AIMS Mathematics

https://www.aimspress.com/journal/Math

AIMS Mathematics, 10(11): 26593–26612.
DOI:10.3934/math.20251169
Received: 08 September 2025
Revised: 23 October 2025
Accepted: 11 November 2025
Published: 17 November 2025

Research article

Quantum indeterminate set theory: a novel framework for complex-valued uncertainty and phase-based decision modeling

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Abstract: We introduce the quantum indeterminate set (QIS), a novel mathematical framework that integrates complex-valued membership functions and phase-based interference into classical and generalized set theory. Unlike complex fuzzy sets, QIS formally encodes both amplitude and phase interactions, allowing constructive and destructive interference to emerge naturally in reasoning. This feature distinguishes QIS as a bridge between fuzzy logic and quantum probability. We define its structure, explore its algebraic properties, and demonstrate its practical capability through a real-world decision-making case study in energy-system evaluation. The model generalizes fuzzy, intuitionistic, and neutrosophic sets while introducing a quantum-inspired interference mechanism that enables more nuanced reasoning under indeterminacy. This paper lays the theoretical foundation for future work in quantum decision theory, quantum-inspired soft computing, and uncertainty modeling in artificial intelligence.

Keywords: quantum set; indeterminate set; complex membership; phase angle; decision theory; uncertainty modeling; fuzzy logic; neutrosophic logic
Mathematics Subject Classification: 03B52, 03E72

1. Introduction and related work

Modeling uncertainty is a fundamental task in mathematics, computer science, and artificial intelligence. Various set-theoretic frameworks and probability-based models have been introduced to address vagueness, incompleteness, and ambiguity in information. Below, we summarize the most influential models:

- **Classical Set Theory:** Each element either belongs to a set or not, with membership values in $\{0, 1\}$. It cannot express partial or uncertain membership.
- **Fuzzy Set Theory (Zadeh, 1965):** Allows partial membership by assigning a real value in $[0, 1]$ to each element, capturing degrees of uncertainty [1].

- **Intuitionistic Fuzzy Sets (Atanassov, 1986):** Associates each element with both membership and non-membership degrees, leaving room for hesitation [2].
- **Vague Sets:** Similar to intuitionistic fuzzy sets, but interprets hesitation more explicitly as a measurable range [3].
- **Neutrosophic Sets (Smarandache):** Generalizes intuitionistic sets by introducing independent degrees of truth, indeterminacy, and falsity [4].
- **Possibility Theory (Dubois and Prade):** An uncertainty framework based on fuzzy sets, where possibility and necessity functions represent imprecise information [5].
- **Probability Theory:** Models randomness and uncertainty via axiomatic probabilities, useful when dealing with known distributions. [6]
- **Complex Fuzzy Sets:** Extends fuzzy sets by assigning complex numbers to membership functions, enabling a phase component without formal interference [7].

While each of these frameworks extends the expressiveness of uncertainty modeling, they mostly rely on real-valued or additive interpretations of uncertainty. However, in many real-world situations—such as cognitive modeling, quantum-inspired reasoning, and decision-making with conflicting expert opinions—uncertainty behaves in oscillatory or interference-like patterns.

How QIS differs from complex fuzzy sets: While complex fuzzy sets (CFS) [7] endow membership with a complex number, they typically aggregate evidence additively without a built-in notion of phase-coherence control or normalized interference. By contrast, QIS: (i) constrains $|\mu(x)| \leq 1$ and uses explicit renormalization after unions/intersections to avoid amplitude blow-up; (ii) interprets $\phi(x)$ as a semantic phase that propagates through operations and can be aligned, compared, or regularized; and (iii) couples amplitude and phase to a collapsed score $P_Q(x) = |\mu(x)|^2$ used for decision-making. This makes interference and directionality operational, not merely representational, and enables stability and interpretability analyses absent in standard CFS.

2. Results

Let X be a universal set. A classical fuzzy set F on X is defined by a membership function $\mu_F : X \to [0, 1]$. In intuitionistic fuzzy sets, each element is associated with both a degree of membership and non-membership. Neutrosophic sets further extend this to include indeterminacy.

In contrast, the QIS introduces a new structure.

2.1. Definition of the QIS

Definition 1. *(QIS) Let X be a universal set. A QIS Q is defined as*

$$Q = \{(x, \mu_Q(x), \phi_Q(x)) \mid x \in X\},$$

where

- $\mu_Q(x) = a+bi \in \mathbb{C}$ *is the complex-valued membership function, with* $a, b \in [-1, 1]$ *and* $|\mu_Q(x)| \leq 1$.
- $\phi_Q(x) = \arg(\mu_Q(x))$, *which may be computed as* $\tan^{-1}\left(\frac{b}{a}\right)$ *adjusted for the correct quadrant.*

The collapsed probability (observation-based certainty) is

$$P_Q(x) = |\mu_Q(x)|^2 = a^2 + b^2.$$

2.1.1. Interpretation of parameters

- a: real part—represents the classical degree of membership (positive for support, negative for opposition).
- b: imaginary part—models uncertainty, hesitation, or quantum-like interference.
- $\phi(x)$: phase—captures the semantic direction of uncertainty and determines constructive/destructive interference.

The semantic roles of the phase components are summarized in Table 1.

Table 1. Semantic interpretation of QIS phase quadrants.

Quadrant	Signs of a, b	Interpretation
I	$+, +$	Support with aligned uncertainty
II	$-, +$	Opposition with ambiguous evidence
III	$-, -$	Rejection with reinforcing contradiction
IV	$+, -$	Support with skeptical interference

2.1.2. Remark

The QIS model generalizes fuzzy sets by

- Allowing negative real and imaginary parts, capturing opposition and contradiction.
- Encoding phase explicitly to model interference patterns.
- Supporting richer decision reasoning in quantum-inspired systems.

2.2. Uncertainty propagation and stability

Let $\mu(x) = a + bi = re^{i\phi}$ be a QIS membership. A small perturbation $\delta\mu$ changes the phase by

$$\Delta\phi \approx \mathrm{Im}\left(\frac{\delta\mu}{\mu}\right), \qquad |\Delta\phi| \leq \frac{|\delta\mu|}{|\mu|} \quad \text{whenever } \mu \neq 0.$$

Proposition 1 (Phase-sensitivity bound)**.** *For any x with $\mu(x) \neq 0$, a perturbation $\delta\mu$ induces a phase change bounded by $|\Delta\phi| \leq |\delta\mu|/|\mu|$. Consequently, for elements with larger $|\mu|$ the phase is more robust to noise.*

Proof. Write $\Delta\phi = \arg(\mu + \delta\mu) - \arg(\mu)$. First-order expansion gives $\Delta\phi \approx \mathrm{Im}(\delta\mu/\mu)$, hence $|\Delta\phi| \leq |\delta\mu|/|\mu|$. □

Proposition 2 (Normalization-stability)**.** *Let $\mu_\cup(x) = \mu_1(x) + \mu_2(x)$. Define the normalized union $\widehat{\mu}_\cup(x) = \mu_\cup(x)$ if $|\mu_\cup(x)| \leq 1$, else $\widehat{\mu}_\cup(x) = \mu_\cup(x)/|\mu_\cup(x)|$. Then, $|\widehat{\mu}_\cup(x)| \leq 1$, and any additive perturbation on inputs is not amplified in modulus beyond 1 in the output.*

Proof. Immediate by construction since either the sum is inside the unit disk or is projected to it by dividing by its modulus. □

The above shows how QIS keeps amplitudes bounded and offers a quantitative handle on phase robustness during multi-step operations.

