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Research article

Algebraic and topological foundations of non-Newtonian analysis via generator functions

Emre Civgin[1] **and Numan Yalcin**[2,*]

[1] Ministry of National Education, Yenikonak Secondary School, Sinop, Türkiye
[2] Department of Electronics and Automation, Vocational School of Gumushane, Gumushane University, Gumushane, Türkiye

* **Correspondence:** Email: numan@gumushane.edu.tr; Tel: +905432615263.

Abstract: In this paper we explored the algebraic and topological underpinnings of non-Newtonian analysis through the framework of a generator function α. We introduced the α-number systems and the associated arithmetic operations, and we established the corresponding algebraic structures, including groups, fields, and vector spaces defined over α-real numbers. On the topological side, we developed the notions of α-metric spaces and α-sequences, thereby extending core concepts of analysis to the non-Newtonian setting. The study culminates with the formulation of star ($\star$) analysis, which provides a systematic mechanism for transitioning between distinct arithmetic systems, together with a rigorous treatment of $\star$-vector spaces and linear operators.

Keywords: alpha generator function; algebraic structures; topological structures; star operators; star vector space
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1. Introduction

Classical Newtonian analysis is built upon well established algebraic and topological structures that provide the foundation for concepts such as limits, continuity, differentiation, and integration. Although this framework has proven highly effective in many areas of mathematics and applied sciences, recent theoretical developments have explored alternative systems of arithmetic and analysis, known collectively as non-Newtonian analysis, which generalize or deviate from the standard real number system [1,2].

Our primary aim of this study is to construct the algebraic and topological foundations of non-Newtonian analysis in a systematic and unified manner by extending the essential structures of Newtonian analysis through the use of generator functions, specifically the bijective and continuous function ($\alpha: \mathbb{R} \to \mathbb{R}_\alpha \subset \mathbb{R}$). These generator functions enable the definition of new number sets, referred to as α-real numbers, over which novel arithmetic operations such as α-addition and α-multiplication are defined [3–7].

We begin by defining the α-number systems and corresponding operations. We then investigate the algebraic properties of these structures, including group and field structures on $\mathbb{R}_\alpha$, and vector spaces constructed over these sets. In parallel, the topological framework is developed, introducing concepts such as α-intervals, α-neighborhoods, and α-metric spaces, which serve as the non-Newtonian analogs of standard topological notions [4–7].

A central contribution of this work is the introduction of star ($\star$) operators, which facilitate the transition different non-Newtonian star vector spaces. Through these operators, we define star limits, derivatives, and integrals, providing generalized tools for analysis that extend beyond the Newtonian paradigm [8–12].

We develop an α-generator based approach in which a bijective, continuous generator α induces arithmetic, order, and topological structures on $\mathbb{R}_\alpha$ and $\star$-operators mediate between α and β systems. In this perspective, multiplicative calculus [8,9] and bigeometric calculus [10,11] arise as special cases obtained by particular choices of the generator(s) (e.g., $\alpha(x) = e^x$ and when applicable, $\beta(x) = e^x$. Thus, the α-generator based formulation subsumes these earlier models within a single algebraic-topological framework: group, field, vector-space, order, metric, and limit notions on $\mathbb{R}_\alpha$ transfer from the classical setting via α, while $\star$-limits/derivatives provide a rigorous bridge between distinct generator systems. This positions the paper's contribution as a unifying α-generator mechanism rather than a separate calculus, clarifying how prior non-Newtonian calculi fit into the same structural pipeline [13–18].

Ultimately, our aim is to demonstrate how the core concepts of classical analysis, both algebraic and topological concepts, can be consistently extended and embedded into a non-Newtonian analytical framework, thus contributing to the growing body of research in generalized mathematical structures [19–21].

2. Materials and methods

In this section, we introduce the foundational concepts and notations used throughout the paper. We begin by defining generator functions, which constitute the basis for the construction of non-Newtonian number systems and operations.

2.1. Generator functions and alpha real numbers

Let $\alpha: \mathbb{R} \to \mathbb{R}_\alpha \subset \mathbb{R}$ be a bijective and continuous function. Such a function is called a generator function, and it enables the construction of a non-Newtonian arithmetic structure. The set $\mathbb{R}_\alpha$ is defined as the image of $\mathbb{R}$ under function α:

$$\mathbb{R}_\alpha := \alpha(\mathbb{R}) = \{\alpha(x) \mid x \in \mathbb{R}\}.$$

The inverse function $\alpha^{-1}: \mathbb{R}_\alpha \to \mathbb{R}$ is also assumed to be continuous, which ensures that the arithmetic operations induced by α are well-defined and structurally consistent.

Definition 2.1.1. Let α be a generator function. Then, the sets of non-Newtonian (or α) real numbers, integers, and natural numbers are respectively defined as follows [1–3]:

$$\mathbb{R}_\alpha = \{\alpha(x) | x \in \mathbb{R}\},$$
$$\mathbb{Z}_\alpha = \{\alpha(x) | x \in \mathbb{Z}\} = \{\cdots, \alpha(-2), \alpha(-1), \alpha(0), \alpha(1), \alpha(2), \cdots\},$$
$$\mathbb{N}_\alpha = \{\alpha(x) \,|\, x \in \mathbb{N}\} = \{\alpha(0), \alpha(1), \alpha(2), \cdots\}.$$

The sets of positive and negative α-numbers are defined as follows [4]:

$$\mathbb{R}_\alpha^+ = \{\alpha(x) \,|\, x \in \mathbb{R}^+\} = \{\alpha(x) \,|\, x > 0\}, \quad \mathbb{Z}_\alpha^+ = \{\alpha(x) | x \in \mathbb{Z}^+\} = \{\alpha(x) | x \in \mathbb{Z}, x > 0\},$$
$$\mathbb{R}_\alpha^- = \{\alpha(x) \,|\, x \in \mathbb{R}^-\} = \{\alpha(x) \,|\, x < 0\}, \quad \mathbb{Z}_\alpha^- = \{\alpha(x) | x \in \mathbb{Z}^-\} = \{\alpha(x) | x \in \mathbb{Z}, x < 0\}.$$

For non-negative and non-positive α-numbers, we adopt the following notations:

$$\mathbb{R}_\alpha^{+,0} = \{\alpha(x) | x \geq 0\}, \quad \mathbb{Z}_\alpha^{+,0} = \{\alpha(x) | x \in \mathbb{Z}, x \geq 0\},$$
$$\mathbb{R}_\alpha^{-,0} = \{\alpha(x) | x \leq 0\}, \quad \mathbb{Z}_\alpha^{-,0} = \{\alpha(x) | x \in \mathbb{Z}, x \leq 0\}.$$

2.2. α-Arithmetic operations

Definition 2.2.1. The arithmetic operations of α-addition ($\dot{+}$), α-subtraction ($\dot{-}$), α-multiplication ($\dot{\times}$), and α-division ($\dot{/}$) on the set $\mathbb{R}_\alpha$ are defined for all $x, y \in \mathbb{R}_\alpha$ as follows:

- $x \dot{+} y = \alpha[\alpha^{-1}(x) + \alpha^{-1}(y)]$,
- $x \dot{-} y = \alpha[\alpha^{-1}(x) - \alpha^{-1}(y)]$,
- $x \dot{\times} y = \alpha[\alpha^{-1}(x) \times \alpha^{-1}(y)]$,
- $x \dot{/} y = \alpha[\alpha^{-1}(x) / \alpha^{-1}(y)]$, if $\alpha^{-1}(y) \neq 0$.

Definition 2.2.2. Let $\alpha: \mathbb{R} \to \mathbb{R}_\alpha \subset \mathbb{R}$ be a bijective generator function. Define the α-order relation ($\dot{<}$), along with the related relations ($\dot{>}, \dot{\leq}, \dot{\geq}$) on $\mathbb{R}_\alpha$ by $\forall x, y \in \mathbb{R}_\alpha$ [1].

- $x \dot{<} y \Leftrightarrow \alpha^{-1}(x) < \alpha^{-1}(y)$,
- $x \dot{>} y \Leftrightarrow \alpha^{-1}(x) > \alpha^{-1}(y)$,
- $x \dot{\leq} y \Leftrightarrow \alpha^{-1}(x) \leq \alpha^{-1}(y)$,
- $x \dot{\geq} y \Leftrightarrow \alpha^{-1}(x) \geq \alpha^{-1}(y)$.

Lemma 2.2.1. Let α be a generator and $\dot{<}$ be the the corresponding α-order relation. Then α is strictly increasing in the sense: $\forall a, b \in \mathbb{R}$,

$$a < b \Rightarrow \alpha(a) \dot{<} \alpha(b).$$

Definition 2.2.3. The 6-tuple consisting of α-operations with α-order relation $\dot{<}$, namely

$(\mathbb{R}_{\alpha}, \dot{+}, \dot{-}, \dot{\times}, \dot{/}, \dot{<})$ is called the α-arithmetic.

α-operations inherit many structural properties (such as associativity and commutativity) from their classical counterparts.

2.3. The Algebraic and Topological foundations of non-Newtonian analysis

In this section, we present some fundamental definitions and theorems concerning the algebraic and topological foundations of non-Newtonian analysis, as discussed in [1–7,10,14,17,18,20].

Proposition 2.3.1. $(\mathbb{R}_{\alpha}, \dot{+})$ is an abelian group [20].

Notation 2.3.1. The inverse of an element $x \in \mathbb{R}_{\alpha}$ with respect to α-addition is denoted by $\dot{-}x$.

Notation 2.3.2. The inverse of an element $x \in \mathbb{R}_{\alpha} \setminus \{\alpha(0)\}$ under α-multiplication is denoted by $x^{-1_{\alpha}}$.

Theorem 2.3.1. $(\mathbb{R}_{\alpha} \setminus \{\alpha(0)\}, \dot{\times})$ is an abelian (commutative) group [20].

Proposition 2.3.2. Let $\alpha: \mathbb{R} \to \mathbb{R}_{\alpha} \subset \mathbb{R}$ be a generator function. Then the α-addition $(\dot{+})$ and α-multiplication$(\dot{\times})$ satisfy the following identities for $\forall x, y \in \mathbb{R}_{\alpha}$ and $\forall a, b \in \mathbb{R}$ [1]:

- $\alpha^{-1}(x \dot{+} y) = \alpha^{-1}(x) + \alpha^{-1}(y)$,
- $\alpha(a) \dot{+} \alpha(b) = \alpha(a + b)$,
- $\alpha^{-1}(x \dot{\times} y) = \alpha^{-1}(x) \times \alpha^{-1}(y)$,
- $\alpha(a) \dot{\times} \alpha(b) = \alpha(a \times b)$.

These identities emphasize that α-operations on $\mathbb{R}_{\alpha}$ correspond directly to classical addition and multiplication through the generator function α, thus providing a consistent algebraic framework for non-Newtonian analysis.

Proposition 2.3.3. Let $\alpha: \mathbb{R} \to \mathbb{R}_{\alpha} \subset \mathbb{R}$ be a generator function. Then, the inverse of $x \in \mathbb{R}_{\alpha}$ with respect to α-addition is given by [20]:

$$\dot{-}x = \alpha[-\alpha^{-1}(x)].$$

Proposition 2.3.4. Let $\alpha: \mathbb{R} \to \mathbb{R}_{\alpha} \subset \mathbb{R}$ be a generator function. The α-subtraction operation on $\mathbb{R}_{\alpha}$ satisfies the following equality: $\forall x, y \in \mathbb{R}_{\alpha}$

$$x \dot{-} y = x \dot{+} (\dot{-} y).$$

Proposition 2.3.5. For all $x, y \in \mathbb{R}_{\alpha}$ and $a, b \in \mathbb{R}$, the following equalities hold [20]:

- $x \dot{-} y = \alpha[\alpha^{-1}(x) - \alpha^{-1}(y)]$,
- $\alpha^{-1}(x \dot{-} y) = \alpha^{-1}(x) - \alpha^{-1}(y)$,
- $\alpha(a) \dot{-} \alpha(b) = \alpha(a - b)$.

