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Research article

Global boundedness of solutions to a quasilinear prey-taxis model with prey-stage structure

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Abstract: This paper investigates the prey-stage structure within a quasilinear prey-taxis model defined in a smooth bounded domain in arbitrary-dimensional spaces. When the exponents of the nonlinear diffusion function and the nonlinear prey-taxis sensitivity function satisfy appropriate conditions, the global boundedness of the solution is established through the application of coupling estimation technique and Moser iteration.

Keywords: predator-prey; prey-stage; boundeness; nonlinear diffusion; nonlinear prey-taxis
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1. Introduction

In natural ecosystems, a considerable number of animal species exhibit life cycles encompassing at least two distinct developmental stages: the immature stage and the mature stage. This stage-structured characteristic exerts a profound influence on key population dynamics, including population growth, interspecific competition, and species interaction patterns. Over the past decades, predator-prey models that integrate stage structure have been the subject of extensive research (see [1–4]). To mathematically characterize such biological phenomena, Zhang et al. [5] proposed the following predator-prey model, which specifically incorporates stage structure in the prey population:

$$\begin{cases} \frac{\mathrm{d}x_1}{\mathrm{d}t} = Bx_2 - Cx_1 - D_1x_1 - \gamma x_1^2 - kx_1y, & t > 0, \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} = Cx_1 - D_2x_2, & t > 0, \\ \frac{\mathrm{d}y}{\mathrm{d}t} = -D_3y + \delta_1 kx_1y - \eta y^2, & t > 0. \end{cases} \tag{1.1}$$

Let the concentrations of immature prey, mature prey, and predators be denoted by $x_1(t)$, $x_2(t)$, and $y(t)$, respectively. The positive parameter B represents the birth rate of immature prey. The parameter $C > 0$ denotes the conversion rate of immature prey to mature prey. The positive constants γ and η stand for the mortality rate caused by intra-specific competition among immature prey and that among predators, respectively. The parameter $k > 0$ describes the consumption rate of immature prey (by predators). The parameters $D_i > 0$ $(i = 1, 2, 3)$ represent the mortality rates of immature prey, mature prey, and predators, respectively. The parameter $\delta_1 > 0$ denotes the predation rate of predators (on prey).

When $\eta = 0$, a more general form of the model can be derived from (1.1) as shown below:

$$\begin{cases} \frac{\mathrm{d}u_1}{\mathrm{d}t} = au_2 - bu_1 - \gamma u_1^2 - g(u_1)v, & t > 0, \\ \frac{\mathrm{d}u_2}{\mathrm{d}t} = u_1 - u_2, & t > 0, \\ \frac{\mathrm{d}v}{\mathrm{d}t} = -\mu v + \delta g(u_1)v, & t > 0. \end{cases} \tag{1.2}$$

The term $g(u_1)v$ represents the interspecific interaction, where $g(u_1)$ denotes the functional response function. This function is commonly formulated in forms such as Holling Type I (i.e., $g(u_1) = u_1$), Holling Type II (i.e., $g(u_1) = \frac{u_1}{\lambda + u_1}$ with a positive constant $\lambda > 0$), and Holling Type III (i.e., $g(u_1) = \frac{u_1^\theta}{\lambda^\theta + u_1^\theta}$ with constants $\theta > 1$ and $\lambda > 0$). In the case $0 < g(u_1) < L$ (where $L > 0$), Xu [6] investigated the stability of equilibrium points and the Hopf bifurcation of model (1.2). Furthermore, considering the random diffusion mechanism, Xu [6] also discussed the following model:

$$\begin{cases} u_{1t} - d_1\Delta u_1 = au_2 - bu_1 - \gamma u_1^2 - g(u_1)v, & x \in \Omega, t > 0, \\ u_{2t} - d_2\Delta u_2 = u_1 - u_2, & x \in \Omega, t > 0, \\ v_t - d_3\Delta v = -\mu v + \delta g(u_1)v, & x \in \Omega, t > 0, \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N (N \geq 1)$, and $u_1(x, t)$ and $u_2(x, t)$ denote the population densities of immature prey and mature prey, respectively. $v(x, t)$ represents the population density of predators. The parameters d_i $(i = 1, 2, 3)$ are positive constants, which stand for the random diffusion rates of the three species, respectively. When $g(u_1)v = u_1v$, Xu [6] established the existence of a globally bounded classical solution to model (1.3). When $g(u_1)v = m\frac{u_1v}{u_1+v}$ (where $m > 0$ is a constant), Wang and Wu [7] established the global asymptotic stability of the unique positive constant equilibrium under appropriate conditions. Additionally, they derived results regarding the existence and non-existence of non-trivial solutions. Considering the impact of prey stage structure on predator-prey models, Li et al. [8] proposed the following model incorporating a tactic term:

$$\begin{cases} u_{1t} = d_1\Delta u_1 - \chi\nabla \cdot (u_1\nabla u_2) + au_2 - bu_1 - \gamma u_1^2 - u_1v, & x \in \Omega, t > 0, \\ u_{2t} = d_2\Delta u_2 + u_1 - u_2, & x \in \Omega, t > 0, \\ v_t = d_3\Delta v - \rho\nabla \cdot (v\nabla u_1) - \mu v + \delta u_1 v, & x \in \Omega, t > 0, \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N (N \geq 1)$. The term $-\chi\nabla \cdot (u_1\nabla u_2)$ describes the directional movement of immature prey toward the density gradient of mature prey. Additionally, the term $-\rho\nabla \cdot (v\nabla u_1)$ denotes the movement of predators toward the density gradient of immature prey. Li et al. [8] established the existence of Hopf bifurcation and steady-state bifurcation for model (1.4). When $d_1 = \chi = 0$ and $au_2 = h_{u_2}$

(where h_{u_2} denotes the non-uniform natural reproduction rate at which mature prey produce immature prey), Mi et al. [9] established the existence of a globally bounded classical solution for (1.4) in the two-dimensional case. In particular, when $\eta > 0$, Wu et al. [10] investigated the boundedness and asymptotic behavior of solutions to (1.4) with intra-specific competition among predators. For more related works, readers can refer to [11, 12].

Inspired by the work of [10], we replace the term $d_1\Delta u_1 - \chi\nabla\cdot(u_1\nabla u_2)$ with $d_1\nabla\cdot(D_1(u_1)\nabla u_1) - \chi\nabla\cdot(S_1(u_1)\nabla u_2)$, and the term $d_3\Delta v - \rho\nabla\cdot(v\nabla u_1)$ with $d_3\nabla\cdot(D_2(v)\nabla v) - \rho\nabla\cdot(S_2(v)\nabla u_1)$. The resulting model is specified as follows:

$$\begin{cases} u_{1t} = d_1\nabla\cdot(D_1(u_1)\nabla u_1) - \chi\nabla\cdot(S_1(u_1)\nabla u_2) + au_2 - bu_1 - \gamma u_1^2 - u_1 v, & x\in\Omega, t>0,\\ u_{2t} = d_2\Delta u_2 + u_1 - u_2, & x\in\Omega, t>0,\\ v_t = d_3\nabla\cdot(D_2(v)\nabla v) - \rho\nabla\cdot(S_2(v)\nabla u_1) - \mu v - \eta v^2 + \delta u_1 v, & x\in\Omega, t>0,\\ \partial_\nu u_1 = \partial_\nu u_2 = \partial_\nu v = 0, & x\in\partial\Omega, t>0,\\ u_1(x,0) = u_{10}(x), u_2(x,0) = u_{20}(x), v(x,0) = v_0(x), & x\in\Omega \end{cases} \tag{1.5}$$

in a bounded domain $\Omega\subset\mathbb{R}^N$ $(N\geq 1)$ with a smooth boundary. When $b<0$, the term $-bu_1 - \gamma u_1^2$ represents the logistic source of immature prey; when $b>0$, it denotes the mortality rate of immature prey. The nonlinear diffusion functions $D_i(s)$ $(i=1,2)$ and nonlinear prey-taxis sensitivity functions $S_i(s)$ $(i=1,2)$ satisfy $D_i(s), S_i(s)\in C^2[0,\infty)$,

$$D_i(s)\geq (s+1)^{\alpha_i} \text{ and } S_i(s)\leq s(s+1)^{\beta_i-1}, \tag{1.6}$$

where $s\geq 0, \alpha_i, \beta_i\in\mathbb{R}$ $(i=1,2)$. The initial values (u_{10}, u_{20}, v_0) satisfy

$$u_{20}\in W^{2,2q}(\Omega),\ (u_{10}, v_0)\in\left[W^{1,2q}(\Omega)\right]^2,\ \text{where } u_{10}, u_{20}, v_0 > 0, q > \frac{N}{2}. \tag{1.7}$$

In the present manuscript, we prove that when the parameters satisfy

$$\begin{cases} 2\beta_1 - \alpha_1 < 1 + \frac{4}{N+2},\\ 2\beta_2 - \alpha_2 < 2, \end{cases} \tag{1.8}$$

the classical solution of this quasilinear prey-taxis model is globally bounded.