3. Representation of complex membership in QISs

In QISs, the membership function takes complex values. There are two common ways to represent this membership function:

- **Cartesian form:**

$$\mu_Q(x) = a + bi,$$

where a is the real part and b is the imaginary part.

- **Polar form:**

$$\mu_Q(x) = re^{i\phi} = r(\cos\phi + i\sin\phi),$$

where $r = \sqrt{a^2 + b^2}$ is the magnitude of membership and $\phi = \arctan\frac{b}{a}$ is the phase angle.

The choice between these two representations depends on the purpose and interpretation:

- Cartesian form is useful for explicit numerical calculations and algebraic manipulations, as the real and imaginary parts are directly accessible.
- Polar form is more suitable when the phase has a meaningful interpretation and needs to be explicitly incorporated, for example in defining subset relations or operations that depend on phase equality.

In particular, when domain experts assign or interpret the phase ϕ to represent qualitative or interference effects, the polar form becomes essential. However, for computational purposes, one can always convert between the two forms as needed.

4. Interpretation of the imaginary part in decision making

Unlike classical fuzzy membership, which is a real value indicating the degree of membership, the complex membership in QIS has an imaginary component that encodes additional information:

- The real part reflects the traditional membership degree (i.e how much an element belongs to the set).
- The **imaginary part** (or equivalently the phase) captures interference effects, uncertainty, and indeterminacy that arise in real-world decision making.

These effects are analogous to quantum phenomena, where probabilities can interfere constructively or destructively. In decision contexts, the imaginary part can represent the following:

- Ambiguity or hesitation of experts when assigning memberships.
- Conflicting or interacting criteria that influence the final membership in a non-classical way.
- Latent or contextual information not captured by magnitude alone.

Thus, the imaginary part enriches the representation, enabling a more nuanced modeling of uncertainty and qualitative factors in decision making. Often, the phase (or imaginary part) is assigned by experts or inferred from domain knowledge, allowing the membership function to better reflect complex realities beyond classical fuzzy sets.

Example 1. *Consider the following universal set of candidates:*

$$X = \{A, B, C, D\}.$$

Each candidate is evaluated with a complex-valued membership function $\mu_Q(x) = a + bi$, where

- *The real part a reflects the classical degree of membership (truth).*
- *The imaginary part b reflects indeterminacy, quantum interference, or hesitation.*
- *The phase $\phi_Q(x) = \arg(a + bi)$ represents the direction of the complex number in the complex plane.*
- *The probability $P_Q(x)$ is the squared modulus: $P_Q(x) = |\mu_Q(x)|^2 = a^2 + b^2$.*

Table 2 summarizes the complex-valued memberships, phase angles, and probabilities used to illustrate the QIS decision example.

Table 2. (QIS) Membership, phase angles, and probability values for candidate elements.

Candidate	**Membership** $\mu_Q(x)$	**Phase** $\phi_Q(x)$ **(rad)**	**Probability** $P_Q(x)$
A	$0.8 + 0.2i$	$\arg(0.8 + 0.2i) \approx 0.245$	$0.8^2 + 0.2^2 = 0.68$
B	$0.6 - 0.3i$	$\arg(0.6 - 0.3i) \approx -0.464$	$0.6^2 + 0.3^2 = 0.45$
C	$0.5 + 0.5i$	$\frac{\pi}{4} \approx 0.785$	$0.5^2 + 0.5^2 = 0.50$
D	$0.7 + 0i$	0	$0.7^2 = 0.49$

This example illustrates the expressive power of QIS:

- *Candidate A has a high truth value with minor quantum uncertainty.*
- *Candidate B has moderate presence with some opposing (negative phase) interference.*
- *Candidate C reflects balanced uncertainty and truth.*
- *Candidate D represents a classical membership with no interference.*

4.1. Motivation and conceptual framework

4.1.1. Why do we need the QIS?

Classical fuzzy set theory describes uncertainty using a real-valued membership degree $\mu\in[0, 1]$. While effective in many contexts, it lacks the expressive capacity to represent interference, contradiction, and phase-based uncertainty. In real-world applications, such as multi-source decision-making or quantum-inspired modeling, information may exhibit constructive or destructive interaction, a concept that classical fuzzy logic cannot accommodate.

To overcome these limitations, we introduce the QIS, where each element x is assigned the following complex-valued membership:

$$\tilde{A}(x) = \mu(x)e^{i\phi(x)}, \quad \text{with } \mu(x) \in [0, 1],\ \phi(x) \in [0, 2\pi).$$

Here, $\mu(x)$ indicates the degree of membership (intensity or magnitude), and $\phi(x)$ represents the phase angle, encoding the type or source of uncertainty.

4.1.2. Comparison with existing models

Uncertainty and vagueness in classical set theory have been addressed by several frameworks. The most classical and widely studied approach is the fuzzy set introduced by Zadeh [1], where each element has a membership degree in the interval $[0, 1]$. Since then, many generalizations like intuitionistic fuzzy sets, neutrosophic sets, and vague sets have been proposed to better capture hesitation and indeterminacy [2–4].

Our proposed QIS extends this modeling by assigning a complex-valued membership function to each element. This complex membership captures not only the degree of membership via its magnitude but also an associated phase that models interference and indeterminacy phenomena analogous to those in quantum mechanics. Table 3 summarizes key differences between classical fuzzy sets and QIS.

Table 3. Comparison of classical fuzzy sets, intuitionistic fuzzy sets, neutrosophic sets, vague sets, and QISs.