Proposition 2.3.6. Let α be a generator function. For every $x \in \mathbb{R}_{\alpha}$, the following identity holds:

$$\dot{-}x = \alpha(0) \dot{-} x.$$

Proof.

$$\alpha(0) \dot{-} x = \alpha(0) \dot{-} \alpha[\alpha^{-1}(x)] = \alpha[0 - \alpha^{-1}(x)] = \alpha[-\alpha^{-1}(x)] = \dot{-}x.$$

Proposition 2.3.7. Let α be a generator function. Then the α-multiplicative inverse of $x \in \mathbb{R}_\alpha \setminus \{\alpha(0)\}$ is given by [20]:

$$x^{-1_\alpha} = \alpha[1/\alpha^{-1}(x)].$$

Proposition 2.3.8. Let α be a generator function. The α-division operation on $\mathbb{R}_\alpha$ satisfies the following equality: $\forall x, y \in \mathbb{R}_\alpha$, with $y \neq \alpha(0)$,

$$x \dot{/} y = x \dot{\times} y^{-1_\alpha}.$$

Proposition 2.3.9. For $\forall x, y \in \mathbb{R}_\alpha$ and $\forall a, b \in \mathbb{R}$, with $\alpha^{-1}(y) \neq 0$ and $b \neq 0$, the following equalities hold [20]:

- $x \dot{/} y = \alpha[\alpha^{-1}(x)/\alpha^{-1}(y)]$,
- $\alpha^{-1}(x \dot{/} y) = \alpha^{-1}(x)/\alpha^{-1}(y)$,
- $\alpha(a/b) = \alpha(a) \dot{/} \alpha(b)$.

Proposition 2.3.10. Let $\alpha: \mathbb{R} \to \mathbb{R}_\alpha \subset \mathbb{R}$ be a generator function. Then, for all $x \in \mathbb{R}_\alpha \setminus \{\alpha(0)\}$, the α-multiplicative inverse satisfies:

$$x^{-1_\alpha} = \alpha(1) \dot{/} x.$$

Proof.

$$\alpha(1) \dot{/} x = \alpha(1) \dot{/} \alpha[\alpha^{-1}(x)] = \alpha[1/\alpha^{-1}(x)] = x^{-1_\alpha}.$$

Proposition 2.3.11. Let $\alpha: \mathbb{R} \to \mathbb{R}_\alpha \subset \mathbb{R}$ be a generator function. The groups $(\mathbb{R}, +)$ and $(\mathbb{R}_\alpha, \dot{+})$ are isomorphic via the mapping $f(x) = \alpha(x)$; that is, $(\mathbb{R}, +) \cong (\mathbb{R}_\alpha, \dot{+})$.

Proof. We show that $f(x) = \alpha(x)$ is a homomorphism and it is surjective and injective:

- *Homomorphism*: For all $a, b \in \mathbb{R}$,

$$\alpha(a + b) = \alpha(a) \dot{+} \alpha(b).$$

- *Surjectivity*: For every $x \in \mathbb{R}_\alpha$, since a generator α is bijective, there exists $y = \alpha^{-1}(x) \in \mathbb{R}$ such that

$$\alpha(y) = x.$$

- *Injectivity*: Since α is bijective it is also injective, $\alpha(a) = \alpha(b)$ implies $a = b$.

Therefore, α establishes an isomorphism.

Proposition 2.3.12. The groups $(\mathbb{R} \setminus \{0\}, \times)$ and $(\mathbb{R}_\alpha \setminus \{\alpha(0)\}, \dot{\times})$ are isomorphic via $f(x) = \alpha(x)$; that is,

$$(\mathbb{R} \setminus \{0\}, \times) \cong (\mathbb{R}_\alpha \setminus \{\alpha(0)\}, \dot{\times}).$$

Proof. For $\forall a, b \in \mathbb{R}$,

$$f(a \times b) = f(a) \dot{\times} f(b).$$

Injectivity and surjectivity were established in Proposition 2.3.11, so f is an isomorphism

Theorem 2.3.2. $(\mathbb{R}_\alpha, \dot{+}, \dot{\times})$ is a field [3].

Proof. Considering the field axioms:
i) The group $(\mathbb{R}_\alpha, \dot{+})$ is an abelian group with the neutral element $\alpha(0)$.
ii) The group $(\mathbb{R}_\alpha \backslash \{\alpha(0)\}, \dot{\times})$ is an abelian group with the multiplicative identity $\alpha(1)$.
iii) The operation $\dot{\times}$ is distributive over $\dot{+}$: For all $\alpha(a), \alpha(b), \alpha(c) \in \mathbb{R}_\alpha$, the following holds:

$$\alpha(a) \dot{\times} [\alpha(b) \dot{+} \alpha(c)] = [\alpha(a) \dot{\times} \alpha(b)] \dot{+} [\alpha(a) \dot{\times} \alpha(c)].$$

This holds because:

$$\begin{aligned} \alpha(a) \dot{\times} [\alpha(b) \dot{+} \alpha(c)] &= \alpha(a) \dot{\times} \alpha(b + c) \\ &= \alpha[a \times (b + c)] \\ &= \alpha[(a \times b) + (a \times c)] \\ &= \alpha(a \times b) \dot{+} \alpha(a \times c). \\ \alpha(a) \dot{\times} [\alpha(b) \dot{+} \alpha(c)] &= [\alpha(a) \dot{\times} \alpha(b)] \dot{+} [\alpha(a) \dot{\times} \alpha(c)]. \end{aligned}$$

Thus, these three properties, i, ii, and iii, prove that $(\mathbb{R}_\alpha, \dot{+}, \dot{\times})$ is a field.

Theorem 2.3.3. $\mathbb{R}_\alpha$ is a vector space over the field $\mathbb{R}$ under the following operations [20]:

1) *Vector addition*: The α-addition,
2) *Scalar multiplication*: For $r \in \mathbb{R}$ and $v \in \mathbb{R}_\alpha$,

$$r \cdot v = \alpha[r \times \alpha^{-1}(v)].$$

Definition 2.3.1. (Natural Powers) For $x \in \mathbb{R}_\alpha$, $n \in \mathbb{N}$:

- $x^{0_\alpha} = \alpha(1)$, $x^{1_\alpha} = x$,
- $x^{(n+1)_\alpha} = x^{n_\alpha} \dot{\times} x$.

Proposition 2.3.13.

$$x^{n_\alpha} = \alpha\{[\alpha^{-1}(x)]^n\}.$$

Definition 2.3.2. (Real Powers) For $x \in \mathbb{R}_\alpha$, $r \in \mathbb{R}$:

$$x^{r_\alpha} = \alpha\{[\alpha^{-1}(x)]^r\}.$$

Definition 2.3.3. (α-Polynomial) Given $n \in \mathbb{N}$, constants $c_0, c_1, \cdots, c_n \in \mathbb{R}_\alpha$, the function

$$P(x) = c_0 \dot{+} (c_1 \dot{\times} x) \dot{+} (c_2 \dot{\times} x^{2_\alpha}) \dot{+} \cdots \dot{+} (c_n \dot{\times} x^{n_\alpha})$$

is called an α-polynomial of degree n on $\mathbb{R}_\alpha$.

Definition 2.3.4. (α-Square Root) For $x \in \mathbb{R}_\alpha^{+,0}$, the α-square root is defined by

$$\sqrt{x}^{\alpha} = \alpha\left([\alpha^{-1}(x)]^{\frac{1}{2}}\right).$$

Definition 2.3.5. (n-th Degree α-root) Let $n \in \mathbb{N}\backslash\{0\}$ and

$$D_n = \begin{cases} \mathbb{R}_\alpha, & \text{if } n \text{ is odd}, \\ \mathbb{R}_\alpha^{+,0}, & \text{if } n \text{ is even}. \end{cases}$$

The function $\sqrt[n]{\cdot}^{\alpha}: D_n \to \mathbb{R}_\alpha$,

$$\sqrt[n]{x}^{\alpha} = \alpha\left([\alpha^{-1}(x)]^{\frac{1}{n}}\right)$$

is called the n-the degree α root function [1,10,11].

Theorem 2.3.4. The pair $(\mathbb{R}_\alpha, \dot{\leq})$ forms a partially ordered set. That is, the relation $\dot{\leq}$ satisfies the following axioms for all $x, y, z \in \mathbb{R}_\alpha$:

- *Reflexivity*: $x \dot{\leq} x$,
- *Antisymmetry*: If $x \dot{\leq} y$ and $y \dot{\leq} x$, then $x = y$,
- *Transitivity*: If $x \dot{\leq} y$ and $y \dot{\leq} z$, then $x \dot{\leq} z$.

Proof.

- *Reflexivity*:

$$\alpha^{-1}(x) \leq \alpha^{-1}(x) \Rightarrow x \dot{\leq} x.$$

- *Antisymmetry*:

If $x \dot{\leq} y$ and $y \dot{\leq} x$, then

$$\alpha^{-1}(x) \leq \alpha^{-1}(y), \alpha^{-1}(y) \leq \alpha^{-1}(x) \Rightarrow \alpha^{-1}(x) = \alpha^{-1}(y) \Rightarrow x = y.$$

- *Transitivity*:

If $x \dot{\leq} y$ and $y \dot{\leq} z$, then

$$\alpha^{-1}(x) \leq \alpha^{-1}(y),\ \alpha^{-1}(y) \leq \alpha^{-1}(z) \Rightarrow \alpha^{-1}(x) \leq \alpha^{-1}(z) \Rightarrow x \dot{\leq} z.$$

Proposition 2.3.14. Any two elements in $\mathbb{R}_\alpha$ are comparable under the α-order: For all $x, y \in \mathbb{R}_\alpha$, either $x \dot{\leq} y$ or $y \dot{\leq} x$.

Proof. Assume $x, y \in \mathbb{R}_\alpha$ and suppose by contradiction that neither $x \dot{\leq} y$ nor $y \dot{\leq} x$ holds. Then,

$$\alpha^{-1}(x) \nleq \alpha^{-1}(y), \alpha^{-1}(y) \nleq \alpha^{-1}(x) \Rightarrow \alpha^{-1}(x) > \alpha^{-1}(y), \alpha^{-1}(y) > \alpha^{-1}(x),$$

which is a contradiction. Therefore, one of $x \dot{\leq} y$ or $y \dot{\leq} x$ must hold.

Theorem 2.3.5. The α-order $\dot{\leq}$ defines a total order on $\mathbb{R}_\alpha$.

Proof. The α-order $\dot{\leq}$ satisfies the axioms of a partial order (Theorem 2.3.4), and every pair of elements in $\mathbb{R}_\alpha$ is comparable (Proposition 2.3.14).

Hence, $(\mathbb{R}_\alpha, \dot{\leq})$ is a totally ordered set.

Proposition 2.3.15. The total α-order $\dot{\leq}$ is compatible with the vector space structure of $\mathbb{R}_\alpha$. That is, for all $x, y, z \in \mathbb{R}_\alpha$ and all $r \in \mathbb{R}$, $r \geq 0$:

- If $x \dot{\leq} y$, then $x \dot{+} z \dot{\leq} y \dot{+} z$,
- If $x \dot{\leq} y$, then $r \cdot x \dot{\leq} r \cdot y$.