Theorem 1.1. *Let $\Omega\subset R^N(N\geq 1)$ be a bounded domain with smooth boundary, and the initial data (u_{10}, u_{20}, v_0) satisfy (1.7). If the condition (1.8) holds, then model (1.5) has a global classical solution*

$$(u_1, u_2, v)\in\left[C^0\left(\bar\Omega\times[0,\infty)\right)\cap C^{2,1}\left(\bar\Omega\times(0,\infty)\right)\right]^3$$

and there exists a constant $M>0$ independent of t such that

$$\|u_1(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|u_2(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} \leq M \quad \textit{for all } t>0.$$

The structure of this paper is organized as follows. Section 2 presents the local existence theorem of solutions and the lemmas required for subsequent calculations. In Section 3, we focus on deriving the global boundedness of classical solutions to model (1.5), together with the proof of Theorem 1.1.

2. Preliminaries

In the following context, for simplicity and to avoid ambiguity, $\int_\Omega \varphi\, dx$ will be abbreviated as $\int_\Omega \varphi$ in subsequent discussions. Additionally, we use c_i and M_i ($i = 1, 2, 3, \cdots$) to denote generic positive constants independent of t, which may vary throughout the paper. The local existence of solutions to model (1.5) follows directly from Amann's theorem; for the sake of conciseness, we omit the detailed proof here and only present the results.

Lemma 2.1. *Assume $\Omega \subset R^N (N \geq 1)$ is a bounded domain with smooth boundary, and condition* (1.7) *holds. Then, there exist $T_{\max} \in (0, \infty]$ and a nonnegative function*

$$(u_1, u_2, v) \in \left[C^0\left(\bar{\Omega} \times [0, T_{\max})\right) \cap C^{2,1}\left(\bar{\Omega} \times (0, T_{\max})\right)\right]^3 \tag{2.1}$$

such that model (1.5) *has a classical solution. Moreover, if $T_{\max} < \infty$, then*

$$\limsup_{t \nearrow T_{\max}} \left(\|u_1(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|u_2(\cdot, t)\|_{W^{1,\infty}(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)}\right) = \infty. \tag{2.2}$$

Proof. Let $Z = (u_2, u_1, v)^T$. Then, model (1.5) can be rewritten as

$$\begin{cases} Z_t = \nabla \cdot (P(Z)\nabla Z) + Q(Z), & (x, t) \in \Omega \times (0, \infty), \\ \frac{\partial Z}{\partial \nu} = 0, & (x, t) \in \partial\Omega \times (0, \infty), \\ Z(\cdot, 0) = Z_0 = (u_{10}, u_{20}, v_0), & x \in \Omega, \end{cases} \tag{2.3}$$

where

$$P(Z) = \begin{pmatrix} d_2 & -\chi S_1(u_1) & 0 \\ 0 & d_1 D_1(u_1) & -\rho S_2(v) \\ 0 & 0 & d_3 D_2(v) \end{pmatrix}, \quad Q(Z) = \begin{pmatrix} u_1 - u_2 \\ au_2 - bu_1 - \gamma u_1^2 - u_1 v \\ -\mu v - \eta v^2 + \delta u_1 v \end{pmatrix}.$$

Clearly, for any initial value Z_0, the matrix $P(Z)$ is positive definite, which implies that Eq (2.3) is uniformly parabolic. As indicated in references [13, 14], there exists a maximal existence time $T_{\max} \in (0, \infty]$ such that Eq (2.3) admits a classical solution

$$(u_1, u_2, v) \in \left[C^0\left(\bar{\Omega} \times [0, T_{\max})\right) \cap C^{2,1}\left(\bar{\Omega} \times (0, T_{\max})\right)\right]^3.$$

Furthermore, by virtue of the strong maximum principle, (u_1, u_2, v) is nonnegative. Subsequently, by applying Amann's theorem, we establish the local existence criterion for the solution. This completes the proof. □

Next, we obtain an inequality from reference [15].

Lemma 2.2. *Let Ω be a bounded domain, and let $\varphi \in C^2(\overline{\Omega})$ satisfy $\partial_\nu \varphi = 0$ on $\partial\Omega$. Then,*

$$\frac{\partial |\nabla \varphi|^2}{\partial \nu} \leq \iota |\nabla \varphi|^2,$$

where $\iota > 0$ depending on Ω is an upper bound of the curvatures of $\partial\Omega$.

Then, we collect an important lemma from [16].

Lemma 2.3. *Suppose that there exist some $p > 1$ and constants $S, H > 0$, and*

$$\int_t^{t+\tau}\int_\Omega |f|^p \le S \text{ and } \int_t^{t+\tau}\int_\Omega |z|^p \le H, \text{ for all } t \in (0, T_{\max} - \tau),$$

where $\tau = \min\left\{1, \frac{T_{\max}}{2}\right\}$, moreover, the function $z \in C^{2,1}(\bar{\Omega} \times (0, T_{\max}))$ solves

$$\begin{cases} z_t = \Delta z + f(x,t), & x \in \Omega, \quad 0 < t < T_{\max}, \\ \frac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega, \quad 0 < t < T_{\max}, \\ z(x,0) = z_0(x), & x \in \Omega, \end{cases}$$

where $z_0 \in W^{2,\infty}(\Omega)$ with $z_0 \ge 0$ and $\frac{\partial z_0}{\partial \nu} = 0$ on $\partial\Omega$. Then, there exists constant $c_1 > 0$ fulfilling

$$\int_t^{t+\tau} \|z_t\|^p_{L^p(\Omega)} + \int_t^{t+\tau} \|z\|^p_{W^{2,p}(\Omega)} \le c_1 \left(\|z_0\|^p_{W^{2,p}(\Omega)} + S + H\right),$$

for all $t \in (0, T_{\max} - \tau)$. Specifically, one also can find a constant $c_2 > 0$ such that

$$\int_t^{t+\tau}\int_\Omega |\Delta z|^p \le c_2 \left(\|z_0\|^p_{W^{2,p}(\Omega)} + S + H\right), \text{ for all } t \in (0, T_{\max} - \tau).$$

We also introduce the following inequality [17].

Lemma 2.4. *Let $a, b > 0$, $T > 0$, and $\tau \in (0, T)$. Assume that $\varphi : [0, T) \to [0, \infty)$ is absolutely continuous and fulfills*

$$\varphi'(t) + a\varphi(t) \le z(t) \quad \text{for all } t \in (0, T),$$

where $z \geqslant 0$, $z \in L^1_{loc}([0, T))$, and

$$\int_t^{t+\tau} z(s)ds \le b \quad \text{for all } t \in [0, T - \tau).$$

Then,

$$\varphi(t) \le \max\left\{\varphi(0) + b, \frac{b}{a\tau} + 2b\right\} \quad \text{for all } t \in (0, T).$$

3. Proof of Theorem 1.1

In the present section, we concentrate on studying that the classical solutions to (1.5) are globally bounded and will build the following energy functional:

$$\int_\Omega (u_1 + 1)^p + \int_\Omega u_2^p + \int_\Omega |\nabla u_2|^{2q}, \quad \text{for all } p > 2, q > 1. \tag{3.1}$$

To lay the foundation for proving Theorem 1.1, the following estimates are needed.

