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Research article

Certain classifications on quadrics in simply isotropic space $\mathbb{I}^3$

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Abstract: In this paper, we deal with the natural geometry of quadric surfaces in the isotropic space $\mathbb{I}^3$. We provided defining equations for the $3^{rd}, 4^{th}$ type ruled surfaces and $1^{st}, 2^{nd}$ type of quadric surfaces. In this paper, we investigated the finite Chen-type of ruled and quadric surfaces in the 3-dimensional simply isotropic space $\mathbb{I}^3$, corresponding to the third fundamental form of the surface.

Keywords: surfaces in the simply isotropic 3-space; surfaces of finite Chen-type; Beltrami operator; quadric surfaces; ruled surfaces
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1. Introduction

Immersions of finite type have been developed by B. Y. Chen in the 1970s. Fortunately, the geometry of spaces equipped with a degraded metric has attracted increasing attention from pure and applied mathematical perspectives. Many results in this area have been collected in [1].

Let S be a connected r-dimensional submanifold in the n-dimensional Euclidean space E^m. Denote by $\triangle^J$ the Laplace operator with respect to the fundamental form $J = I, II, III$ [2]. An isometric immersion $\boldsymbol{X} : S \to E^m$ is said to be of k-type if the position vector $\boldsymbol{X}$ of S can be written as a finite sum of nonconstant eigenvectors of the Laplacian $\triangle^J$, that is

$$\boldsymbol{X} = \boldsymbol{X_0} + \sum_{j=1}^{k} \boldsymbol{X}_j, \qquad \triangle^J \boldsymbol{X}_j = \zeta_i \boldsymbol{X}_j, j = 1, \dots, k, \tag{1.1}$$

where X_0 is a fixed vector and $\boldsymbol{X}_j, (j = 1, 2, ..., k)$ are non-constant E^m-valued functions on M^r [3].

When S is of finite type k, then from (1.1), there exists a polynomial, $h(x) \neq 0$, such that $h(\Delta^J)(\boldsymbol{x}) = \boldsymbol{0}$. If $h(x) = x^{k+1} + \mu_1 x^k + ... + \mu_{k-1} x^2 + \mu_k x$, then coefficients μ_i are given by

$$\begin{array}{rcl} \mu_1 & = & -(\zeta_1+\zeta_2+...+\zeta_k), \\ \mu_2 & = & (\zeta_1\zeta_2+\zeta_1\zeta_3+...+\zeta_1\zeta_k+\zeta_2\zeta_3+...+\zeta_2\zeta_k+...+\zeta_{k-1}\zeta_k), \\ \mu_3 & = & -(\zeta_1\zeta_2\zeta_3+\zeta_1\zeta_2\zeta_4+...+\zeta_1\zeta_2\zeta_k+...+\zeta_{k-2}\zeta_{k-1}\zeta_k), \\ \vdots & & \\ \mu_k & = & (-1)^k\zeta_1\zeta_2...\zeta_k. \end{array}$$

Thus, position vector $\boldsymbol{X}$ satisfies the following relation (see [4])

$$(\triangle^J)^{k+1}\boldsymbol{X}+\mu_1(\triangle^J)^k\boldsymbol{X}+...+\mu_k\triangle^J\boldsymbol{X}=\boldsymbol{0}, \quad J=I,II,III. \tag{1.2}$$

2. Preliminaries

To commence, we would like to give a concise overview encompassing fundamental definitions, essential facts, and relations in the theory of surfaces in $\mathbb{I}^3$ (see [5]).

Space $\mathbb{I}^3$ corresponds to the classic real vector space $\mathbb{R}^3$ with coordinates (X_1, X_2, X_3) equipped with the degenerate metric

$$<\boldsymbol{n},\boldsymbol{m}>=n_1m_1+n_2m_2,$$

where $\boldsymbol{n}=(n_1,n_2,n_3)$ and $\boldsymbol{m}=(m_1,m_2,m_3)$. Let $\boldsymbol{n}\neq\boldsymbol{0}$. Then $\boldsymbol{n}$ is called isotropic if $<\boldsymbol{n},\boldsymbol{n}>=0$. In addition, if $\boldsymbol{n}\in\mathbb{I}^3:\boldsymbol{n}=(0,0,n_3)$, we use the calculation

$$<\boldsymbol{n},\boldsymbol{m}>=n_3m_3.$$

K. Strubecker [6], D. Palman [7], and H. Sachs [8–10] were the first to establish the concept of differential geometry of isotropic spaces. A thorough bibliography of isotropic planes and isotropic 3-spaces is available to the reader in [9,10]. An isotropic space, as defined by K. Strubecker, is a three-dimensional Cartesian space (X_1, X_2, X_3), whose square of the line element (arc element) is defined by the quadratic differential form of rank two:

$$ds^2=dX_1^2+dX_2^2 \tag{2.1}$$

where, as it is known, the arc element ds measures how much the length of the curve changes when the independent parameter changes.

In this space, the lengths and angles on curves and surfaces can be directly derived from the normal projection of these figures onto the X_1X_2-plane, known as the ground plan.

The affine transformations that leave the arc element square (2.1) invariant form a seven-parameter group G_7. Now, two points (X_1, X_2, X_3) and (X_1, X_2, X_3') whose ground plans coincide, have, according to (2.1), a vanishing distance. Likewise, two lines $X_3 = aX_2, X_1 = 0$ and $X_3 = bX_2, X_1 = 0$, whose ground plans coincide, have, according to (2.1), a vanishing inclination angle. We call $s = X_3' - X_3$ the span between the points, and $\sigma = b - a$ the deviation (or separation) of the lines. Those affinities that leave invariant not only the arc element square but also the span between two points (or the deviation between two lines) form a six-parameter subgroup of G_7, namely:

$$X_1' = a + X_1 cos\phi - X_2 sin\phi,$$

$$X_2' = b + X_1 sin\phi + X_2 cos\phi,$$

$$X_3' = c + dX_1 + eX_2 + X_3,$$

where $\phi, a, b, c, d, e \in \mathbb{R}$.