Proof. Let $x, y, z \in \mathbb{R}_\alpha$ such that $x \dot{\leq} y \Rightarrow \alpha^{-1}(x) \leq \alpha^{-1}(y)$.

- For any $z \in \mathbb{R}_\alpha$,

$$\alpha^{-1}(x) + \alpha^{-1}(z) \leq \alpha^{-1}(y) + \alpha^{-1}(z) \Rightarrow \alpha^{-1}(x \dot{+} z) \leq \alpha^{-1}(y \dot{+} z) \Rightarrow x \dot{+} z \leq y \dot{+} z.$$

- For any $r \in \mathbb{R}$, $r \geq 0$,

$$r \times \alpha^{-1}(x) \leq r \times \alpha^{-1}(y) \Rightarrow \alpha^{-1}(r \cdot x) \leq \alpha^{-1}(r \cdot y) \Rightarrow r \cdot x \dot{\leq} r \cdot y.$$

Theorem 2.3.6. $(\mathbb{R}_\alpha, \dot{\leq})$ is a totally ordered vector space.

Proof. Since $\dot{\leq}$ is a total order on $\mathbb{R}_\alpha$ (Theorem 2.3.5) and the order is compatible with the vector space operations (Proposition 2.3.15), it follows that $(\mathbb{R}_\alpha, \dot{\leq})$ is a totally ordered vector space.

Definition 2.3.6. Let $u, v \in \mathbb{R}_\alpha$. The set

- $[u, v]_\alpha = \{x \in \mathbb{R}_\alpha | u \dot{\leq} x \dot{\leq} v\}$ is called an α-closed interval,
- $(u, v)_\alpha = \{x \in \mathbb{R}_\alpha | u \dot{<} x \dot{<} v\}$ is called an α-open interval,
- $[u, v)_\alpha = \{x \in \mathbb{R}_\alpha | u \dot{\leq} x \dot{<} v\}$ is called an α-half-open interval [1].

Notation 2.3.3. Let α be a generator function and $r \in \mathbb{R}$ a real number. Then the α-real number corresponding to r is denoted by

$$\dot{r} = \alpha(r).$$

Definition 2.3.7. The α-absolute value function is defined as

$$|\cdot|_\alpha: \mathbb{R}_\alpha \to \mathbb{R}_\alpha^{+,0}, \qquad |x|_\alpha = \begin{cases} x, & x \dot{\geq} \dot{0}, \\ \dot{-}x, & x \dot{<} \dot{0}, \end{cases}$$

where $\dot{-}x$ denotes the additive inverse of x with respect to the α-addition. [1].

Proposition 2.3.16. For the α-absolute value function, the following equality holds:

$$|x|_\alpha = \alpha(|\alpha^{-1}(x)|).$$

Proposition 2.3.17. For $x, y \in \mathbb{R}_\alpha$, the α-absolute value function satisfies the following properties [3]:

i) *Non-negativity*: $|x|_\alpha \dot{\geq} \dot{0}$.
ii) *Separation of points*: $|x|_\alpha = \dot{0} \Leftrightarrow x = \dot{0}$.
iii) *Multiplicativity*: $|x \dot{\times} y|_\alpha = |x|_\alpha \dot{\times} |y|_\alpha$.
iv) *Triangle inequality*: $|x \dot{+} y|_\alpha \leq |x|_\alpha \dot{+} |y|_\alpha$.

Definition 2.3.8. (α-Metric Space) Let X be a nonempty set and

$$d_\alpha: X \times X \to \mathbb{R}_\alpha^{+,0}$$

be a function defined on X. If for every $x, y, z \in X$, the function d_α satisfies the following axioms [9], then d_α is called an α-metric:

- *Identity of indiscernibles*:

$$d_\alpha(x, y) = \dot{0} \Leftrightarrow x = y.$$

- *Symmetry*:

$$d_\alpha(x, y) = d_\alpha(y, x).$$

- *Triangle inequality*:

$$d_\alpha(x, y) \dot{\leq} d_\alpha(x, z) \dot{+} d_\alpha(z, y).$$

Definition 2.3.9. If X is a nonempty set and d_α is an α-metric on X, then the pair (X, d_α) is called an α-metric space [3].

Proposition 2.3.18. Define

$$d_\alpha: \mathbb{R}_\alpha \times \mathbb{R}_\alpha \to \mathbb{R}_\alpha^{+,0}, \qquad d_\alpha(x, y) = \begin{cases} \dot{0}, & x = y, \\ \dot{1}, & x \neq y. \end{cases}$$

Then d_α is an α-metric on $\mathbb{R}_\alpha$ and it is called the α-discrete metric.

Proof.

- *Identity of indiscernibles*:

$$d_\alpha(x,y)=\dot{0}\Leftrightarrow y=x.$$

- *Symmetry*:

$$d_\alpha(x,y)=\begin{cases}\dot{0}, & x=y\\ \dot{1}, & x\neq y\end{cases}=\begin{cases}\dot{0}, & y=x\\ \dot{1}, & y\neq x\end{cases}=d_\alpha(y,x).$$

- *Triangle inequality*: There are two cases:

1) If $y=x$, then

$$d_\alpha(x,y)=d_\alpha(x,x)=\dot{0}\dot{\leq} d_\alpha(x,z)\dot{+}d_\alpha(z,y),$$

which holds trivially.

2) If $y\neq x$, then either $y\neq z$ or $x\neq z$, so

$$d_\alpha(x,y)=\dot{1}\dot{\leq} d_\alpha(x,z)\dot{+}d_\alpha(z,y),$$

which also holds.

Proposition 2.3.19. Define

$$d_\alpha\colon \mathbb{R}_\alpha\times\mathbb{R}_\alpha\to\mathbb{R}_\alpha^{+,0},\qquad d_\alpha(x,y)=\begin{cases}\alpha(0), & x=y,\\ |x|_\alpha\dot{+}|y|_\alpha, & x\neq y.\end{cases}$$

Then d_α is an α-metric on $\mathbb{R}_\alpha$ and it is called the α-post office metric.

Proof. For every $x,y,z\in\mathbb{R}_\alpha$:

- *Identity of indiscernibles*:

$$d_\alpha(x,y)=\alpha(0)\Leftrightarrow x=y.$$

- *Symmetry*:

$$d_\alpha(x,y)=\begin{cases}\alpha(0), & x=y\\ |x|_\alpha\dot{+}|y|_\alpha, & x\neq y\end{cases}=\begin{cases}\alpha(0), & y=x\\ |y|_\alpha\dot{+}|x|_\alpha, & y\neq x\end{cases}=d_\alpha(y,x).$$

- *Triangle inequality*:

$$d_\alpha(x,y)\dot{\leq} d_\alpha(x,z)\dot{+}d_\alpha(z,y)$$

holds in the following cases:

a) If $x=y$

$$\alpha(0)=d_\alpha(x,y)\dot{\leq} d_\alpha(x,z)\dot{+}d_\alpha(z,y).$$

b) If $x\neq y, x\neq z$ ve $y\neq z$

$$|x|_\alpha\dot{+}|y|_\alpha=d_\alpha(x,y)\dot{\leq} d_\alpha(x,z)\dot{+}d_\alpha(z,y)=|x|_\alpha\dot{+}|y|_\alpha\dot{+}\alpha(2)\dot{\times}|z|_\alpha.$$

c) If $z=x\neq y$

$$d_\alpha(x,z)=\alpha(0),\qquad d_\alpha(x,y)=d_\alpha(z,y),$$

thus the triangle inequality holds.

d) The case $x = y \neq z$ is similar to (c).

Definition 2.3.10. Let

$$d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+ \cup \{0\}, \qquad d(a, b) = |a - b|$$

be the classical distance function on the metric space $(\mathbb{R}, d)$. Define

$$d_\alpha: \mathbb{R}_\alpha \times \mathbb{R}_\alpha \to \mathbb{R}_\alpha^{+,0}$$

by

$$d_\alpha(x, y) = \alpha\{d[\alpha^{-1}(x), \alpha^{-1}(y)]\}.$$

Then d_α is called the α-distance function in the α-metric space $(\mathbb{R}_\alpha, d_\alpha)$. The value $d_\alpha(x, y)$ is called the α-distance between x and y [3].

Proposition 2.3.20. In a one-dimensional space, the α-distance function satisfies [20]:

- $\forall x, y \in \mathbb{R}_\alpha,\ d_\alpha(x, y) = |x \dot{-} y|_\alpha,$
- $\forall a, b \in \mathbb{R},\ d_\alpha(\dot{a}, \dot{b}) = \alpha(|a - b|).$

Proof.

a) $\forall x, y \in \mathbb{R}_\alpha$

$$\begin{aligned} d_\alpha(x, y) &= \alpha\{d[\alpha^{-1}(x), \alpha^{-1}(y)]\} \\ &= \alpha(|\alpha^{-1}(x) - \alpha^{-1}(y)|) \\ &= \begin{cases} \alpha\big(\alpha^{-1}(x) - \alpha^{-1}(y)\big), & \alpha^{-1}(x) \geq \alpha^{-1}(y) \\ \alpha\big(\alpha^{-1}(y) - \alpha^{-1}(x)\big), & \alpha^{-1}(y) < \alpha^{-1}(x) \end{cases} \\ &= \begin{cases} x \dot{-} y, & x \dot{>} y \text{ or } x = y\,, \\ y \dot{-} x, & y \dot{<} x, \end{cases} \\ &= \begin{cases} x \dot{-} y, & x \dot{-} y \dot{>} \dot{0} \text{ or } x \dot{-} y = \dot{0}\,, \\ y \dot{-} x, & x \dot{-} y \dot{<} \dot{0}, \end{cases} \\ d_\alpha(x, y) &= |x \dot{-} y|_\alpha. \end{aligned}$$

b) $\forall a, b \in \mathbb{R},$

$$d_\alpha(\dot{a}, \dot{b}) = \alpha\{d[\alpha^{-1}(\dot{a}), \alpha^{-1}(\dot{b})]\} = \alpha(|a - b|).$$

Proposition 2.3.21. The α-distance function

$$d_\alpha: \mathbb{R}_\alpha \times \mathbb{R}_\alpha \to \mathbb{R}_\alpha^{+,0}$$

is an α-metric on $\mathbb{R}_\alpha$ [20].

Definition 2.3.11. The closed α-ball $B_\alpha[x_0, \dot{1}]$ centered at x_0 with radius $\dot{1}$ is called the unit α-ball centered at x_0.

Definition 2.3.12. Let (X, d_α) be an α-metric space $x_0 \in X$, and $A \subset X$.

- If there exists $\varepsilon \dot{>} \dot{0}$, such that

$$B_\alpha(x_0, \varepsilon) \subset A,$$

then x_0 is called an interior point of A. The set of all interior points of A is called the interior of A and it is denoted by $\mathrm{int}(A)$.

- If there exists $\varepsilon \dot{>} \dot{0}$, such that

$$B_\alpha(x_0, \varepsilon) \cap A = \emptyset,$$

then x_0 is called an exterior point of A. The set of all exterior points of A is called the exterior of A and it is denoted by $\mathrm{ext}(A)$.

- If for every $\varepsilon \dot{>} \dot{0}$

$$B_\alpha(x_0, \varepsilon) \cap A \neq \emptyset \text{ and } B_\alpha(x_0, \varepsilon) \cap (X \backslash A) \neq \emptyset,$$

then x_0 is called a boundary point of A [3].

