Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 10(11): 26545–26567.
DOI:10.3934/math.20251167
Received: 07 September 2025
Revised: 29 October 2025
Accepted: 10 November 2025
Published: 17 November 2025

Research article

Closed-form solutions of a nonlinear bidimensional difference system via generalized Fibonacci sequences

Ahmed A. Al Ghafli*

Department of Mathematics and Statistics, College of Science, King Faisal University Hofuf 31982, Alahsa, Saudi Arabia

* **Correspondence:** Email: aamgkafli@kfu.edu.sa.

Abstract: This paper presents a new model for a two-dimensional nonlinear difference system that incorporates symmetric interactions between two sequences through a scaling parameter d and a continuous one-to-one transformation function f. Explicit analytical solutions are derived, establishing a direct connection with the d-Fibonacci sequence. The transformation function f plays a crucial role: It accommodates diverse nonlinear iteration patterns and provides a natural mechanism for regulating both growth dynamics and sequence interactions. Moreover, the use of a continuous one-to-one function guarantees that the analytical solutions of transformed systems can be recovered through its inverse mapping. The approach highlights a unified framework linking generalized Fibonacci-type recursions with nonlinear transformations, offering new insights into the structure and solvability of higher-order discrete systems. Several illustrative examples are provided to support the theoretical findings.

Keywords: nonlinear difference equations; closed form solutions; d-Fibonacci sequence; system of difference equations
Mathematics Subject Classification: 39A10, 40A05

1. Introduction

Nonlinear difference equations and systems are an essential tool in the analysis of complex dynamical systems, allowing the description of systems whose behavior is difficult to express using standard linear models. These equations capture the complex dynamical patterns that arise in various disciplines and provide a mathematical framework for analyzing stability, bifurcations, and chaotic behaviors. Their importance lies in their ability to simulate dynamic phenomena in many fields, such as biology, economics, physics, and neuroscience (see, e.g., [1–5]), where nonlinear effects lead to the emergence of complex behavioral patterns that go beyond simple linear behaviors. In particular, the study of higher-order difference equations has received increasing attention due to their fundamental and practical role in modeling periodic oscillations, phase transitions, and emergent phenomena in nonlinear systems. Building on this, the works in [6–8] make significant contributions to understanding the internal structure of rational and recursive systems, revealing complex periodic behaviors that arise from interdependencies on past values. Their methodologies provide precise tools for examining stability and convergence, thereby deepening our theoretical insight into nonlinear patterns in multi-variable discrete systems. References [9–11] further explore explicit expressions of solutions for certain higher-order rational sequences, including the characterization of forbidden sets and global dynamics. These studies highlight the significant influence of delays on system behavior, offering a rigorous framework to understand how past values shape long-term evolution in nonlinear discrete systems. Finally, the research in [12–14] expands the analysis of nonlinear systems by investigating periodicity and global behavior in second- and higher-order difference equations. They demonstrate how the interaction between nonlinear terms and delays can generate rich dynamical patterns, ranging from stability to quasi-periodic behaviors, emphasizing the importance of higher-order equations in capturing systems with long-term dependencies and multi-scale temporal structures. Such equations help describe population dynamics, financial time series, and neural activity, where long-term dependencies and recurring structures greatly influence the evolution of the system. In recent years, the study of nonlinear systems of difference equations has expanded considerably, with several models developed to explore complex dynamic interactions from various perspectives. Ghezal and Attia (2025, [15]) introduced a new three-dimensional nonlinear differential system and derived explicit closed-form solutions using characteristic root analysis, thereby emphasizing the system's sensitivity to initial conditions. In another recent study, Balegh and Ghezal (2025, [16]) examined a fuzzy difference system involving third-order nonlinear terms, focusing on the existence, boundedness, and stability of positive solutions within a fuzzy framework that accounts for uncertainty in the coefficients. Elsayed and Alharthi (2024, [17]) also conducted a qualitative analysis of a tenth-order difference system, highlighting the complex behaviors arising from multi-order coefficients and interrelated terms. However, a careful review of the existing literature reveals that most studies have primarily concentrated on one-dimensional nonlinear difference equations, while models involving interacting sequences or functional transformations that modify the system's internal dynamics have been largely overlooked. In particular, the interaction between two nonlinear components governed by a scaling parameter and a general transformation function has not been adequately explored in previous research. Hence, this study is significant as it introduces a novel two-dimensional nonlinear difference system that symmetrically couples two sequences through their shared algebraic and functional dependencies. By deriving explicit closed-form analytic solutions and examining their connection to the d-Fibonacci sequence, this research aims to address this theoretical gap and offer new analytical tools for investigating complex nonlinear interactions. The motivation for adopting this model arises from the need to establish a mathematical framework that unites analytical rigor with structural simplicity in the study of two-dimensional nonlinear systems. The objective is not merely to extend existing models, but to develop a flexible formulation that enables the analysis of interrelationships between sequences within a unified framework, linking them to well-known numerical sequences, such as the augmented Fibonacci sequence. The novelty of the proposed system lies in its ability to illustrate how scaling coefficients d and continuous transformation functions f can provide a natural mechanism for regulating growth and interaction patterns, thereby opening new analytical avenues for exploring complex dynamical phenomena. From this standpoint, the present work seeks to introduce a model that contributes both theoretically and practically, offering a versatile mathematical tool applicable across multiple scientific domains. Recent studies show that many of these systems relate to well-known numerical series, such as the Fibonacci sequence, which is defined by the relationship:

$$\mathbf{F}_{m+2} = \mathbf{F}_{m+1} + \mathbf{F}_m, m \in \mathbb{N},$$

with initial values $\mathbf{F}_0 = \mathbf{F}_1 = 1$. The Fibonacci sequence exhibits fundamental repetitive properties that are deeply connected to various combinatorial structures and growth patterns. Here, the term "d-Fibonacci sequence" refers to a generalized form of the classical Fibonacci sequence, obtained by introducing a real scaling parameter d that modifies the rate of growth. When $d = 1$, the sequence reduces to the standard Fibonacci sequence, making it a natural generalization. It is represented by the following relationship:

$$F_{m+2} = F_{m+1} + dF_m, \ m \in \mathbb{N},$$

with initial values $F_0 = F_1 = 1$. This generalization introduces an additional scaling factor, d, which modifies the sequence growth rate and extends its application to broader dynamical systems. A connection has been shown between the solutions of some nonlinear difference equations and the d-Fibonacci sequence, as in the model studied by Merve Kara et al. [18],

$$\alpha_m = \frac{\alpha_{m-2}^{d}\alpha_{m-3}\alpha_{m-4}}{\alpha_{m-1}\left(\sigma\alpha_{m-5}^{d} + \rho\alpha_{m-3}\alpha_{m-4}\right)}, \ d, \ m \in \mathbb{N}, \tag{A.1}$$

with initial values α_{-1}, α_{-2}, α_{-3}, α_{-4}, α_{-5}. This model is considered a generalization of a nonlinear difference equation of the fourth order of the form:

$$\alpha_m = \frac{\alpha_{m-2}\alpha_{m-3}}{\alpha_m\left(\pm 1 \pm \alpha_{m-2}\alpha_{m-3}\right)}, \ d, \ m \in \mathbb{N},$$

with initial values α_{-1}, α_{-2}, α_{-3}. These models illustrate how nonlinear recurrence relationships can lead to complex solution structures, which are influenced by the choice of parameters and initial conditions. To facilitate a deeper understanding of the mathematical foundations underlying generalized Fibonacci sequences, readers are encouraged to consult the original references that have extensively discussed this topic, particularly the work by Kara et al. ([18] and the references cited therein). Based on these results, this research aims to present a new model that generalizes these equations, where we formulate the following system:

$$\alpha_m = \frac{\alpha_{m-2}^{d}\beta_{m-3}\beta_{m-4}}{\alpha_{m-1}\left(\sigma\beta_{m-5}^{d} + \rho\beta_{m-3}\beta_{m-4}\right)}, \quad \beta_m = \frac{\beta_{m-2}^{d}\alpha_{m-3}\alpha_{m-4}}{\beta_{m-1}\left(\sigma\alpha_{m-5}^{d} + \rho\alpha_{m-3}\alpha_{m-4}\right)}, d, m \in \mathbb{N}, \tag{A.2}$$

