Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(11): 26697–26716.
DOI:10.3934/math.20251174
Received: 16 July 2025
Revised: 01 October 2025
Accepted: 28 October 2025
Published: 18 November 2025

*Research article*

# Core of cooperative fuzzy games with coalition structures

**Fengye Wang**[1,*], **Yuanyuan Huang**[1], **Youlin Shang**[1] **and Zhihua Zheng** [2]

[1] School of Mathematics and Statistics, Henan University of Science and Technology, Luoyang 471023, China

[2] School of Logistics and Management Engineering, Yunnan University of Finance and Economics, Yunnan, 650221, China

* **Correspondence:** Email: fengyewang@haust.edu.cn.

**Abstract:** This paper develops an extended model of cooperative fuzzy games with coalition structures, where fuzzy coalitions are characterized by real-valued functions. Moving beyond existing studies on crisp coalition structures, our framework integrates fuzzy set principles with coalitional cooperation to model partial participation in structured games. Our contributions are fourfold: (1) a formal core definition for such games, extending the Aubin core; (2) construction of the superadditive cover and establishment of core non-emptiness conditions; (3) an axiomatic characterization showing the core is the unique solution that satisfies non-emptiness, individual rationality, weak reduced game property, and superadditivity; and (4) a domination-core concept proven equivalent to the conventional core under specific conditions. A concise numerical example validates these theoretical findings, illustrating core allocation, non-emptiness, and the core-cover equivalence in scenarios of partial cooperation. These results unify fuzzy games with structured coalitions, offering new insights into the stability and allocation in games with graded participation and advancing the theory of cooperative decision-making under uncertainty.

**Keywords:** cooperative fuzzy game; coalition structure; core; domination-core
**Mathematics Subject Classification:** 91A12, 91A35

## 1. Introduction

Cooperative game theory offers a principled framework for analyzing settings where players can form coalitions to improve collective outcomes. A well-known solution concept in this field is the core—a set of payoff distributions that are stable against coalitional deviations. Foundational studies by Peleg [1], Tadenuma [2], Serrano [3], and Hwang [4] established core properties in classical transferable utility (TU) games with crisp coalitions, where participation is all-or-nothing.

Many real-world collaborations, however, involve partial or graded participation. To capture such settings, Aubin (1974) introduced cooperative fuzzy games, in which each coalition is represented by a membership vector in $[0, 1]^N$, and defined the Aubin core, in which payoffs are proportional to participation levels [5,6]. Subsequent research has expanded this line of research along several themes. In the direction of core characterization and non-emptiness, Tsurumi et al. [7] studied core functions and dominance cores in fuzzy games, while Yu et al. [8] derived explicit non-emptiness conditions for the fuzzy core by exploring balancedness conditions in fuzzy settings. In the axiomatic domain, Hwang [9] developed two types of reduced games for fuzzy coalitions and used them to provide a full axiomatic characterization of the fuzzy core, establishing uniqueness under a set of natural properties. Muto et al. [10] further generalized core concepts by introducing stable sets for fuzzy games and examining their relationship with core allocations under various convexity assumptions. Recent research shows that in Aubin's fuzzy extension of almost positive cooperative games, the core is a singleton containing only the Shapley value [11].

Further extensions of fuzzy cooperative games have been proposed to address more complex scenarios. For instance, fuzzy configuration structure games allow not only fuzzy participation but also intersections of fuzzy a priori coalitions, with the fuzzy value expressed through linear combinations of classical cooperative game values, providing a generalized allocation rule [12]. In applied contexts, fuzzy cooperative games have been utilized for specialist role determination in emergency assessments, incorporating solutions such as the C-core, Neumann-Morgenstern solution, and nucleolus [13].

In parallel, another significant branch of research has incorporated coalition structures—partitions of players into disjoint groups—reflecting organizational or institutional constraints. Aumann and Drèze [14] systematically extended solution concepts such as the core, kernel, and nucleolus to games with coalition structures, establishing fundamental conditions for stability under predefined partitions. Owen [15, 16] introduced the Owen value, a coalition-structured extension of the Shapley value that accounts for a priori unions. Recent work has continued to enrich this domain through several innovative extensions. Under uncertainty settings, Yu et al. [17] studied interval-valued coalition structure games to handle payoff uncertainty using Hukuhara difference and interval arithmetic. In environments with limited cooperation, Sun et al. [18] introduced a probabilistic Owen value for games on matroids, while Yu [19] extended the Owen value to settings with limited feasible coalitions by proposing two distinct allocation approaches. Most relevant to our work, Meng et al. [20] combined fuzzy coalition structures with interval payoffs and studied the corresponding Owen value, marking an initial step toward integrating fuzzy participation with coalitional constraints.

Despite these advances, the literature remains bifurcated. Fuzzy game research focuses on graded participation but seldom incorporates coalitional constraints, while coalition structure studies assume crisp membership. This leaves a gap, as real-world scenarios (e.g., cross-organizational projects, multi-agent systems) often involve graded participation within pre-existing groups. The integration of these lines, particularly concerning core stability and dominance, remains underexplored. Although [20] made initial progress, their focus on the Owen value with interval payoffs leaves open fundamental questions about core stability and non-emptiness.

This paper bridges this gap by developing a unified framework for cooperative fuzzy games with coalition structures. Our specific contributions are fourfold. First, we formally define the core for such games, extending the Aubin core. Second, we construct the superadditive cover and establish equivalence conditions for non-emptiness of the core, generalizing classical results by [14]. Third, we provide an axiomatic characterization, proving the core is the unique solution satisfying non-emptiness, individual rationality, weak reduced game property, and superadditivity. Finally, we propose a domination-core (D-core) concept and demonstrate its equivalence to the conventional core under specific value-bound conditions.

Our work differs from [20] by addressing core stability with real-valued payoffs, not value allocation with intervals. Unlike fuzzy core studies [7–10], we explicitly incorporate coalition constraints via a structured partition system, offering a more realistic model for cooperative decision-making under fuzziness and coalitional constraints.

The paper is organized as follows: Section 2 reviews basic concepts of crisp and fuzzy games with coalition structures. Section 3 introduces the superadditive cover and presents core non-emptiness results. Section 4 provides the axiomatic characterization of the core. Section 5 defines the D-core and establishes its equivalence to the core. Section 6 concludes and suggests future research directions.

## 2. Preliminaries

In this section, some important definitions of crisp cooperative games with coalition structures and their generalized forms of fuzzy cooperative games with coalition structures are given.

### *2.1. Concepts on crisp games with coalition structures*

In crisp cooperative game theory, a coalition structure represents a partition of players into mutually exclusive groups. This organizational framework reflects situations where players belong to different organizations, departments, or teams before engaging in cooperation.

Let $\mathcal{U}$ be a universe of players. A crisp coalition refers to any nonempty subset of $\mathcal{U}$. For a specific crisp coalition $N = \{1, 2, \ldots, n\}$, a coalition structure is defined as a partition $\mathcal{R} = \{R_1, R_2, \ldots, R_m\}$ satisfying

$$\bigcup_{k=1}^{m} R_k = N, \tag{2.1}$$

$$R_i \cap R_j = \emptyset \quad \forall i, j \in \{1, 2, \ldots, m\},\ i \neq j. \tag{2.2}$$

This means that every player in $N$ must belong to exactly one coalition in the partition $\mathcal{R}$, ensuring a complete and non-overlapping grouping of all players.

Given these definitions, we can now formally define a cooperative game in this context. This leads us to the concept of a payoff vector that is considered attainable under the given coalition structure $\mathcal{R}$.

**Definition 2.1.** *Let $(N, v)$ be a cooperative game and $\mathcal{R}$ be a coalition structure for $N$, then the triple $(N, v, \mathcal{R})$ is called a crisp game with coalition structure. The set of feasible payoff vectors for this game is defined as:*

$$X^*(N, v, \mathcal{R}) = \left\{ x \in \mathbb{R}^N \;\middle|\; \sum_{i \in R} x_i \leq v(R),\ \forall R \in \mathcal{R} \right\}. \tag{2.3}$$

This definition establishes that a payoff vector $x$ is feasible if, within each formed coalition $R \in \mathcal{R}$, the total distribution to its members does not exceed the worth that coalition $R$ can generate on its own.

This can be interpreted as a form of budget constraint or collective rationality within each pre-existing group.

While the set $X^*(N, v, \mathcal{R})$ defines which payoff vectors are feasible under the coalition structure, it does not guarantee that these distributions are stable. A feasible vector could, for instance, unfairly disadvantage a subset of players, giving them an incentive to break away and form their own coalition. To address this issue of stability, we now introduce the core of a game with a coalition structure, a solution concept developed by Aumann [14] and Peleg [1,22].

**Definition 2.2.** *Let $(N, v, \mathcal{R})$ be a game with coalition structure. The core $C(N, v, \mathcal{R})$ of $(N, v, \mathcal{R})$ is the set*

$$C(N, v, \mathcal{R}) = \{x \in X^*(N, v, \mathcal{R}) \Big| \sum_{i \in S} x_i \geq v(S) \text{ for each } S \subseteq N\}. \tag{2.4}$$

Clearly, $C(N, v) = C(N, v, \{N\})$.