Definition 2.3.13. Let (X, d_α) be an α-metric space, $x_0 \in X$, and $A \subset X$.

- If every point of A is an interior point, then A is called an α-open set in X
- If the complement $X \backslash A$ is α-open, then A is called an α-closed set.
- The set of all boundary points of A is called the α-boundary of A, denoted by $\partial_\alpha A$ [4].

Definition 2.3.14. Let (X, d_α) be an α-metric space and $A \subset X$. The union of A with its α-boundary $\partial_\alpha A$ is called the α-closure of A, denoted by $\bar{A}^\alpha$ [4].

Definition 2.3.15. Let $A \subset X$ be a subset of the α-space X. If the α-closure of A equals X, i.e., $\bar{A}^\alpha = X$, then A is called α-dense in X.

If A is α-dense in X, then for every $x \in X$, every neighborhood of x contains at least one point of A.

Example 2.3.1. Let $\mathbb{Q}_\alpha = \{\alpha(r) | r \in \mathbb{Q}\}$ be the set of α-rational numbers. Then $\mathbb{Q}_\alpha$ is α-dense in the space $\mathbb{R}_\alpha$.

Solution. To show α-density of $\mathbb{Q}_\alpha$ in $\mathbb{R}_\alpha$, we must prove that for every $x, y \in \mathbb{R}_\alpha$ with $x \dot{<} y$, there exists $\rho \in \mathbb{Q}_\alpha$, such that

$$x \dot{<} \rho \dot{<} y.$$

From the definition of the α-order,

$$x \dot{<} \ y \Rightarrow \alpha^{-1}(x) < \alpha^{-1}(y).$$

Since $\alpha^{-1}(x), \alpha^{-1}(y) \in \mathbb{R}$, and the rationals are dense in the reals, there exists $r \in \mathbb{Q}$, such that

$$\alpha^{-1}(x) < r < \alpha^{-1}(y).$$

From the α-order, we have

$$\alpha\left(\alpha^{-1}(x)\right) \dot{<} \alpha(r) \dot{<} \alpha\left(\alpha^{-1}(y)\right),$$

i.e.,

$$x \dot{<} \rho \dot{<} y,$$

where $\rho = \alpha(r) \in \mathbb{Q}_\alpha$. Hence, $\mathbb{Q}_\alpha$ is α-dense in $\mathbb{R}_\alpha$. Here, we want to show the existence of ρ and we find that there exist an α-rational number ρ, which is equal to $\rho = \alpha(r)$.

Definition 2.3.16. Let (X, d_α) be an α-metric space. A function defined from the set of natural numbers $\mathbb{N}$ into a subset $X \subset \mathbb{R}_\alpha$ is called an α-sequence, and it is denoted by $(x_n)_{n\in\mathbb{N}}$, where each $x_n \in X$ [3].

Definition 2.3.17. Let (X, d_α) be an α-metric space and $(x_n)_{n\in\mathbb{N}}$ an α-sequence in this space.

- The sequence (x_n) is said to α-converge to a point $x_n \in X$ if for every $\varepsilon \dot{>} \dot{0}$, there exists a natural number $N_0 \in \mathbb{N}$, such that for all $n \geq N_0$,

$$d_\alpha(x_n, x) \dot{<} \varepsilon.$$

- The sequence (x_n) is called an α-Cauchy sequence if for every $\varepsilon \dot{>} \dot{0}$, there exists $N_0 \in \mathbb{N}$, such that for all $n, m \geq N_0$,

$$d_\alpha(x_n, x_m) \dot{<} \varepsilon.$$

- The α-metric space (X, d_α) is said to be complete if every α-Cauchy sequence in X α-converges to a point $x \in X$ [3].

Theorem 2.3.7. Let d_α be an α-metric defined on the totally ordered field $\mathbb{R}_\alpha$. Then, $\mathbb{R}_\alpha$ is Dedekind complete with respect to the α-metric d_α.

Proof. The set $\mathbb{R}_\alpha$ is a totally ordered field. Let $S \subset \mathbb{R}_\alpha$ be a nonempty subset that is bounded above. Let $x \in \mathbb{R}_\alpha$ be an arbitrary upper bound of S, i.e.,

$$s \dot{\leq} x, \qquad \forall s \in S.$$

Since $S \subset \mathbb{R}_\alpha$, the set

$$\alpha^{-1}(S) = \{\alpha^{-1}(s) \mid s \in S\} \subset \mathbb{R}$$

is well-defined. By the definition of α-order, we have

$$\alpha^{-1}(s) \leq \alpha^{-1}(x), \qquad \forall \alpha^{-1}(s) \in \alpha^{-1}(S).$$

Thus, $\alpha^{-1}(x)$ is an upper bound of $\alpha^{-1}(S)$ in $\mathbb{R}$. Since $\mathbb{R}$ is Dedekind complete, the set $\alpha^{-1}(S)$ has a supremum, say r. Therefore,

$$\alpha^{-1}(s) \leq r, \text{ and } r \leq \alpha^{-1}(x).$$

From the α-order, we have

$$s \dot{\leq} \alpha(r), \qquad \forall s \in S \text{ and } \alpha(r) \dot{\leq} x.$$

Hence, $\alpha(r)$ is an upper bound of S in $\mathbb{R}_\alpha$, and it is less than or equal to any other upper bound of S. Therefore, $\alpha(r)$ is the supremum of S. Consequently, any nonempty subset $S \subset \mathbb{R}_\alpha$ that is bounded above has a supremum in $\mathbb{R}_\alpha$. This proves that $\mathbb{R}_\alpha$ is Dedekind complete.

Definition 2.3.18. Let $\dot{A} \subset \mathbb{R}_\alpha$**.** If there exists $m \in \dot{A}$, such that $x \dot{\leq} m$ for all $x \in \dot{A}$, then m is called the α-maximum element of $\dot{A}$ and is denoted by $m = \overset{\alpha}{\max} \dot{A}$.

Lemma 2.3.1. Let $\dot{A} \subset \mathbb{R}_\alpha$. If the α-maximum element of $\dot{A}$ exists, then

$$\overset{\alpha}{\max} \dot{A} = \alpha\left(\max\left[\alpha^{-1}(\dot{A})\right]\right).$$

Proof. Suppose $m = \overset{\alpha}{\max} \dot{A}$ exists. Since $\alpha^{-1}(\dot{A}) = \{\alpha^{-1}(x) | x \in \dot{A}\}$ and $\forall x \in \dot{A}$

$$x \dot{\leq} m \Rightarrow \alpha^{-1}(x) \leq \alpha^{-1}(m),$$

we can say that $\alpha^{-1}(m) \in \dot{A}$ and it is the maximum element of $\alpha^{-1}(\dot{A})$. Hence,

$$\alpha^{-1}(m) = \max[\alpha^{-1}(\dot{A})].$$

Applying α to both sides gives

$$m = \alpha(\max[\alpha^{-1}(\dot{A})]),$$

i.e.,

$$\overset{\alpha}{\max}\dot{A} = \alpha(\max[\alpha^{-1}(\dot{A})]).$$

2.4. Star analysis

Definition 2.4.1. ($\star$-Analysis) Let us consider two arbitrary arithmetics: α-arithmetic $= (\mathbb{R}_\alpha, \dot{+}, \dot{-}, \dot{\times}, \dot{/}, \dot{<})$ and β-arithmetic $= (\mathbb{R}_\beta, \ddot{+}, \ddot{-}, \ddot{\times}, \ddot{/}, \ddot{<})$, and denote the pair by $\star = (\alpha\text{ arithmetic}, \beta\text{ arithmetic})$. In $\star$-analysis, the domain of a function is a subset of $\mathbb{R}_\alpha$, and the codomain is a subset of $\mathbb{R}_\beta$. We denote a general function in this context as:

$$f: \mathbb{R}_\alpha \supset A \to B \subset \mathbb{R}_\beta.$$

Since both $(\mathbb{R}_\alpha, \dot{+}, \dot{-}, \dot{\times}, \dot{/}, \dot{<})$ and $(\mathbb{R}_\beta, \ddot{+}, \ddot{-}, \ddot{\times}, \ddot{/}, \ddot{<})$ are totally ordered fields, they are isomorphic. The operation symbols in $\star$-analysis are given in Table 1.

Table 1. The operation symbols in $\star$-analysis.

	α-arithmetic	β-arithmetic
Universe / Realm	$\mathbb{R}_\alpha$	$\mathbb{R}_\beta$
Addition operation	$\dot{+}$	$\ddot{+}$
Subtraction operation	$\dot{-}$	$\ddot{-}$
Multiplication operation	$\dot{\times}$	$\ddot{\times}$
Division operation	$\dot{/}$	$\ddot{/}$
Order relation	$\dot{<}$	$\ddot{<}$

Definition 2.4.2. Let $f: \mathbb{R}_\alpha \to \mathbb{R}_\beta$. $|f|_\beta: \mathbb{R}_\alpha \to \mathbb{R}_\beta$, the absolute value function of f is defined by

$$|f|_\beta(x) = |f(x)|_\beta.$$

Lemma 2.4.1. Let $f: \mathbb{R}_\alpha \to \mathbb{R}_\beta$. Then the absolute value function of f satisfies

$$|f|_\beta = \beta \circ |\beta^{-1} \circ f|.$$

Proof. For all $x \in \mathbb{R}_\alpha$, we have

$$\begin{aligned} |f|_\beta(x) &= |f(x)|_\beta \\ &= \beta\left(\left|\beta^{-1}\left(f(x)\right)\right|\right) \\ &= \beta(|(\beta^{-1} \circ f)(x)|) \\ &= \beta\left(|(\beta^{-1} \circ f)|(x)\right). \end{aligned}$$

$$|f|_\beta(x) = (\beta \circ |\beta^{-1} \circ f|)(x).$$

Hence, the lemma is proved.

Definition 2.4.3. ($\star$-Limit) Let $f: \mathbb{R}_\alpha \supset A \to B \subset \mathbb{R}_\beta$ be a function. If for every $\varepsilon_\beta \ddot{>} \ddot{0}$ $\left(\varepsilon_\beta \in \mathbb{R}_\beta\right)$ there exists $\delta_\alpha \dot{>} \dot{0}$ $(\delta_\alpha \in \mathbb{R}_\alpha)$, such that whenever $\dot{0} \dot{<} |x \dot{-} x_0|_\alpha \dot{<} \delta_\alpha$, we have $|f(x) \ddot{-} L|_\beta \ddot{<} \varepsilon_\beta$, then the $\star$-limit of f at $x_0 \in \mathbb{R}_\alpha$ exists and it is equal to L. This is denoted as [17]

$$\star \lim_{x \to x_0} f(x) = L\,.$$

Proposition 2.4.1. Let there exist $\delta_\alpha \in \mathbb{R}_\alpha^+$ and $\varepsilon_\beta \in \mathbb{R}_\beta^+$, such that

$$\dot{0} \dot{<} |x \dot{-} x_0|_\alpha \dot{<} \delta_\alpha \;\Rightarrow\; |f(x) \ddot{-} L|_\beta \ddot{<} \varepsilon_\beta.$$

Then, for $\delta := \alpha^{-1}(\delta_\alpha) > 0$ and $\varepsilon := \beta^{-1}\left(\varepsilon_\beta\right) > 0$, we have

$$0 < |\alpha^{-1}(x) - \alpha^{-1}(x_0)| < \delta \Rightarrow |(\beta^{-1} \circ f)(x) - \beta^{-1}(L)| < \varepsilon.$$

Proof.