with initial values α_{-1}, α_{-2}, α_{-3}, α_{-4}, α_{-5}, β_{-1}, β_{-2}, β_{-3}, β_{-4}, β_{-5}. System (A.2) can be regarded as a natural generalization of previously studied models for several reasons. First, it preserves the fundamental recursive nonlinear structure in which each new term depends on the products and ratios of lagged terms, as in the fourth-order Kara et al. [18] equation (A.1). However, our formulation extends this structure in two essential ways: (i) by introducing two correlated sequences, α_m and β_m, that interact symmetrically with one another, and (ii) by incorporating the scalar parameter d, which influences the growth rate of the sequences and enables a wider range of dynamical behaviors. This extension preserves the algebraic framework of earlier models while encompassing them as special cases; for instance, it reduces to the scalar equation when $\alpha_m = \beta_m$. Thus, system (A.2) is a natural and coherent extension rather than an arbitrary modification of the original recurrence, offering a richer setting for exploring connections with the d-Fibonacci sequence and related nonlinear phenomena. Our choice to investigate a nonlinear system of rational form was not arbitrary but rather motivated by the distinctive ability of such models to capture complex dynamic behaviors. Rational systems exhibit evolutionary patterns that are highly sensitive to growth coefficients and initial parameters, providing a natural framework for analyzing phenomena such as periodic oscillations, phase transitions, and chaotic dynamics. Moreover, the rational structure facilitates a conceptual link between difference equations and well-known numerical sequences, such as the d-Fibonacci sequence, since the recurrence of algebraic ratios among terms generates a recursive structure similar to that observed in sequences with controlled growth. Therefore, adopting the rational form in this research aims to establish a more adaptable and analytically tractable model, capable of describing mutual nonlinear interactions between variables while preserving the feasibility of deriving precise closed-form solutions. For a further natural generalization of system (A.2), we introduce a new formulation that extends the classical fourth-order iteration by incorporating two symmetrically interacting sequences together with a scaling parameter d. The resulting system integrates these dynamics into a functional framework through a continuous one-to-one transformation $f : \mathbb{R} \longrightarrow \mathbb{R}$. Specifically, we consider

$$\begin{aligned} \alpha_m &= f^{-1}\left(\frac{(f(\alpha_{m-2}))^d f(\beta_{m-3}) f(\beta_{m-4})}{f(\alpha_{m-1})\left(\sigma (f(\beta_{m-5}))^d + \rho f(\beta_{m-3}) f(\beta_{m-4})\right)}\right), \\ \beta_m &= f^{-1}\left(\frac{(f(\beta_{m-2}))^d f(\alpha_{m-3}) f(\alpha_{m-4})}{f(\beta_{m-1})\left(\sigma (f(\alpha_{m-5}))^d + \rho f(\alpha_{m-3}) f(\alpha_{m-4})\right)}\right), m \in \mathbb{N}, \end{aligned} \tag{A.3}$$

with initial values $\alpha_{-1}, \alpha_{-2}, \alpha_{-3}, \alpha_{-4}, \alpha_{-5}, \beta_{-1}, \beta_{-2}, \beta_{-3}, \beta_{-4}, \beta_{-5}$. This functional formulation significantly broadens the scope of the model. The function f allows the system to encompass a wide range of nonlinear iterations and provides a natural mechanism for altering the interaction and growth patterns of the sequences. Importantly, when f is chosen as the identity function, the system reduces precisely to (A.2), showing that (A.3) is not a random modification but a coherent unification and extension. Thus, (A.3) establishes a flexible framework that preserves the underlying nonlinear iterative structure while enabling richer dynamical behavior and broader applicability. Moreover, it opens the way to the study of closed-form solutions, periodicity, and potential links with generalized Fibonacci sequences and other nonlinear phenomena.

This paper aims to present a new model for a system of nonlinear difference equations that generalizes previous models, focusing on the association of the solutions of this system with the d-Fibonacci sequence. The main goal is to provide closed-form solutions that reflect the periodic structure of the system. Through theoretical analysis and numerical examples, the paper provides deep insights into how the system evolves over time and the influence of initial parameters on its behavior.