Given two vectors $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{v} = (v_1, v_2, v_3)$, the isotropic inner product is given by

$$< \boldsymbol{u}, \boldsymbol{v} >_{\mathbb{I}^3} = u_1 v_1 + u_2 v_2.$$

The isotropic distance between two points $Q_1 = (X_1, Y_1, Z_1)$ and $Q_2 = (X_2, Y_2, Z_2)$ is defined by

$$d(Q_1, Q_2) = \sqrt{(X_1 - X_2)^2 + (Y_1 - Y_2)^2}.$$

If two points have the same top views, then they are said to be parallel. Therefore, the position vectors of these points are said to be isotropic vectors (i.e., parallel to the z-axis). The isotropic inner product between two isotropic vectors vanishes identically. In this case, we introduce the isotropic co-distance $co - d((a, b, X_3), (a, b, Y_3)) = |X_3 - Y_3|$.

Two situations need to be distinguished when working with surfaces S in isotropic geometry. When the metric in S induced by the isotropic scalar product has rank 2, we say that S is an admissible surface. If S is parameterized by a C^2 map $\boldsymbol{X}(u^1, u^2) = (X_1(u^1, u^2), X_2(u^1, u^2), X_3(u^1, u^2))$, then it is admissible if and only if $rank(C) \neq 0$, where

$$C = \begin{bmatrix} X_1^1 & X_1^2 & X_1^3 \\ X_2^1 & X_2^2 & X_2^3 \end{bmatrix},$$

and $X_i^k = \frac{\partial X_i}{\partial u^k}$.

Consequently, every admissible C^2 surface S can be locally parameterized as $\boldsymbol{X}(u^1, u^2) = (u^1, u^2, g(u^1, u^2))$, and we say that S is in its regular form.

3. Quadric surfaces in $\mathbb{I}^3$

Let S be a non-degenerate quadric surface in the isotropic 3-space $\mathbb{I}^3$. Then S is either a ruled surface or one of the types listed below [11]:

$$Z^2 - A\,X^2 - B\,Y^2 = C, \tag{3.1}$$

where $A, B, C \in \mathbb{R}, \quad A\,B \neq 0, \quad C > 0$, or

$$Z = \frac{A}{2}\,X^2 + \frac{B}{2}\,Y^2, \tag{3.2}$$

where $A, B \in \mathbb{R}, \quad A, B > 0$.

Let v^1, v^2 be a local coordinate system of S. Let e_{km} be the matrix of the components of the non-degenerate third fundamental form III of S. We denote by e^{km} the inverse matrix of e_{km}. The second differential parameter of Beltrami corresponding to the fundamental form III of S is defined by [12,13]

$$\triangle^{III} p := -\frac{1}{\sqrt{e}}(\sqrt{e}e^{km} p_{/m})_{/k}, \tag{3.3}$$

where $p_{/m} := \frac{\partial p}{\partial v^m}$,and $e = \det(e_{km})$. For simplicity, we use the symbol $\triangle$ instead of $\triangle^{III}$.

3.1. Ruled surfaces in $\mathbb{I}^3$

Let S be ruled surface in $\mathbb{I}^3$ given by the parametrization [14]

$$\boldsymbol{X}(s,t) = \boldsymbol{\alpha}(s) + t\boldsymbol{\beta}(s), \quad (s,t)\epsilon I \times R \longrightarrow \mathbb{I}^3.$$

We refer to the base curve $\boldsymbol{\alpha}$ and the director curve $\boldsymbol{\beta}$. The former is a differentiable curve parametrized by its arc length, meaning that $< \boldsymbol{\alpha}', \boldsymbol{\alpha}' >_{\mathbb{I}^3} = 1$ and $< \boldsymbol{\beta}, \boldsymbol{\beta} >_{\mathbb{I}^3} = 1$. The director curve $\boldsymbol{\beta}$ is orthogonal to the tangent vector field T_α of the base curve $\boldsymbol{\alpha}$, i.e., $< \boldsymbol{\beta}', T_\alpha >_{\mathbb{I}^3} = 0$. First, we consider non isotropic plane curves $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ parametrized by $\boldsymbol{\alpha}(s) = (s, 0, f(s))$ and $\boldsymbol{\beta}(s) = (0, 1, g(s))$. Then the surface S is parametrized by

$$\boldsymbol{X}(s,t) = (s, t, f(s) + tg(s)). \tag{3.4}$$

The matrices of the coefficients of the first and second fundamental forms are, respectively [15]

$$(g_{ij}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and

$$(b_{ij}) = \begin{bmatrix} 0 & g' \\ g' & f'' + tg'' \end{bmatrix}.$$

We have $G = Det\left(g_{ij}\right) = 1$, and $B = Det\left(b_{ij}\right) = -g'^2$. The Gaussian and the mean curvature are respectively

$$K_{\mathbb{I}^3} = -g'^2, \quad 2H_{\mathbb{I}^3} = f'' + tg''. \tag{3.5}$$

From (3.5), it can be verified that

Proposition 1. *[15] In the simply isotropic 3-space* $\mathbb{I}^3$*, the ruled surfaces given by (3.4) are isotropic flat or developable* ($K_{\mathbb{I}^3} = 0$)*, iff* $g(s) = c$ *for constant* c*.*

From (3.5), $H_{\mathbb{I}^3} = 0$ if $f'' + tg'' = 0$, which is a linear equation regarding parameter t. Since it valids for all t, we must have $f'' = 0$, and $g'' = 0$. By integrating these two equations, we have $f = as + b$ and $g = cs + d$. Thus, we have the following:

Proposition 2. *The ruled surfaces given by (3.4) in the simply isotropic 3-space* $\mathbb{I}^3$ *are isotropic minimals* ($H_{\mathbb{I}^3} = 0$)*, iff* $g(s)$ *and* $f(s)$ *are polynomials in* s *of degree at most 1.*

Let $\boldsymbol{\alpha} = (0, 0, f(s))$ be an isotropic curve, and $\boldsymbol{\beta}$ parametrized by $\boldsymbol{\beta} = (\cos s, \sin s, g(s))$ be a non isotropic space curve, where $< \boldsymbol{\beta}, \boldsymbol{\beta} >_{\mathbb{I}^3} = 1$. Then a parametrization of S is given by

$$\boldsymbol{X}(s,t) = (t\cos s, t\sin s, f(s) + tg(s)). \tag{3.6}$$

The coefficients of the first and second fundamental forms are

$$(g_{ij}) = \begin{bmatrix} t^2 & 0 \\ 0 & 1 \end{bmatrix},$$

and

$$(b_{ij}) = \begin{bmatrix} -f'' - tg'' - tg & g' \\ g' & 0 \end{bmatrix}.$$

The Gaussian and the mean curvature are respectively.