*2.2. Concepts on the cooperative fuzzy games with coalition structures*

In classical cooperative game theory, coalitions are crisp—a player is either fully in or fully out of a coalition. Fuzzy cooperative games extend this framework by allowing for partial participation, a concept known as a fuzzy coalition. A fuzzy coalition $S$ is defined as a fuzzy subset on the player set $N$, mathematically represented as a participation vector $S = (S_1, S_2, \cdots, S_n)$ with $0 \leq S_i \leq 1$, where each component of $S \in [0, 1]^N$ quantifies the membership level of player $i$ in the fuzzy coalition $S$. The set of all fuzzy coalitions on $N$, denoted by $\mathcal{F}^N$, constitutes an infinite space. A canonical correspondence exists between crisp coalitions (crisp subsets of $N$) and their fuzzy counterparts: For any crisp coalition $T \subseteq N$, its associated fuzzy coalition $e^T = (\delta_T(1), \delta_T(2), \ldots, \delta_T(n)) \in \mathcal{F}^N$, where

$$\delta_T(i) = \begin{cases} 1, & \text{if } i \in T; \\ 0, & \text{otherwise.} \end{cases} \tag{2.5}$$

Within this framework, the grand coalition $e^N = (1, 1, \ldots, 1)$ represents full participation from all players, while the empty coalition $\emptyset = (0, 0, \ldots, 0)$ indicates null participation. Single-player coalitions $S = \{i\}$ are denoted by $e^i$. For any fuzzy coalition $S \in \mathcal{F}^N$, two key concepts help us understand its composition: the support of $S$ and its cardinality. The support of $S$, denoted $\text{Supp}S = \{i \in N \mid S_i > 0\}$, is the set of players who have a non-zero participation level. It represents the set of players who are, to some extent, actively involved in the coalition. The cardinality is simply the size of its support, $|\text{Supp}S|$, which counts the number of players with positive participation.

Operations on fuzzy coalitions are defined componentwise. For any $S, T \in \mathcal{F}^N$:

$$(S \vee T)_i = max\{S_i, T_i\}, (S \wedge T)_i = min\{S_i, T_i\} \text{ for all } i \in N.$$

These operations intuitively correspond to combining the maximum or minimum participation levels of each player across the two coalitions. It follows logically that $\text{Supp}(S \vee T) = \text{Supp}S \cup \text{Supp}T$ and $\text{Supp}(S \wedge T) = \text{Supp}S \cap \text{Supp}T$.

Having defined fuzzy coalitions, we can now introduce a cooperative fuzzy game.

**Definition 2.3.** *A cooperative fuzzy game is a function $v : \mathcal{F}^N \to \mathbb{R}$ satisfying $v(\emptyset) = 0$.*

The value $v(S)$ represents the total payoff that coalition $S$ can achieve given its specific membership levels. The central problem is to distribute the value $v(e^N)$ of the grand coalition among the players in a fair and stable way. This leads to the concept of imputation.

**Definition 2.4.** *The imputation set of a cooperative fuzzy game v is defined as the set of efficient and individually rational payoff vectors*

$$I(v) = \left\{ x \in \mathbb{R}^N \;\middle|\; \sum_{i=1}^{n} x_i = v(e^N),\ and\ x_i \geq v(e^i)\ \forall i \in N \right\}. \tag{2.6}$$

Efficiency ensures the entire value of the grand coalition is distributed, while individual rationality guarantees each player receives at least what they could get alone.

**Definition 2.5.** *The Core of a cooperative fuzzy game v consists of all imputations satisfying coalitional rationality*

$$C(v) = \left\{ x \in I(v) \;\middle|\; \sum_{i \in SuppS} S_i x_i \geq v(S),\ \forall S \in \mathcal{F}^N \right\}. \tag{2.7}$$

Note that the sum $\sum_{i \in SuppS} S_i x_i$ represents the total payoff received by coalition $S$, weighted by each player's participation rate. This condition ensures that for every possible fuzzy coalition $S$, its weighted payoff is at least as large as the value $v(S)$ it can generate on its own.

The theory is further extended by introducing fuzzy coalition structures, which partition a fuzzy universe of players into groups.

**Definition 2.6.** *Let $U \in \mathcal{F}^N$, $\mathcal{T} = \{T_1, T_2, \cdots, T_m\}$ is said to be a fuzzy coalition structure on U, if $\mathcal{T}$ satisfies*

(1) $T_1 \vee T_2 \vee \cdots \vee T_m = U$;
(2) $T_i \wedge T_j = \emptyset \quad \forall i, j \in \{1, 2, \cdots, m\} \quad s.t. \quad i \neq j.$

We denote a fuzzy coalition structure $\mathcal{T}$ on $U$ by the pair $(U, \mathcal{T})$. The set of all feasible coalitions within this structure is denoted by $L(U, \mathcal{T})$.

**Definition 2.7.** *A cooperative fuzzy game with coalition structure on U is a triple $(U, v, \mathcal{T})$, where $U \in \mathcal{F}^N$, $\mathcal{T}$ is a fuzzy coalition structure on U, $v : L(U, \mathcal{T}) \to \mathbb{R}$ is a function assigning a value to each feasible coalition in the structure, satisfying $v(\emptyset) = 0$.*

This model provides a general framework for analyzing cooperation in environments with complex, graded membership structures across multiple overlapping groups.

**Example 2.1.** *Consider three enterprises $N = \{1, 2, 3\}$ forming a fuzzy coalition structure based on their resource participation levels (e.g., manpower, capital, equipment), represented by values in $[0, 1]$. Let $U = (0.6, 0.8, 1.0)$ be a fuzzy coalition on $\mathcal{F}^N$, $\mathcal{T} = \{T_1, T_2\}$ be a fuzzy coalition structure on U, where $T_1 = (0.6, 0.8, 0)$ represents a technical collaboration between enterprises 1 and 2; $T_2 = (0, 0, 1.0)$ represents independent financial support from enterprise 3. The characteristic function $v$ :*

*$L(U, \mathcal{T}) \to \mathbb{R}$ is defined as follows:*

$$v(S) = \begin{cases} 0.5, & \text{if } S = T_1, \\ 0.4, & \text{if } S = T_2, \\ 0.2, & \text{if } S = (0.6, 0, 0) \text{ or } (0, 0.8, 0), \\ 0.7, & \text{if } S = U, \\ 0, & \text{otherwise}. \end{cases} \tag{2.8}$$

Then, $v$ constitutes a cooperative fuzzy game with coalition structure on $U$.

Similar to the definition of core for the crisp case [1, 14, 22], we give the definition of core with fuzzy coalition structure as follows:

**Definition 2.8.** *Let $U \in \mathcal{F}^N$, $(U, v, \mathcal{T})$ be a cooperative fuzzy game with coalition structure on $U$. The set $I^*(U, v, \mathcal{T})$ of feasible payoff vectors for $(U, v, \mathcal{T})$ is defined by*

$$I^*(U, v, \mathcal{T}) = \{x \in \mathbb{R}^N | \sum_{i \in SuppT} T_i x_i \leq v(T) \text{ for each } T \in \mathcal{T}\}. \tag{2.9}$$

**Definition 2.9.** *Let $U \in \mathcal{F}^N$, $(U, v, \mathcal{T})$ be a cooperative fuzzy game with coalition structure on $U$. The Core $C(U, v, \mathcal{T})$ of $(U, v, \mathcal{T})$ is defined by*

$$C(U, v, \mathcal{T}) = \{x \in I^*(U, v, \mathcal{T}) | \sum_{i \in SuppS} S_i x_i \geq v(S) \text{ for each } S \in L(U, \mathcal{T})\}. \tag{2.10}$$

**Example 2.2.** *For Example 2.1, the set $I^*(U, v, \mathcal{T})$ of feasible payoff vectors is given by:*

$$I^*(U, v, \mathcal{T}) = \left\{ x \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} 0.6x_1 + 0.8x_2 \leq 0.5, \\ 1.0x_3 \leq 0.4 \end{array} \right\}.$$

For a payoff vector $x = (x_1, x_2, x_3)$, the core $C(U, v, \mathcal{T})$ consists of all $x \in I^*(U, v, \mathcal{T})$ satisfying coalitional rationality:

$$\begin{aligned} &0.6x_1 \geq 0.2, \quad 0.8x_2 \geq 0.2, \quad 1.0x_3 \geq 0.4, \\ &0.6x_1 + 0.8x_2 + 1.0x_3 \geq 0.7, \\ &0.6x_1 + 0.8x_2 \geq 0.5. \end{aligned}$$

Therefore, the core is

$$C(U, v, \mathcal{T}) = \left\{ x \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} 0.6x_1 + 0.8x_2 \leq 0.5, \\ 1.0x_3 \leq 0.4, \\ 0.6x_1 \geq 0.2, \quad 0.8x_2 \geq 0.2, \quad 1.0x_3 \geq 0.4, \\ 0.6x_1 + 0.8x_2 + 1.0x_3 \geq 0.7, \\ 0.6x_1 + 0.8x_2 \geq 0.5 \end{array} \right\}.$$

This simplifies to

$$C(U, v, \mathcal{T}) = \left\{ x \in \mathbb{R}^3 \;\middle|\; \begin{aligned} & 0.6x_1 + 0.8x_2 = 0.5, \\ & x_3 = 0.4, \\ & x_1 \geq 1/3, \\ & x_2 \geq 0.25 \end{aligned} \right\}.$$

Geometrically, the core is represented by a line segment in three-dimensional space, where the payoffs to enterprises 1 and 2 lie on the line $0.6x_1 + 0.8x_2 = 0.5$ with individual rationality constraints, while enterprise 3 receives a fixed payoff of 0.4. The specific geometric representation is shown in Figure 1.