Lemma 3.1. *Suppose conditions* (1.6) *and* (1.7) *hold. Then, for* $p > 2$ *and all* $t \in (0, T_{\max})$*, the following estimates hold:*

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_\Omega (u_1+1)^p + \frac{d_1(p-1)}{2}\int_\Omega (u_1+1)^{p+\alpha_1-2}|\nabla u_1|^2\\&\le \frac{\chi^2(p-1)}{2d_1}\int_\Omega (u_1+1)^{p-\alpha_1+2\beta_1-2}|\nabla u_2|^2 + \frac{2(p-1)a^{\frac{p}{p-1}}}{p}\int_\Omega (u_1+1)^p\\&\quad+\frac{1}{2^{p-1}}\cdot\frac{1}{p}\int_\Omega u_2^p - \frac{\gamma}{2}\int_\Omega (u_1+1)^{p+1} + c_1\end{aligned}\tag{3.2}$$

and

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u_2^p + d_2(p-1)\int_\Omega u_2^{p-2}|\nabla u_2|^2 \le \frac{2^{p-1}}{p}\int_\Omega (u_1+1)^p - \frac{1}{p}\int_\Omega u_2^p.\tag{3.3}$$

Proof. Multiplying the first equation of (1.5) by $(u_1+1)^{p-1}$ and integrating the result over Ω, gives

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_\Omega (u_1+1)^p + d_1(p-1)\int_\Omega D_1(u_1)(u_1+1)^{p-2}|\nabla u_1|^2\\&\le \chi(p-1)\int_\Omega S_1(u_1)(u_1+1)^{p-2}\nabla u_1\cdot\nabla u_2 + a\int_\Omega (u_1+1)^{p-1}u_2\\&\quad- b\int_\Omega (u_1+1)^{p-1}u_1 - \gamma\int_\Omega u_1^2(u_1+1)^{p-1}.\end{aligned}\tag{3.4}$$

Next, we estimate the terms on the right-hand side of (3.4). Applying Young's inequality and the fact $s^p \le (s+1)^p$, one obtains

$$\begin{aligned}&\chi(p-1)\int_\Omega S_1(u_1)(u_1+1)^{p-2}\nabla u_1\cdot\nabla u_2\\&\le \frac{d_1(p-1)}{2}\int_\Omega (u_1+1)^{p+\alpha_1-2}|\nabla u_1|^2 + \frac{\chi^2(p-1)}{2d_1}\int_\Omega S_1^2(u_1)(u_1+1)^{p-\alpha_1-2}|\nabla u_2|^2\\&\le \frac{d_1(p-1)}{2}\int_\Omega (u_1+1)^{p+\alpha_1-2}|\nabla u_1|^2 + \frac{\chi^2(p-1)}{2d_1}\int_\Omega (u_1+1)^{p-\alpha_1+2\beta_1-2}|\nabla u_2|^2\end{aligned}\tag{3.5}$$

and

$$a\int_\Omega (u_1+1)^{p-1}u_2 \le \frac{2(p-1)a^{\frac{p}{p-1}}}{p}\int_\Omega (u_1+1)^p + \frac{1}{2^{p-1}}\cdot\frac{1}{p}\int_\Omega u_2^p.\tag{3.6}$$

Then, the inequality $(s+1)^2 \le 2(s^2+1)$ and Young's inequality ensure that

$$\begin{aligned}-\gamma\int_\Omega u_1^2(u_1+1)^{p-1} &\le -\frac{\gamma}{2}\int_\Omega (u_1+1)^{p+1} + \gamma\int_\Omega (u_1+1)^{p-1}\\&\le -\frac{\gamma}{2}\int_\Omega (u_1+1)^{p+1} + \frac{1}{p}\int_\Omega (u_1+1)^p + c_1,\end{aligned}\tag{3.7}$$

where $c_1 > 0$ depends only on p. Combining (3.4)–(3.7), we can obtain (3.2). Multiplying the second equation of (1.5) by u_2^{p-1}, an integration by parts shows that

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u_2^p + d_2(p-1)\int_\Omega u_2^{p-2}|\nabla u_2|^2 = \int_\Omega u_2^{p-1}u_1 - \int_\Omega u_2^p.\tag{3.8}$$

Young's inequality implies that

$$\int_\Omega u_2^{p-1}u_1 \le \frac{2^{p-1}}{p}\int_\Omega (u_1+1)^p + \frac{p-1}{2p}\int_\Omega u_2^p. \tag{3.9}$$

Then the combination of (3.8) and (3.9) gives (3.3). □

Next, we shall establish the estimate of $\int_\Omega |\nabla u_2|^{2q}$.

Lemma 3.2. *For any $N \ge 1$, suppose conditions* (1.6) *and* (1.7) *hold. Then, for $p > 1$ and all $t \in (0, T_{\max})$, there exists a positive constant c_1 dependent on q such that*

$$\begin{aligned}&\frac{1}{2q}\frac{d}{dt}\int_\Omega |\nabla u_2|^{2q} + \frac{q-1}{2q^2}\int_\Omega |\nabla|\nabla u_2|^q|^2\\ &\le c_1\int_\Omega |\nabla u_2|^{2q} + \left(2(q-1)+\frac{N}{2d_2}\right)\int_\Omega (u_1+1)^2|\nabla u_2|^{2q-2}.\end{aligned} \tag{3.10}$$

Proof. Using the second equation of (1.5), we can derive that

$$\begin{aligned}\frac{1}{2q}\frac{d}{dt}\int_\Omega |\nabla u_2|^{2q} &= \int_\Omega |\nabla u_2|^{2q-2}\nabla u_2\cdot\nabla u_{2t}\\ &= d_2\int_\Omega |\nabla u_2|^{2q-2}\nabla u_2\cdot\nabla\Delta u_2 + \int_\Omega |\nabla u_2|^{2q-2}\nabla u_1\cdot\nabla u_2 - \int_\Omega |\nabla u_2|^{2q}.\end{aligned} \tag{3.11}$$

Noting the fact $2\nabla u_2\cdot\nabla\Delta u_2 = \Delta|\nabla u_2|^2 - 2|D^2u_2|^2$, we can obtain

$$\begin{aligned}d_2\int_\Omega |\nabla u_2|^{2q-2}\nabla u_2\cdot\nabla\Delta u_2 &= \frac{d_2}{2}\int_\Omega |\nabla u_2|^{2q-2}\Delta|\nabla u_2|^2 - d_2\int_\Omega |\nabla u_2|^{2q-2}|D^2u_2|^2\\ &= -\frac{d_2(q-1)}{2}\int_\Omega |\nabla u_2|^{2q-4}|\nabla|\nabla u_2|^2|^2\\ &\quad+\frac{d_2}{2}\int_{\partial\Omega}|\nabla u_2|^{2q-2}\frac{\partial|\nabla u_2|^2}{\partial\nu} - d_2\int_\Omega |\nabla u_2|^{2q-2}|D^2u_2|^2.\end{aligned} \tag{3.12}$$

By the pointwise inequality, the Sobolev embedding theorem and Lemma 2.2, one gets

$$\begin{aligned}&\frac{d_2}{2}\int_{\partial\Omega}|\nabla u_2|^{2q-2}\frac{\partial|\nabla u_2|^2}{\partial\nu} - \frac{d_2(q-1)}{2}\int_\Omega |\nabla u_2|^{2q-4}\left|\nabla|\nabla u_2|^2\right|^2\\ &\le -\frac{3(q-1)}{2q^2}\int_\Omega \left|\nabla|\nabla u_2|^q\right|^2 + c_1\int_\Omega |\nabla u_2|^{2q},\end{aligned} \tag{3.13}$$

where the constant $c_1 > 0$ depends only on q and Ω [18]. Using Young's inequality again and $|\Delta u_2|^2 \le N|D^2u_2|^2$, we can obtain

$$\begin{aligned}\int_\Omega |\nabla u_2|^{2q-2}\nabla u_1\cdot\nabla u_2 &= -\int_\Omega u_1|\nabla u_2|^{2q-2}\Delta u_2 - (q-1)\int_\Omega u_1|\nabla u_2|^{2q-4}\nabla|\nabla u_2|^2\cdot\nabla u_2\\ &\le \frac{q-1}{2q^2}\int_\Omega |\nabla|\nabla u_2|^q|^2 + 2(q-1)\int_\Omega (u_1+1)^2|\nabla u_2|^{2q-2}\\ &\quad+\frac{d_2}{2}\int_\Omega |\nabla u_2|^{2q-2}|D^2u_2|^2 + \frac{N}{2d_2}\int_\Omega (u_1+1)^2|\nabla u_2|^{2q-2}.\end{aligned} \tag{3.14}$$