$$K_{\mathbb{I}^3} = -\frac{g'^2}{t^2}, \quad 2H_{\mathbb{I}^3} = -\frac{f'' + tg'' + tg}{t^2}. \tag{3.7}$$

From (3.7) we have

Proposition 3. *[15] The ruled surfaces defined by (3.6) in the simply isotropic 3-space $\mathbb{I}^3$ are isotropic flat or developable ($K_{\mathbb{I}^3} = 0$), iff $g(s) = c$ for constant c.*

Let the ruled surface given by (3.6) be isotropic minimal. Then from (3.7), we must have

$$f'' + t(g'' + g) = 0.$$

Since the above equation holds for all values of t, we must have $f'' = 0$, and $g'' + g = 0$. Solving these two differential equations, we finally have

Proposition 4. *The Ruled surfaces defined by (3.6) in the three dimensional simply isotropic space $\mathbb{I}^3$ are isotropic minimals ($H_{\mathbb{I}^3} = 0$), iff $g(s) = c_1 coss + c_2 sins$ and $f(s)$ is a polynomial in s of degree at most 1.*

Let f and g be smooth functions of a single variable. We refer to the surfaces given by (3.4) and (3.6) as ruled surfaces of Type 3 and Type 4 in the simply isotropic 3-space $\mathbb{I}^3$, respectively.

3.2. Ruled surfaces in $\mathbb{I}^3$ of Type 3

Recalling the parametrization (3.4) of ruled surfaces, then on account of (3.5), the matrix of the coefficients of the third fundamental form is

$$(e_{ij}) = \begin{bmatrix} g'^2 & g'(f'' + tg'') \\ g'(f'' + tg'') & g'^2 + (f'' + tg'')^2 \end{bmatrix}. \tag{3.8}$$

Therefore, from (3.3) and (3.8), we get [12]

$$\triangle \;=\; \frac{1}{g'^2}\Big[-\Big(1 + (\frac{f'' + tg''}{g'})^2\Big)\frac{\partial^2}{\partial s^2} + (\frac{f'' + tg''}{g'})\frac{\partial^2}{\partial s \partial t} - \frac{\partial^2}{\partial t^2}$$

$$-\Big(1+(\frac{f''+tg''}{g'})^2\Big)_s\frac{\partial}{\partial s}+\frac{g''}{g'}\frac{\partial}{\partial s}+(\frac{f''+tg''}{g'})_s\frac{\partial}{\partial t}\Big]. \quad (3.9)$$

We classify the non-developable ruled surfaces of Type 3, that is, $g'(s) \neq 0$. Applying (3.9) for the components of (3.4), we obtain

$$\triangle s = \frac{1}{g'^2}\Big[-\Big(1+(\frac{f''+tg''}{g'})^2\Big)_s+\frac{g''}{g'}\Big],$$

$$\triangle t = \frac{1}{g'^2}\Big[(\frac{f''+tg''}{g'})_s\Big],$$

$$\begin{aligned}\triangle(f(s)+tg(s)) &= \frac{1}{g'^2}\Big[-\frac{(f''+tg'')^3}{g'^2}-2(f'+tg')(\frac{f''+tg''}{g'})(\frac{f''+tg''}{g'})_s \\ &+g(\frac{f''+tg''}{g'})_s+\frac{f'g''}{g'}+tg''+f'+tg'\Big].\end{aligned}$$

We write the last three equations as polynomials of t with coefficients, functions of s, as follows

$$\triangle s = A_{01}(s)+A_{11}(s)t+A_{21}(s)t^2,$$

$$\triangle t = B_{01}(s)+B_{11}(s)t,$$

$$\triangle(f(s)+tg(s)) = C_{01}(s)+C_{11}(s)t+C_{21}(s)t^2+C_{31}(s)t^3.$$

Lemma 1. *Let Q be a polynomial in t with functions in s as coefficients and $deg(Q)=n$. Then $\triangle Q = \hat{Q}$, where $\hat{Q}$ is a polynomial in t with functions in s as coefficients and $deg(\hat{Q}) \leq n+2$.*

Remark 1. *The proof of Lemma 1 can be obtained from the coefficient function of the first term and the fourth term of the relation (3.9), which includes the variable t to the power 2.*

From the above lemma, one can obtain that

$$\triangle^2 s = A_{02}(s)+A_{12}(s)t+A_{22}(s)t^2+A_{32}(s)t^3+A_{42}(s)t^4,$$

$$\triangle^2 t = B_{02}(s)+B_{12}(s)t+B_{22}(s)t^2+B_{32}(s)t^3,$$

$$\triangle^2(f(s)+tg(s)) = C_{02}(s)+C_{12}(s)t+C_{22}(s)t^2+C_{32}(s)t^3+C_{42}(s)t^4+C_{52}(s)t^5.$$

In general, we have

$$\triangle^k s = A_{0k}(s)+A_{1k}(s)t+\ldots+A_{k+2,k}(s)t^{k+2},$$

$$\triangle^k t = B_{0k}(s)+B_{1k}(s)t+\ldots+B_{k+1,k}(s)t^{k+1},$$

$$\triangle^k(f(s)+tg(s)) = C_{0k}(s) + C_{1k}(s)t + \ldots + C_{k+3,k}(s)t^{k+3}.$$

We suppose that S is of finite III-type k. Then from (2.1), there exist real numbers $\mu_1, \ldots, \mu_k$, such that

$$\triangle^{k+1}\boldsymbol{X} + \mu_1\triangle^k\boldsymbol{X} + \ldots + \mu_k\triangle\boldsymbol{X} = \boldsymbol{0}. \tag{3.10}$$

Applying the above equation to each component of $\boldsymbol{X}$, we get

$$\triangle^{k+1}s + \mu_1\triangle^k s + \ldots + \mu_k\Delta s = 0, \tag{3.11}$$

$$\triangle^{k+1}t + \mu_1\triangle^k t + \ldots + \mu_k\Delta t = 0. \tag{3.12}$$

$$\triangle^{k+1}(f+tg) + \mu_1\triangle^k(f+tg) + \ldots + \mu_k\triangle(f+tg) = 0. \tag{3.13}$$