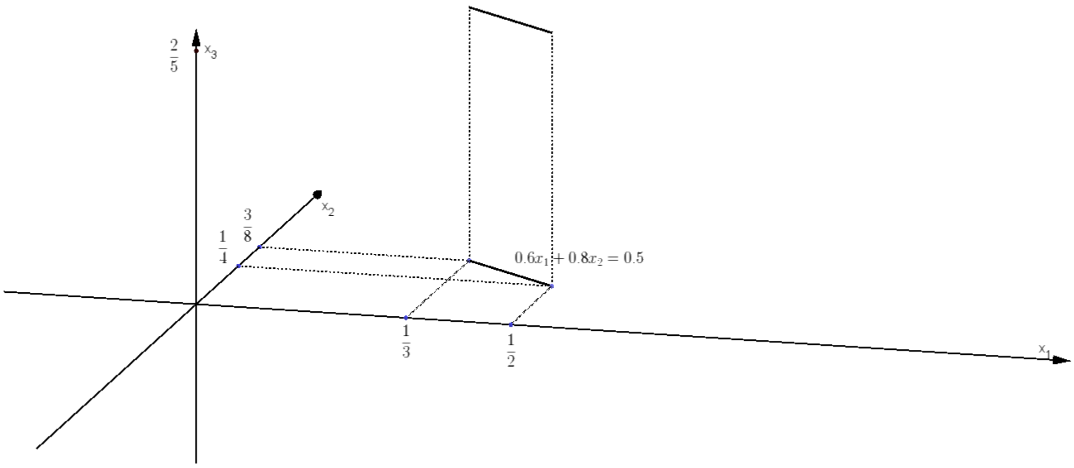


**Figure 1.** Core solution set projection on the $x_1$-$x_2$ plane with constraint $0.6x_1 + 0.8x_2 = 0.5$.

## 3. The superadditive cover and core non-emptiness

This section presents a fundamental construction in the theory of cooperative fuzzy games with coalition structures: the superadditive cover. This transformation converts a game into a superadditive set function, enabling us to establish equivalence conditions for the existence of stable allocations. Specifically, we investigate when the core of such games is non-empty, extending the classical work of Aumann and Dreze [14] to the fuzzy setting.

In many cooperative scenarios, players can generate greater value through coordinated cooperation than through separate efforts. The superadditive cover formally captures this potential by considering all possible ways to partition a coalition optimally.

**Definition 3.1.** *Let $U \in \mathcal{F}^N$ and $(U, v, \mathcal{T})$ be a cooperative fuzzy game with coalition structure on $U$. The superadditive cover $\hat{v}$ of $v$ is defined as:*

$$\hat{v}(T) = \max \left\{ \sum_{R \in \mathcal{R}} v(R) \;\middle|\; \mathcal{R} \text{ is a partition of } T \right\} \tag{3.1}$$

for all $\emptyset \neq T \in L(U, \mathcal{T})$, with $\hat{v}(\emptyset) = 0$.

Intuitively, $\hat{v}(T)$ represents the maximum total value achievable by optimally partitioning $T$ into subcoalitions within the structure $\mathcal{T}$. This construction effectively "completes" the game to satisfy superadditivity, a property crucial for many solution concepts.

**Example 3.1.** *Consider the cooperative fuzzy game from Example 2.1. For coalition $T_1 = (0.6, 0.8, 0)$, the possible partitions are Partition $\mathcal{R}_1 = \{T_1\}$ with $\sum_{R\in\mathcal{R}_1} v(R) = v(T_1) = 0.5$, and Partition $\mathcal{R}_2 = \{(0.6, 0, 0), (0, 0.8, 0)\}$ with $\sum_{R\in\mathcal{R}_2} v(R) = 0.2 + 0.2 = 0.4$. Thus, $\hat{v}(T_1) = \max\{0.5, 0.4\} = 0.5$.*

For $T_2 = (0, 0, 1.0)$, the only partition is $\mathcal{R} = \{T_2\}$, giving $\hat{v}(T_2) = v(T_2) = 0.4$.

For the grand coalition $U = (0.6, 0.8, 1.0)$, we consider five possible partitions: $\mathcal{R}_1 = \{U\}$ with $\sum_{R\in\mathcal{R}_1} v(R) = v(U) = 0.7$; $\mathcal{R}_2 = \{T_1, T_2\}$ with $\sum_{R\in\mathcal{R}_2} v(R) = v(T_1) + v(T_2) = 0.5 + 0.4 = 0.9$; $\mathcal{R}_3 = \{(0.6, 0, 0), (0, 0.8, 0), (0, 0, 1.0)\}$ with $\sum_{R\in\mathcal{R}_3} v(R) = 0.2 + 0.2 + 0.4 = 0.8$; $\mathcal{R}_4 = \{(0.6, 0, 0), (0, 0.8, 1.0)\}$ with $\sum_{R\in\mathcal{R}_4} v(R) = 0.2 + 0 = 0.2$; and $\mathcal{R}_5 = \{(0, 0.8, 0), (0.6, 0, 1.0)\}$ with $\sum_{R\in\mathcal{R}_5} v(R) = 0.2 + 0 = 0.2$. The maximum value is $\hat{v}(U) = \max\{0.7, 0.9, 0.8, 0.2, 0.2\} = 0.9$.

For coalition $T = (0.6, 0, 1.0)$, the partitions yield: $\mathcal{R}_1 = \{T\}$ with $v(T) = 0$ and $\mathcal{R}_2 = \{(0.6, 0, 0), (0, 0, 1.0)\}$ with $\sum_{R\in\mathcal{R}_2} v(R) = 0.2 + 0.4 = 0.6$. Thus, $\hat{v}(T) = \max\{0, 0.6\} = 0.6$.

Similarly, for $T = (0, 0.8, 1.0)$: $\mathcal{R}_1 = \{T\}$ with $v(T) = 0$; and $\mathcal{R}_2 = \{(0, 0.8, 0), (0, 0, 1.0)\}$ with $\sum_{R\in\mathcal{R}_2} v(R) = 0.2 + 0.4 = 0.6$, giving $\hat{v}(T) = \max\{0, 0.6\} = 0.6$.

The superadditive cover $\hat{v}$ of the cooperative fuzzy game $(U, v, \mathcal{T})$ is therefore:

$$\hat{v}(T) = \begin{cases} 0.9, & \text{if } T = U, \\ 0.5, & \text{if } T = T_1, \\ 0.4, & \text{if } T = T_2, \\ 0.6, & \text{if } T = (0.6, 0, 1.0) \text{ or } (0, 0.8, 1.0), \\ 0.2, & \text{if } T = (0.6, 0, 0) \text{ or } (0, 0.8, 0), \\ 0, & \text{otherwise.} \end{cases}$$

The superadditive cover indeed possesses the desired mathematical property, as established in the following proposition.

**Proposition 3.1.** *Let $U \in \mathcal{F}^N$ and $(U, v, \mathcal{T})$ be a cooperative fuzzy game with coalition structure on $U$, then the superadditive cover $\hat{v}$ of $(U, v, \mathcal{T})$ is superadditive, i.e.,*

$$\hat{v}(S \vee T) \geq \hat{v}(S) + \hat{v}(T), \tag{3.2}$$

for all $S, T \in L(U, \mathcal{T})$ with $S \wedge T = \emptyset$.

*Proof.* For $S, T \in L(U, \mathcal{T})$ with $S \wedge T = \emptyset$, let $\mathcal{R}_S$ and $\mathcal{R}_T$ be optimal partitions achieving $\hat{v}(S)$ and $\hat{v}(T)$, respectively. Since $S$ and $T$ have disjoint supports, their union $\mathcal{R}_S \cup \mathcal{R}_T$ forms a valid partition of $S \vee T$. Consequently,

$$\hat{v}(S \vee T) \geq \sum_{R\in\mathcal{R}_S\cup\mathcal{R}_T} v(R) = \hat{v}(S) + \hat{v}(T),$$

establishing the superadditivity property. □

Building on the foundational work of Aumann and Dreze [14], we now present the central theorem characterizing non-empty cores in fuzzy games with coalition structures.

**Theorem 3.1.** *Let $(U, v, \mathcal{T})$ be a cooperative fuzzy game with coalition structure. Then,*

(1) $C(U, v, \mathcal{T}) \neq \emptyset$ if and only if $C(\hat{v}) \neq \emptyset$ and $\hat{v}(U) = \sum_{T\in\mathcal{T}} v(T)$;

(2) If $C(U, v, \mathcal{T}) \neq \emptyset$, then $C(U, v, \mathcal{T}) = C(\hat{v})$.