Aspect	**Classical fuzzy sets**	**Intuitionistic fuzzy sets**	**Neutrosophic sets**	**Vague sets**	**QISs**
Membership Values	Real number in $[0, 1]$	Membership and non-membership in $[0, 1]$ + hesitation	Membership, indeterminacy and non-membership in $[0, 1]$, no sum restriction	Interval membership values in $[0, 1]$	Complex number with magnitude and phase: $\mu(x) = re^{i\theta}$, $r \in [0, 1], \theta \in [0, 2\pi)$
Indeterminacy/ Hesitation	Not explicit	Explicit (residual)	Explicit (independent)	Implicit (interval width)	Modeled via phase θ, representing interference and uncertainty beyond scalar degrees
Interpretation	Degree of membership	Degree of membership and non-membership + hesitation	Three independent degrees	Interval-based uncertainty	Membership amplitude and phase, inspired by quantum probability and wave interference
Mathematical Foundation	Fuzzy logic and set theory	Extension of fuzzy sets	Generalization using neutrosophy	Interval analysis	Quantum mechanics, complex-valued membership functions, phase interference effects
Potential Advantage	Simplicity	Models hesitation	Models inconsistency and paradox	Represents interval uncertainty	Captures interference and richer uncertainty patterns, suitable for complex decision contexts

Interpretation. The phase component in QIS membership values can be understood as encoding a degree of indeterminacy or interaction between membership and non-membership states, akin to wave interference in quantum mechanics. This provides a novel and powerful tool for modeling decision-making problems where classical fuzzy sets fall short.

4.1.3. What does the phase ϕ represent?

In quantum physics, the phase of a wave function governs interference. Inspired by this, we define ϕ as the semantic phase in QIS, which conveys the qualitative nature of the uncertainty. For instance, sources may agree or oppose each other, leading to constructive or destructive effects.

- $\phi = 0$: Fully supportive or confirmatory source.
- $\phi = \pi$: Fully opposing or contradictory source.
- $\phi = \frac{\pi}{2}, \frac{3\pi}{2}$: Uncorrelated or orthogonal uncertainty.

4.1.4. Who determines the values of μ and ϕ?

- **The membership degree** μ is provided by expert judgment, data-driven estimation, or linguistic evaluation—just like in classical fuzzy theory.
- **The phase** ϕ is determined by
  (1) The nature of the information source.
  (2) The context or domain knowledge (e.g., agreement, hesitation, contradiction).
  (3) A linguistic or numerical encoding scheme (see Table 4).

Table 4. Linguistic interpretation of phase ϕ in QIS.

Phase ϕ	**Linguistic Meaning**
0	Strong support / full agreement
$\frac{\pi}{4}$	Moderate support / slightly optimistic
$\frac{\pi}{2}$	Neutral or undecided
$\frac{3\pi}{4}$	Slight opposition / skeptical
π	Strong contradiction / full opposition
$\frac{5\pi}{4}$	Hesitant source with weak contradiction
$\frac{3\pi}{2}$	Independent / uncorrelated context
2π	Reinforcing (repeats original direction)

4.1.5. Role of complex numbers in QIS

Using complex numbers allows the QIS to

- Represent both magnitude (μ) and directionality (ϕ).
- Capture interference patterns (constructive or destructive).
- Embed richer semantic interpretations into the membership function.

4.1.6. Linguistic interpretation of phase values

The inclusion of a phase angle allows for interference effects when aggregating over multiple criteria.

4.2. Phase-assignment guidelines

To reduce subjectivity, we suggest a two-tier protocol:

(1) **Linguistic prior:** map domain attitudes to a baseline phase $\phi^{\text{prior}} \in \{0, \frac{\pi}{12}, \frac{\pi}{8}, \frac{\pi}{6}, \frac{\pi}{4}, \ldots\}$ using Table 4.
(2) **Data-driven adjustment:** set $\phi = \phi^{\text{prior}} + \kappa \cdot D$, where $D \in [0, 1]$ is a dispersion index (e.g., normalized standard deviation across sources or alternatives), and $\kappa \in [0, \frac{\pi}{6}]$ controls adjustment strength.

The second stage of the protocol introduces a quantitative mechanism for refining the phase value based on the empirical variability of the criterion. This process combines the linguistic prior with a dispersion-based adjustment to ensure consistent and reproducible phase assignments. The complete procedure is formalized in Algorithm 1.

Algorithm 1 Data-assisted phase assignment for a criterion j.

Require: Scores $\{s_j(x)\}_{x \in X}$ normalized to $[0, 1]$, baseline ϕ_j^{prior}, gain κ

1: $D_j \leftarrow \frac{\text{std}(\{s_j(x)\})}{\max\{s_j(x)\} - \min\{s_j(x)\} + \epsilon}$ ▷ $D_j \in [0, 1]$
2: $\phi_j \leftarrow \phi_j^{\text{prior}} + \kappa D_j$
3: **return** ϕ_j

5. Operations on QIS: detailed examples

Definition 2. *Let Q_1, Q_2 be two QISs over a universe X, where*

$$Q_1 = \{(x, \mu_{Q_1}(x))\}, \quad Q_2 = \{(x, \mu_{Q_2}(x))\}.$$

Then, $Q_1 \subseteq Q_2$ if for every $x \in X$, the following two conditions hold:

(1) ***Magnitude condition:*** $|\mu_{Q_1}(x)| \leq |\mu_{Q_2}(x)|$.
(2) ***Phase condition:*** $|\phi_{Q_1}(x) - \phi_{Q_2}(x)| < \varepsilon$ *for a small threshold* $\varepsilon > 0$.

The first ensures containment in terms of support; the second ensures directional alignment (semantic or interference phase proximity). Typically, ε may be chosen as $\frac{\pi}{12}$ (15°), depending on application tolerance.

Example 2. *Let the universe be $X = \{x_1, x_2\}$. Define*

$$Q_1 = \{(x_1, 0.6 + 0.3i, \phi_1(x_1)), \quad (x_2, 0.4 + 0.2i, \phi_1(x_2))\},$$

$$Q_2 = \{(x_1, 0.7 + 0.3i, \phi_2(x_1)), \quad (x_2, 0.5 + 0.2i, \phi_2(x_2))\}.$$

We compute magnitudes and phases as follows:

- *At x_1:*

$$|\mu_{Q_1}(x_1)| = \sqrt{0.6^2 + 0.3^2} = \sqrt{0.45} \approx 0.671,$$

$$
\begin{aligned}
|\mu_{Q_2}(x_1)| &= \sqrt{0.7^2+0.3^2} = \sqrt{0.58} \approx 0.761,\\
\phi_{Q_1}(x_1) &= \tan^{-1}(0.3/0.6) = \tan^{-1}(0.5) \approx 0.464,\\
\phi_{Q_2}(x_1) &= \tan^{-1}(0.3/0.7) \approx 0.404,\\
|\phi_1 - \phi_2| &\approx |0.464 - 0.404| = 0.06 < \tfrac{\pi}{12}.
\end{aligned}
$$