By applying Lemma 1, we conclude that there is a polynomial P_{1k} in the variable t with some functions in s as coefficients, such that

$$\triangle^k s = P_{1k}, \quad deg(P_{1k}) \leq 2k.$$

Similarly, for the second and third components of $\boldsymbol{x}$, there are polynomials P_{2k}, P_{3k} in the variable t with some functions in s as coefficients, such that

$$\triangle^k t = P_{2k}, \quad deg(P_{2k}) \leq 2k-1.$$

$$\triangle^k(f+tg) = P_{3k}, \quad deg(P_{3k}) \leq 2k+1.$$

Now, if k goes up by one, the degree of the components P_{1k}, P_{2k}, and P_{3k} goes up at most by 2. Hence, the sums (3.11)–(3.13) can never be zeros, unless, of course, we have [16]

$$\triangle s = \frac{1}{g'^2}\left[-\left(1+(\frac{f''+tg''}{g'})^2\right)_s + \frac{g''}{g'}\right] = 0, \tag{3.14}$$

$$\triangle t = \frac{1}{g'^2}\left[(\frac{f''+tg''}{g'})_s\right] = 0, \tag{3.15}$$

$$\begin{aligned}\triangle(f(s)+tg(s)) \quad = \quad & \frac{1}{g'^2}\left[-\frac{(f''+tg'')^3}{g'^2} - 2(f'+tg')(\frac{f''+tg''}{g'})(\frac{f''+tg''}{g'})_s\right. \\ & \left. +g(\frac{f''+tg''}{g'})_s + \frac{f'g''}{g'} + tg'' + f' + tg'\right] = 0.\end{aligned} \tag{3.16}$$

From (3.15), we find that $(\frac{f''+tg''}{g'})_s = 0$, so relation (3.14) becomes $\frac{g''}{g'^3} = 0$, from which we obtain that $g'' = 0$. Hence, Eq (3.16) reduces to

$$-\frac{(f'')^3}{g'^4}+\frac{f'+tg'}{g'^2}=0,$$

or

$$f'g'^2-f''^3+tg'^3=0.$$

This equation holds true for all values of t when $g'=0$, a case which is excluded. Thus, we have the following theorem:

Theorem 1. *Non-developable ruled surfaces of type 3 in the simply isotropic space $\mathbb{I}^3$ are of infinite III-type.*

3.3. Ruled surfaces in $\mathbb{I}^3$ of Type 4

In this section, we classify the non-developable ruled surface of Type 4 in $\mathbb{I}^3$ given by (3.6). The Laplacian is

$$\begin{aligned}\triangle &= -\frac{1}{g'^2}\Big[\frac{\partial^2}{\partial s^2}+\frac{2(f''+tg''+tg)}{g'}\frac{\partial^2}{\partial s\partial t}-\frac{f''}{g't}\frac{\partial}{\partial s}\\&\quad+\Big((\frac{f''+tg''+tg}{g'})_s+t-\frac{f''^2}{g'^2t}+\frac{g''^2+g^2}{g'^2}t\Big)\frac{\partial}{\partial t}\\&\quad+\Big(t^2+\frac{(f''+g''t+gt)^2}{g'^2}\frac{\partial^2}{\partial t^2}\Big].\end{aligned}\tag{3.17}$$

Applying (3.17) for the components of (3.6), we obtain

$$\begin{aligned}\triangle t\cos s &= \frac{1}{g'^3}\Big((f''+2(g''+g)t)\Big)\sin s\\&\quad-\frac{1}{g'^3}\Big((f''+g''t+gt)_s-\frac{f''^2}{g't}-\frac{f''g''}{g'}-\frac{g''g}{g'}t+\frac{g^2}{g'}t\Big)\cos s,\end{aligned}$$

$$\begin{aligned}\triangle t\sin s &= -\frac{1}{g'^3}\Big((f''+2(g''+g)t)\Big)\cos s\\&\quad-\frac{1}{g'^3}\Big((f''+g''t+gt)_s-\frac{f''^2}{g't}-\frac{f''g''}{g'}-\frac{g''g}{g'}t+\frac{g^2}{g'}t\Big)\sin s,\end{aligned}$$

$$\begin{aligned}\triangle(f(s)+tg(s)) &= -\frac{1}{g'^2}\Big(3f''+3tg''+3tg-\frac{f'f''}{g't}-2f''-\frac{f''}{t}\\&\quad+g(\frac{f''+g''t+gt}{g'})_s+g(\frac{g''^2}{g'^2}t+\frac{g^2}{g'^2}t-\frac{f''^2}{g'^2t})\Big).\end{aligned}$$

We rewrite the above three equations as follows

$$\triangle(t\cos s)=A_{11}(s)+A_{12}(s)t+A_{13}(s)\frac{1}{t},\tag{3.18}$$

$$\triangle(t\sin s) = B_{11}(s) + B_{12}(s)t + B_{13}(s)\frac{1}{t},$$

$$\triangle(f(s) + tg(s)) = C_{11}(s) + C_{12}(s)t + C_{13}(s)\frac{1}{t}.$$

From relation (3.17), it can be obtained the following:

Lemma 2. *For any non-negative integer number n, and some functions $D_{11}(s), D_{12}(s)$, and $E_{11}(s)$ of the variable s, we have*

$$\triangle(D_{11}(s) + D_{12}(s)t) = D_{21}(s) + D_{22}(s)t + D_{23}(s)\frac{1}{t}, \tag{3.19}$$

and

$$\triangle(E_{11}(s)\frac{1}{t^n}) = E_{21}(s)\frac{1}{t^n} + E_{22}(s)\frac{1}{t^{n+1}} + E_{23}(s)\frac{1}{t^{n+2}}, \tag{3.20}$$

where D_{ij}, E_{ij} are functions of the variable s.