*Proof.* (1) Let $x \in C(U, v, \mathcal{T})$, $\emptyset \neq S \in L(U, \mathcal{T})$, and let $\mathcal{R}$ be a fuzzy coalition structure of $S$, from the definition of the core of fuzzy game with coalition structure, we have $\sum_{i\in SuppT} T_i x_i \geq v(T)$ for each $T \in \mathcal{T}$. Therefore, $\sum_{i\in SuppS} S_i x_i \geq \sum_{R\in\mathcal{R}} \sum_{i\in SuppR} R_i x_i \geq \sum_{R\in\mathcal{R}} v(R)$, it follows that $\sum_{i\in SuppS} S_i x_i \geq \hat{v}(S)$. In particular, $\sum_{T\in\mathcal{T}} v(T) = \sum_{T\in\mathcal{T}} \sum_{i\in SuppT} T_i x_i = \sum_{i\in \bigcup_{T\in\mathcal{T}} SuppT} T_i x_i = \sum_{i\in SuppU} U_i x_i \geq \hat{v}(U)$ and $x_i \geq \hat{v}(e^i)$. By Definition 3.1, we have $\hat{v}(U) \geq \sum_{T\in\mathcal{T}} v(T)$. Hence, $\sum_{i\in SuppU} U_i x_i = \hat{v}(U) = \sum_{T\in\mathcal{T}} v(T)$. Specially, $\sum_{i\in N} x_i = \hat{v}(e^N)$. By the definition of 2.5, we get $x \in C(\hat{v})$. Conversely, assume that $x \in C(\hat{v})$ and $\hat{v}(U) = \sum_{T\in\mathcal{T}} v(T)$. From the definition of $C(\hat{v})$, we can get $\sum_{i\in SuppS} S_i x_i \geq \hat{v}(S) \geq v(S)$ for each $\emptyset \neq S \in L(U, \mathcal{T})$. Since $\sum_{i\in SuppU} U_i x_i = \sum_{T\in\mathcal{T}} \sum_{i\in SuppT} T_i x_i = \sum_{T\in\mathcal{T}} v(T)$ and $\sum_{i\in SuppT} T_i x_i \geq v(T)$, we can get $\sum_{i\in SuppT} T_i x_i = v(T)$ for each $T \in \mathcal{T}$. Therefore, $x \in C(U, v, \mathcal{T})$.

(2) Assume that $C(U, v, \mathcal{T}) \neq \emptyset$, by (1), we can get $C(\hat{v}) \neq \emptyset$ and $\hat{v}(U) = \sum_{T\in\mathcal{T}} v(T)$, By the proof of (1), we know $C(U, v, \mathcal{T}) = C(\hat{v})$. □

**Example 3.2.** *Obviously, the superadditive cover $\hat{v}$ in Example 3.1 satisfies superadditivity. For a payoff vector $x = (x_1, x_2, x_3)$, the core of fuzzy cooperative game $\hat{v}$ satisfy the following inequalities:*

$$\begin{aligned} 0.6x_1 + 0.8x_2 + x_3 &= 0.9, \\ 0.6x_1 + 0.8x_2 &\geq 0.5, \\ x_3 &\geq 0.4, \\ 0.6x_1 + x_3 &\geq 0.6, \\ 0.8x_2 + x_3 &\geq 0.6, \\ 0.6x_1 &\geq 0.2, \\ 0.8x_2 &\geq 0.2. \end{aligned}$$

This simplifies to

$$C(\hat{v}) = \left\{ x \in \mathbb{R}^3 \;\middle|\; \begin{array}{l} 0.6x_1 + 0.8x_2 = 0.5, \\ x_3 = 0.4, \\ x_1 \geq 1/3, \\ x_2 \geq 0.25 \end{array} \right\}.$$

Hence, the equality $C(U, v, \mathcal{T}) = C(\hat{v})$ is confirmed, in agreement with Theorem 3.1.

We conclude by examining how symmetry between players manifests in core allocations through the substitution property.

**Definition 3.2.** *Let $v$ be a cooperative fuzzy game on $\mathcal{F}^N$. Players $i, j \in N$ are substitutes if for any $S \in \mathcal{F}^N$ with $i, j \notin \mathrm{Supp}S$ and any $a \in (0, 1]$,*

$$v(S \vee ae^i) = v(S \vee ae^j). \tag{3.3}$$

This definition captures situations where players contribute equally to any coalition they join, making them interchangeable from a strategic perspective.

**Theorem 3.2.** *Let $(U, v, \mathcal{T})$ be a cooperative fuzzy game with coalition structure where players i and j are substitutes. If $i \notin \mathrm{Supp}T$, $j \notin \mathrm{Supp}R$, and $R_i = T_j \neq 0$ for some $R, T \in \mathcal{T}$, then for any $x \in C(U, v, \mathcal{T})$,*

$$x_i = x_j.$$

*Proof.* Assume without loss of generality that $R_i = T_j \neq 0$, $R_j = 0$, and $T_i = 0$. For $x \in C(U, v, \mathcal{T})$, consider the deviation where player $i$' participation in $R$ is replaced by player $j$ at the same level. Coalitional rationality yields

$$0 \geq v((R - R_i e^i) \vee R_i e^j) - \sum_{k \in (SuppR \setminus \{i\}) \cup \{j\}} ((R - R_i e^i) \vee R_i e^j)_k x_k.$$

Using the substitution property and simplifying, we obtain $0 \geq R_i x_i - R_i x_j$, implying $x_j \geq x_i$. The reverse inequality follows symmetrically, thus $x_i = x_j$. □

This result shows that the core respects symmetric contributions between substitute players, even when they belong to different coalitions within the structure, provided their participation levels are equal.

## 4. Axiomatic characterization of the core of fuzzy games with coalition structures

In this section, we formally define the reduced game for a cooperative fuzzy game with a coalition structure and analyze key properties of the core in such games. We focus on several important properties: non-emptiness (NE), individual rationality (IR), the weak reduced game property (WRGP), superadditivity (SUPA), and the converse reduced game property (CRGP). These properties help characterize the behavior and rationality of solution concepts under coalitional constraints and fuzzy preferences.

Let $U \in \mathcal{F}^N$, and let $\mathcal{T}$ be a coalition structure on $U$. For any non-empty fuzzy coalition $S \in \mathcal{F}^N$ such that $\mathrm{Supp}\, S \subseteq \mathrm{Supp}\, U$, we denote by $\mathcal{T}_{|S}$ the restriction of the coalition structure to $S$, defined as: $\mathcal{T}_{|S} = \{T \wedge S \mid T \in \mathcal{T} \text{ and } T \wedge S \neq \emptyset\}$.

**Definition 4.1.** *Let $(U, v, \mathcal{T})$ be a cooperative fuzzy game with coalition structure, $\emptyset \neq S \subseteq U$, and let $x \in I^*(U, v, \mathcal{T})$. The reduced game $(S, v^{\mathcal{T}}_{S,x}, \mathcal{T}_{|S})$ with respect to S and x is defined as:*

$$v^{\mathcal{T}}_{S,x}(R) = \begin{cases} 0, & if\ R = \emptyset, \\ v(T) - \sum\limits_{i \in SuppT \setminus SuppR} T_i x_i, & if\ \emptyset \neq R = S \wedge T\ for\ some\ T \in \mathcal{T}, \\ \max\limits_{SuppQ \subseteq SuppU \setminus SuppS} (v(R \vee Q) - \sum\limits_{i \in SuppQ} Q_i x_i), & if\ R \subset S \wedge T. \end{cases}$$

The reduced game models how a sub-coalition $S$ would evaluate its worth after the original game is resolved by payoff vector $x$. The first case handles the empty coalition. The second case applies when $R$ corresponds to the intersection of $S$ with some original coalition $T$; here, the value adjusts for the payoffs allocated to players outside $R$. The third case, for proper sub-coalitions of $S \wedge T$, reflects the maximum surplus that $R$ can achieve by collaborating with any external coalition $Q$, after compensating $Q$ according to $x$.

If the members of $SuppT \setminus SuppR$ receive the payoff $\sum_{i \in SuppT \setminus SuppR} T_i x_i$, then the remaining members in $SuppT \cap SuppS$ may receive $v(T) - \sum_{i \in SuppT \setminus SuppR} T_i x_i$. Moreover, for every non-empty subset of $SuppT \cap SuppS$, the reduced game value represents the maximal payoff that the coalition $R$ can expect to secure.