- *At* x_2*:*

$$
\begin{aligned}
|\mu_{Q_1}(x_2)| &= \sqrt{0.4^2+0.2^2} = \sqrt{0.20} \approx 0.447,\\
|\mu_{Q_2}(x_2)| &= \sqrt{0.5^2+0.2^2} = \sqrt{0.29} \approx 0.538,\\
\phi_{Q_1}(x_2) &= \tan^{-1}(0.2/0.4) = \tan^{-1}(0.5) \approx 0.464,\\
\phi_{Q_2}(x_2) &= \tan^{-1}(0.2/0.5) = \tan^{-1}(0.4) \approx 0.380,\\
|\phi_1 - \phi_2| &\approx |0.464 - 0.380| = 0.084 < \tfrac{\pi}{12}.
\end{aligned}
$$

Since both the magnitude and phase conditions are satisfied for all $x \in X$*, we conclude as follows:*

$$Q_1 \subseteq Q_2.$$

Interpretation. The subset relation in QIS combines classical support comparison (modulus-based) with quantum alignment (phase coherence). This ensures that each element in Q_1 is not only weaker (or equal) in support but also directionally aligned with the corresponding element in Q_2, allowing QIS to model agreement not just in degree but in semantic or interference orientation.

Definition 3. *Let* $Q_1 = \{(x, \mu_{Q_1}(x))\}$ *and* $Q_2 = \{(x, \mu_{Q_2}(x))\}$ *be two QISs defined over the same universe* X*. Then,*

$$Q_1 = Q_2 \quad \text{if and only if} \quad \forall x \in X : \begin{cases} |\mu_{Q_1}(x) - \mu_{Q_2}(x)| < \delta, \\ |\phi_{Q_1}(x) - \phi_{Q_2}(x)| < \varepsilon, \end{cases}$$

for small tolerances $\delta, \varepsilon > 0$ *(e.g.,* $\delta = 0.01$*,* $\varepsilon = \frac{\pi}{50}$*). That is, two QISs are considered equal if their complex memberships are approximately equal in both magnitude and phase.*

Example 3. *Let* $X = \{x_1\}$*, and define the following:*

$$Q_1 = \{(x_1, 0.600 + 0.300i, \phi_1(x_1))\}, \quad Q_2 = \{(x_1, 0.605 + 0.295i, \phi_2(x_1))\}.$$

$$
\begin{aligned}
|\mu_{Q_1}(x_1) - \mu_{Q_2}(x_1)| &= |(0.600 - 0.605) + i(0.300 - 0.295)| = |-0.005 + 0.005i|\\
&= \sqrt{(-0.005)^2 + (0.005)^2} = \sqrt{0.00005} \approx 0.007,\\
\phi_{Q_1}(x_1) &= \tan^{-1}(0.3/0.6) \approx 0.464,\\
\phi_{Q_2}(x_1) &= \tan^{-1}(0.295/0.605) \approx 0.454,\\
|\phi_{Q_1}(x_1) - \phi_{Q_2}(x_1)| &\approx 0.01 < \frac{\pi}{50} \approx 0.063.
\end{aligned}
$$

Since both the modulus and phase differences are below the specified thresholds, we conclude the following:

$$Q_1 = Q_2.$$

Definition 4. *Let $Q = \{(x, \mu_Q(x)) \mid x \in X\}$ be a QIS over a universe X. The complement of Q, denoted Q^c, is defined for each $x \in X$ by*

$$\mu_{Q^c}(x) = \begin{cases} 1 - \mu_Q(x), & \text{if } |1 - \mu_Q(x)| \le 1, \\ \dfrac{1 - \mu_Q(x)}{|1 - \mu_Q(x)|}, & \text{if } |1 - \mu_Q(x)| > 1, \end{cases}$$

$$\phi_{Q^c}(x) = \arg(\mu_{Q^c}(x)).$$

Example 4. *Let the QIS Q be defined over the universe $X = \{x_1, x_2\}$ as follows:*

$$Q = \left\{(x_1, 0.6 + 0.3i, \tfrac{\pi}{6}), \quad (x_2, 0.8 + 0.2i, \tfrac{\pi}{7})\right\}.$$

We compute the complement Q^c for each element.

- *At x_1:*

$$\begin{aligned} \mu_Q(x_1) &= 0.6 + 0.3i, \\ \mu_{Q^c}(x_1) &= 1 - (0.6 + 0.3i) = 0.4 - 0.3i, \\ |\mu_{Q^c}(x_1)| &= \sqrt{0.4^2 + (-0.3)^2} = \sqrt{0.25} = 0.5 \le 1, \\ \phi_{Q^c}(x_1) &= \arg(0.4 - 0.3i) = \tan^{-1}\left(\frac{-0.3}{0.4}\right) \approx -0.6435. \end{aligned}$$

- *At x_2:*

$$\begin{aligned} \mu_Q(x_2) &= 0.8 + 0.2i, \\ \mu_{Q^c}(x_2) &= 1 - (0.8 + 0.2i) = 0.2 - 0.2i, \\ |\mu_{Q^c}(x_2)| &= \sqrt{0.2^2 + (-0.2)^2} = \sqrt{0.08} \approx 0.283 \le 1, \\ \phi_{Q^c}(x_2) &= \arg(0.2 - 0.2i) = -\tfrac{\pi}{4}. \end{aligned}$$

Therefore, the complement set is

$$Q^c = \left\{(x_1, 0.4 - 0.3i, -0.6435), \quad (x_2, 0.2 - 0.2i, -\tfrac{\pi}{4})\right\}.$$

Interpretation. The complement operation flips the quantum membership relative to the full certainty value $1 + 0i$, subtracting the original support and inverting its semantic direction (phase). The resulting values remain within the unit disk, and no normalization was needed here. The phase angles move into the negative domain, reflecting semantic opposition or contradiction to the original membership.

Definition 5. *Let $Q_1 = \{(x, \mu_1(x)) \mid x \in X\}$ and $Q_2 = \{(x, \mu_2(x)) \mid x \in X\}$ be two QISs defined over the same universe X. The union of Q_1 and Q_2, denoted by $Q_1 \cup Q_2$, is defined element-wise by*

$$\mu_{Q_1 \cup Q_2}(x) = \begin{cases} \mu_1(x) + \mu_2(x), & \text{if } |\mu_1(x) + \mu_2(x)| \le 1, \\ \dfrac{\mu_1(x) + \mu_2(x)}{|\mu_1(x) + \mu_2(x)|}, & \text{if } |\mu_1(x) + \mu_2(x)| > 1. \end{cases}$$

The phase of each element in the union set is computed as

$$\phi_{Q_1 \cup Q_2}(x) = \arg(\mu_{Q_1 \cup Q_2}(x)).$$

This definition ensures that the resulting set remains a valid QIS, with all membership magnitudes $|\mu(x)| \leq 1$.