The paper is divided into several main sections to facilitate the understanding of the presented results and analyses. In Section 2, the derivation of closed-form solutions is presented using variable and linear transformations of the system (A.2). Section 3 extends this analysis to system (A.3), where closed-form solutions are obtained by employing nonlinear transformations of the variables. Finally, the main conclusions and suggestions for future work are presented.

2. Main results

This section is dedicated to the formal presentation of the principal theoretical results. Building upon the auxiliary lemmas and fundamental transformations, we now provide a rigorous formulation of the main theorems that characterize the explicit behavior of the nonlinear system under investigation. The results presented herein not only establish the existence of the system's solution structure but also yield its closed-form representations, thereby unveiling the intrinsic relationship between the system's dynamics and generalized Fibonacci-type recursions. We now state the following theorem, which provides a closed-form solution for the nonlinear system (A.2) under the initial conditions $\left(\alpha_{-j},\beta_{-j}\right)$, $j=1,...5$. This result offers an explicit expression for the values α_{12s+l} and β_{12s+l}, $l\in\{0,...,11\}$, using constructions based on Fibonacci sequences and the previous values of the system.

Theorem 2.1. *Consider the nonlinear system (A.2) where the initial values are given by* $\left(\alpha_{-j},\beta_{-j}\right)$, $j=1,...5$. *Then the following relations hold:*

$$\alpha_{12s}=\frac{\alpha_{-4}^{F_{12s+2}}}{\alpha_{-3}^{dF_{12s+3}}\widehat{\alpha}_{12s}^{F_0}}\left\{\prod_{j=0}^{s}M_{\widehat{\alpha},0}(s,j)\right\}\left\{\prod_{j=0}^{s-1}\widehat{M}_{\widehat{\alpha},0}(s,j)\right\},\qquad \alpha_{12s+1}=\frac{\alpha_{-3}^{F_{12s+4}}}{\alpha_{-4}^{dF_{12s+3}}}\left\{\prod_{j=0}^{s}\widetilde{M}_{\widehat{\alpha},1}(s,j)\right\}\left\{\prod_{j=0}^{s-1}\overline{M}_{\widehat{\alpha},1}(s,j)\right\},$$

$$\alpha_{12s+2}=\frac{\alpha_{-4}^{F_{12s+4}}}{\alpha_{-3}^{dF_{12s+5}}\widehat{\alpha}_{12s+2}^{F_0}}\left\{\prod_{j=0}^{s}M_{\widehat{\alpha},1}(s,j)\right\}\left\{\prod_{j=0}^{s-1}\widehat{M}_{\widehat{\alpha},1}(s,j)\right\},\qquad \alpha_{12s+3}=\frac{\alpha_{-3}^{F_{12s+6}}}{\alpha_{-4}^{dF_{12s+5}}}\left\{\prod_{j=0}^{s}\widetilde{M}_{\widehat{\alpha},2}(s,j)\right\}\left\{\prod_{j=0}^{s-1}\overline{M}_{\widehat{\alpha},2}(s,j)\right\},$$

$$\alpha_{12s+4}=\frac{\alpha_{-4}^{F_{12s+6}}}{\alpha_{-3}^{dF_{12s+7}}\widehat{\alpha}_{12s+4}^{F_0}}\left\{\prod_{j=0}^{s}M_{\widehat{\alpha},2}(s,j)\right\}\left\{\prod_{j=0}^{s-1}\widehat{M}_{\widehat{\alpha},2}(s,j)\right\},\qquad \alpha_{12s+5}=\frac{\alpha_{-3}^{F_{12s+8}}}{\alpha_{-4}^{dF_{12s+7}}}\left\{\prod_{j=0}^{s}\widetilde{M}_{\widehat{\alpha},3}(s,j)\right\}\left\{\prod_{j=0}^{s-1}\overline{M}_{\widehat{\alpha},3}(s,j)\right\},$$