Note that if $\mathcal{T} = \{U\}$, the game reduces to a standard cooperative fuzzy game. For $\emptyset \neq S \subseteq U \in \mathcal{F}^N$ and $x \in I(v)$, the reduced game with respect to $S$ and $x$ is the fuzzy game $v_{S,x}$, defined by

$$v_{S,x}(T) = \begin{cases} 0, & if\ T = \emptyset, \\ v(U) - \sum_{i \in SuppU \setminus SuppT} U_i x_i, & if\ \emptyset \neq T = S, \\ \max_{SuppQ \subseteq SuppU \setminus SuppS} (v(T \vee Q) - \sum_{i \in SuppQ} Q_i x_i), & if\ T \subset S. \end{cases}$$

**Example 4.1.** *In Example 2.1, suppose firm 3 exits, consider the reduced game with respect to* $S = T_1 = (0.6, 0.8, 0)$ *and payoff vector* $x = (0.5, 0.25, 0.4)$. *The reduced game* $v^{\mathcal{T}}_{S,x}$ *is defined as follows:*

For $R = T_1$:

$$v^{\mathcal{T}}_{S,x}(R) = v(T_1) - \sum_{i \in SuppT_1 \setminus SuppR} T_i x_i = 0.5 - 0 = 0.5.$$

For $R = (0.6, 0, 0)$:

$$v^{\mathcal{T}}_{S,x}(R) = \max_{SuppQ \subseteq SuppU \setminus SuppS} \left[ v(R \vee Q) - \sum_{i \in SuppQ} Q_i x_i \right] = v(R) = 0.2.$$

Similarly, for $R = (0, 0.8, 0)$:

$$v^{\mathcal{T}}_{S,x}(R) = 0.2.$$

**Definition 4.2.** *Let* $\Delta_U$ *be a set of cooperative fuzzy games with coalition structures. A solution* $\sigma$ *on* $\Delta_U$ *is a function that maps each fuzzy game with coalition structure* $(U, v, \mathcal{T})$ *to a subset* $\sigma(U, v, \mathcal{T})$ *of the feasible payoff set* $I^*(U, v, \mathcal{T})$*, i.e.,*

$$\sigma(U, v, \mathcal{T}) \subseteq I^*(U, v, \mathcal{T}).$$

For $(U, v, \mathcal{T}) \in \Delta_U$, $x \in \sigma(U, v, \mathcal{T})$, and $S \in L(U, \mathcal{T})$, we denote by $x^S \in \mathbb{R}^N$ the restriction of $x$ to $S$, where $x_i = 0$ for $i \in \text{Supp}\ U \setminus \text{Supp}\ S$.

**Definition 4.3.** *A solution* $\sigma$ *on* $\Delta_U$ *satisfies the* reduced game property (RGP) *if for all* $(U, v, \mathcal{T}) \in \Delta_U$, *all non-empty* $S \subseteq U$, *and all* $x \in \sigma(U, v, \mathcal{T})$, *we have:*

$$(S, v^{\mathcal{T}}_{S,x}, \mathcal{T}_{|S}) \in \Delta_U \quad and \quad x^S \in \sigma(S, v^{\mathcal{T}}_{S,x}, \mathcal{T}_{|S}).$$

RGP ensures that if a payoff vector $x$ is considered a solution in the original game, then its restriction to any subgame $S$ should also be a solution in the corresponding reduced game. This property embodies consistency across different levels of coalitional granularity.

**Lemma 4.1.** *The core on* $\triangle_U^C = \{(U, v, \mathcal{T}) \in \triangle_U \mid C(U, v, \mathcal{T}) \neq \emptyset\}$ *satisfies RGP.*

*Proof.* Assume that $(U, v, \mathcal{T}) \in \triangle_U^C$, $x \in C(U, v, \mathcal{T})$, $T \in \mathcal{T}$, $\emptyset \neq S \subseteq U$, and $R \subseteq T \wedge S$ satisfying $R \neq \emptyset$. If $R = T \wedge S$, then,

$$v^{\mathcal{T}}_{S,x}(R) - \sum_{i \in SuppR} R_i x_i = v(T) - \sum_{i \in SuppT \backslash SuppR} T_i x_i - \sum_{i \in SuppR} R_i x_i$$
$$\geq v(T) - \sum_{i \in SuppT} (T \vee R)_i x_i = v(T) - \sum_{i \in SuppT} T_i x_i = 0.$$

Therefore, $v^{\mathcal{T}}_{S,x}(R) = \sum_{i \in SuppR} R_i x_i$.

If $R \subset T \wedge S$, then

$$v^{\mathcal{T}}_{S,x}(R) - \sum_{i \in SuppR} R_i x_i = \max_{SuppQ \subseteq SuppU \backslash SuppS} (v(R \vee Q) - \sum_{i \in SuppQ} Q_i x_i) - \sum_{i \in SuppR} R_i x_i$$
$$= \max_{SuppQ \subseteq SuppU \backslash SuppS} [v(R \vee Q) - (\sum_{i \in SuppQ} Q_i x_i + \sum_{i \in SuppR} R_i x_i)]$$
$$= \max_{SuppQ \subseteq SuppU \backslash SuppS} [v(R \vee Q) - \sum_{i \in SuppR \cup SuppQ} (R \vee Q)_i x_i] \leq 0.$$

Therefore, $v^{\mathcal{T}}_{S,x}(R) \leq \sum_{i \in SuppR} R_i x_i$. By the definition of core, we can get $x^S \in C(S, v^{\mathcal{T}}_{S,x}, \mathcal{T}_{|S})$. □

In example 4.1, the restricted payoff vector $x^S = (0.5, 0.25, 0)$ is in the core of the reduced game, illustrating the reduced game property (RGP).

For $(U, v, \mathcal{T}) \in \triangle_U$, two distinct players $i, j \in N$ are *partners* in $\mathcal{T}$ if there exists $T \in \mathcal{T}$ such that $T_i \cdot T_j \neq 0$.

We denote

$$\mathcal{P}(\mathcal{T}) = \{S \in \mathbb{R}^{\{i,j\}} | i \neq j, \ i \text{ and } j \text{ are partners in } \mathcal{T}, S_i = T_i, S_j = T_j\}.$$

$$I(U, v, \mathcal{T}) = \{x \in \mathbb{R}^N | \sum_{i \in SuppT} T_i x_i = v(T) \text{ for each } T \in \mathcal{T}\}.$$

**Definition 4.4.** *A solution* $\sigma$ *on* $\triangle_U$ *is said to has the converse reduced game property (CRGP) if the following condition holds: Let* $(U, v, \mathcal{T}) \in \triangle_U, \mathcal{P}(\mathcal{T}) \neq \emptyset, x \in I(U, v, \mathcal{T})$, *if* $(S, v_{S,x}) \in \triangle_U$, *and* $x^S \in \sigma(S, v_{S,x})$ *for each* $S \in \mathcal{P}(\mathcal{T})$, *then* $x \in \sigma(U, v, \mathcal{T})$.

CRGP ensures that if a payoff vector $x$ is locally consistent (i.e., it is a solution in every two-player partner reduced game), then it must also be globally consistent. This prevents situations where a payoff is justified in every bilateral relation but leads to contradictions in the full game.

**Lemma 4.2.** *The core on* $\triangle_U^C$ *satisfies CRGP.*

*Proof.* Let $(U, v, \mathcal{T}) \in \triangle_U^C, \mathcal{P}(\mathcal{T}) \neq \emptyset, y \in I(U, v, \mathcal{T})$ satisfy $y^S \in C(S, v_{S,y})$ for every $S \in \mathcal{P}(\mathcal{T})$. In order to show $y \in C(U, v, \mathcal{T})$, we only prove $\sum\limits_{i \in SuppR} R_i y_i \geq v(R)$ for each $R \in L(U, \mathcal{T})$.

Let $\mathcal{F} = \{\bigvee\limits_{T \in T^*} T | T^* \subseteq \mathcal{T}\}$, and $R = \bigvee\limits_{T \in T^*} T \in \mathcal{F}$. If $x \in C(U, v, \mathcal{T})$, then

$$v(R) \leq \sum_{i \in SuppR} R_i x_i = \sum_{i \in \bigcup\limits_{T \in T^*} SuppT} T_i x_i = \sum_{T \in T^*} \sum_{i \in SuppT} T_i x_i = \sum_{T \in T^*} v(T). \tag{4.1}$$

In view of (4.1), $\sum\limits_{i \in SuppR} R_i y_i \geq v(R)$ for every $T \in \mathcal{F}$.

Now, let $R \in L(U, \mathcal{T}) \backslash \mathcal{F}$, then there exists $T \in \mathcal{T}$ such that $\emptyset \neq R \wedge T \neq T$, let $i \in SuppR \cap SuppT$, $j \in SuppT \backslash SuppR$ and $S \in \mathcal{F}^{\{i,j\}}$ satisfy $S_i = R_i, S_j = R_j$, then $S \in \mathcal{P}(\mathcal{T})$. Therefore, by the assumption, $y^S \in C(S, v_{S,y})$. So,

$$v(R) - \sum_{k \in SuppR} R_k y_k = v(R_i e^i \vee Q) - \sum_{k \in SuppR \backslash SuppS} R_k y_k - \sum_{k \in SuppR \cap SuppS} R_k y_k \leq v_{S,y}(R_i e^i) - R_i y_i \leq 0,$$

where $Q = R - R_i e^i$. Hence, $\sum\limits_{i \in SuppR} R_i y_i \geq v(R)$ and the proof is complete. □

**Definition 4.5.** *Let $(U, v, \mathcal{T})$ be an arbitrary cooperative fuzzy game with coalition structure on $\triangle_U$, if $\sigma(U, v, \mathcal{T}) \neq \emptyset$ for each $(U, v, \mathcal{T}) \in \triangle_U$, then the solution $\sigma$ on $\triangle_U$ is called nonempty.*

**Definition 4.6.** *Let $(U, v, \mathcal{T}) \in \triangle_U$ and $x \in \sigma(U, v, \mathcal{T})$, if $x_i \geq v(e^i)$ for all $i \in N$, then the solution $\sigma$ on $\triangle_U$ is said to satisfy individually rational (IR).*

**Definition 4.7.** *Let $(U, v, \mathcal{T}) \in \triangle_U, x \in \sigma(U, v, \mathcal{T})$, if $(S, v^{\mathcal{T}}_{S,x}, \mathcal{T}_{|S}) \in \triangle_U$ and $x^S \in \sigma(S, v^{\mathcal{T}}_{S,x}, \mathcal{T}_{|S})$ for each $S \in L(U, \mathcal{T})$ and $1 \leq |SuppS| \leq 2$, then the solution $\sigma$ on $\triangle_U$ is said to has the weak reduced game property (WRGP).*

WRGP is a weaker form of RGP that only requires consistency for sub-coalitions of size at most 2. While RGP implies WRGP, the converse is not generally true.