Applying (3.17) for the components of (3.18), and on account of Lemma 2, we obtain

$$\triangle^2(t\cos s) \;=\; A_{21}(s) + A_{22}(s)t + A_{23}(s)\frac{1}{t} + A_{24}(s)\frac{1}{t^2} + A_{25}(s)\frac{1}{t^3}.$$

Similarly, we get

$$\triangle^2(t\sin s) \;=\; B_{21}(s) + B_{22}(s)t + B_{23}(s)\frac{1}{t} + B_{24}(s)\frac{1}{t^2} + B_{25}(s)\frac{1}{t^3},$$

and

$$\begin{aligned}\triangle^2(f(s) + tg(s)) &= C_{21}(s) + C_{22}(s)t + C_{23}(s)\frac{1}{t}\\ &+ C_{24}(s)\frac{1}{t^2} + C_{25}(s)\frac{1}{t^3}.\end{aligned}$$

Applying the mathematical induction (see [11]), one can prove the following:

Lemma 3. *For any natural number k, the following relations hold true*

$$\begin{aligned}\triangle^k(t\cos s) &= A_{k1}(s) + A_{k2}(s)t + A_{k3}(s)\frac{1}{t} + A_{k4}(s)\frac{1}{t^2}\\ &+ \ldots + A_{k,2k-1}(s)\frac{1}{t^{2k-1}}\end{aligned} \tag{3.21}$$

$$\begin{aligned}\triangle^k(t\sin s) &= B_{k1}(s) + B_{k2}(s)t + B_{k3}(s)\frac{1}{t} + B_{k4}(s)\frac{1}{t^2}\\ &+ \ldots + B_{k,2k-1}(s)\frac{1}{t^{2k-1}}\end{aligned} \tag{3.22}$$

$$\begin{aligned}\triangle^k(f(s) + tg(s)) &= C_{k1}(s) + C_{k2}(s)t + C_{k3}(s)\frac{1}{t}\\ &+ C_{k4}(s)\frac{1}{t^2} + \ldots + C_{k,2k-1}(s)\frac{1}{t^{2k-1}}.\end{aligned} \tag{3.23}$$

Assume that S is of finite III-type k. Applying (3.10) to the coordinate functions X_1, X_2, and X_3 of the position vector (3.6) of the quadric S, we obtain

$$\triangle^{k+1} t\cos s + \mu_1 \triangle^{k} t\cos s + \ldots + \mu_k \Delta t\cos s = 0,$$

$$\triangle^{k+1} t\sin s + \mu_1 \triangle^{k} t\sin s + \ldots + \mu_k \Delta t\sin s = 0.$$

$$\triangle^{k+1}(f+tg) + \mu_1 \triangle^{k}(f+tg) + \ldots + \mu_k \triangle(f+tg) = 0.$$

Taking into account Lemmas 2 and 3, we conclude

$$\begin{aligned} & A_{k+1,1}(s) + A_{k+1,2}(s)t + \sum_{k=1}^{2k+1} A_{k+1,k+2}(s)\frac{1}{t^k} + \mu_1\Big(A_{k1}(s) + A_{k2}(s)t \\ + & \sum_{k=1}^{2k-1} A_{k+1,k+2}(s)\frac{1}{t^k}\Big) + \ldots + \mu_{k-1}\Big(A_{21}(s) + A_{22}(s)t + A_{23}(s)\frac{1}{t} + A_{24}(s)\frac{1}{t^2} + A_{25}(s)\frac{1}{t^3}\Big) \\ + & \mu_k(A_{11}(s) + A_{12}(s)t + A_{13}(s)\frac{1}{t}) = 0. \end{aligned} \tag{3.24}$$

$$\begin{aligned} & B_{k+1,1}(s) + B_{k+1,2}(s)t + \sum_{k=1}^{2k+1} B_{k+1,k+2}(s)\frac{1}{t^k} + \mu_1\Big(B_{k1}(s) + B_{k2}(s)t \\ + & \sum_{k=1}^{2k-1} B_{k+1,k+2}(s)\frac{1}{t^k}\Big) + \ldots + \mu_{k-1}\Big(B_{21}(s) + B_{22}(s)t + B_{23}(s)\frac{1}{t} + B_{24}(s)\frac{1}{t^2} + B_{25}(s)\frac{1}{t^3}\Big) \\ + & \mu_k(B_{11}(s) + B_{12}(s)t + B_{13}(s)\frac{1}{t}) = 0. \end{aligned} \tag{3.25}$$

$$\begin{aligned} & C_{k+1,1}(s) + C_{k+1,2}(s)t + \sum_{k=1}^{2k+1} C_{k+1,k+2}(s)\frac{1}{t^k} + \mu_1\Big(C_{k1}(s) + C_{k2}(s)t \\ + & \sum_{k=1}^{2k-1} C_{k+1,k+2}(s)\frac{1}{t^k}\Big) + \ldots + \mu_{k-1}\Big(C_{21}(s) + C_{22}(s)t + C_{23}(s)\frac{1}{t} + C_{24}(s)\frac{1}{t^2} + C_{25}(s)\frac{1}{t^3}\Big) \\ + & \mu_k(C_{11}(s) + C_{12}(s)t + C_{13}(s)\frac{1}{t}) = 0. \end{aligned} \tag{3.26}$$

Now, if k goes up by one, the degree of the variable t in the denominator of Eqs (3.21)–(3.23) goes up at most by 2. Hence, the sums (3.24)–(3.26) can never be zeros, unless, we have

$$\begin{aligned} \triangle t\cos s = & \frac{1}{g'^3}\Big((f'' + 2(g''+g)t)\Big)\sin s \\ & -\frac{1}{g'^3}\Big((f'' + g''t + gt)_s - \frac{f''^2}{g't} - \frac{f''g''}{g'} - \frac{g''g}{g'}t + \frac{g^2}{g'}t\Big)\cos s = 0, \end{aligned} \tag{3.27}$$

$$\triangle t \sin s = -\frac{1}{g'^3}\Big((f'' + 2(g'' + g)t)\Big)\cos s$$
$$-\frac{1}{g'^3}\Big((f'' + g''t + gt)_s - \frac{f''^2}{g't} - \frac{f''g''}{g'} - \frac{g''g}{g'}t + \frac{g^2}{g'}t\Big)\sin s = 0, \tag{3.28}$$

$$\triangle(f(s) + tg(s)) = -\frac{1}{g'^2}\Big(3f'' + 3tg'' + 3tg - \frac{f'f''}{g't} - 2f'' - \frac{f''}{t}$$
$$+g(\frac{f'' + g''t + gt}{g'})_s + g(\frac{g''^2}{g'^2}t + \frac{g^2}{g'^2}t - \frac{f''^2}{g'^2 t})\Big) = 0. \tag{3.29}$$

From (3.27) and (3.28), $\cos s$ and $\sin s$ are linearly independent functions. Therefore, we have

$$f'' + 2(g'' + g)t = 0, \tag{3.30}$$

and

$$(f'' + g''t + gt)_s - \frac{f''^2}{g't} - \frac{f''g''}{g'} - \frac{g''g}{g'}t + \frac{g^2}{g'}t = 0. \tag{3.31}$$

From (3.30), we must have $f'' = 0$ and $g'' + g = 0$. Thus, relations (3.29) and (3.31) on account of the above two equations reduce to $\frac{2g^2}{g'}t = 0$, from which we must have $g(s) \equiv 0$, a case that has been excluded since III is non-degenerate.