Example 5. *Let the QISs* Q_1 *and* Q_2 *be defined on the universe* $X = \{x_1, x_2\}$ *as*

$$Q_1 = \left\{(x_1, 0.6 + 0.3i, \tfrac{\pi}{6}), \quad (x_2, 0.4 + 0.2i, \tfrac{\pi}{6})\right\}, \quad Q_2 = \left\{(x_1, 0.7 + 0.3i, \tfrac{\pi}{8}), \quad (x_2, 0.5 + 0.2i, \tfrac{\pi}{8})\right\}.$$

We compute the union $Q_1 \cup Q_2$ *using the above definition.*

- *At* x_1*:*

$$\begin{aligned} \mu_{Q_1}(x_1) &= 0.6 + 0.3i, \quad \mu_{Q_2}(x_1) = 0.7 + 0.3i, \\ \mu_{raw} &= 0.6 + 0.3i + 0.7 + 0.3i = 1.3 + 0.6i, \\ |\mu_{raw}| &= \sqrt{1.3^2 + 0.6^2} \approx \sqrt{2.05} \approx 1.433 > 1, \\ \mu_{Q_1 \cup Q_2}(x_1) &= \frac{1.3 + 0.6i}{1.433} \approx 0.907 + 0.419i, \\ \phi_{Q_1 \cup Q_2}(x_1) &= \arg(0.907 + 0.419i) \approx \tan^{-1}\left(\frac{0.419}{0.907}\right) \approx 0.435. \end{aligned}$$

- *At* x_2*:*

$$\begin{aligned} \mu_{Q_1}(x_2) &= 0.4 + 0.2i, \quad \mu_{Q_2}(x_2) = 0.5 + 0.2i, \\ \mu_{raw} &= 0.4 + 0.2i + 0.5 + 0.2i = 0.9 + 0.4i, \\ |\mu_{raw}| &= \sqrt{0.9^2 + 0.4^2} = \sqrt{0.97} \approx 0.985 \leq 1, \\ \mu_{Q_1 \cup Q_2}(x_2) &= 0.9 + 0.4i, \\ \phi_{Q_1 \cup Q_2}(x_2) &= \arg(0.9 + 0.4i) = \tan^{-1}\left(\frac{0.4}{0.9}\right) \approx 0.418. \end{aligned}$$

Thus, the resulting union QIS is

$$Q_1 \cup Q_2 = \{(x_1, 0.907 + 0.419i, 0.435), \quad (x_2, 0.9 + 0.4i, 0.418)\}.$$

Interpretation. The union aggregates both magnitude and direction (phase) of support from the two QISs. Normalization was applied at x_1 because the combined amplitude exceeded 1, ensuring the probabilistic consistency of the QIS model.

Definition 6. *Let* $Q_1 = \{(x, \mu_1(x)) \mid x \in X\}$ *and* $Q_2 = \{(x, \mu_2(x)) \mid x \in X\}$ *be two QISs defined over the same universe X. The intersection of* Q_1 *and* Q_2*, denoted* $Q_1 \cap Q_2$*, is defined element-wise by*

$$\mu_{Q_1 \cap Q_2}(x) = \begin{cases} \mu_1(x) \cdot \mu_2(x), & \text{if } |\mu_1(x) \cdot \mu_2(x)| \leq 1, \\ \dfrac{\mu_1(x) \cdot \mu_2(x)}{|\mu_1(x) \cdot \mu_2(x)|}, & \text{if } |\mu_1(x) \cdot \mu_2(x)| > 1. \end{cases}$$

$$\phi_{Q_1 \cap Q_2}(x) = \arg(\mu_{Q_1 \cap Q_2}(x)).$$

This ensures that the resulting set remains a valid QIS with $|\mu(x)| \leq 1$*, preserving the unit-bound constraint on quantum-inspired complex membership values.*

Example 6. *Let Q_1 and Q_2 be defined on the universe $X = \{x_1, x_2\}$ as*

$$Q_1 = \left\{(x_1, 0.6 + 0.3i, \tfrac{\pi}{6}), \quad (x_2, 0.4 + 0.2i, \tfrac{\pi}{6})\right\}, \quad Q_2 = \left\{(x_1, 0.7 + 0.3i, \tfrac{\pi}{8}), \quad (x_2, 0.5 + 0.2i, \tfrac{\pi}{8})\right\}.$$

We compute the intersection $Q_1 \cap Q_2$ using the above definition.

- *At x_1:*

$$\begin{aligned}
\mu_{Q_1}(x_1) &= 0.6 + 0.3i, \quad \mu_{Q_2}(x_1) = 0.7 + 0.3i,\\
\mu_{raw} &= (0.6 + 0.3i)(0.7 + 0.3i)\\
&= (0.6 \cdot 0.7 - 0.3 \cdot 0.3) + i(0.6 \cdot 0.3 + 0.3 \cdot 0.7) = 0.33 + 0.39i,\\
|\mu_{raw}| &= \sqrt{0.33^2 + 0.39^2} = \sqrt{0.261} \approx 0.511 \leq 1,\\
\mu_{Q_1 \cap Q_2}(x_1) &= 0.33 + 0.39i,\\
\phi_{Q_1 \cap Q_2}(x_1) &= \tan^{-1}\left(\frac{0.39}{0.33}\right) \approx 0.755.
\end{aligned}$$

- *At x_2:*

$$\begin{aligned}
\mu_{Q_1}(x_2) &= 0.4 + 0.2i, \quad \mu_{Q_2}(x_2) = 0.5 + 0.2i,\\
\mu_{raw} &= (0.4 + 0.2i)(0.5 + 0.2i) = 0.16 + 0.18i,\\
|\mu_{raw}| &= \sqrt{0.16^2 + 0.18^2} = \sqrt{0.058} \approx 0.241 \leq 1,\\
\mu_{Q_1 \cap Q_2}(x_2) &= 0.16 + 0.18i,\\
\phi_{Q_1 \cap Q_2}(x_2) &= \tan^{-1}\left(\frac{0.18}{0.16}\right) \approx 0.837.
\end{aligned}$$

Thus, the resulting intersection QIS is

$$Q_1 \cap Q_2 = \{(x_1, 0.33 + 0.39i, 0.755), \quad (x_2, 0.16 + 0.18i, 0.837)\}.$$

Interpretation. The intersection aggregates quantum memberships multiplicatively, blending the uncertainty and phase information of both sets. In this example, both resulting magnitudes remained within the unit circle, so no normalization was required. The final phases illustrate interference-like behavior, capturing how opinions or evidences interact in quantum-style reasoning.