i. $$\begin{aligned} \dot{0} \dot{<} |x \dot{-} x_0|_\alpha \dot{<} \delta_\alpha &\Leftrightarrow \alpha(0) \dot{<} \left|\alpha\left(\alpha^{-1}(x) - \alpha^{-1}(x_0)\right)\right|_\alpha \dot{<} \alpha\left(\alpha^{-1}(\delta_\alpha)\right) \\ &\Leftrightarrow \alpha(0) \dot{<} \alpha\left\{\left|\alpha^{-1}\left[\alpha\left(\alpha^{-1}(x) - \alpha^{-1}(x_0)\right)\right]\right|\right\} \dot{<} \alpha(\delta) \\ &\Leftrightarrow 0 < |\alpha^{-1}(x) - \alpha^{-1}(x_0)| < \delta. \end{aligned}$$

ii. $$\left|f(x) \ddot{-} L\right|_\beta \ddot{<} \varepsilon_\beta \Leftrightarrow \left|\beta\left(\beta^{-1}\left(f(x)\right) - \beta^{-1}(L)\right)\right|_\beta \ddot{<} \beta\left(\beta^{-1}\left(\varepsilon_\beta\right)\right)$$

$$\Leftrightarrow \beta\left\{\left|\beta^{-1}\left[\beta\left(\beta^{-1}(f(x)) - \beta^{-1}(L)\right)\right]\right|\right\} \ddot{<} \beta(\varepsilon)$$

$$\Leftrightarrow \beta\left(\left|\beta^{-1}(f(x)) - \beta^{-1}(L)\right|\right) \ddot{<} \beta(\varepsilon)$$

$$\Leftrightarrow \left|\beta^{-1}(f(x)) - \beta^{-1}(L)\right| < \varepsilon.$$

iii. From i), ii) and $\dot{0} \dot{<} |x \dot{-} x_0|_\alpha \dot{<} \delta_\alpha \Rightarrow \left|f(x) \ddot{-} L\right|_\beta \ddot{<} \varepsilon_\beta$, we have

$$0 < |\alpha^{-1}(x) - \alpha^{-1}(x_0)| < \delta \Rightarrow |(\beta^{-1} \circ f)(x) - \beta^{-1}(L)| < \varepsilon.$$

Proposition 2.4.2. Let $f: \mathbb{R}_\alpha \supset A \to B \subset \mathbb{R}_\beta$ and suppose that for $x \to x_0 \in$ A, the ⋆-limit

$$\star \lim_{x \to x_0} f(x) = L$$

exists. Also assume that the classical limit

$$\lim_{t \to \alpha^{-1}(x_0)} (\beta^{-1} \circ f \circ \alpha)(t)$$

exists. Then the following equality holds:

$$\lim_{t \to \alpha^{-1}(x_0)} (\beta^{-1} \circ f \circ \alpha)(t) = \beta^{-1}(L).$$

Proof. Let $\varepsilon > 0$ be arbitrary and define $\varepsilon_\beta = \beta(\varepsilon) \in \mathbb{R}_\beta$, so that $\varepsilon_\beta \ddot{>} 0$. Since the ⋆-limit exists, there is $\delta_\alpha = \delta_\alpha(\varepsilon_\beta) \dot{>} 0$, such that if $\dot{0} \dot{<} |x \dot{-} x_0|_\alpha \dot{<} \delta_\alpha$, then $|f(x) \ddot{-} L|_\beta \ddot{<} \varepsilon_\beta$. From Proposition 2.4.1 for $\delta = \alpha^{-1}(\delta_\alpha) > 0$, we have

$$0 < |\alpha^{-1}(x) - \alpha^{-1}(x_0)| < \delta \Rightarrow |(\beta^{-1} \circ f)(x) - \beta^{-1}(L)| < \varepsilon.$$

Now we use change of variables, namely $t = \alpha^{-1}(x)$ so that

$$0 < |t - \alpha^{-1}(x_0)| < \delta \Rightarrow \left|(\beta^{-1} \circ f)\left(\alpha(t)\right) - \beta^{-1}(L)\right| < \varepsilon,$$

$$0 < |t - \alpha^{-1}(x_0)| < \delta \Rightarrow |(\beta^{-1} \circ f \circ \alpha)(t) - \beta^{-1}(L)| < \varepsilon.$$

Thus, the proposition is proved.

Theorem 2.4.1. Let $f: \mathbb{R}_\alpha \supset A \to B \subset \mathbb{R}_\beta$ be a function. Suppose both the ⋆-limit

$$\star \lim_{x \to x_0} f(x)$$

and the classical limit

$$\lim_{t \to \alpha^{-1}(x_0)} (\beta^{-1} \circ f \circ \alpha)(t)$$

exist. Then the following equality holds:

$$\star \lim_{x \to x_0} f(x) = \beta\left[\lim_{t \to \alpha^{-1}(x_0)} (\beta^{-1} \circ f \circ \alpha)(t)\right].$$

Proof. From Proposition 2.4.2, we have

$$\lim_{t \to \alpha^{-1}(x_0)} (\beta^{-1} \circ f \circ \alpha)(t) = \beta^{-1}(L).$$

Applying β to both sides yields:

$$\star \lim_{x \to x_0} f(x) = \beta\left[\lim_{t \to \alpha^{-1}(x_0)} (\beta^{-1} \circ f \circ \alpha)(t)\right].$$

Proposition 2.4.3. Let $f, g: \mathbb{R}_\alpha \to \mathbb{R}_\beta$ be given functions that have $\star$ limits at the point x_0, and let $c \in \mathbb{R}_\beta$ be a constant. Then the following equalities hold:

a) $\star \lim_{x \to x_0} [f(x) \ddot{+} g(x)] = \star \lim_{x \to x_0} f(x) \ddot{+} \star \lim_{x \to x_0} g(x)$.

b) $\star \lim_{x \to x_0} [f(x) \ddot{-} g(x)] = \star \lim_{x \to x_0} f(x) \ddot{-} \star \lim_{x \to x_0} g(x)$.

c) $\star \lim_{x \to x_0} [f(x) \ddot{\times} g(x)] = \star \lim_{x \to x_0} f(x) \ddot{\times} \star \lim_{x \to x_0} g(x)$.

d) $\star \lim_{x \to x_0} \left[f(x) \ddot{/} g(x)\right] = \star \lim_{x \to x_0} f(x) \ddot{/} \star \lim_{x \to x_0} g(x)$, whenever $\star \lim_{x \to x_0} g(x) \neq 0$.

e) $\star \lim_{x \to x_0} [c \ddot{\times} f(x)] = c \ddot{\times} \star \lim_{x \to x_0} f(x)$.

Proof.

a)

$$\star \lim_{x \to x_0} [f(x) \ddot{+} g(x)] =$$

$$= \beta\left\{\lim_{t \to \alpha^{-1}(x_0)} (\beta^{-1} \circ (f \ddot{+} g) \circ \alpha)(t)\right\}$$

$$= \beta\left\{\lim_{t \to \alpha^{-1}(x_0)} ([\beta^{-1} \circ (f \ddot{+} g)] \circ \alpha)(t)\right\}$$

$$= \beta\left\{\lim_{t \to \alpha^{-1}(x_0)} ([(\beta^{-1} \circ f) + (\beta^{-1} \circ g)] \circ \alpha)(t)\right\}$$

$$= \beta\left\{\lim_{t \to \alpha^{-1}(x_0)} ([(\beta^{-1} \circ f) + (\beta^{-1} \circ g)] \circ \alpha)(t)\right\}$$

$$= \beta\left\{\lim_{t \to \alpha^{-1}(x_0)} ([\beta^{-1} \circ f] \circ \alpha)(t) + \lim_{t \to \alpha^{-1}(x_0)} ([\beta^{-1} \circ g] \circ \alpha)(t)\right\}$$

$$= \beta\left\{\lim_{t \to \alpha^{-1}(x_0)} (\beta^{-1} \circ f \circ \alpha)(t) + \lim_{t \to \alpha^{-1}(x_0)} (\beta^{-1} \circ g \circ \alpha)(t)\right\}$$

$$= \beta\left\{\lim_{t\to\alpha^{-1}(x_0)}(\beta^{-1}\circ f\circ\alpha)\,(t)\right\}\dot{+}\beta\left\{\lim_{t\to\alpha^{-1}(x_0)}(\beta^{-1}\circ g\circ\alpha)\,(t)\right\}.$$

$$\star\lim_{x\to x_0}[f(x)\dot{+}g(x)] = \star\lim_{x\to x_0}f(x)\dot{+}\star\lim_{x\to x_0}g(x).$$

b), c), d) Proofs are done similar to a)

e)

$$\star\lim_{x\to x_0}[c\,\ddot{\times}\,f(x)] = \beta\left\{\lim_{t\to\alpha^{-1}(x_0)}(\beta^{-1}\circ(c\,\ddot{\times}\,f)\circ\alpha)\,(t)\right\}$$
$$= \beta\left\{\lim_{t\to\alpha^{-1}(x_0)}([\beta^{-1}\circ(c\,\ddot{\times}\,f)]\circ\alpha)\,(t)\right\}$$
$$= \beta\left\{\lim_{t\to\alpha^{-1}(x_0)}([\beta^{-1}(c)\times(\beta^{-1}\circ f)]\circ\alpha)\,(t)\right\}.$$

Since $\beta^{-1}(c)$ is a real constant, we have

$$\star\lim_{x\to x_0}[c\,\ddot{\times}\,f(x)] = \beta\left\{\lim_{t\to\alpha^{-1}(x_0)}(\beta^{-1}(c)\times[(\beta^{-1}\circ f)\circ\alpha])\,(t)\right\}$$
$$= \beta\left\{\beta^{-1}(c)\lim_{t\to\alpha^{-1}(x_0)}(\beta^{-1}\circ f\circ\alpha)(t)\right\}$$
$$= c\,\ddot{\times}\,\beta\left\{\lim_{t\to\alpha^{-1}(x_0)}(\beta^{-1}\circ f\circ\alpha)(t)\right\}.$$
$$\star\lim_{x\to x_0}[c\,\ddot{\times}\,f(x)] = c\,\ddot{\times}\star\lim_{x\to x_0}f(x).$$

Definition 2.4.4. ($\star$-Continuity) A function $f:\mathbb{R}_\alpha\supset A\to B\subset\mathbb{R}_\beta$ is said to be $\star$-continuous at $x_0\in\mathbb{R}_\alpha$ if [1]:

$$\star\lim_{x\to x_0}f(x) = f(x_0).$$

Proposition 2.4.4. Let $f:\mathbb{R}_\alpha\supset A\to B\subset\mathbb{R}_\beta$ be a function. If f is $\star$-continuous at point $x_0\in A$, then the function $(\beta^{-1}\circ f\circ\alpha):\alpha^{-1}(A)\to\mathbb{R}$ is continuous at point $\alpha^{-1}(x_0)\in\alpha^{-1}(A)$ in the classical sense.

Proof. f is $\star$-continuous at the point $x_0\in A$, so

$$\star\lim_{x\to x_0}f(x) = f(x_0)$$
$$\Rightarrow \beta\left[\lim_{t\to\alpha^{-1}(x_0)}(\beta^{-1}\circ f\circ\alpha)\,(t)\right] = f(x_0)$$
$$\Rightarrow \beta\left[\lim_{t\to\alpha^{-1}(x_0)}(\beta^{-1}\circ f\circ\alpha)\,(t)\right] = \beta\left[(\beta^{-1}\circ f\circ\alpha)\left(\alpha^{-1}(x_0)\right)\right]$$
$$\Rightarrow \lim_{t\to\alpha^{-1}(x_0)}(\beta^{-1}\circ f\circ\alpha)\,(t) = (\beta^{-1}\circ f\circ\alpha)\left(\alpha^{-1}(x_0)\right).$$

Thus, $(\beta^{-1}\circ f\circ\alpha)$ is continuous at point $\alpha^{-1}(x_0)$.