$$\alpha_{12s+6}=\frac{\alpha_{-4}^{F_{12s+8}}}{\alpha_{-3}^{dF_{12s+9}}\widehat{\alpha}_{12s+6}^{F_0}}\left\{\prod_{j=0}^{s}M_{\widehat{\alpha},3}(s,j)\right\}\left\{\prod_{j=0}^{s-1}\widehat{M}_{\widehat{\alpha},3}(s,j)\right\},\qquad \alpha_{12s+7}=\frac{\alpha_{-3}^{F_{12s+10}}}{\alpha_{-4}^{dF_{12s+9}}}\left\{\prod_{j=0}^{s}\widetilde{M}_{\widehat{\alpha},4}(s,j)\right\}\left\{\prod_{j=0}^{s-1}\overline{M}_{\widehat{\alpha},3}(s,j)\right\},$$

$$\alpha_{12s+8}=\frac{\alpha_{-4}^{F_{12s+10}}}{\alpha_{-3}^{dF_{12s+11}}\widehat{\alpha}_{12s+8}^{F_0}}\left\{\prod_{j=0}^{s}M_{\widehat{\alpha},4}(s,j)\right\}\left\{\prod_{j=0}^{s-1}\widehat{M}_{\widehat{\alpha},4}(s,j)\right\},\qquad \alpha_{12s+9}=\frac{\alpha_{-3}^{F_{12(s+1)}}}{\alpha_{-4}^{dF_{12s+11}}}\left\{\prod_{j=0}^{s}\widetilde{M}_{\widehat{\alpha},5}(s,j)\right\},$$

$$\alpha_{12s+10}=\frac{\alpha_{-4}^{F_{12(s+1)}}}{\alpha_{-3}^{dF_{12s+13}}\widehat{\alpha}_{12s+10}^{F_0}}\left\{\prod_{j=0}^{s}M_{\widehat{\alpha},5}(s,j)\right\},$$

$$\alpha_{12s+11}=\frac{\alpha_{-3}^{F_{12(s+1)+2}}}{\alpha_{-4}^{dF_{12(s+1)+1}}}\left\{\prod_{j=0}^{s+1}\widetilde{M}_{\widehat{\alpha},0}(s+1,j)\right\}\left\{\prod_{j=0}^{s}\overline{M}_{\widehat{\alpha},0}(s+1,j)\right\},$$

and

$$\beta_{12s} = \frac{\beta_{-4}^{F_{12s+2}}}{\beta_{-3}^{dF_{12s+3}}\widehat{\beta}_{12s}^{F_0}} \left\{\prod_{j=0}^{s} M_{\widehat{\beta},0}(s,j)\right\}\left\{\prod_{j=0}^{s-1} \widehat{M}_{\widehat{\beta},0}(s,j)\right\}, \qquad \beta_{12s+1} = \frac{\beta_{-3}^{F_{12s+4}}}{\beta_{-4}^{dF_{12s+3}}} \left\{\prod_{j=0}^{s} \widetilde{M}_{\widehat{\beta},1}(s,j)\right\}\left\{\prod_{j=0}^{s-1} \overline{M}_{\widehat{\beta},1}(s,j)\right\},$$

$$\beta_{12s+2} = \frac{\beta_{-4}^{F_{12s+4}}}{\beta_{-3}^{dF_{12s+5}}\widehat{\beta}_{12s+2}^{F_0}} \left\{\prod_{j=0}^{s} M_{\widehat{\beta},1}(s,j)\right\}\left\{\prod_{j=0}^{s-1} \widehat{M}_{\widehat{\beta},1}(s,j)\right\}, \qquad \beta_{12s+3} = \frac{\beta_{-3}^{F_{12s+6}}}{\beta_{-4}^{dF_{12s+5}}} \left\{\prod_{j=0}^{s} \widetilde{M}_{\widehat{\beta},2}(s,j)\right\}\left\{\prod_{j=0}^{s-1} \overline{M}_{\widehat{\beta},2}(s,j)\right\},$$