Substituting (3.12)–(3.14) into (3.11), then the proof is completed. □

Building on the above estimates, this section investigates how nonlinear diffusion functions and nonlinear sensitivity functions affect the global boundedness of the model. To proceed, several parameters required for proving the global boundedness of solutions are defined. For any $p \geq 1$, $q \geq 1$, and $m \geq 1$, we define

$$\theta_1 := \theta_1(p, q) = \frac{2(p+1)}{3 + \alpha_1 - 2\beta_1}, \tag{3.15}$$

$$\eta_1 := \eta_1(p, q) = \frac{2(q-1)(p+1)}{p-1}, \tag{3.16}$$

$$\kappa_1 := \kappa_1(p, q; m) = \frac{\frac{q}{m} - \frac{q}{\theta_1}}{\frac{q}{m} - \left(\frac{1}{2} - \frac{1}{N}\right)}, \tag{3.17}$$

$$\kappa_2 := \kappa_2(p, q; m) = \frac{\frac{q}{m} - \frac{q}{\eta_1}}{\frac{q}{m} - \left(\frac{1}{2} - \frac{1}{N}\right)}, \tag{3.18}$$

$$f_1(p, q; m) := \frac{\theta_1}{q}\kappa_1(p, q; m) = \frac{\frac{\theta_1}{m} - 1}{\frac{q}{m} - \left(\frac{1}{2} - \frac{1}{N}\right)}, \tag{3.19}$$

$$f_2(p, q; m) := \frac{\eta_1}{q}\kappa_2(p, q; m) = \frac{\frac{\eta_1}{m} - 1}{\frac{q}{m} - \left(\frac{1}{2} - \frac{1}{N}\right)}. \tag{3.20}$$

Next, we shall add some additional restrictions to the above parameters to prove the global boundedness of the solution for (1.5).

Lemma 3.3. *For any $N \geq 2$ and sufficiently large $p > 1$, if*

$$1 + \frac{4}{N+2} > 2\beta_1 - \alpha_1, \tag{3.21}$$

then there exists $q > 1$ fulfilling

$$\kappa_1(p, q; 2) \in (0, 1) \text{ and } \kappa_2(p, q; 2) \in (0, 1) \tag{3.22}$$

and

$$f_1(p, q; 2) < 2 \text{ and } f_2(p, q; 2) < 2. \tag{3.23}$$

Proof. Assume

$$\theta_1 > m, \quad \eta_1 > m, \quad q > \frac{\theta_1}{2} - \frac{m}{N}, \quad q > \frac{\eta_1}{2} - \frac{m}{N}. \tag{3.24}$$

In fact, by a direct calculation, we deduce

$$\frac{d}{dt}\int_\Omega (u_1 + 2au_2) \leq (2a + |b_1|)\int_\Omega u_1 - \gamma\int_\Omega u_1^2 - a\int_\Omega u_2,$$

which gives

$$\frac{d}{dt}\int_\Omega (u_1 + 2au_2) + \int_\Omega u_1 + a\int_\Omega u_2 + \frac{\gamma}{2}\int_\Omega u_1^2 \leq c_1. \tag{3.25}$$

Then, applying the Grönwall inequality, one deduces that

$$\|u_1\|_{L^1(\Omega)} + 2a\|u_2\|_{L^1(\Omega)} \le c_2.$$

Integrating (3.25) over the interval $(t, t+\tau)$, we infer that

$$\int_t^{t+\tau} \int_\Omega u_1^2 \le c_3. \tag{3.26}$$

We integrate by parts in the second equation from (1.5) and use Young's inequality to see that

$$\frac{d}{dt}\int_\Omega |\nabla u_2|^2 + d_2 \int_\Omega |\Delta u_2|^2 + 2\int_\Omega |\nabla u_2|^2 \le \frac{1}{d_2}\int_\Omega u_1^2.$$

Using Lemma 2.4 and (3.26), one has

$$\|\nabla u_2\|_{L^2(\Omega)} \le c_4.$$

Therefore, we only need to consider the case when $m = 2$. Due to the assumption in (3.24), we can prove that (3.22) and (3.23) hold by direct calculation. It should be noted that, if

$$p > 2 + \alpha_1 - 2\beta_1 \text{ and } q > 2 \tag{3.27}$$

and

$$\frac{p+1}{\alpha_1 - 2\beta_1 + 3} - \frac{2}{N} < q < \frac{(N+2)(p+1)}{2N} - \frac{2}{N}, \tag{3.28}$$

then θ_1 and η_1 also satisfy (3.24). Without loss of generality, we assume

$$q \in \left(\frac{p+1}{\alpha_1 - 2\beta_1 + 3} - \frac{2}{N}, \frac{(N+2)(p+1)}{2N} - \frac{2}{N} \right), \tag{3.29}$$

if

$$2 - \frac{4}{N+2} < \alpha_1 - 2\beta_1 + 3, \tag{3.30}$$

and hence, for sufficiently large $p > 2$, there always exists an appropriate q satisfying (3.29). Let

$$p^* := \max\left\{1, -\frac{N\alpha_1}{2}, 2 + \alpha_1 - 2\beta_1, \frac{2(N+1)(\alpha_1 - 2\beta_1 + 3)}{N} - 1\right\}, \tag{3.31}$$

which, for any $p > p^*$ and some appropriate q, (3.22) and (3.23) hold. □

Based on Lemmas 3.1–3.3, we can establish the boundedness of u_1, u_2, and v in $L^p(\Omega)$, and thus obtain the following lemma.

Lemma 3.4. *Suppose conditions* (1.6) *and* (1.7) *hold. For any $N \ge 2$ and sufficiently large $p > 2$, if*

$$2\beta_1 - \alpha_1 < 1 + \frac{4}{N+2},$$

then there exists a constant $M_1 > 0$ such that for all $t \in (0, T_{\max})$,

$$\int_\Omega (u_1+1)^p + \int_\Omega u_2^p + \int_\Omega |\nabla u_2|^{2q} \le M_1.$$

Proof. The demonstration will be carried out in two steps. The first step, combining Lemma 3.1 and 3.2, gives

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{p}\int_\Omega (u_1+1)^p+\frac{1}{p}\int_\Omega u_2^p+\frac{1}{2q}\int_\Omega |\nabla u_2|^{2q}\right)+\frac{4d_2(p-1)}{p^2}\int_\Omega \left|\nabla u_2^{\frac{p}{2}}\right|^2\\&+\frac{2d_1(p-1)}{(p+\alpha_1)^2}\int_\Omega \left|\nabla (u_1+1)^{\frac{p+\alpha_1}{2}}\right|^2+\frac{q-1}{q^2}\int_\Omega |\nabla|\nabla u_2|^q|^2\\&\le I_1+I_2+c_1\int_\Omega |\nabla u_2|^{2q}+c_2,\end{aligned}\tag{3.32}$$

where

$$I_1:=\frac{\chi^2(p-1)}{2d_1}\int_\Omega (u_1+1)^{p-\alpha_1+2\beta_1-2}|\nabla u_2|^2+\left(2(q-1)+\frac{N}{2d_2}\right)\int_\Omega (u_1+1)^2|\nabla u_2|^{2q-2}\tag{3.33}$$

and

$$I_2:=\left(\frac{2(p-1)a^{\frac{p}{p-1}}}{p}+\frac{2^{p-1}}{p}+\frac{1}{p}\right)\int_\Omega (u_1+1)^p+\left(\frac{1}{2^{p-1}}\cdot\frac{1}{p}-\frac{1}{p}\right)\int_\Omega u_2^p-\frac{\gamma}{2}\int_\Omega (u_1+1)^{p+1}.\tag{3.34}$$