**Definition 4.8.** *Let $(U, v_1, \mathcal{T}), (U, v_2, \mathcal{T})$, and $(U, v_1 + v_2, \mathcal{T})$ are in $\triangle_U$. The solution $\sigma$ on $\triangle_U$ is superadditive (SUPA) if*

$$\sigma(U, v_1, \mathcal{T}) + \sigma(U, v_2, \mathcal{T}) \subseteq \sigma(U, v_1 + v_2, \mathcal{T}),$$

where $\sigma(U, v_1, \mathcal{T}) + \sigma(U, v_2, \mathcal{T}) = \{a + b | a \in \sigma(U, v_1, \mathcal{T}) \textit{ and } b \in \sigma(U, v_2, \mathcal{T})\}$.

Superadditivity captures the idea that combining two games should not reduce the set of acceptable payoffs. The core satisfies superadditivity because the sum of two core allocations remains efficient and coalitionally rational in the combined game.

**Definition 4.9.** *Let $\sigma$ be a solution on $\triangle_U$. If $\sigma(U, v, \mathcal{T}) \subseteq I(U, v, \mathcal{T})$ for each cooperative fuzzy game with coalition structure $(U, v, \mathcal{T}) \in \triangle_U$, then $\sigma$ is Pareto optimal (PO).*

**Lemma 4.3.** *Let $\sigma$ be a solution on $\triangle_U$. If $\sigma$ satisfies IR, and WRGP, then it also satisfies PO.*

*Proof.* Assume that there exist $(U, v, \mathcal{T}) \in \triangle_U$, and $x \in \sigma(U, v, \mathcal{T})$ such that

$$\sum_{i \in SuppT} T_i x_i < v(T)$$

for each $T \in \mathcal{T}$. let $T_i = 1, T \in \mathcal{T}, S = e^i$, by WRGP, we can get $(S, v^{\mathcal{T}}_{S,x}, \mathcal{T}_{|S}) \in \triangle$ and $x_i \in \sigma(S, v^{\mathcal{T}}_{S,x}, \mathcal{T}_{|S})$. By IR, we have $x_i \geq v(e^i)$. On the other hand, by the Definition 4.1, we can obtain

$$v^{\mathcal{T}}_{S,x}(e^i) = v(T) - \sum_{k \in SuppT \setminus \{i\}} T_k x_k > x_i.$$

Thus, it contradicts the property of IR. So, $\sigma$ satisfies PO. □

**Lemma 4.4.** *Let $(U, v, \mathcal{T}) \in \triangle_U$, $\sigma$ be a solution on $\triangle_U$. For each $(U, v, \mathcal{T}) \in \triangle_U$, $\sigma$ satisfies IR and WRGP, then $\sigma(U, v, \mathcal{T}) \subseteq C(U, v, \mathcal{T})$.*

*Proof.* Let $(U, v, \mathcal{T}) \in \triangle_U$ and $n = \max\{|SuppT||T \in \mathcal{T}\}$. Let $n = 1$, $x \in \sigma(U, v, \mathcal{T})$, then $x_i \leq v(e^i)$. By IR, we can get $x_i \geq v(e^i)$. Therefore, $x \in C(U, v, \mathcal{T})$.

By Lemma 4.3, we obtain that $\sigma$ satisfies PO. Therefore, let $n = 2$, then

$$\sigma(U, v, \mathcal{T}) \subseteq \{x \in I(U, v, \mathcal{T})|x_i \geq v(e^i) \text{ for all } i \in SuppU\} = C(U, v, \mathcal{T}).$$

Let $n \geq 3$, and $x \in \sigma(U, v, \mathcal{T})$. By WRGP, we can get that $x^S \in \sigma(S, v^{\mathcal{T}}_{S,x}, \mathcal{T}_{|S})$ for all $S \in \mathcal{P}(\mathcal{T})$. Therefore $x^S \in C(S, v^{\mathcal{T}}_{S,x}, \mathcal{T}_{|S})$ for every $S \in \mathcal{P}(\mathcal{T})$. By the proof of Lemma 4.2, we can get $\sigma(U, v, \mathcal{T}) \subseteq C(U, v, \mathcal{T})$. □

By the proof of Lemma 4.4, we can get the following corollary.

**Corollary 4.1.** *Let $\sigma$ be a solution on $\triangle_U$ that satisfies NE, IR, and WRGP. If the core of a cooperative fuzzy game with coalition structure $(U, v, \mathcal{T})$ consists of a single unique point, then $\sigma(U, v, \mathcal{T}) = C(U, v, \mathcal{T})$ for each $(U, v, \mathcal{T}) \in \triangle_U$.*

**Theorem 4.1.** *The core is the unique solution on $\triangle^C_U$ that satisfies NE, IR, WRGP, and SUPA.*

*Proof.* Clearly, the core on $\triangle^C_U$ satisfies NE, IR, and SUPA. From Lemma 4.1, we can get the core on $\triangle^C_U$ satisfies RGP. RGP implies WRGP, so the core on $\triangle^C_U$ satisfies WRGP. We only prove the uniqueness of the core. Let $(U, v, \mathcal{T}) \in \triangle_U$, $\sigma$ be a solution on $\triangle^C_U$ that satisfies NE, IR,WRGP, SUPA. By the Lemma 4.4, we can get $\sigma(U, v, \mathcal{T}) \subseteq C(U, v, \mathcal{T})$. Therefore, we only show that $C(U, v, \mathcal{T}) \subseteq \sigma(U, v, \mathcal{T})$. Assume that $x \in C(U, v, \mathcal{T})$.

(1) $|SuppU| \geq 3$. We define a fuzzy coalition function $w\colon (U, v, \mathcal{T}) \to \mathbb{R}$ as follows: $w(e^i) = v(e^i)$ for all $i \in SuppU$ and $w(S) = \sum_{i \in SuppS} S_i x_i$ for all $S \in L(U, \mathcal{T})$ with $|SuppS| \neq 1$. Since $|SuppU| \geq 3$, we can obtain $C(U, v, \mathcal{T}) = \{x\}$. Therefore, by Corollary 4.1, we can get $\sigma(U, v, \mathcal{T}) = \{x\}$. Let $t = v - w$. Then, $t(e^i) = 0$ for all $i \in SuppU$, $t(R) = 0$ for all $T \in \mathcal{T}$, and $t(S) = v(S) - w(S) = v(S) - \sum_{i \in SuppS} S_i x_i \leq 0$ for all $S \in L(U, \mathcal{T})$. Thus, $C(U, v, \mathcal{T}) = \{0\}$. By Corollary 4.1, we get $\sigma(U, v, \mathcal{T}) = \{0\}$. By SUPA, we have $\{x\} = \sigma(U, t, \mathcal{T}) + \sigma(U, w, \mathcal{T}) \subseteq \sigma(U, v, \mathcal{T})$. Thus, $x \in \sigma(U, v, \mathcal{T})$, and it implies that $C(U, v, \mathcal{T}) \subseteq \sigma(U, v, \mathcal{T})$.

(2) $|SuppU| \le 2$. If $|SuppU| = 1$, by NE and IR, we can get $x \in \sigma(U, v, \mathcal{T})$ . If $|SuppU| = 2$, then assume that $SuppU = \{i, j\}$, $k \in N\backslash SuppU$, $M = \{i, j, k\}$. We define a fuzzy coalition function $u$ on $\mathcal{F}^M$ as follows:

$$u(S) = \begin{cases} v(T), & if\ |SuppS| = 3, \\ v(S), & if\ |SuppS| = 2\ and\ SuppS = SuppT \\ S_h v(e^h), & otherwise, \end{cases}$$

where $SuppS \cap SuppT \neq \emptyset$, $h \in SuppS \cap SuppT$ and $T \in \mathcal{T}$. We define $y \in \mathbb{R}^M$ by $y^k = 0$ and $y^{SuppU} = x^{SuppU}$, we can get $y \in C(W, u, \mathcal{T}')$ for $W \in \mathcal{F}^M$, $\mathcal{T}'|_U = \mathcal{T}$ and $|SuppW| = 3$. Let $S \in L(W, \mathcal{T}')$. If $|SuppS| = 3$, then $u(S) = v(T) = \sum_{i \in SuppT} T_i x_i = \sum_{i \in SuppT} T_i y_i = \sum_{i \in SuppS} S_i y_i$. If $|SuppS| = 2$ and $SuppS = SuppT$, then $u(S) = v(S) \le \sum_{i \in SuppS} S_i x_i = \sum_{i \in SuppS} S_i y_i$. If $SuppS \neq SuppT$, then $u(S) = S_h v(e^h) \le S_h x(e^h) = \sum_{h \in SuppS} S_h y_h$, where $h \in SuppS \cap SuppT$. So, $y \in C(W, u, \mathcal{T}')$. Since $|SuppW| = 3$, $C(W, u, \mathcal{T}') \subseteq \sigma(W, u, \mathcal{T}')$. Therefore, $y \in \sigma(W, u, \mathcal{T}')$. Again, $(U, u^{\mathcal{T}}_{U,y}, T'_{|U}) = (U, v, \mathcal{T})$. Thus, by WRGP, $x \in \sigma(U, v, \mathcal{T})$, and we conclude that $C(U, v, \mathcal{T}) \subseteq \sigma(U, v, \mathcal{T})$. □

This theorem establishes the core as the unique solution satisfying non-emptiness, individual rationality, weak reduced game property, and superadditivity. It highlights the robustness and structural consistency of the core in fuzzy games with coalition structures.