Proposition 2.4.5. Let $g: \mathbb{R} \supset A \to B \subset \mathbb{R}$ be a function and $\star = (\alpha, \beta)$. If g is continuous in the classical sense at the point $x_0 \in A$, then the function $(\beta \circ g \circ \alpha^{-1}): \mathbb{R}_\alpha \supset \alpha(A) \to \beta(B) \subset \mathbb{R}_\beta$ is $\star$-continuous at point $\alpha(x_0) \in \alpha(A)$.

Proof. The proof proceeds in the same way as the preceding proposition.

Lemma 2.4.2. Let $\star_1 = (\alpha, \gamma)$, $\star_2 = (\gamma, \beta)$, and $\star_3 = (\alpha, \beta)$. Additionally, suppose $f: \mathbb{R}_\alpha \to \mathbb{R}_\gamma$ is $\star_1$-continuous and $g: \mathbb{R}_\gamma \to \mathbb{R}_\beta$ is $\star_1$-continuous, then $g \circ f: \mathbb{R}_\alpha \to \mathbb{R}_\beta$ is $\star_3 = (\alpha, \beta)$ continuous.

Proof. f is$\star_1$-continuous $\Rightarrow$ $F = \gamma^{-1} \circ f \circ \alpha$ is continuous in the classical sense.

g is$\star_2$-continuous $\Rightarrow$ $G = \beta^{-1} \circ g \circ \gamma$ is continuous in the classical sense.

Thus $G \circ F = \beta^{-1} \circ (g \circ f) \circ \alpha$ is continuous in the classical sense. Let us define this function as

$$H = \beta^{-1} \circ (g \circ f) \circ \alpha.$$

Since H is is continuous in the classical sense, $g \circ f = \beta \circ H \circ \alpha^{-1}$ is $\star_3$-continuous.

Theorem 2.4.2. The α-absolute value function is continuous.

Proof. Let $\star = (\alpha,\ \alpha)$ and consider $f = I$ (the identity function). Then for any $c \in \mathbb{R}_\alpha$, we have

$$\begin{aligned}
\star \lim_{x \to c} |x|_\alpha &= \star \lim_{x \to c} (\alpha \circ |\alpha^{-1} \circ I|)(x) \\
&= \alpha \left[\lim_{t \to \alpha^{-1}(c)} (\alpha^{-1} \circ \alpha \circ |\alpha^{-1}| \circ \alpha)(t) \right] \\
&= \alpha \left[\lim_{t \to \alpha^{-1}(c)} (|\alpha^{-1}| \circ \alpha)(t) \right] \\
&= \alpha \left[\left(|\alpha^{-1}| \alpha\left(\alpha^{-1}(c)\right) \right) \right] \\
&= \alpha[|\alpha^{-1}|(c)] = \alpha(|\alpha^{-1}(c)|). \\
\star \lim_{x \to c} |x|_\alpha &= |c|_\alpha.
\end{aligned}$$

Hence, the α-absolute value function is continuous.

Definition 2.4.5. ($\star$-Derivative) Let α, β be two generators, and $\iota: \mathbb{R}_\alpha \to \mathbb{R}_\beta, \iota = \beta \circ \alpha^{-1}$.

Additionally, let $f: \mathbb{R}_\alpha \supset A \to B \subset \mathbb{R}_\beta$ be a function. If the following $\star$-limit exists, then the $\star$-derivative of f at $x_0 \in A$ is defined by [1]:

$$f^\star(x_0) = \star \lim_{x \to x_0} \left\{ [f(x) \ddot{-} f(x_0)] \ddot{/} [\iota(x) \dot{-} \iota(x_0)] \right\}.$$

Theorem 2.4.3. Let $f: \mathbb{R}_\alpha \supset A \to B \subset \mathbb{R}_\beta$ be a function. If f is $\star$-differentiable at $x_0 \in A$ and the composite function $(\beta^{-1} \circ f \circ \alpha)$ is classically differentiable at $\alpha^{-1}(x_0)$, then:

$$f^\star(x_0) = [\beta \circ (\beta^{-1} \circ f \circ \alpha)' \circ \alpha^{-1}](x_0).$$

Here, the classical derivative of $f(x)$ is represented by $f'(x)$.

Proof. Let us define

$$g(x) = [f(x) \ddot{-} f(x_0)] \ddot{/} [\iota(x) \ddot{-} \iota(x_0)].$$

Then $f^{\star}(x_0) = \star \lim_{x \to x_0} g(x)$. First, note that

$$\begin{aligned} g(x) &= [f(x) \ddot{-} f(x_0)] \ddot{/} [\iota(x) \ddot{-} \iota(x_0)] \\ &= [f(x) \ddot{-} f(x_0)] \ddot{/} [(\beta \circ \alpha^{-1})(x) \ddot{-} (\beta \circ \alpha^{-1})(x_0)] \\ &= \beta \left\{ \frac{\beta^{-1}\big(f(x)\big) - \beta^{-1}\big(f(x_0)\big)}{\alpha^{-1}(x) - \alpha^{-1}(x_0)} \right\} \end{aligned}$$

and

$$\begin{aligned} g\big(\alpha(t)\big) &= \beta \left\{ \frac{\beta^{-1}\Big(f\big(\alpha(t)\big)\Big) - \beta^{-1}\big(f(x_0)\big)}{\alpha^{-1}\big(\alpha(t)\big) - \alpha^{-1}(x_0)} \right\} \\ &= \beta \left\{ \frac{(\beta^{-1} \circ f \circ \alpha)(t) - (\beta^{-1} \circ f \circ \alpha)\big(\alpha^{-1}(x_0)\big)}{t - \alpha^{-1}(x_0)} \right\}. \end{aligned}$$

Therefore,

$$\begin{aligned} f^{\star}(x_0) &= \star \lim_{x \to x_0} g(x) \\ &= \beta \left[\lim_{t \to \alpha^{-1}(x_0)} (\beta^{-1} \circ g \circ \alpha)\,(t) \right] \\ &= \beta \left\{ \lim_{t \to \alpha^{-1}(x_0)} \beta^{-1}[g(\alpha(t)] \right\} \\ &= \beta \left\{ \lim_{t \to \alpha^{-1}(x_0)} \frac{(\beta^{-1} \circ f \circ \alpha)(t) - (\beta^{-1} \circ f \circ \alpha)\big(\alpha^{-1}(x_0)\big)}{t - \alpha^{-1}(x_0)} \right\}. \end{aligned}$$

$$f^{\star}(x_0) = \beta\big\{(\beta^{-1} \circ f \circ \alpha)'\big(\alpha^{-1}(x_0)\big)\big\}.$$

Thus, the result follows:

$$f^{\star}(x_0) = [\beta \circ (\beta^{-1} \circ f \circ \alpha)' \circ \alpha^{-1}](x_0).$$

2.4.1. Star ($\star$) integral

Definition 2.4.1.1. ($\star$-Riemann Integral) Let α and β be given generator functions, $\mathbb{R}_\alpha$ and $\mathbb{R}_\beta$ be the corresponding vector spaces, and $[a, b]_\alpha$ be an α-interval. Additionally, let $P = \{x_0 = a, x_1, \ldots, x_n = b\}$ be a partition of the interval $[a, b]_\alpha$, such that

$$h = \alpha\left(\frac{\alpha^{-1}(b) - \alpha^{-1}(a)}{n}\right) \text{ and } x_k = x_0 \dot{+} [\alpha(k) \dot{\times} h].$$

Consider a function $f: [a,b]_\alpha \subset \mathbb{R}_\alpha \to \mathbb{R}_\beta$.

If the limit

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} \left[(\beta^{-1} \circ f)(x_k) \times \left(\alpha^{-1}(x_{k+1}) - \alpha^{-1}(x_k)\right)\right]$$

exists, then the function f is said to be ⋆-Riemann integrable on the interval $[a,b]_\alpha$. In this case, the corresponding ⋆-Riemann integral is given by

$$\beta\left[\lim_{n\to\infty} \sum_{k=0}^{n-1} \left[(\beta^{-1} \circ f)(x_k) \times \left(\alpha^{-1}(x_{k+1}) - \alpha^{-1}(x_k)\right)\right]\right]$$

and it is denoted by the symbolic expression

$$\int_{[a,b]_\alpha}^{\star} f(x)dx = \beta\left[\lim_{n\to\infty} \sum_{k=0}^{n-1} \left[(\beta^{-1} \circ f)(x_k) \times \left(\alpha^{-1}(x_{k+1}) - \alpha^{-1}(x_k)\right)\right]\right].$$

This integral generalizes the classical Riemann integral within the ⋆-analysis framework, where the domain and codomain of the function are defined over distinct arithmetic structures governed by the α and β generator functions, respectively.

2.4.2. Star vector spaces

Theorem 2.4.2.1. Let β be a generator, and let $\mathbb{R}_\beta$ be a corresponding set equipped with the operations defined below. Then $\mathbb{R}_\beta$ forms a vector space over the field $(\mathbb{R}_\alpha, \dot{+}, \dot{\times})$:

- The vector addition is given by the operation $\ddot{+}$, referred to as β-addition.
- The scalar multiplication is defined as follows:

Let $v \in \mathbb{R}_\beta$ and $r \in \mathbb{R}_\alpha$, then the scalar multiple of v by r is defined by

$$(r, v) \mapsto r \cdot v = \beta[\alpha^{-1}(r) \times \beta^{-1}(v)] = (\beta \circ \alpha^{-1})(r) \ddot{\times} v.$$

Proof. The set $\left(\mathbb{R}_\beta, \ddot{+}\right)$ forms an abelian group. Hence, it suffices to verify the vector space axioms related to scalar multiplication.