We shall estimate I_1. Thanks to Young's inequality, one has

$$I_1\le \frac{\gamma}{4}\int_\Omega (u_1+1)^{p+1}+c_3\int_\Omega |\nabla u_2|^{\theta_1}+c_4\int_\Omega |\nabla u_2|^{\eta_1},$$

where θ_1 and η_1 are defined in (3.15) and (3.16). From the Gagliardo-Nirenberg inequality, we have

$$c_3\int_\Omega |\nabla u_2|^{\theta_1}=c_3\,\||\nabla u_2|^q\|^{\frac{\theta_1}{q}}_{L^{\frac{\theta_1}{q}}(\Omega)}\le c_5\left(\|\nabla|\nabla u_2|^q\|^{\kappa_1}_{L^2(\Omega)}\cdot\||\nabla u_2|^q\|^{1-\kappa_1}_{L^{\frac{m}{q}}(\Omega)}+\||\nabla u_2|^q\|_{L^{\frac{m}{q}}(\Omega)}\right)^{\frac{\theta_1}{q}}$$

and

$$c_4\int_\Omega |\nabla u_2|^{\eta_1}=c_4\,\||\nabla u_2|^q\|^{\frac{\eta_1}{q}}_{L^{\frac{\eta_1}{q}}(\Omega)}\le c_6\left(\|\nabla|\nabla u_2|^q\|^{\kappa_2}_{L^2(\Omega)}\cdot\||\nabla u_2|^q\|^{1-\kappa_2}_{L^{\frac{m}{q}}(\Omega)}+\||\nabla u_2|^q\|_{L^{\frac{m}{q}}(\Omega)}\right)^{\frac{\eta_1}{q}},$$

where κ_1 and κ_2 are given by (3.17) and (3.18). Assuming $m=2$ and choosing a suitable q from (3.29), one has

$$c_3\int_\Omega |\nabla u_2|^{\theta_1}+c_4\int_\Omega |\nabla u_2|^{\eta_1}\le \frac{q-1}{4q^2}\int_\Omega |\nabla|\nabla u_2|^q|^2+c_7.\tag{3.35}$$

Next, we can estimate I_2. For sufficiently large $p>2$, there exists a positive constant c_8 such that

$$\left(\frac{2(p-1)a^{\frac{p}{p-1}}}{p}+\frac{2^{p-1}}{p}+\frac{1}{p}\right)\int_\Omega (u_1+1)^p-\frac{\gamma}{4}\int_\Omega (u_1+1)^{p+1}\le c_8.\tag{3.36}$$

After simple calculations, we can obtain

$$\frac{1}{2^{p-1}}\cdot\frac{1}{p}-\frac{1}{p}<0.\tag{3.37}$$

Substituting (3.36) and (3.37) into (3.34), there exists a constant $c_9>0$ such that

$$I_2\le c_9,\quad \text{for all } t\in(0,T_{\max}).\tag{3.38}$$

In the second step, we shall use the Gagliardo-Nirenberg inequality multiple times to obtain the following result:

$$
\begin{aligned}
\frac{1}{p}\int_\Omega (u_1+1)^p &= \frac{1}{p}\left\|(u_1+1)^{\frac{p+\alpha_1}{2}}\right\|_{L^{\frac{2p}{p+\alpha_1}}(\Omega)}^{\frac{2p}{p+\alpha_1}} \\
&\le c_{10}\left\|\nabla(u_1+1)^{\frac{p+\alpha_1}{2}}\right\|_{L^2(\Omega)}^{\frac{2p\kappa_3}{p+\alpha_1}}\left\|(u_1+1)^{\frac{p+\alpha_1}{2}}\right\|_{L^{\frac{2}{p+\alpha_1}}(\Omega)}^{\frac{2p(1-\kappa_3)}{p+\alpha_1}} \\
&\quad + c_{10}\left\|(u_1+1)^{\frac{p+\alpha_1}{2}}\right\|_{L^{\frac{2}{p+\alpha_1}}(\Omega)}^{2} \\
&\le c_{11}\left\|\nabla(u_1+1)^{\frac{p+\alpha_1}{2}}\right\|_{L^2(\Omega)}^{\frac{2p\kappa_3}{p+\alpha_1}} + c_{12}
\end{aligned}
\tag{3.39}
$$

and

$$
\begin{aligned}
\frac{1}{p}\int_\Omega u_2^p &\le c_{13}\left\|\nabla u_2^{\frac{p}{2}}\right\|_{L^2(\Omega)}^{2\kappa_4}\left\|u_2^{\frac{p}{2}}\right\|_{L^{\frac{2}{p}}(\Omega)}^{2(1-\kappa_4)} + c_{13}\left\|u_2^{\frac{p}{2}}\right\|_{L^{\frac{2}{p}}(\Omega)}^{2} \\
&\le c_{14}\left\|\nabla u_2^{\frac{p}{2}}\right\|_{L^2(\Omega)}^{2\kappa_4} + c_{14} \\
&\le \frac{4d_2(p-1)}{p^2}\left\|\nabla u_2^{\frac{p}{2}}\right\|_{L^2(\Omega)}^{2} + c_{15},
\end{aligned}
\tag{3.40}
$$

where

$$
\begin{aligned}
\kappa_3 &= \frac{\frac{N(p+\alpha_1)}{2}\left(1-\frac{1}{p}\right)}{1-\frac{N}{2}+\frac{N(p+\alpha_1)}{2}} \in (0,1), \\
\kappa_4 &= \frac{\frac{Np}{2}\left(1-\frac{1}{p}\right)}{1-\frac{N}{2}+\frac{Np}{2}} \in (0,1).
\end{aligned}
\tag{3.41}
$$

Moreover, applying the Gagliardo-Nirenberg inequality and Young's inequality again, one can show that

$$
\begin{aligned}
&\left(\frac{1}{2q}+c_1\right)\||\nabla u_2|^q\|_{L^2(\Omega)}^2 \\
&\le c_{16}\,\|\nabla|\nabla u_2|^q\|_{L^2(\Omega)}^{2\kappa_5}\,\||\nabla u_2|^q\|_{L^{\frac{s}{q}}(\Omega)}^{2(1-\kappa_5)} + c_{16}\,\||\nabla u_2|^q\|_{L^{\frac{s}{q}}(\Omega)}^{2} \\
&\le c_{17}\,\|\nabla|\nabla u_2|^q\|_{L^2(\Omega)}^{2\kappa_5} + c_{17} \\
&\le \frac{q-1}{4q^2}\,\|\nabla|\nabla u_2|^q\|_{L^2(\Omega)}^{2} + c_{18},
\end{aligned}
\tag{3.42}
$$

where

$$
\kappa_5 = \frac{\frac{Nq}{s}\left(1-\frac{s}{2q}\right)}{1-\frac{N}{2}+\frac{Nq}{s}} \in (0,1).
$$

Combining (3.32), (3.35), and (3.39)–(3.42), we obtain

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{p}\int_\Omega (u_1+1)^p+\frac{1}{p}\int_\Omega u_2^p+\frac{1}{2q}\int_\Omega |\nabla u_2|^{2q}\right)\\&+\frac{1}{p}\int_\Omega u_2^p+c_{19}\left(\int_\Omega (u_1+1)^p\right)^{\frac{p+\alpha_1}{p\kappa_3}}+\frac{1}{2q}\int_\Omega |\nabla u_2|^{2q}\le c_{20}.\end{aligned}$$

For every $t\in(0,T_{\max})$, we define

$$y(t):=\frac{1}{p}\int_\Omega (u_1+1)^p+\frac{1}{p}\int_\Omega u_2^p+\frac{1}{2q}\int_\Omega |\nabla u_2|^{2q}.$$

Then, there exist $c_{21}>0$ and $c_{22}>0$ such that

$$y(t)+c_{21}y^k(t)\le c_{22},$$

where $k=\min\left\{1,\frac{p+\alpha_1}{p\kappa_3}\right\}>0$. Thanks to the ODE comparison argument, one has

$$y(t):=\frac{1}{p}\int_\Omega (u_1+1)^p+\frac{1}{p}\int_\Omega u_2^p+\frac{1}{2q}\int_\Omega |\nabla u_2|^{2q}\le c_{23},\quad \text{for all } t>0.$$