## 5. D-core

In this section, we define the domination core for a cooperative fuzzy game with coalition structure, based on the definition of domination, and investigate the equivalence between the core and the domination core.

**Definition 5.1.** *Let $(U, v, \mathcal{T})$ be a cooperative fuzzy game with coalition structure, $x, y \in I(U, v, \mathcal{T}), \emptyset \neq S \in L(U, \mathcal{T})$. Then $x$ is said to dominate $y$ via fuzzy coalition $S$, denoted by $x\ dom_S\ y$, if the following conditions hold:*

(1) $x_i > y_i$ for all $i \in SuppS$;
(2) $\sum_{i \in SuppS} S_i x_i \le v(S)$.

This definition captures the idea that a payoff vector $x$ can dominate another vector $y$ through a coalition $S$ if all members of $S$ strictly prefer $x$ over $y$, and the total payoff allocated to $S$ under $x$ does not exceed the value of the coalition. Domination reflects both individual incentives and coalitional feasibility.

**Definition 5.2.** *The domination core (D-core) of a cooperative fuzzy game with coalition structure $(U, v, \mathcal{T})$ is the set*

$$DC(U, v, \mathcal{T}) = I(U, v, \mathcal{T}) \backslash \bigcup_{\emptyset \neq S \in L(U, \mathcal{T})} D(S), \tag{5.1}$$

where $D(S) = \{y \in I(U, v, \mathcal{T}) | there\ exists\ x \in I(U, v, \mathcal{T})\ with\ x\ dom_S\ y\}$, i.e., the set of all undominated elements in $I(U, v, \mathcal{T})$.

The D-core represents the set of stable payoff distributions that cannot be improved upon by any coalition through a feasible and unanimously preferred alternative. It is a solution concept based on pairwise comparisons and veto power of coalitions.

**Example 5.1.** *Consider the cooperative fuzzy game with coalition structure from Example 2.1. We examine whether the payoff vector* $x = (0.5, 0.25, 0.4) \in I(U, v, \mathcal{T})$ *belongs to the domination core. To establish* $x \in DC(U, v, \mathcal{T})$*, we must verify that no allocation* $y \in I(U, v, \mathcal{T})$ *dominates* $x$ *via any non-empty feasible coalition* $S \in L(U, \mathcal{T})$*.*

First, consider $S = T_1 = (0.6, 0.8, 0)$. For $y$ to dominate $x$ via $S$, we require $y_1 > 0.5$, $y_2 > 0.25$, and $0.6y_1 + 0.8y_2 \leq v(T_1) = 0.5$. However, the allocation constraints impose $0.6y_1 + 0.8y_2 = 0.5$. The conditions $y_1 > 0.5$ and $y_2 > 0.25$ would imply $0.6y_1 + 0.8y_2 > 0.6 \times 0.5 + 0.8 \times 0.25 = 0.5$, contradicting the necessary equality. Thus, no such $y$ exists.

Next, take $S = T_2 = (0, 0, 1.0)$. Domination requires $y_3 > 0.4$ and $y_3 \leq v(T_2) = 0.4$. The allocation constraint fixes $y_3 = 0.4$, making the inequality $y_3 > 0.4$ impossible. Hence, domination cannot occur.

Now examine $S = (0.6, 0, 0)$, where $v(S) = 0.2$. Domination requires $y_1 > 0.5$ and $0.6y_1 \leq 0.2$, implying $y_1 \leq 0.2/0.6 \approx 0.333$. This contradicts $y_1 > 0.5$, excluding any dominating $y$.

Consider $S = (0, 0.8, 0)$, with $v(S) = 0.2$. Domination requires $y_2 > 0.25$ and $0.8y_2 \leq 0.2$, forcing $y_2 \leq 0.25$. This contradicts $y_2 > 0.25$, making domination impossible.

For $S = U = (0.6, 0.8, 1.0)$, domination requires $y_1 > 0.5$, $y_2 > 0.25$, $y_3 > 0.4$, and $0.6y_1 + 0.8y_2 + 1.0y_3 \leq v(U) = 0.7$. The allocation constraint $y_3 = 0.4$ violates $y_3 > 0.4$, precluding domination.

Finally, for any coalition $S$ with $v(S) = 0$, the condition $\sum_{i \in \mathrm{Supp}\, S} S_i y_i \leq 0$ cannot be satisfied while maintaining $y_i > x_i \geq 0$. Thus, domination is impossible through these coalitions.

Since no feasible coalition $S$ permits domination of $x$ by any $y \in I(U, v, \mathcal{T})$, we conclude that $x \in DC(U, v, \mathcal{T})$. This demonstrates the stability of the D-core: the allocation $x$ cannot be improved upon by any coalition through a mutually preferred and feasible alternative, confirming its stability within the given coalition structure.

**Theorem 5.1.** *Let* $(U, v, \mathcal{T})$ *be a cooperative fuzzy game with coalition structure satisfying* $v(T) \geq v(S) + \sum_{i \in SuppT \setminus SuppS} T_i v(e^i)$ *for all* $T \in \mathcal{T}$ *and* $S_i = T_i$ *for* $i \in SuppS \cap SuppT$*. Then,*

$$DC(U, v, \mathcal{T}) = C(U, v, \mathcal{T}). \tag{5.2}$$

*Proof.* Assume that $x \in C(U, v, \mathcal{T})$, then $x \in I(U, v, \mathcal{T})$. Let $x \notin DC(U, v, \mathcal{T})$. Then, there exist a $y \in I(U, v, \mathcal{T})$ and a fuzzy coalition $\emptyset \neq S \in L(U, \mathcal{T})$ such that $v(S) \geq \sum_{i \in SuppS} S_i y_i$ and $y\ dom_S\ x$, i.e., $y_i > x_i$ for all $i \in SuppS$. Therefore, $v(S) \geq \sum_{i \in SuppS} S_i y_i > \sum_{i \in SuppS} S_i x_i$, which implies that $x \notin C(U, v, \mathcal{T})$.

Assume that $x \in DC(U, v, \mathcal{T})$, then $x \in I(U, v, \mathcal{T})$, thus $v(T) = \sum_{i \in SuppT} T_i x_i$. Asumme that there exists a fuzzy coalition $S \in L(U, \mathcal{T})$ satisfying $SuppS \subseteq SuppT$ and $S_i = T_i$ for $i \in SuppS, T \in \mathcal{T}$ such that $\sum_{i \in SuppS} S_i x_i < v(S)$. Let

$$\epsilon = \frac{v(S) - \sum_{i \in SuppS} S_i x_i}{\sum_{i \in SuppT \cap SuppS} T_i} > 0, \alpha = \frac{1}{\sum_{i \in SuppT \setminus SuppS} T_i}(v(T) - v(S) - \sum_{i \in SuppT \setminus SuppS} T_i v(e^i)) \geq 0.$$

We define $y$ as follows:

$$y_i = \begin{cases} x_i + \epsilon, & if\ i \in SuppS, \\ v(e^i) + \alpha, & if\ i \notin SuppS. \end{cases}$$

Then,

$$\begin{aligned} \sum_{i \in SuppT} T_i y_i &= \sum_{i \in SuppS} T_i(x_i + \epsilon) + \sum_{i \in SuppT \setminus SuppS} T_i(v(e^i) + \alpha) \\ &= \sum_{i \in SuppS} T_i x_i + \epsilon \sum_{i \in SuppS} T_i + \sum_{i \in SuppT \setminus SuppS} T_i v(e^i) + \alpha \sum_{i \in SuppT \setminus SuppS} T_i \alpha \\ &= \sum_{i \in SuppS} T_i x_i + \frac{v(S) - \sum_{i \in SuppS} S_i x_i}{\sum_{i \in SuppS} T_i} \sum_{i \in SuppT \cap SuppS} T_i + \\ &\quad \frac{1}{\sum_{i \in SuppT \setminus SuppS} T_i}(v(T) - v(S) - \sum_{i \in SuppT \setminus SuppS} T_i v(e^i)) \sum_{i \in SuppT \setminus SuppS} T_i \end{aligned}$$

$$= \sum_{i \in SuppS} T_i x_i + v(S) - \sum_{i \in SuppS} T_i x_i + \sum_{i \in SuppT \setminus SuppS} T_i v(e^i) + v(T) - v(S) - \sum_{i \in SuppT \setminus SuppS} T_i v(e^i) = v(T). \quad \square$$

Thus, $y \in I(U, v, \mathcal{T})$, and $y\ dom_S\ x$, which implies $x \notin DC(U, v, \mathcal{T})$.

This theorem establishes that under the given condition—which ensures that the value of any coalition $T$ is sufficiently large to cover the value of a sub-coalition $S$ and the individual values of the remaining players—the core and the domination core are identical. This result generalizes the classical equivalence between the core and the domination core to the fuzzy coalition structure setting, highlighting the stability and rationality of the core under well-defined value constraints.