Let $r, s \in \mathbb{R}_\alpha$ and $u, v \in \mathbb{R}_\beta$. Then, the following identities hold:

- *Associativity of scalar multiplication*:

$$s \cdot (r \cdot v) = (s \dot{\times} r) \cdot v.$$

To prove this identity, we use Proposition 2.3.5

$$s \cdot (r \cdot v) = s \cdot \beta[\alpha^{-1}(r) \times \beta^{-1}(v)]$$

$$
\begin{aligned}
&= (\beta \circ \alpha^{-1})(s) \ddot{\times} \{(\beta \circ \alpha^{-1})(r) \ddot{\times} v\} \\
&= \beta\{\alpha^{-1}(s) \times \alpha^{-1}(r) \times \beta^{-1}(v)\} \\
&= \beta\{\alpha^{-1}(s \dot{\times} r) \times \beta^{-1}(v)\}.
\end{aligned}
$$

$$s \cdot (r \cdot v) = (s \dot{\times} r) \cdot v.$$

- *Existence of multiplicative identity*:

Let $\dot{1} \in \mathbb{R}_\alpha$ be the multiplicative identity with respect to α-multiplication. Then $\forall v \in \mathbb{R}_\beta$

$$\dot{1} \cdot v = v$$

is satisfied since for all $v \in \mathbb{R}_\beta$

$$\dot{1} \cdot v = \beta\left[\alpha^{-1}(\dot{1}) \times \beta^{-1}(v)\right] = \beta[1 \times \beta^{-1}(v)] = \beta[\beta^{-1}(v)] = v.$$

- *Distributivity over vector addition*: For all $r \in \mathbb{R}_\alpha, u, v \in \mathbb{R}_\beta$ the following equality holds:

$$r \cdot (u \ddot{+} v) = (r \cdot u) \ddot{+} (r \cdot v),$$

since for all $r \in \mathbb{R}_\alpha, u, v \in \mathbb{R}_\beta$, we have

$$
\begin{aligned}
r \cdot (u \ddot{+} v) &= \beta[\alpha^{-1}(r) \times \beta^{-1}(u \ddot{+} v)] \\
&= \beta\{\alpha^{-1}(r) \times [\beta^{-1}(u) + \beta^{-1}(v)]\} \\
&= \beta\{[\alpha^{-1}(r) \times \beta^{-1}(u)] + [\alpha^{-1}(r) \times \beta^{-1}(v)]\} \\
&= \beta[\alpha^{-1}(r) \times \beta^{-1}(u)] \ddot{+} \beta[\alpha^{-1}(r) \times \beta^{-1}(v)] \\
r \cdot (u \ddot{+} v) &= (r \cdot u) \ddot{+} (r \cdot v).
\end{aligned}
$$

- *Distributivity over scalar addition*: For all $r, s \in \mathbb{R}_\alpha, v \in \mathbb{R}_\beta$ the following equality holds:

$$(r \dot{+} s) \cdot v = (r \cdot v) \ddot{+} (s \cdot v)$$

since for all $r, s \in \mathbb{R}_\alpha, v \in \mathbb{R}_\beta$, we have

$$
\begin{aligned}
(r \dot{+} s) \cdot v &= \beta[\alpha^{-1}(r \dot{+} s) \times \beta^{-1}(v)] \\
&= \beta\{[\alpha^{-1}(r) + \alpha^{-1}(s)] \times \beta^{-1}(v)\} \\
&= \beta\{[\alpha^{-1}(r) \times \beta^{-1}(v)] + [\alpha^{-1}(s) \times \beta^{-1}(v)]\} \\
&= \beta[\alpha^{-1}(r) \times \beta^{-1}(v)] \ddot{+} \beta[\alpha^{-1}(s) \times \beta^{-1}(v)] \\
(r \dot{+} s) \cdot v &= (r \cdot v) \ddot{+} (s \cdot v).
\end{aligned}
$$

Definition 2.4.2.1. ($\star$-Vector Space) Let $\star = (\alpha\text{ arithmetic}, \beta\text{ arithmetic})$. Then the vector space defined over the field $(\mathbb{R}_\alpha, \dot{+}, \dot{\times})$ with the operations of β-addition $\ddot{+}$ and scalar multiplication $\cdot$ on $\mathbb{R}_\beta$ is called a $\star$-vector space.

Definition 2.4.2.2. (α-Normed Vector Space) Let $X \subset \mathbb{R}_\alpha$ be a vector space over $\mathbb{R}_\alpha$. A function

$\|\cdot\|_\alpha : X \rightarrow \mathbb{R}_\alpha$ is called an α-*norm* on X if for $\forall x, y \in X$ and $\forall \lambda \in \mathbb{R}_\alpha$, and the following conditions are satisfied:

- $\|x\|_\alpha = \dot{0} \Leftrightarrow x = \dot{0}$,
- $\|\lambda \dot{\times} x\|_\alpha = |\lambda|_\alpha \dot{\times} \|x\|_\alpha$,
- $\|x \dot{+} y\|_\alpha \dot{\leq} \|x\|_\alpha \dot{+} \|y\|_\alpha$.

In this case, the pair $(X, \|\cdot\|_\alpha)$ is called an α-normed vector space. Moreover, every α-norm $\|\cdot\|_\alpha$ on X induces an α-metric d_α, defined by

$$d_\alpha(x, y) = \|x \dot{-} y\|_\alpha,$$

which equips X with a corresponding α-metric space structure [9].
Definition 2.4.2.3. (α-Banach Space) Let $(X, \|\cdot\|_\alpha)$ be an α-normed vector space. If every α-Cauchy sequence in X converges to a limit in X, then the space $(X, \|\cdot\|_\alpha)$ is called an α-Banach space (i.e., a complete α-normed space).

To prove that an α-normed space is an α-Banach space, it is sufficient to show that every α-Cauchy sequence in X converges in X. If $d_\alpha : X \times X \rightarrow \mathbb{R}_\alpha$ is the α-metric induced by the norm $\|\cdot\|_\alpha$, and if the metric space (X, d_α) is complete, then the α-normed space $(X, \|\cdot\|_\alpha)$ is also an α-Banach space [3].

2.4.3. ⋆-Linear operators

Definition 2.4.3.1. Let $\mathbb{R}_\alpha$ and $\mathbb{R}_\beta$ be two α-normed vector spaces, and let $\star = (\mathbb{R}_\alpha, \ \mathbb{R}_\beta)$. A mapping $T: \mathbb{R}_\alpha \rightarrow \mathbb{R}_\beta$ is called a ⋆-operator from $\mathbb{R}_\alpha$ to $\mathbb{R}_\beta$.

Definition 2.4.3.2. (⋆-Linear Operator) Let $\mathbb{R}_\alpha$ and $\mathbb{R}_\beta$ be two α-normed vector spaces defined over a common field $\mathbb{R}_\gamma$, and let $\star = (\mathbb{R}_\alpha, \ \mathbb{R}_\beta)$. If $T: \mathbb{R}_\alpha \rightarrow \mathbb{R}_\beta$ satisfies

$$T(a \cdot x \dot{+} b \cdot y) = a \cdot T(x) \ddot{+} b \cdot T(y),$$

for all $x, y \in \mathbb{R}_\alpha$ and $\forall a, b \in \mathbb{R}_\gamma$, then T is called a ⋆-linear operator.

2.4.4. ⋆-Differentiation operator

Notation 2.4.4.1. Let $C_\beta[a, b]_\alpha$ denote the set of ⋆-continuous functions from the α-interval $[a, b]_\alpha$ to $\mathbb{R}_\beta$.

Theorem 2.4.4.1. If $f \in C_\beta[a, b]_\alpha$ then $(\beta^{-1} \circ f \circ \alpha) \in C[\alpha^{-1}(a), \alpha^{-1}(b)]$.

Proof. Suppose $f \in C_\beta[a, b]_\alpha$. Since f is ⋆-continuous on $[a, b]_\alpha$, for every $c \in [a, b]_\alpha$ we have:

$$\star \lim_{x \to c} f(x) = f(c).$$

$$\beta \left[\lim_{t \to \alpha^{-1}(c)} (\beta^{-1} \circ f \circ \alpha)\,(t) \right] = f(c).$$

$$\lim_{t \to \alpha^{-1}(c)} (\beta^{-1} \circ f \circ \alpha)\,(t) = \beta^{-1}\big(f(c)\big).$$

$$\lim_{t \to \alpha^{-1}(c)} (\beta^{-1} \circ f \circ \alpha)\,(t) = (\beta^{-1} \circ f)(c).$$

$$\lim_{t \to \alpha^{-1}(c)} (\beta^{-1} \circ f \circ \alpha)\,(t) = (\beta^{-1} \circ f \circ \alpha)\big(\alpha^{-1}(c)\big).$$

Therefore, $(\beta^{-1} \circ f \circ \alpha)$ is continuous at the point $\alpha^{-1}(c)$. Since we take c to be arbitrary and $\alpha^{-1}([a,b]_\alpha) = [\alpha^{-1}(a),\ \alpha^{-1}(b)]$, $\beta^{-1} \circ f \circ \alpha$ is continuous on the interval $[\alpha^{-1}(a),\ \alpha^{-1}(b)]$.

Notation 2.4.4.2. Let $C_\beta^{\star(r)}[a,b]_\alpha$ denote the set of functions that are $\star$-differentiable up to order r on $[a,b]_\alpha$, where $r \geq 1$ and $r \in \mathbb{N}$.

Theorem 2.4.4.2. Let $\mathbb{R}_\alpha$ and $\mathbb{R}_\beta$ be two α-normed vector spaces, and let $\star = \big(\mathbb{R}_\alpha,\ \mathbb{R}_\beta\big)$. The set $C_\beta[a,b]_\alpha$ forms a vector space over the field $\mathbb{R}_\beta$ under the operations:

- $(f \ddot{+} g)(t) = f(t) \ddot{+} g(t)$,
- $(c \cdot f)(t) = c \ddot{\times} f(t)$, for $t \in [a,b]_\alpha$, $c \in \mathbb{R}_\beta$.

This space is denoted by $\big(C_\beta[a,b]_\alpha, \mathbb{R}_\beta\big)$.

Proof. For all $f, g, h \in C_\beta[a,b]_\alpha$; $c, c_1, c_2 \in \mathbb{R}_\beta$ and $t \in [a,b]_\alpha$

- *Commutativity of Addition*:

$$f \ddot{+} g = g \ddot{+} f.$$

Since addition in $\mathbb{R}_\beta$ is commutative: $\forall t \in [a,b]_\alpha$

$$f(t) \ddot{+} g(t) = g(t) \ddot{+} f(t).$$

- *Associativity of Addition*:

$$f \ddot{+} (g \ddot{+} h) = (f \ddot{+} g) \ddot{+} h.$$

Since addition in $\mathbb{R}_\beta$ is associative: $\forall t \in [a,b]_\alpha$

$$f(t) \ddot{+} \big(g(t) \ddot{+} h(t)\big) = \big(f(t) \ddot{+} g(t)\big) \ddot{+} h(t).$$

- *Existence of Zero Vector*:

Let $0_\beta(t) = \ddot{0}$ for all $t \in [a,b]_\alpha$. Then,

$$f \ddot{+} 0_\beta = f.$$

- *Existence of Additive Inverse*:

For each $f \in C_\beta[a,b]_\alpha$, define $\ddot{-}f = 0_\beta \ddot{-} f$. Then,

$$f \ddot{+} (\ddot{-}f) = 0_\beta.$$

- *Scalar Multiplication Identity*:

Let $1_\beta(t) = \ddot{1}$ for all $t \in [a,b]_\alpha$. Then,

$$1_\beta \ddot{\times} f = f.$$

- *Distributivity of Scalar Multiplication over Vector Addition*:

$$c \cdot (f \ddot{+} g) = (c \cdot f) \ddot{+} (c \cdot g).$$

- *Distributivity over Field Addition*:

$$(c_1 \ddot{+} c_2) \cdot f = (c_1 \cdot f) \ddot{+} (c_2 \cdot f).$$

- *Compatibility of Scalar Multiplication*:

$$(c_1 \ddot{\times} c_2) \cdot f = c_1 \ddot{\times} (c_2 \cdot f).$$

Hence, all vector space axioms are satisfied. Therefore,

$$\left(C_\beta[a,b]_\alpha, \mathbb{R}_\beta\right)$$

is a vector space.

Notation 2.4.4.3. Let $g: [a,b]_\alpha \to \mathbb{R}_\beta$ be β continuous and $h: [c,d] \to \mathbb{R}$ be continuous in a classical sense. Thus, we use the following notations

$$\overset{\beta}{\max_{t \in [a,b]_\alpha}} g(t) = \overset{\beta}{\max}\{g(t) \mid t \in [a,b]_\alpha\}.$$

$$\max_{x \in [c,d]} h(x) = \max\{h(x) \mid x \in [c,d]\}.$$

Lemma 2.4.4.1. Let $f \in C_\beta[a,b]_\alpha$. Then the β-maximum value of $|f|_\beta$ on $[a,b]_\alpha$ exists.