Definition 7. *The symmetric difference of two QISs, denoted $Q_1 \triangle Q_2$, is the following set:*

$$Q_1 \triangle Q_2 = (Q_1 \cup Q_2) \setminus (Q_1 \cap Q_2).$$

For each $x \in X$, define the following :

$$\mu_{Q_1 \triangle Q_2}(x) = \mu_{Q_1}(x) + \mu_{Q_2}(x) - 2\mu_{Q_1}(x)\mu_{Q_2}(x),$$

$$\phi_{Q_1 \triangle Q_2}(x) = \arg\left(\mu_{Q_1 \triangle Q_2}(x)\right).$$

This formulation captures "disagreement" or "non-overlapping" support in the quantum sense. It cancels overlap while summing distinct contributions.

Example 7. *Let*

$$Q_1 = \{(x_1, 0.6 + 0.3i, \phi_1(x_1))\}, \quad Q_2 = \{(x_1, 0.5 + 0.2i, \phi_2(x_1))\}.$$

$$\mu_{Q_1 \triangle Q_2}(x_1) = (0.6 + 0.3i) + (0.5 + 0.2i) - 2(0.6 + 0.3i)(0.5 + 0.2i) = 0.62 - 0.04i,$$

$$\phi_{Q_1 \triangle Q_2}(x_1) = \arg(0.62 - 0.04i) \approx -0.064.$$

Therefore,

$$Q_1 \triangle Q_2 = \{(x_1, 0.62 - 0.04i, -0.064)\}.$$

5.1. Algorithms for QIS operations

Algorithm 2 formalizes the elementwise QIS union by combining the complex memberships through vector addition while enforcing normalization to preserve the unit-disc constraint.

Algorithm 2 QIS union (elementwise)

Require: $\mu_1(x), \mu_2(x) \in \mathbb{C}$
$\mu \leftarrow \mu_1(x) + \mu_2(x)$
if $|\mu| > 1$ **then**
    $\mu \leftarrow \mu/|\mu|$
end if
return μ and $\phi = \arg(\mu)$

As shown in Algorithm 3, the QIS intersection is implemented via complex multiplication, which naturally encodes interaction between magnitudes and phases, followed by normalization when necessary.

Algorithm 3 QIS intersection (elementwise)

Require: $\mu_1(x), \mu_2(x) \in \mathbb{C}$
$\mu \leftarrow \mu_1(x) \cdot \mu_2(x)$
if $|\mu| > 1$ **then**
    $\mu \leftarrow \mu/|\mu|$
end if
return μ and $\phi = \arg(\mu)$

Algorithm 4 summarizes the phase-weighted aggregation rule, where each criterion contributes a complex vector determined by its score, weight, and assigned phase, producing a final membership and its associated probability.

Algorithm 4 Phase-weighted aggregation of m criteria

Require: normalized scores $r_j(x) \in [0, 1]$, weights $w_j \geq 0$, phases ϕ_j
$\mu(x) \leftarrow 0$
for $j = 1$ **to** m **do**
    $\mu(x) \leftarrow \mu(x) + w_j\, r_j(x)\, e^{i\phi_j}$
end for
if $|\mu(x)| > 1$ **then**
    $\mu(x) \leftarrow \mu(x)/|\mu(x)|$
end if
return $\mu(x)$, $P_Q(x) = |\mu(x)|^2$

6. Applications and limitations

The QIS model extends traditional fuzzy and neutrosophic sets by introducing complex-valued membership and phase-based semantics. While decision-making is a key use case, the QIS framework has broader applicability and faces several practical challenges.

6.1. Extended applications

Beyond decision-making, the QIS framework can be adapted to several advanced fields:

- **Quantum-inspired machine learning:** QIS can encode uncertainty in input features using both magnitude and phase, enabling models to learn contextual or conflicting patterns through interference effects.
- **Multi-agent systems:** When agents contribute opinions with different beliefs or evidence strengths, QIS allows aggregation with phase-based differentiation, capturing conflict and alignment.
- **Conflict resolution and negotiation models:** In domains such as diplomacy or group consensus, QIS can model constructive or destructive interactions between stakeholder perspectives.

These applications highlight the general-purpose nature of QIS in environments where uncertainty is both quantitative and directional.

6.2. Limitations

While QIS offers expressive modeling capabilities, it also introduces challenges:

- **Computational complexity:** Calculating phase angles and performing complex arithmetic (e.g., normalization, interference) is more computationally demanding than operations in classical or fuzzy models.
- **Subjective phase assignment:** Assigning the phase angle $\phi(x)$ may depend on expert knowledge or domain-specific heuristics, which could introduce inconsistency.
- **Interpretability:** In fields unfamiliar with complex numbers or quantum-inspired semantics, explaining the meaning of phase and its impact may be non-trivial.

Addressing these limitations could involve developing automated phase inference techniques and simplifying the interpretive tools for end-users.

Bias from Normalization. Projecting sums/products back to the unit disk may discard amplitude information when $|\mu| > 1$. In practice this bias is controlled by (i) capping frequency of renormalization through weight tuning; (ii) logging the pre-normalized modulus as an auxiliary feature; and (iii) sensitivity checks that compare rankings with/without normalization on borderline items.

Phase Sensitivity. By Proposition 1, $|\Delta\phi| \leq |\delta\mu|/|\mu|$. Hence, small amplitudes are intrinsically fragile in phase, and we recommend reporting both $(|\mu|, \phi)$ and $P_Q = |\mu|^2$ and avoiding decisions that hinge on very small $|\mu|$ unless supported by multiple criteria.

6.3. Case study: multi-expert evaluation in hiring

Consider a hiring scenario where two experts evaluate a candidate x_1 based on two criteria: leadership and academic excellence. The evaluation is expressed using the QIS framework over the universe $X = \{x_1\}$. Each expert provides a complex-valued membership degree with an associated phase angle:

$$Q_1 = \{(x_1, 0.7 + 0.4i, \phi_{Q_1}(x_1) = \arg(0.7 + 0.4i) \approx 0.519 \text{ rad})\},$$

$$Q_2 = \{(x_1, 0.6 + 0.5i, \phi_{Q_2}(x_1) = \arg(0.6 + 0.5i) \approx 0.694 \text{ rad})\}.$$

We combine the evaluations using the QIS union definition as follows:

$$\mu_{Q_1 \cup Q_2}(x_1) = \mu_{Q_1}(x_1) + \mu_{Q_2}(x_1) = (0.7 + 0.4i) + (0.6 + 0.5i) = 1.3 + 0.9i,$$

$$|\mu_{Q_1 \cup Q_2}(x_1)| = \sqrt{1.3^2 + 0.9^2} = \sqrt{2.5} \approx 1.58.$$

Since the magnitude exceeds 1, we apply normalization (optional in certain applications):

$$\mu^*_{Q_1 \cup Q_2}(x_1) = \frac{1.3 + 0.9i}{\sqrt{2}} \approx 0.919 + 0.636i$$

$$\phi_{Q_1 \cup Q_2}(x_1) = \arg(\mu^*_{Q_1 \cup Q_2}(x_1)) \approx 0.608 \text{ rad}.$$

$$Q_1 \cup Q_2 = \{(x_1, 0.919 + 0.636i, \phi = 0.608 \text{ rad})\}.$$

Interpretation. The resulting complex membership degree after normalization reflects the cumulative evaluation of the candidate. The increased magnitude indicates stronger overall agreement, while the phase angle (approx. 0.608 radians) represents a compromise direction between the experts' perspectives (e.g., balancing leadership and academic strength).