## 6. Conclusions

This paper has systematically investigated the core and domination core of cooperative fuzzy games with coalition structures, extending the classical framework of the Aubin core to an environment involving both graded membership and pre-defined coalitional organization. The main contributions and findings of this research are summarized as follows: First, we introduced a formal model for cooperative fuzzy games incorporating coalition structures and proposed a generalized core concept within this setting. This model effectively captures settings where players exhibit partial participation and are constrained by a coalition structure, thereby offering a more realistic representation of many practical cooperative decision-making scenarios under uncertainty. Second, we constructed the superadditive cover for such games and established key equivalence conditions under which the core remains non-empty. It was shown that when the core is non-empty, it coincides with the core of the superadditive cover, providing a significant bridge between fuzzy games and their crisp counterparts. Third, an axiomatic characterization of the core was presented. We proved that the core is the unique solution concept satisfying the properties of non-emptiness, individual rationality, weak reduced game property, and superadditivity. This characterization not only clarifies the structural essence of the core but also offers a solid theoretical foundation for stability analysis in fuzzy coalitional games. Finally, a domination-core (D-core) was proposed based on a dominance relation among payoff vectors. Under certain reasonable conditions, the D-core was shown to be equivalent to the conventional core, further reinforcing the robustness and soundness of the core concept in fuzzy environments with coalition constraints.

Despite these theoretical advances, there remain several promising directions for further research. First, the current model is restricted to fuzzy games with real-valued payoffs. Extending the framework to more general uncertainty representations—such as interval-valued, intuitionistic, or spherical fuzzy payoffs—would enhance its applicability to complex decision-making environments. Such extensions would allow the model to integrate richer forms of ambiguity and linguistic information, as seen in recent ELECTRE-based and aggregation-operator-driven approaches (e.g., spherical fuzzy ELECTRE [23] and related fuzzy linguistic operators [24]). Second, the present study assumes a static and fully known coalition structure. Relaxing this assumption to incorporate dynamic, evolving, or adaptive coalition formations represents another meaningful direction. This would enable the model to better capture real-world settings where interaction patterns change over time, such as knowledge graph completion [25], multi-agent systems [26], and renewable energy planning. Developing a framework that accommodates such dynamic structures would significantly broaden the applicability of cooperative fuzzy games with coalition constraints.

In summary, this study enriches the theory of cooperative fuzzy games with coalition structures and offers a solid basis for further theoretical exploration and practical applications in multi-agent decision-making under fuzziness and uncertainty.

## Author contributions

Fengye Wang: Conceptualization, methodology, formal analysis, writing–original draft, writing–review & editing, project administration; Yuanyuan Huang: Methodology, formal analysis, writing–review & editing; Youlin Shang: Formal analysis, supervision, writing–review & editing; Zhihua Zheng: Validation, writing–review & editing. All the authors of this article have read and approved the final version of the manuscript for publication.

## Use of Generative-AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

## Acknowledgments

This work is supported by National Natural Science Foundation of China (Grants No. 12071112) and Key Science and Technology Research Project of Henan Province of China (Grants No. 252102240125).

## Conflict of interest

The authors declare that they have no conflicts of interest regarding this research.

## References

1. B. Peleg, An axiomatization of the core of cooperative games without side payments, *J. Math. Econ.*, **14** (1985), 203–214. http://doi.org/10.1016/0304-4068(85)90020-5
2. K. Tadenuma, Reduced games, consistency, and the core, *Int. J. Game Theory*, **20** (1992), 325–334. http://doi.org/10.1007/BF01271129

3. R. Serrano, O. Volij, Axiomatizations of neoclassical concepts for economies, *J. Math. Econ.*, **30** (1998), 87–108. http://doi.org/10.1016/S0304-4068(97)00025-6

4. Y. A. Hwang, P. Sudhölter, Axiomatizations of the core on the universal domain and other natural domains, *Int. J. Game Theory*, **29** (2001), 597–623. http://doi.org/10.1007/s001820100060

5. J. P. Aubin, Cooperative fuzzy games, *Math. Oper. Res.*, **6** (1981), 1–13. https://doi.org/10.1287/moor.6.1.1

6. J. P. Aubin, *Mathematical methods of game and economic theory*, 2 Eds., Amsterdam: North-Holland, 1982.

7. M. Tsurumi, T. Tanino, M. Inuiguchi, The core and the related solution concepts in cooperative fuzzy games, *Journal of Japan Society for Fuzzy Theory and Systems*, **12** (2000), 193–202. https://doi.org/10.3156/jfuzzy.12.1_193

8. X. H. Yu, Q. Zhang, The fuzzy core in games with fuzzy coalitions, *J. Comput. Appl. Math.*, **230** (2009), 173–186. http://doi.org/10.1016/j.cam.2008.11.004

9. Y. A. Hwang, Fuzzy games: A characterization of the core, *Fuzzy Set. Syst.*, **158** (2007), 2480–2493. http://doi.org/10.1016/j.fss.2007.03.009

10. S. Muto, S. Ishihara, E. Fukuda, S. H. Tijs, R. Branzei, Generalized cores and stable sets for fuzzy games, *Int. Game Theory Rev.*, **8** (2006), 95–109. http://doi.org/10.1142/s0219198906000801

11. V. A. Vasil'Ev, On the core of the Aubin extension of an almost positive game, *Sib. Adv. Math.*, **35** (2025), 79–86. http://doi.org/10.1134/S1055134425010067

12. X. H. Yu, Fuzzy cooperative game with intersecting priori coalition: A generalized configuration value, *Fuzzy Set. Syst.*, **471** (2023), 108684. http://doi.org/10.1016/j.fss.2023.108684

13. V. B. Vilkov, A. I. Dergachev, A. K. Chernykh, M. S. Abu-Khasan, On the concept of solving a fuzzy cooperative game with side payments, *2020 International Multi-Conference on Industrial Engineering and Modern Technologies (FarEastCon)*, Vladivostok, Russia, 2020, 1–8. http://doi.org/10.1109/FarEastCon50210.2020.9271558

14. R. J. Aumann, J. H. Drèze, Cooperative games with coalition structures, *Int. J. Game Theory*, **3** (1974), 217–237. http://doi.org/10.1007/BF01766876

15. G. Owen, Values of games with a priori unions, In: *Mathematical economics and game theory*, Heidelberg: Springer, 1977, 76–88. https://doi.org/10.1007/978-3-642-45494-3_7

16. G. Owen, Characterization of the Banzhaf-Coleman index, *SIAM J. Appl. Math.*, **35** (1978), 315–327. http://doi.org/10.1137/0135026

17. X. H. Yu, M. K. He , H. X. Sun, Z. Zhou, Uncertain coalition structure game with payoff of belief structure, *Appl. Math. Comput.*, **372** (2020), 125000. https://doi.org/10.1016/j.amc.2019.125000

18. H. X. Sun, Q. Zhang, An axiomatization of probabilistic Owen value for games with coalition structure, *Acta Math. Appl. Sin. Engl. Ser.*, **30** (2014), 571–582. http://doi.org/10.1007/s10255-014-0403-y

19. X. H. Yu, Extension of owen value for the game with a coalition structure under the limited feasible coalition, *Soft Comput.*, **25** (2021), 6139–6156. http://doi.org/10.1007/s00500-021-05604-9

20. F. Y. Meng, Q. Zhang, Y. Wang, Cooperative fuzzy games with a coalition structure and interval payoffs, *Int. J. Comput. Intell. Syst.*, **6** (2013), 548–558. http://doi.org/10.1080/18756891.2013.795395
21. M. Maschler, B. Peleg, The structure of the kernel of a cooperative game, *SIAM J. Appl. Math.*, **15** (1967), 569–604. https://doi.org/10.1137/0115050
22. B. Peleg, P. Sudhölter, *Introduction to the theory of cooperative games*, 2 Eds., Berlin Heidelberg: Springer, 2007. http://doi.org/10.1007/978-3-540-72945-7
23. M. Akram, F. Ilyas, M. Deveci, Interval rough integrated SWARA-ELECTRE model: An application to machine tool remanufacturing, *Expert Syst. Appl.*, **238** (2024), 122067. https://doi.org/10.1016/j.eswa.2023.122067
24. L. Y. Zhou, S. Abdullah, H. Zafar, S. Muhammad, A. Qadir, H. S. Huang, Analysis of artificial neural network based on pq-rung Orthopair fuzzy linguistic muirhead mean operators, *Expert Syst. Appl.*, **276** (2025), 127157. https://doi.org/10.1016/j.eswa.2025.127157
25. Z. F. Li, W. Huang, X. C. Gong, X. Y. Luo, K. Xiao, H. L. Deng, et al., Decoupled semantic graph neural network for knowledge graph embedding, *Neurocomputing*, **611** (2025), 128614. https://doi.org/10.1016/j.neucom.2024.128614
26. J. Sun, G. P. Wang, J. Yun, L. Liu, Q. H. Shan, J. X. Zhang, Cooperative output regulation of multi-agent systems via energy-dependent intermittent event-triggered compensator approach, *J. Franklin I.*, **362** (2025), 107868. https://doi.org/10.1016/j.jfranklin.2025.107868