□

Lemma 3.5. *Suppose conditions* (1.6) *and* (1.7) *hold. For any $N\ge 2$ and sufficiently large $p>2$, if*

$$2\beta_1-\alpha_1<1+\frac{4}{N+2},$$

then there exists a constant $M_2>0$ such that for any $t\in(0,T_{\max})$,

$$\left\|u_1(\cdot,t)\right\|_{L^\infty(\Omega)}+\left\|u_2(\cdot,t)\right\|_{W^{1,\infty}(\Omega)}\le M_2.$$

Proof. For sufficiently large $p>2$, according to the inequality $s^p\le(s+1)^p$, and together with Lemma 3.4, we can derive that

$$\int_\Omega u_1^p+\int_\Omega u_2^p+\int_\Omega |\nabla u_2|^{2q}\le M_1,\quad \text{for all } t\in(0,T_{\max}).$$

Combining the standard Moser-Alikakos iteration theory, we can complete the proof of Lemma 3.5. □

Lemma 3.6. *Suppose conditions* (1.6) *and* (1.7) *hold. For any $N\ge 2$ and sufficiently large $p>2$, if*

$$2\beta_1-\alpha_1<1+\frac{4}{N+2},$$

then there exists a constant $c>0$ such that for every $t\in(0,T_{\max})$,

$$\begin{aligned}&\frac{d}{dt}\int_\Omega\left(|\nabla u_1|^{2q}+(v+1)^p\right)+\int_\Omega |\nabla u_1|^{2q}+\int_\Omega (v+1)^p\\&\le -\frac{1}{32}\int_\Omega |\nabla u_1|^{2q-2}|D^2u_1|^2+6M_2^2(4(2q-2)^2+N)\int_\Omega v^2|\nabla u_1|^{2q-2}\\&\quad-\frac{d_3(p-1)}{2}\int_\Omega (v+1)^{p+\alpha_2-2}|\nabla v|^2+\frac{\rho^2(p-1)}{2d_3}\int_\Omega (v+1)^{p-\alpha_2+2\beta_2-2}|\nabla u_1|^2\\&\quad-\frac{\eta}{4}\int_\Omega (v+1)^{p+1}+c.\end{aligned}\tag{3.43}$$

Proof. Applying the first equation of (1.5), we can conclude with

$$
\begin{aligned}
\frac{1}{2q}\frac{d}{dt}\int_\Omega |\nabla u_1|^{2q} &= \int_\Omega |\nabla u_1|^{2q-2}\nabla u_1\cdot\nabla u_{1t} \\
&= \int_\Omega |\nabla u_1|^{2q-2}\nabla u_1\cdot\nabla\Big(d_1\nabla\cdot((u_1+1)^{\alpha_1}\nabla u_1) \\
&\quad -\chi\nabla\cdot(u_1(1+u_1)^{\beta_1-1}\nabla u_2)+au_2-bu_1-\gamma u_1^2-u_1v\Big) \\
&= d_1\int_\Omega |\nabla u_1|^{2q-2}\nabla u_1\cdot\nabla\Big(\alpha_1(u_1+1)^{\alpha_1-1}|\nabla u_1|^2+(u_1+1)^{\alpha_1}\Delta u_1\Big) \\
&\quad +\chi\int_\Omega \nabla\cdot(|\nabla u_1|^{2q-2}\nabla u_1)\nabla\cdot(u_1(1+u_1)^{\beta_1-1}\nabla u_2) \\
&\quad +\int_\Omega |\nabla u_1|^{2q-2}\nabla u_1\cdot\nabla\Big(au_2-bu_1-\gamma u_1^2-u_1v\Big) \\
&:= I(t)+J(t)+Q(t).
\end{aligned}
\tag{3.44}
$$

We will estimate the first term on the right-hand side of (3.44):

$$
\begin{aligned}
& d_1\int_\Omega |\nabla u_1|^{2q-2}\nabla u_1\cdot\Big(\alpha_1(\alpha_1-1)(u_1+1)^{\alpha_1-2}|\nabla u_1|^2\nabla u_1+\alpha_1(u_1+1)^{\alpha_1-1}\nabla|\nabla u_1|^2\Big) \\
&\quad + d_1\int_\Omega |\nabla u_1|^{2q-2}\nabla u_1\cdot\Big(\alpha_1(u_1+1)^{\alpha_1-1}\Delta u_1\nabla u_1+(u_1+1)^{\alpha_1}\nabla\Delta u_1\Big) \\
&= d_1\alpha_1(\alpha_1-1)\int_\Omega (u_1+1)^{\alpha_1-2}|\nabla u_1|^{2q+2} \\
&\quad + d_1\alpha_1\int_\Omega (u_1+1)^{\alpha_1-1}|\nabla u_1|^{2q-2}\nabla u_1\cdot\nabla|\nabla u_1|^2 + d_1\alpha_1\int_\Omega (u_1+1)^{\alpha_1-1}|\nabla u_1|^{2q}\Delta u_1 \\
&\quad + d_1\int_\Omega (u_1+1)^{\alpha_1}|\nabla u_1|^{2q-2}\nabla u_1\cdot\nabla\Delta u_1 \\
&:= I_1(t)+I_2(t)+I_3(t)+I_4(t).
\end{aligned}
\tag{3.45}
$$

According to Lemma 3.5, there exists a constant $c_1>0$ such that

$$
I_1(t)\le c_1\int_\Omega |\nabla u_1|^{2q+2}. \tag{3.46}
$$

To proceed, applying the identity $\nabla|\nabla u_1|^2 = 2D^2u_1\cdot\nabla u_1$ and Lemma 3.5, one has

$$
I_2(t) = 2d_1\alpha_1\int_\Omega (u_1+1)^{\alpha_1-1}|\nabla u_1|^{2q-1}\cdot(D^2u_1\cdot\nabla u_1)\le c_2\int_\Omega |\nabla u_1|^{2q}\big|D^2u_1\big|. \tag{3.47}
$$

Then, combining Lemma 3.5 again and the inequality $|\Delta u_1|^2\le N|D^2u_1|^2$, we can derive that

$$
I_3(t)\le d_1\alpha_1\sqrt{N}\int_\Omega (u_1+1)^{\alpha_1-1}|\nabla u_1|^{2q}|D^2u_1|\le c_3\int_\Omega |\nabla u_1|^{2q}|D^2u_1|. \tag{3.48}
$$

Furthermore, we shall estimate the last term on the right-hand side of (3.45),

$$
\begin{aligned}
I_4(t) &= d_1 \int_\Omega (u_1+1)^{\alpha_1} |\nabla u_1|^{2q-2} \nabla u_1 \cdot \nabla \Delta u_1 \\
&= \frac{d_1}{2} \int_\Omega (u_1+1)^{\alpha_1} |\nabla u_1|^{2q-2} \Delta |\nabla u_1|^2 - d_1 \int_\Omega (u_1+1)^{\alpha_1} |\nabla u_1|^{2q-2} |D^2 u_1|^2 \\
&= \frac{d_1}{2} \int_{\partial\Omega} (u_1+1)^{\alpha_1} |\nabla u_1|^{2q-2} \frac{\partial |\nabla u_1|^2}{\partial \nu} ds \\
&\quad - \frac{d_1 \alpha_1}{2} \int_\Omega (u_1+1)^{\alpha_1 - 1} |\nabla u_1|^{2q-2} \nabla u_1 \cdot \nabla |\nabla u_1|^2 \\
&\quad - \frac{d_1 (q-1)}{2} \int_\Omega (u_1+1)^{\alpha_1} |\nabla u_1|^{2q-4} |\nabla |\nabla u_1|^2| - d_1 \int_\Omega (u_1+1)^{\alpha_1} |\nabla u_1|^{2q-2} |D^2 u_1|^2 \\
&:= I_{41} - I_{42} - I_{43} - I_{44}.
\end{aligned}
\tag{3.49}
$$