6.4. Case study: expert opinion aggregation with assigned phases

Consider a decision problem where two experts evaluate a product x_1 for market readiness. Each expert provides:
- A complex-valued membership $\mu_Q(x_1) = a + bi \in \mathbb{C}$ representing support and uncertainty.
- An assigned phase angle $\phi_Q(x_1) \in [0, 2\pi)$, interpreted as the expert's subjective orientation or context emphasis (e.g., technical vs. financial).

Let the QISs be

$$Q_1 = \left\{(x_1, 0.6 + 0.4i, \phi_{Q_1}(x_1) = \frac{\pi}{4})\right\}, \quad Q_2 = \left\{(x_1, 0.5 + 0.5i, \phi_{Q_2}(x_1) = \frac{\pi}{6})\right\}.$$

These phase values are expert-assigned, not derived from $\mu_Q(x_1)$. They reflect different interpretation strategies (e.g., $\frac{\pi}{4}$: risk-balanced optimism, $\frac{\pi}{6}$: cost-focused analysis).

$$\mu_{Q_1 \cup Q_2}(x_1) = (0.6 + 0.4i) + (0.5 + 0.5i) = 1.1 + 0.9i, \quad |\mu_{Q_1 \cup Q_2}(x_1)| \approx 1.42.$$

Since the magnitude exceeds 1, we apply normalization as follows:

$$\mu^*_{Q_1\cup Q_2}(x_1) = \frac{1.1 + 0.9i}{\sqrt{2}} \approx 0.778 + 0.636i.$$

For the final phase, use the following magnitude-weighted average:

$$\phi_{Q_1\cup Q_2}(x_1) = \frac{|\mu_{Q_1}| \cdot \phi_{Q_1} + |\mu_{Q_2}| \cdot \phi_{Q_2}}{|\mu_{Q_1}| + |\mu_{Q_2}|} \approx 0.655 \text{ rad}.$$

$$Q_1 \cup Q_2 = \{(x_1, 0.778 + 0.636i, \phi = 0.655 \text{ rad})\}.$$

6.5. Application: multi-expert decision-making using QIS score

Three experts evaluate two candidates x_1 and x_2 for a research position. Their opinions are modeled using QIS, where each expert provides $\mu_Q(x) = a + bi$ and $\phi_Q(x) = \arg(\mu_Q(x))$.

Score function: $\text{Score}_Q(x) = |\mu_Q(x)|^2 = a^2 + b^2$.

Expert input (QIS memberships):

$$Q_1 = \{(x_1, 0.6 + 0.3i),\ (x_2, 0.5 + 0.2i)\}, \quad Q_2 = \{(x_1, 0.5 + 0.4i),\ (x_2, 0.3 + 0.5i)\}.$$

Aggregation via union:

$$\mu_Q(x_1) = 1.1 + 0.7i \Rightarrow \phi_Q(x_1) \approx 0.567, \qquad \mu_Q(x_2) = 0.8 + 0.7i \Rightarrow \phi_Q(x_2) \approx 0.718.$$

Scores:

$$\text{Score}_Q(x_1) = 1.70, \qquad \text{Score}_Q(x_2) = 1.13.$$

Hence, x_1 is preferred; $\phi_Q(x_1) \approx 32.5^\circ$ indicates moderate alignment.

6.6. Application: QIS-based MCDM for optimal energy system selection

We evaluate four alternatives: A_1 Solar PV (utility-scale), A_2 onshore wind, A_3 gas turbine (CHP), A_4 diesel genset. Criteria: LCOE (\$/MWh, ↓), CO_2 (kg/MWh, ↓), Availability (%, ↑), CAPEX (\$/kW, ↓), Land (m^2/MW, ↓). Weights $(w_1, \ldots, w_5) = (0.30, 0.25, 0.20, 0.15, 0.10)$. The techno-economic indicators are based on industrial and research reports from Lazard [8] and IRENA [9, 10].

Before applying the QIS-based normalization and aggregation, the raw techno-economic performance values for the four alternatives are summarized in Table 5. These values reflect realistic order-of-magnitude benchmarks widely reported in recent energy studies.

Table 5. Raw performance (illustrative, realistic order-of-magnitude).

Alternative	LCOE	CO_2	Avail.	CAPEX	Land
A_1 PV	45	40	96.0	900	80,000
A_2 Wind	38	12	95.0	1400	5,000
A_3 Gas (CHP)	70	400	99.5	1000	1,000
A_4 Diesel	150	700	99.0	500	800

Data sources: Lazard levelized cost of energy+ (2025) [8] and IRENA renewable power generation costs (2024) [10].

We min–max normalize each criterion to the interval $[0, 1]$, where cost-type attributes use $N = (\max - x)/(\max - \min)$ and benefit-type attributes use $N = (x - \min)/(\max - \min)$. The resulting normalized decision matrix, which serves as the input for the QIS aggregation stage, is provided in Table 6.

Table 6. Normalized scores $r_j(x) \in [0, 1]$.

Alternative	LCOE	CO_2	Avail.	CAPEX	Land
A_1 PV	0.9375	0.9593	0.2222	0.5556	0.0000
A_2 Wind	1.0000	1.0000	0.0000	0.0000	0.9470
A_3 Gas (CHP)	0.7143	0.4360	1.0000	0.4444	0.9975
A_4 Diesel	0.0000	0.0000	0.8889	1.0000	1.0000

We assign criterion phases (linguistic prior + mild dispersion):

$$(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5) = \left(\tfrac{\pi}{12}, \tfrac{\pi}{8}, \tfrac{\pi}{6}, \tfrac{\pi}{12}, \tfrac{\pi}{5}\right).$$

Aggregate complex membership per alternative using Algorithm 4:

$$\mu(A) = \sum_{j=1}^{5} w_j\, r_j(A)\, e^{i\phi_j}, \qquad P_Q(A) = |\mu(A)|^2.$$

Using the worked complex-valued scores shown above, we compute the QIS probabilities $P_Q(A) = |\mu(A)|^2$ for each alternative. These values form the basis for the final ranking obtained through the QIS aggregation model, as reported in Table 7.

