigations is consideration of linear difference inequalities with nonconstant coefficients.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflicts of interest.

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