Applying Lemma 2.4 in [19], one has

$$
I_{41} \le c_4 \int_{\partial\Omega} |\nabla u_1|^{2q-2} \frac{\partial |\nabla u_1|^2}{\partial \nu} ds \le \frac{d_1(q-1)}{2} \int_\Omega |\nabla u_1|^{2q-4} \left|\nabla |\nabla u_1|^2\right|^2 + c_5 \int_\Omega |\nabla u_1|^{2q}. \tag{3.50}
$$

Using Young's inequality again, we can derive that

$$
|I_{42}| \le c_6 \int_\Omega |\nabla u_1|^{2q} |D^2 u_1| \le \frac{1}{2} \int_\Omega |\nabla u_1|^{2q-2} |D^2 u_1|^2 + c_7 \int_\Omega |\nabla u_1|^{2q+2}, \tag{3.51}
$$

where $c_6, c_7 > 0$ are constants. Meanwhile,

$$
I_{43} \ge \frac{d_1(q-1)}{2} \int_\Omega |\nabla u_1|^{2q-4} \left|\nabla |\nabla u_1|^2\right|^2 \tag{3.52}
$$

and

$$
I_{44} \ge \int_\Omega |\nabla u_1|^{2q-2} \left|D^2 u_1\right|^2. \tag{3.53}
$$

Then, substituting (3.46)–(3.53) into (3.45) and combining with Young's inequality gives

$$
\begin{aligned}
I(t) &\le -\frac{1}{2} \int_\Omega |\nabla u_1|^{2q-2} \left|D^2 u_1\right|^2 + (c_1 + c_7) \int_\Omega |\nabla u_1|^{2q+2} + (c_2 + c_3) \int_\Omega |\nabla u_1|^{2q} \left|D^2 u_1\right| \\
&\quad + c_5 \int_\Omega |\nabla u_1|^{2q} \\
&\le -\frac{7}{16} \int_\Omega |\nabla u_1|^{2q-2} \left|D^2 u_1\right|^2 + c_8 \int_\Omega |\nabla u_1|^{2q+2} + c_8.
\end{aligned}
\tag{3.54}
$$

Moreover,

$$
\begin{aligned}
J(t) &= \chi\int_\Omega \nabla\cdot(|\nabla u_1|^{2q-2}\nabla u_1)(\nabla\cdot(u_1(u_1+1)^{\beta_1-1}\nabla u_2))\\
&= \chi\int_\Omega\Big(2(q-1)|\nabla u_1|^{2q-4}\nabla|\nabla u_1|^2\cdot\nabla u_1+|\nabla u_1|^{2q-2}\Delta u_1\Big)\Big((u_1+1)^{\beta_1-1}\nabla u_1\cdot\nabla u_2\\
&\quad+(\beta_1-1)u_1(u_1+1)^{\beta_1-2}\nabla u_1\cdot\nabla u_2+u_1(u_1+1)^{\beta_1-1}\Delta u_2\Big)\\
&= 2(q-1)\chi\int_\Omega(u_1+1)^{\beta_1-1}|\nabla u_1|^{2q-2}\nabla|\nabla u_1|^2\cdot\nabla u_2\\
&\quad+\chi\int_\Omega(u_1+1)^{\beta_1-1}|\nabla u_1|^{2q-2}\nabla u_1\cdot\nabla u_2\Delta u_1\\
&\quad+2(q-1)(\beta_1-1)\chi\int_\Omega u_1(u_1+1)^{\beta_1-2}|\nabla u_1|^{2q-2}\nabla|\nabla u_1|^2\cdot\nabla u_2\\
&\quad+(\beta_1-1)\chi\int_\Omega u_1(u_1+1)^{\beta_1-2}|\nabla u_1|^{2q-2}\nabla u_1\cdot\nabla u_2\Delta u_1\\
&\quad+2(q-1)\chi\int_\Omega u_1(u_1+1)^{\beta_1-1}|\nabla u_1|^{2q-4}\nabla|\nabla u_1|^2\nabla u_1\Delta u_2\\
&\quad+\chi\int_\Omega u_1(u_1+1)^{\beta_1-1}|\nabla u_1|^{2q-2}\Delta u_1\Delta u_2\\
&:= J_1(t)+J_2(t)+J_3(t)+J_4(t)+J_5(t)+J_6(t).
\end{aligned}
\tag{3.55}
$$

Together with Lemma 3.5, the inequality $|\Delta u_1|^2 \le N|D^2u_1|^2$ and the identity $\nabla|\nabla u_1|^2 = 2D^2u_1\cdot\nabla u_1$, we have

$$J_1(t)\le c_9\int_\Omega|\nabla u_1|^{2q-2}\,|D^2u_1|\,\nabla u_1\cdot\nabla u_2\le\frac{1}{24}\int_\Omega|\nabla u_1|^{2q-2}\,|D^2u_1|^2+c_{10},\tag{3.56}$$

$$J_2(t)\le c_{11}\int_\Omega|\nabla u_1|^{2q-2}\,|D^2u_1|\,\nabla u_1\cdot\nabla u_2\le\frac{1}{24}\int_\Omega|\nabla u_1|^{2q-2}\,|D^2u_1|^2+c_{12},\tag{3.57}$$

$$J_3(t)\le c_{13}\int_\Omega|\nabla u_1|^{2q-2}|D^2u_1|\,\nabla u_1\cdot\nabla u_2\le\frac{1}{24}\int_\Omega|\nabla u_1|^{2q-2}|D^2u_1|^2+c_{14},\tag{3.58}$$

$$J_4(t)\le c_{15}\int_\Omega|\nabla u_1|^{2q-2}|D^2u_1|\,\nabla u_1\cdot\nabla u_2\le\frac{1}{24}\int_\Omega|\nabla u_1|^{2q-2}|D^2u_1|^2+c_{16},\tag{3.59}$$

$$J_5(t)\le c_{17}\int_\Omega|\nabla u_1|^{2q-2}|D^2u_1|\,|\Delta u_2|\le\frac{1}{24}\int_\Omega|\nabla u_1|^{2q-2}|D^2u_1|^2+c_{18}\int_\Omega|\Delta u_2|^{q+1}\tag{3.60}$$

and

$$J_6(t)\le c_{19}\int_\Omega|\nabla u_1|^{2q-2}|D^2u_1|\,|\Delta u_2|\le\frac{1}{24}\int_\Omega|\nabla u_1|^{2q-2}|D^2u_1|^2+c_{20}\int_\Omega|\Delta u_2|^{q+1}.\tag{3.61}$$

Substituting (3.56)–(3.61) into (3.55), one gets

$$J(t)\le\frac{1}{4}\int_\Omega|\nabla u_1|^{2q-2}\left|D^2u_1\right|^2+c_{21}\left(\int_\Omega|\Delta u_2|^{q+1}+1\right).\tag{3.62}$$

Finally,

$$Q(t)\le a\int_\Omega|\nabla u_1|^{2q-2}\nabla u_1\cdot\nabla u_2-\int_\Omega|\nabla u_1|^{2q-2}\nabla u_1\cdot\nabla(u_1v):=Q_1(t)+Q_2(t).\tag{3.63}$$

According to Lemma 3.5 and Young's inequality, we have

$$Q_1(t) \le \frac{1}{24}\int_\Omega |\nabla u_1|^{2q-2}\left|D^2u_1\right|^2 + c_{22}. \tag{3.64}$$

Applying Lemma 3.5, Young's inequality, and the identity $\nabla|\nabla u_1|^2 = 2D^2u_1\cdot\nabla u_1$ again, one has

$$\begin{aligned}
Q_2(t) &= \int_\Omega u_1 v\nabla\cdot(|\nabla u_1|^{2q-2}\nabla u_1)\\
&= \int_\Omega u_1 v((2q-2)|\nabla u_1|^{2q-4}\nabla|\nabla u_1|^2\cdot\nabla u_1 + |\nabla u_1|^{2q-2}\Delta u_1)\\
&\le 2(2q-2)M_2\int_\Omega v\,|\nabla u_1|^{2q-2}\,|D^2u_1| + M_2\sqrt{N}\int_\Omega v\,|\nabla u_1|^{2q-2}\,|D^2u_1|\\
&\le \frac{1}{12}\int_\Omega |\nabla u_1|^{2q-2}\,|D^2u_1|^2 + 6M_2^2(4(2q-2)^2+N)\int_\Omega v^2\,|\nabla u_1|^{2q-2}.
\end{aligned}\tag{3.65}$$