Worked numbers (rounded): Using $\cos(\frac{\pi}{12}) \approx 0.9659$, $\sin(\frac{\pi}{12}) \approx 0.2588$, $\cos(\frac{\pi}{8}) \approx 0.9239$, $\sin(\frac{\pi}{8}) \approx 0.3827$, $\cos(\frac{\pi}{6}) = 0.8660$, $\sin(\frac{\pi}{6}) = 0.5$, $\cos(\frac{\pi}{5}) \approx 0.8090$, $\sin(\frac{\pi}{5}) \approx 0.5878$, we obtain the following:

$$\begin{aligned} \mu(A_1) &\approx 0.612 + 0.208\, i, & P_Q(A_1) &\approx 0.418, \\ \mu(A_2) &\approx 0.597 + 0.229\, i, & P_Q(A_2) &\approx 0.409, \\ \mu(A_3) &\approx 0.626 + 0.273\, i, & P_Q(A_3) &\approx 0.466, \\ \mu(A_4) &\approx 0.380 + 0.187\, i, & P_Q(A_4) &\approx 0.179. \end{aligned}$$

Table 7. QIS scores and ranking.

Alternative	$P_Q(A) = \lvert\mu(A)\rvert^2$	Rank
A_3 Gas (CHP)	0.466	1
A_1 PV	0.418	2
A_2 Wind	0.409	3
A_4 Diesel	0.179	4

Interpretation. Under the chosen weights, the high availability and compact land footprint lift A_3. If environmental weight increases (e.g., $w_{CO_2} \uparrow$), A_1–A_2 quickly overtake. QIS makes this trade-off transparent via phase-aware aggregation and the scalar P_Q for ranking, consistent with the latest global cost trends reported by IRENA [9].

6.7. Uncertainty propagation experiment

To examine the robustness of the QIS-based decision under small perturbations, we vary:

- Criterion weights by ±5% around their baseline values.
- Phases by ±5% (i.e., $\phi_j \mapsto \phi_j(1 \pm 0.05)$).

For each perturbed configuration, we recompute $P_Q(A) = |\mu(A)|^2$ using the same normalization rules as before.

Perturbation of Weights. Let $w'_j = w_j(1 + \eta_j)$ with $\eta_j \in \{-0.05, 0, +0.05\}$ independently sampled (renormalized so that $\sum_j w'_j = 1$). Monte–Carlo simulation with 100 draws produces the mean and standard deviation of $P_Q(A)$ shown in Table 8.

Table 8. Sensitivity of $P_Q(A)$ to ±5% weight perturbations (100 trials).

Alternative	$\mathbb{E}[P_Q(A)]$	$\mathrm{Std}(P_Q(A))$
A_3 Gas (CHP)	0.465	0.009
A_1 PV	0.418	0.011
A_2 Wind	0.407	0.013
A_4 Diesel	0.179	0.007

The ranking remains invariant across all perturbations, demonstrating low weight sensitivity.

Perturbation of Phases. For phase variation, $\phi'_j = \phi_j(1 + \zeta_j)$ with $\zeta_j \in \{-0.05, 0, +0.05\}$. The resulting deviations are summarized in Table 9.

Table 9. Sensitivity of $P_Q(A)$ to ±5% phase perturbations.

Alternative	$\mathbb{E}[P_Q(A)]$	$\mathrm{Std}(P_Q(A))$
A_3 Gas (CHP)	0.466	0.005
A_1 PV	0.417	0.006
A_2 Wind	0.410	0.006
A_4 Diesel	0.179	0.003

Interpretation. Both experiments show that $P_Q(A)$ exhibits sub-linear sensitivity to small input perturbations. By Proposition 1, since $|\mu(A)| \approx 0.6$ for leading alternatives, the expected phase deviation $|\Delta\phi| \lesssim 0.05/0.6 \approx 0.083$ rad ($\approx 5^\circ$), which explains the minor score fluctuation. Thus, QIS-based ranking is numerically and semantically stable for realistic uncertainty levels.

7. Discussion

The QIS model offers a powerful extension to classical fuzzy and soft set theories by incorporating both magnitude and phase as fundamental dimensions of uncertainty. Unlike traditional models that rely solely on scalar degrees of truth, QIS expresses membership through complex numbers $\mu_Q(x) = a + bi$, with a corresponding phase angle $\phi_Q(x) = \arg(\mu_Q(x))$. This phase-based modeling allows QIS to naturally capture interference, opposition, and constructive ambiguity—properties commonly found in quantum logic and human expert reasoning.

Through the use of normalized operations, QIS maintains stability even under cumulative uncertainty. A key feature is the ability to support decision-making without reliance on external parameter sets, making it suitable for contexts with minimal or ambiguous information. The score function derived from complex membership also offers an interpretable scalar for ranking or evaluation, while preserving the original semantic orientation of the data through its phase.

Furthermore, the example of multi-expert hiring demonstrated how QIS can model consensus and divergence in expert judgments, using both modulus (confidence) and phase (semantic alignment). Compared to fuzzy sets, QIS handles contradiction more gracefully, offering a richer structure to encode and manipulate uncertainty.

8. Conclusions

This paper introduced the QIS, a new set-theoretic framework designed to capture both the magnitude and phase of uncertainty through complex-valued membership functions. By extending membership values into the complex plane, QIS incorporates quantum-like interference, enabling the modeling of constructive and destructive interactions among uncertain elements.

We defined the fundamental operations of QIS, provided formal proofs of their properties, and illustrated their interpretability through numerical examples and a real-world multi-criteria decision-making application in energy-system evaluation. Compared with fuzzy, intuitionistic, and neutrosophic sets, QIS offers richer semantics, a stronger interpretive phase mechanism, and more flexible aggregation under contradictory evidence.

Overall, QIS establishes a mathematical foundation for integrating phase-based reasoning into uncertainty modeling, marking a potential paradigm shift toward quantum-inspired intelligent systems and phase-dependent decision analytics.

9. Future work

The QIS framework opens several promising research directions:

- **Uncertainty propagation and phase stability:** Investigate cumulative phase effects in sequential operations and large-scale decision systems to formally quantify how interference evolves and stabilizes across iterative reasoning steps.
- **Learning-based phase assignment:** Develop data-driven and neural learning approaches for automatic phase inference, linking QIS semantics with observable features to reduce subjectivity.
- **Quantum-inspired machine learning:** Integrate QIS into neural and kernel architectures, enabling phase-aware learning and interpretable uncertainty modeling.
- **Multi-agent reasoning:** Extend QIS to model cooperative and competitive decision systems where each agent carries distinct phase orientations.
- **Computational optimization:** Design efficient numerical and matrix-based implementations for QIS operations to enhance scalability in AI and uncertainty analytics.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgment

This research was funded by Jadara University, Jordan under its internal research grant scheme.

The author gratefully acknowledges the support provided by Jadara University, which made this research possible.

Conflict of interest

The author declares that there is no conflict of interest regarding the publication of this paper. The author declares no financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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