Proof. We have

$$\begin{aligned}\overset{\beta}{\max_{t \in [a,b]_\alpha}} |f|_\beta(t) &= \beta\left(\max\left[\beta^{-1}\left\{|f|_\beta(t): t \in [a,b]_\alpha\right\}\right]\right) \\ &= \beta\left(\max\left[\beta^{-1}\left\{|f|_\beta(t): t \in [a,b]_\alpha\right\}\right]\right) \\ &= \beta\left(\max\left\{\left(\beta^{-1} \circ |f|_\beta\right)(t): t \in [a,b]_\alpha\right\}\right).\end{aligned}$$

Let us perform the change of variables $x = \alpha^{-1}(t)$. Since $a \dot{\leq} t \dot{\leq} b$, it follows that $\alpha^{-1}(a) \leq x \leq \alpha^{-1}(b)$ from the definition of α-order. Therefore,

$$\overset{\beta}{\max_{x\in[a,b]_\alpha}} |f|_\beta(t) = \beta\big(\max\{(\beta^{-1} \circ |f|_\beta)(\alpha(x)) : x \in [\alpha^{-1}(a), \alpha^{-1}(b)]\}\big)$$
$$= \beta\left(\max_{x\in[\alpha^{-1}(a),\alpha^{-1}(b)]} (\beta^{-1} \circ |f|_\beta \circ \alpha)(x)\right).$$

Since $|f|_\beta = \beta \circ |\beta^{-1} \circ f|$ we can rewrite this as

$$\overset{\beta}{\max_{x\in[a,b]_\alpha}} |f|_\beta(t) = \beta\left(\max_{x\in[\alpha^{-1}(a),\alpha^{-1}(b)]} (\beta^{-1} \circ \beta \circ |\beta^{-1} \circ f| \circ \alpha)(x)\right),$$
$$\overset{\beta}{\max_{x\in[a,b]_\alpha}} |f|_\beta(t) = \beta\left(\max_{x\in[\alpha^{-1}(a),\alpha^{-1}(b)]} (|\beta^{-1} \circ f| \circ \alpha)(x)\right).$$

Since $\beta^{-1} \circ f$ and $|\beta^{-1} \circ f|$ are $\star_1 = (\alpha, I)$ continuous, α is $\star_2 = (I, \alpha)$ continuous, we have that $|\beta^{-1} \circ f| \circ \alpha$ is $\star_3 = (I, I)$ continuous, i.e., $|\beta^{-1} \circ f| \circ \alpha$ a is continuous on the compact set $[\alpha^{-1}(a), \alpha^{-1}(b)]$ in the classical sense. By the Weierstrass extreme value theorem, $|\beta^{-1} \circ f| \circ \alpha$ attains a maximum on this interval. Hence, the β-maximum of $|f|_\beta$ on $[a, b]_\alpha$ exists.

Theorem 2.4.4.2. The vector space $C_\beta[a, b]_\alpha$ equipped with the α-norm

$$\|f\|_{C_\beta[a,b]_\alpha} = \max\{|f(t)|_\beta : t \in [a, b]_\alpha\}$$

is an α-normed vector space.

Theorem 2.4.4.3. The set $C_\beta^{\star(r)}[a, b]_\alpha$ forms a vector space over $\mathbb{R}_\beta$ with operations:

- $(f \ddot{+} g)(t) = f(t) \ddot{+} g(t)$,
- $(a \cdot f)(t) = a \ddot{\times} f(t)$, for $t \in [a, b]_\alpha$, $a \in \mathbb{R}_\beta$.

This space is denoted by $\left(C_\beta^{\star(r)}[a, b]_\alpha, \mathbb{R}_\beta\right)$.

Definition 2.4.4.1. (α-Summation Operator) For $x_i \in \mathbb{R}_\alpha$, $0 \leq i \leq n$, the α-summation is defined by:

$$\overset{\alpha}{\sum_{0\leq i\leq n}} x_i = x_0 \dot{+} x_1 \dot{+} \cdots \dot{+} x_n.$$

Theorem 2.4.4.4. Let $C_\beta^{\star(r)}[a, b]_\alpha$ be the vector space of function defined on the closed interval $[a, b]_\alpha$ with values in the field $\mathbb{R}_\beta$, and suppose this space is equipped with the following α-norm:

$$\|f\|_{C_\beta^{\star(r)}[a,b]_\alpha} = \overset{\beta}{\sum_{0\leq i\leq r}} \left\|f^{\star(i)}\right\|_{C_\beta[a,b]_\alpha}.$$

Here, the α-summation with respect to the generator β is used. Then $\left(C_\beta^{\star(r)}[a,b]_\alpha,\ \|\cdot\|_{C_\beta^{\star(r)}[a,b]_\alpha}\right)$ forms a α-normed vector space.

Definition 2.4.4.2. Let $A: D(A) = \left(C_\beta^\star[a,b]_\alpha, \|\cdot\|_{C_\beta^\star[a,b]_\alpha}\right) \to \left(C_\beta[a,b]_\alpha, \|\cdot\|_{C_\beta[a,b]_\alpha}\right)$ be an operator defined by

$$Af(t) = f^\star(t), t \in [a,b]_\alpha.$$

This operator is called the $\star$-derivative operator.

Proposition 2.4.4.1. The $\star$-derivative operator defined above is linear over $\mathbb{R}_\beta$.

Proof. Let $A: C_\beta^\star[a,b]_\alpha \to C_\beta[a,b]_\alpha$ be a $\star$-derivative operator. Additionally, let $f, g \in C_\beta^\star[a,b]_\alpha$ and $c_1, c_2 \in \mathbb{R}_\beta$

$$\begin{aligned}
&A\big(c_1 \ddot{\times} f(t) \ddot{+} c_2 \ddot{\times} g(t)\big)\\
&=\star \lim_{x\to t}\big\{\big[\big(c_1 \ddot{\times} f(x) \ddot{+} c_2 \ddot{\times} g(x)\big) \ddot{-} \big(c_1 \ddot{\times} f(t) \ddot{+} c_2 \ddot{\times} g(t)\big)\big] \ddot{/} [\iota(x) \ddot{-} \iota(t)]\big\}\\
&=\star \lim_{x\to t}\big\{\big(\big[\big(c_1 \ddot{\times} f(x)\big) \ddot{-} \big(c_1 \ddot{\times} f(t)\big)\big] \ddot{+} \big[\big(c_2 \ddot{\times} g(x)\big) \ddot{-} \big(c_2 \ddot{\times} g(t)\big)\big]\big) \ddot{/} [\iota(x) \ddot{-} \iota(t)]\big\}\\
&=\star \lim_{x\to t}\big\{\big[\big(c_1 \ddot{\times} f(x)\big) \ddot{-} \big(c_1 \ddot{\times} f(t)\big)\big] \ddot{/} [\iota(x) \ddot{-} \iota(t)]\big\} \ddot{+} \star \lim_{x\to t}\big\{\big[\big(c_2 \ddot{\times} g(x)\big) \ddot{-} \big(c_2 \ddot{\times} g(t)\big)\big] \ddot{/} [\iota(x) \ddot{-} \iota(t)]\big\}\\
&= \Big[c_1 \ddot{\times} \star \lim_{x\to t}\big\{[f(x) \ddot{-} f(t)] \ddot{/} [\iota(x) \ddot{-} \iota(t)]\big\}\Big] \ddot{+} \Big[c_2 \ddot{\times} \star \lim_{x\to t}\big\{[g(x) \ddot{-} g(t)] \ddot{/} [\iota(x) \ddot{-} \iota(t)]\big\}\Big]\\
&= [c_1 \ddot{\times} f^\star(t)] \ddot{+} [c_2 \ddot{\times} g^\star(t)]\\
&= c_1 \ddot{\times} A(f) + c_2 \ddot{\times} A(g),
\end{aligned}$$

which is needed to prove that $\star$-derivative operator is linear.

2.4.5. $\star$-Integral operators

Notation 2.4.5.1. Let $\mathcal{R}_\beta[a,b]_\alpha$ denote the set of all functions that are $\star$-Riemann integrable over the interval $[a,b]_\alpha$.

Theorem 2.4.5.1. Let $\mathbb{R}_\alpha$ and $\mathbb{R}_\beta$ be two α-normed vector spaces, and let $\star = (\mathbb{R}_\alpha,\ \mathbb{R}_\beta)$. Then the set $\mathcal{R}_\beta[a,b]_\alpha$, equipped with the operations

- $(f \ddot{+} g)(t) = f(t) \ddot{+} g(t)$,
- $(c \cdot f)(t) = c \ddot{\times} f(t)$, for all $t \in [a,b]_\alpha$, $c \in \mathbb{R}_\beta$,

forms a vector space over the field $\mathbb{R}_\beta$. This space is denoted by $(\mathcal{R}_\beta[a,b]_\alpha, \mathbb{R}_\beta)$.

Theorem 2.4.5.2. The vector space $\mathcal{R}_\beta[a,b]_\alpha$, defined over the field $\mathbb{R}_\beta$, becomes an α-normed vector space under the α-norm given by

$$\|f\|_{\mathcal{R}_\beta[a,b]_\alpha} = \int_{[a,b]_\alpha}^{\star} |f(x)|_\beta dx.$$

This normed space is denoted by $\left(\mathcal{R}_\beta[a,b]_\alpha, \|\cdot\|_{\mathcal{R}_\beta[a,b]_\alpha}\right)$.

Definition 2.4.5.1. Let $\mathcal{J}: \mathcal{R}_\beta[a,b]_\alpha \to \mathbb{R}_\beta$ be the operator defined by

$$\mathcal{J}_{[a,b]} f(x) = \int_{[a,b]_\alpha}^{\star} f(t)dt.$$

This operator is called the $\star$-Fredholm integral operator.

Definition 2.4.5.2. Let $x \in [a,b]_\alpha$, and define the operator $\mathrm{K}: \mathcal{R}_\beta[a,b]_\alpha \to \mathcal{R}_\beta[a,b]_\alpha$ by

$$\mathrm{K} f(x) = \int_{[a,x]_\alpha}^{\star} f(t)dt.$$

This operator is called the $\star$-Volterra integral operator.

3. Conclusions

In this study, we systematically examined the algebraic and topological foundations of non-Newtonian analysis, an extension of classical mathematical structures achieved through generator functions and generalized arithmetic operations. The introduction of alpha-arithmetic led to the construction of a novel numerical framework in which arithmetic, topology, and analysis were redefined through generator-based transformations.

The investigation began with defining the set of alpha real numbers ($\mathbb{R}_\alpha$), obtained from the classical real numbers through a bijective and continuous generator function α. Based on this foundation, group and field structures were established, and vector spaces were defined over $\mathbb{R}_\alpha$ by means of alpha linear operations.

From a topological perspective, we introduced concepts such as α-intervals, α-neighborhoods, and α-open sets, along with an α-metric that established a meaningful and coherent topological structure on $\mathbb{R}_\alpha$. These constructions constitute adaptations of classical topological notions within the non-Newtonian framework.

A key contribution of this study is the formulation of $\star$-analysis, a methodological framework enabling systematic transitions between arithmetic systems. Through $\star$-operators that map between α and β systems, the framework enables structural compatibility across analytical settings.

Overall, the findings demonstrated that non-Newtonian analysis provides a robust and extensible theoretical framework for generalizing classical algebraic and analytical structures. Its theoretical flexibility and operational adaptability make it a promising direction for future research in pure and applied mathematics.

Author contributions

Emre Çıvgın: Conceptualization, methodology, formal analysis, investigation, visualization, writing original draft; Numan Yalçın: Supervision, validation, project administration, writing, review and editing. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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