Substituting (3.64) and (3.65) into (3.63) gives

$$Q(t) \le \frac{1}{8}\int_\Omega |\nabla u_1|^{2q-2}\left|D^2u_1\right|^2 + 6M_2^2(4(2q-2)^2+N)\int_\Omega v^2|\nabla u_1|^{2q-2} + c_{22}. \tag{3.66}$$

Then, substituting (3.54), (3.62), and (3.66) into (3.44), it holds that

$$\begin{aligned}
\frac{1}{2q}\frac{d}{dt}\int_\Omega |\nabla u_1|^{2q} \le& -\frac{1}{16}\int_\Omega |\nabla u_1|^{2q-2}|D^2u_1|^2 + c_8\int_\Omega |\nabla u_1|^{2q+2} + c_{21}\int_\Omega |\Delta u_2|^{q+1}\\
&+ 6M_2^2(4(2q-2)^2+N)\int_\Omega v^2|\nabla u_1|^{2q-2} + c_{23}.
\end{aligned}\tag{3.67}$$

Next, we estimate $\|v\|_{L^p(\Omega)}$. There exists a positive constant $c_{24} > 0$, and through calculations of the following expression, we can find that

$$\begin{aligned}
-\eta\int_\Omega s^2(s+1)^{p-1} &= -\eta\int_\Omega (s+1)^{p+1} + 2\eta\int_\Omega (s+1)^p - \eta\int_\Omega (s+1)^{p-1}\\
&\le -\eta\int_\Omega (s+1)^{p+1} + \left(2\eta + \frac{1}{p}\right)\int_\Omega (s+1)^p + c_{24},
\end{aligned}$$

which gives

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega (v+1)^p =& \int_\Omega (v+1)^{p-1}\Big[d_3\nabla\cdot((v+1)^{\alpha_2}\nabla v) - \rho\nabla\cdot(v(v+1)^{\beta_2-1}\nabla u_1)\Big]\\
&+ \int_\Omega (v+1)^{p-1}v(-\mu - \eta v + \delta u_1)\\
\le& -\frac{d_3(p-1)}{2}\int_\Omega (v+1)^{p+\alpha_2-2}\,|\nabla v|^2 + \frac{\rho^2(p-1)}{2d_3}\int_\Omega (v+1)^{p-\alpha_2+2\beta_2-2}\,|\nabla u_1|^2\\
&- \mu\int_\Omega v(v+1)^{p-1} - \eta\int_\Omega v^2(v+1)^{p-1} + \delta\int_\Omega u_1 v(v+1)^{p-1}\\
\le& -\frac{d_3(p-1)}{2}\int_\Omega (v+1)^{p+\alpha_2-2}\,|\nabla v|^2 + \frac{\rho^2(p-1)}{2d_3}\int_\Omega (v+1)^{p-\alpha_2+2\beta_2-2}\,|\nabla u_1|^2\\
&- \frac{\eta}{4}\int_\Omega (v+1)^{p+1} + c_{25}.
\end{aligned}\tag{3.68}$$

Combining (3.67), (3.68), and Lemmas 2.3 and 3.5, we have

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega(|\nabla u_1|^{2q}+(v+1)^p)+\int_\Omega|\nabla u_1|^{2q}+\int_\Omega(v+1)^p\\
&\le -\frac{1}{32}\int_\Omega|\nabla u_1|^{2q-2}\,|D^2u_1|^2+6M_2^2(4(2q-2)^2+N)\int_\Omega v^2\,|\nabla u_1|^{2q-2}\\
&\quad-\frac{d_3(p-1)}{2}\int_\Omega(v+1)^{p+\alpha_2-2}\,|\nabla v|^2+\frac{\rho^2(p-1)}{2d_3}\int_\Omega(v+1)^{p-\alpha_2+2\beta_2-2}\,|\nabla u_1|^2\\
&\quad-\frac{\eta}{4}\int_\Omega(v+1)^{p+1}+c_{25}.
\end{aligned}\tag{3.69}$$

□

Lemma 3.7. *Suppose conditions* (1.6) *and* (1.7) *hold. For any* $N\ge 2$ *and sufficiently large* $p>2$*, if*

$$\begin{cases}2\beta_1-\alpha_1<1+\dfrac{4}{N+2},\\ 2\beta_2-\alpha_2<2,\end{cases}$$

then there exist $q>1$ *and two constants* $M_3, M_4>0$ *such that for all* $t\in(0,T_{\max})$*,*

$$\|\nabla u_1\|_{L^{2q}(\Omega)}+\|v\|_{L^p(\Omega)}\le M_3.\tag{3.70}$$

Furthermore,

$$\|\nabla u_1\|_{L^\infty(\Omega)}+\|v\|_{L^\infty(\Omega)}\le M_4.\tag{3.71}$$

Proof. Suppose that $p>2$ and $q>1$. There exist positive constants c_1 and c_2 such that

$$\begin{aligned}
&\frac{\rho^2(p-1)}{2d_3}\int_\Omega(v+1)^{p-\alpha_2+2\beta_2-2}\,|\nabla u_1|^2+6M_2^2(4(2q-2)^2+N)\int_\Omega v^2\,|\nabla u_1|^{2q-2}\\
&\le\frac{1}{64}\int_\Omega|\nabla u_1|^{2q-2}\,|D^2u_1|^2+c_1\int_\Omega(v+1)^{\theta_2}+c_2\int_\Omega v^{q+1}
\end{aligned}\tag{3.72}$$

with $\theta_2=(p-\alpha_2+2\beta_2-2)\frac{q+1}{q}$, then, taking appropriate $p-\alpha_2+2\beta_2-2<q<p$, we have

$$c_1\int_\Omega(v+1)^{\theta_2}+c_2\int_\Omega v^{q+1}\le\frac{\eta}{8}\int_\Omega(v+1)^{p+1}+c_3.\tag{3.73}$$

Substituting (3.72) and (3.73) into (3.69), we end up with

$$\frac{d}{dt}\int_\Omega(|\nabla u_1|^{2q}+(v+1)^p)+\int_\Omega|\nabla u_1|^{2q}+\int_\Omega(v+1)^p\le c_4,\tag{3.74}$$

which together with Lemma 2.4 and the standard Moser-Alikakos iteration method [20, Lemma A.1] indicates (3.71). □

Proof of Theorem 1.1. When $\Omega\subset\mathbb{R}^N$ ($N\ge 1$), if $2\beta_1-\alpha_1<1+\frac{4}{N+2}$ and $2\beta_2-\alpha_2<2$, using Lemmas 3.1–3.7, for all $t\in(0,T_{\max})$, there exists a constant $K>0$ independent of t such that

$$\|u_1(\cdot,t)\|_{W^{1,\infty}(\Omega)}+\|u_2(\cdot,t)\|_{W^{1,\infty}(\Omega)}+\|v(\cdot,t)\|_{L^\infty(\Omega)}\le K,$$

which together with the extension criterion in Lemma 2.1 proves Theorem 1.1. □

4. Conclusions

In this work, we investigate a predator-prey model with prey-stage structure. Compared with the predator-prey model, one feature of this model is the inclusion of the prey-stage structure, dividing the prey into two stages: the immature stage and the mature stage. Another feature is the intra-specific competition among immature prey and that among predators, respectively. Through mathematical analysis, we mainly discuss the dynamic properties of the quasilinear prey-taxis model with prey-stage structure in any dimensional space, and demonstrate that the classical solution of the model is globally bounded when a certain relationship is satisfied between the exponent of the nonlinear diffusion function and the exponent of the nonlinear trophic sensitivity function.

Author contributions

L. Xu provided essential guidance on the theoretical calculations and mathematical framework of this paper. Z.Y. Wei and G.L. Wu were primarily responsible for drafting the manuscript. All authors of this article have been contributed equally. All authors have read and approved the final version of the manuscript for publication.

Use of Generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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