3.4. *Quadrics of type (3.1)*

A parametrization of this kind is given by [17]

$$\boldsymbol{X}(s,t) = \Big(s, t, \sqrt{A\,s^2 + B\,t^2 + C}\Big). \tag{3.32}$$

For simplicity, we put

$$A\,s^2 + B\,t^2 + C: = \varpi.$$

Denote by (g_{ij}) and (b_{ij}) the components of the first and second fundamental forms. Then we find

$$(g_{ij}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and for components (b_{ij}), we have

$$(b_{ij}) = \begin{bmatrix} \frac{A(Bt^2+C)}{\varpi^{\frac{3}{2}}} & -\frac{ABst}{\varpi^{\frac{3}{2}}} \\ -\frac{ABst}{\varpi^{\frac{3}{2}}} & \frac{B(As^2+C)}{\varpi^{\frac{3}{2}}} \end{bmatrix}.$$

The relative curvature $K_{\mathbb{I}^3}$ and isotropic mean curvature $H_{\mathbb{I}^3}$ are defined by

$$K_{\mathbb{I}^3} = \frac{Det(b_{ij})}{Det(g_{ij})} = \frac{ABC}{\varpi^2},$$

$$H_{\mathbb{I}^3} = \frac{g_{11}b_{22} - 2g_{12}b_{12} + g_{22}b_{11}}{2det(g_{ij})} = \frac{B(As^2 + C) + A(Bt^2 + C)}{\varpi^{\frac{3}{2}}}.$$

The third fundamental form III is given by

$$III = \frac{A^2}{\varpi^3}F_3(s,t)ds^2 - 2\frac{A\,B}{\varpi^3}F_2(s,t)dsdt + \frac{B^2}{\varpi^3}F_1(s,t)dt^2,$$

where

$$\begin{aligned} F_1(s,t) &= A^2\,s^2\,t^2 + (A\,s^2 + C)^2, \\ F_2(s,t) &= s\,t\left[C(A+B) + A\,B(s^2 + t^2 + \varpi)\right], \\ F_3(s,t) &= B^2\,s^2\,t^2 + (B\,t^2 + C)^2. \end{aligned}$$

Then the Laplacian $\triangle$ of S is given by

$$\begin{aligned} \triangle \quad = \quad & -\frac{\varpi}{A^2\,B^2\,C^2}\left[B^2F_1\frac{\partial^2}{\partial s^2} + 2A\,B\,F_2\frac{\partial^2}{\partial s\partial t} + A^2F_3\frac{\partial^2}{\partial t^2}\right] \\ & -\frac{\varpi}{A^2\,B^2\,C^2}\left[B\left(B\frac{\partial F_1}{\partial s} + A\frac{\partial F_2}{\partial t}\right)\frac{\partial}{\partial s} + A\left(A\frac{\partial F_3}{\partial t} + B\frac{\partial F_2}{\partial s}\right)\frac{\partial}{\partial t}\right] \\ & +\frac{1}{A^2\,B^2\,C^2}\left[A\,B^2\,(s\,F_1 + t\,F_2)\,\frac{\partial}{\partial s} + A^2\,B\,(s\,F_2 + t\,F_3)\,\frac{\partial}{\partial t}\right]. \end{aligned} \tag{3.33}$$

We write (3.33) as follows:

$$\begin{aligned} \triangle^{III} \quad = \quad & -\frac{As^5}{C^2}\left[s\frac{\partial^2}{\partial s^2} + 4\frac{\partial}{\partial s}\right] + f_1(s,t)\frac{\partial^2}{\partial s^2} \\ & -\frac{Bt^5}{C^2}\left[t\frac{\partial^2}{\partial t^2} + 4\frac{\partial}{\partial t}\right] + f_2(s,t)\frac{\partial^2}{\partial t^2} \\ & +f_3(s,t)\frac{\partial^2}{\partial s\partial t} + f_4(s,t)\frac{\partial}{\partial s} + f_5(s,t)\frac{\partial}{\partial t}, \end{aligned} \tag{3.34}$$

where

$$f_1(s,t) \quad = \quad -\frac{A+B}{C^2}s^4t^2 - \frac{B}{C^2}s^2t^4 - \frac{3}{C}s^4 - \frac{3}{A}s^2 \tag{3.35}$$

$$-\frac{2B+A}{AC}s^2t^2 - \frac{B}{A^2}t^2 - \frac{C}{A^2}, \tag{3.36}$$

$$\tag{3.37}$$

$$\begin{aligned} f_2(s,t) \quad = \quad & -\frac{A+B}{C^2}s^2t^4 - \frac{A}{C^2}s^4t^2 - \frac{3}{C}t^4 - \frac{3}{B}t^2 \\ & -\frac{2A+B}{bC}s^2t^2 - \frac{A}{B^2}s^2 - \frac{C}{B^2}, \end{aligned}$$

$$
\begin{aligned}
f_3(s,t) &= -2st\Big[\frac{A}{C^2}s^4+\frac{(A+2B)}{BC}s^2+\frac{B}{C^2}t^4+\frac{(B+2A)}{AC}t^2\\
&\quad+\frac{(A+B)}{C^2}s^2t^2+\frac{A+B}{AB}\Big],
\end{aligned}
$$

$$
f_4(s,t) = -\frac{(A+8B)}{BC}s^3-\frac{(A+4B)}{AB}s-\frac{4(A+B)}{C^2}s^3t^2-\frac{4B}{C^2}st^4-\frac{(5A+4B)}{AC}st^2,
$$

$$
f_5(s,t) = -\frac{(8A+B)}{AC}s^3-\frac{(4A+B)}{AB}t-\frac{4A}{C^2}s^4t-\frac{4(A+B)}{C^2}s^2t^3-\frac{(4A+5B)}{BC}s^2t.
$$

Here, the functions $f_i, i=1,\ldots,5$, are polynomials in s and t with $\deg(f_i)\le 6$. We consider a function $g(s)\in C^\infty(U)$. By means of (3.34), we find

$$
\triangle g = -\frac{As^5}{C^2}\left(s\frac{\partial^2 g}{\partial s^2}+4\frac{\partial g}{\partial s}\right)+f_1(s,t)\frac{\partial^2 g}{\partial s^2}+f_4(s,t)\frac{\partial g}{\partial s}. \tag{3.38}
$$

Substituting $t=0$, then the functions f_1 and f_4 are polynomials of the single parameter s of degree ≤ 4. Now we need the following:

Lemma 4. *The relation*

$$
(\triangle)^k\, s = (-1)^k\,(4^k)\left(\prod_{i=1}^{k}(4i-3)\,i\right)\left(\frac{A^k s^{4k+1}}{C^{2k}}\right)+P_{4k}(s,t),
$$

is valid, where $\deg(P_{4k}(s,0))\le 4k$.

Proof. Use the induction method on k. From (3.38), and putting $g=s$, then the formula follows immediately for $k=1$ since the component function $f_1(s,0)$ in relation (3.35) with respect to the variable s is of degree 4. Suppose the Lemma is true for $k-1$. Then

$$
(\triangle)^{k-1}\, s = (-1)^{k-1}\,(4^{k-1})\left(\prod_{i=1}^{k-1}(4i-3)\,i\right)\left(\frac{A^{k-1} s^{4k-3}}{c^{2k-2}}\right)+P_{4k-4}(s,t).
$$

On account of (3.38), we obtain

$$
\begin{aligned}
(\triangle)^k\, s &= \triangle\Big((\triangle)^{k-1}\, s\Big) = -\frac{As^5}{C^2}\,(-4)^{k-1}\left(\prod_{i=1}^{k-1}(4i-3)\,i\right)\\
&\quad\left(\frac{A^{k-1}}{C^{2k-2}}\right)\left(s\frac{\partial^2}{\partial s^2}\left(s^{4k-3}\right)+4\frac{\partial}{\partial s}\left(s^{4k-3}\right)\right)\\
&\quad+(-4)^k\left(\prod_{i=1}^{k-1}(4i-3)\,i\right)\left(\frac{A^{k-1}}{C^{2k-2}}\right)f_1(s,t)\frac{\partial^2}{\partial s^2}\left(s^{4k-3}\right)\\
&\quad+(-4)^k\left(\prod_{i=1}^{k-1}(4i-3)\,i\right)\left(\frac{A^{k-1}}{C^{2k-2}}\right)f_4(s,t)\frac{\partial}{\partial s}\left(s^{4k-3}\right)
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{As^5}{C^2}\left(s\frac{\partial^2}{\partial s^2}\left(P_{4k-4}\right)+4\frac{\partial}{\partial s}\left(P_{4k-4}\right)\right) \\
& +f_1(s,t)\frac{\partial^2}{\partial s^2}\left(P_{4k-4}\right)+f_4(s,t)\frac{\partial}{\partial s}\left(P_{4k-4}\right) \\
= \; & (-4)^k\left(\prod_{i=1}^{k}(4i-3)\,i\right)\left(\frac{A^k s^{4k+1}}{C^{2k}}\right)+P_{4k}(s,t),
\end{aligned}
$$

where

$$
\begin{aligned}
P_{4k}(s,t) = \; & -\frac{As^5}{C^2}\left(s\frac{\partial^2}{\partial s^2}\left(P_{4k-4}\right)+4\frac{\partial}{\partial s}\left(P_{4k-4}\right)\right) \\
& +(-4)^k\left(\prod_{i=1}^{k-1}(4i-3)\,i\right)\left(\frac{A^{k-1}}{C^{2k-2}}\right)f_1(s,t)\frac{\partial^2}{\partial s^2}\left(s^{4k-3}\right) \\
& +(-4)^k\left(\prod_{i=1}^{k-1}(4i-3)\,i\right)\left(\frac{A^{k-1}}{C^{2k-2}}\right)f_4(s,t)\frac{\partial}{\partial s}\left(s^{4k-3}\right) \\
& +f_1(s,t)\frac{\partial^2}{\partial s^2}\left(P_{4k-4}\right)+f_4(s,t)\frac{\partial}{\partial s}\left(P_{4k-4}\right).
\end{aligned} \tag{3.39}
$$

For $t=0$, the degree of $P_{4k-4}(s,0)$ is less than or equal $4k-4$ and the functions $f_2(s,0)$ and $f_4(s,0)$ are of degree less than or equal 4. Therefore, from (3.39), we find that $P_{4k}(s,0)$ is of degree less than or equal $4k$. □

Applying (3.34) for a function $h(t)\in C^\infty$, we obtain

$$
\triangle^{III}h=-\frac{Bt^5}{C^2}\left(t\frac{\partial^2 h}{\partial t^2}+4\frac{\partial h}{\partial t}\right)+f_2(s,t)\frac{\partial^2 h}{\partial t^2}+f_5(s,t)\frac{\partial h}{\partial t}.
$$

Taking $s=0$, then the polynomials $f_3(0,t)$ and $f_5(0,t)$ are of degree ≤ 4. Following the same procedure as in Lemma 4, we demonstrate the following:

Lemma 5. *The equation*

$$
(\triangle)^k\,t=(-4)^k\left(\prod_{i=1}^{k}(4i-3)\,i\right)\left(\frac{B^k t^{4k+1}}{C^{2k}}\right)+Q_{4k}(s,t)
$$

holds true, where $\deg(Q_{4k}(0,t))\leq 4k$.

Assume that S is of finite III-type k. Then, for a constant numbers $c_1,\dots,c_k$, we must have

$$
\triangle^{k+1}\boldsymbol{X}+c_1\,\triangle^k\boldsymbol{X}+\ldots+c_k\triangle\boldsymbol{X}=\boldsymbol{0}. \tag{3.40}
$$

Applying (3.40) to the coordinate functions $X_1=s$ and $X_2=t$ of the position vector (3.32) of S, we get

$$
\triangle^{k+1}s+c_1\triangle^k s+\cdots+c_k\triangle s=0, \tag{3.41}
$$

$$
\triangle^{k+1}t+c_1\triangle^k t+\cdots+c_k\triangle t=0. \tag{3.42}
$$

On account of relation (3.41) and Lemma 4, it follows that there exists a polynomial $P_{4k+4}(s,t)$ of degree at most $4k+4$, satisfying

$$(-4)^{k+1}\left(\prod_{i=1}^{k+1}(4i-3)\,i\right)\left(\frac{A^{k+1}s^{4k+5}}{C^{2k+2}}\right)+P_{4k+4}(s,t)=0. \tag{3.43}$$

Similarly, from relation (3.42) and Lemma 5, we obtain a polynomial $Q_{4k+4}(s,t)$ of degree at most $4k+4$, satisfying

$$(-4)^{k+1}\left(\prod_{i=1}^{k+1}(4i-3)\,i\right)\left(\frac{B^{k+1}v^{4k+5}}{C^{2k+2}}\right)+Q_{4k+4}(s,t)=0. \tag{3.44}$$

Putting $t = 0$ in (3.43), and $s = 0$ in (3.44), then relations (3.43) and (3.44) are nontrivial polynomials in s and t, respectively, with constant coefficients. Since these two equations must hold true for all values of s and t, then we must have $A = B = 0$, which is a contradiction. Thus, we have the following;

Theorem 2. *Quadric surfaces of the first kind (3.1) in the simply isotropic space $\mathbb{I}^3$ are of infinite III-type.*

3.5. Quadrics of type (3.2)

A parametrization of this type is given by

$$\boldsymbol{X}(s,t)=\left(s,t,\frac{A}{2}s^2+\frac{B}{2}t^2\right). \tag{3.45}$$

The matrix of the metric I of S is

$$(g_{ij})=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},$$

and of the second fundamental form II of S is given as follows:

$$(b_{ij})=\begin{bmatrix}A & 0\\ 0 & B\end{bmatrix}.$$

The relative curvature $K_{\mathbb{I}^3}$ and isotropic mean curvature $H_{\mathbb{I}^3}$ are defined by

$$K_{\mathbb{I}^3}=AB,\quad H_{\mathbb{I}^3}=A+B.$$

One can prove the following:

Corollary 1. *The surface S is isotropic minimal if and only if A = -B.*

The metric III of S is

$$III=A^2ds^2+B^2dt^2.$$

Hence, the Laplacian $\triangle$ of S becomes

$$\triangle = -\frac{1}{A^2}\frac{\partial^2}{\partial s^2} - \frac{1}{B^2}\frac{\partial^2}{\partial t^2}. \tag{3.46}$$

For the position vector $\boldsymbol{X}(s,t)$, it is well known that

$$\triangle\boldsymbol{X}(s,t) = (\triangle X_1, \triangle X_2, \triangle X_3).$$

Applying the operator $\triangle$ to the component functions of $\boldsymbol{X}(s,t)$, we find

$$\begin{aligned}\triangle X_1 &= \triangle s = 0,\\ \triangle X_2 &= \triangle t = 0,\\ \triangle X_3 &= -\frac{1}{A} - \frac{1}{B}.\end{aligned}$$

We distinguish the following two cases:

CaseI. $A = -B$. From the last equation, we get $\triangle X_3 = 0$. Thus, we find that $\triangle\boldsymbol{X}(s,t) = \boldsymbol{0}$, that is, S is of 0-type 1.

CaseII. $A \neq B$. Then, $(\triangle^{III})^2 X_3 = 0$, so the position vector $\boldsymbol{X}(s,t)$ can be written as a sum of two nonconstant vectors as follows:

$$\boldsymbol{X}(s,t) = \boldsymbol{X_1}(s,t) + \boldsymbol{X_2}(s,t)$$

where

$$\boldsymbol{X_1}(s,t) = (s,t,0), \quad \boldsymbol{X_2}(s,t) = (0,0,\frac{A}{2}s^2 + \frac{B}{2}t^2),$$

and

$$\begin{aligned}\triangle\boldsymbol{X_1}(s,t) &= \zeta_1\boldsymbol{X_1},\\ \triangle\boldsymbol{X_2}(s,t) &= \zeta_2\boldsymbol{X_2},\end{aligned}$$

where $\zeta_1 = 0$, and $\zeta_2 \neq 0$. When $\zeta_2 = 0$, case II reduces to case I. Thus, we provide this case.

Theorem 3. *All quadric surfaces of the second kind (3.2) in the simply isotropic space $\mathbb{I}^3$ are of finite 0-type 2.*

In the special case where S is isotropic minimal, we have the following:

Corollary 2. *Hyperbolic paraboloid of the form $z = A(X^2 - Y^2)$ is of 0-type 1, with corresponding eigenvalue $\zeta = 0$.*

4. Conclusions

The classification of ruled surfaces of the $3^{rd}, 4^{th}$ type and quadric surfaces of first kind and second kind in the simply isotropic 3-space $\mathbb{I}^3$ within the limitations of finite Chen type concerning the third fundamental form was investigated. It was proved that among these classes, only quadric surfaces of the second kind in the simply isotropic space $\mathbb{I}^3$ are of finite 0-type 2. In a special case, when the surface is isotropically minimal, then it is a hyperbolic paraboloid of 0-type 1. Thus, the finite type classification concerning the third fundamental form not only provides a meaningful extension of Chen's theory into the realm of isotropic geometry but also serves as an effective tool for identifying and characterizing geometrically distinguished surfaces in $\mathbb{I}^3$ [18, 19].

In the future, researchers may extend these results to higher-dimensional isotropic spaces or explore analogous classifications regarding the first and second fundamental forms. Additionally, one can use the definition of the fractional vector operators so that new forms of first and second Beltrami operators can be found and applied to any class of surfaces [20, 21].

Author contributions

H. Alzaareer and H. Al-Zoubi: Conceptualization, methodology, investigation, resources, writing original draft; F. Abdel-Fattah: Software, data curation, writing original draft, funding acquisition. All authors have read and approved the final version of the manuscript.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflict of interest in this paper.

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