$$\beta_{12s+4} = \frac{\beta_{-4}^{F_{12s+6}}}{\beta_{-3}^{dF_{12s+7}}\widehat{\beta}_{12s+4}^{F_0}} \left\{\prod_{j=0}^{s} M_{\widehat{\beta},2}(s,j)\right\}\left\{\prod_{j=0}^{s-1} \widehat{M}_{\widehat{\beta},2}(s,j)\right\}, \qquad \beta_{12s+5} = \frac{\beta_{-3}^{F_{12s+8}}}{\beta_{-4}^{dF_{12s+7}}} \left\{\prod_{j=0}^{s} \widetilde{M}_{\widehat{\beta},3}(s,j)\right\}\left\{\prod_{j=0}^{s-1} \overline{M}_{\widehat{\beta},3}(s,j)\right\},$$

$$\beta_{12s+6} = \frac{\beta_{-4}^{F_{12s+8}}}{\beta_{-3}^{dF_{12s+9}}\widehat{\beta}_{12s+6}^{F_0}} \left\{\prod_{j=0}^{s} M_{\widehat{\beta},3}(s,j)\right\}\left\{\prod_{j=0}^{s-1} \widehat{M}_{\widehat{\beta},3}(s,j)\right\}, \qquad \beta_{12s+7} = \frac{\beta_{-3}^{F_{12s+10}}}{\beta_{-4}^{dF_{12s+9}}} \left\{\prod_{j=0}^{s} \widetilde{M}_{\widehat{\beta},4}(s,j)\right\}\left\{\prod_{j=0}^{s-1} \overline{M}_{\widehat{\beta},3}(s,j)\right\},$$

$$\beta_{12s+8} = \frac{\beta_{-4}^{F_{12s+10}}}{\beta_{-3}^{dF_{12s+11}}\widehat{\beta}_{12s+8}^{F_0}} \left\{\prod_{j=0}^{s} M_{\widehat{\beta},4}(s,j)\right\}\left\{\prod_{j=0}^{s-1} \widehat{M}_{\widehat{\beta},4}(s,j)\right\}, \qquad \beta_{12s+9} = \frac{\beta_{-3}^{F_{12(s+1)}}}{\beta_{-4}^{dF_{12s+11}}} \left\{\prod_{j=0}^{s} \widetilde{M}_{\widehat{\beta},5}(s,j)\right\},$$

$$\beta_{12s+10} = \frac{\beta_{-4}^{F_{12(s+1)}}}{\beta_{-3}^{dF_{12s+13}}\widehat{\beta}_{12s+10}^{F_0}} \left\{\prod_{j=0}^{s} M_{\widehat{\beta},5}(s,j)\right\},$$

$$\beta_{12s+11} = \frac{\beta_{-3}^{F_{12(s+1)+2}}}{\beta_{-4}^{dF_{12(s+1)+1}}} \left\{\prod_{j=0}^{s+1} \widetilde{M}_{\widehat{\beta},0}(s+1,j)\right\}\left\{\prod_{j=0}^{s} \overline{M}_{\widehat{\beta},0}(s+1,j)\right\},$$

where

$$M_{h,r}(s,j) = \prod_{l=0}^{r} \frac{h_{6(2j)+2l-1}^{F_{12(s-j)+2(r-l)+1}}}{h_{6(2j)+2l-2}^{F_{12(s-j)+2(r-l+1)}}}, \qquad \widetilde{M}_{h,r}(s,j) = \prod_{l=0}^{r} \frac{h_{6(2j)+2l-2}^{F_{12(s-j)+2(r-l)+1}}}{h_{6(2j)+2l-1}^{F_{12(s-j)+2(r-l)}}},$$

$$\widehat{M}_{h,r}(s,j) = \prod_{l=1}^{5-r} \frac{h_{6(2j)+2(l+r)+1}^{F_{12(s-j)-2l+1}}}{h_{6(2j)+2(l+r-1)}^{F_{12(s-j)-2(l-1)}}}, \qquad \overline{M}_{h,r}(s,j) = \prod_{l=1}^{5-r} \frac{h_{6(2j)+2(l+r-1)}^{F_{12(s-j)-2l+1}}}{h_{6(2j)+2(l+r)+1}^{F_{12(s-j)-2l}}},$$

and the values $\left(\widehat{\alpha}_{6s+l}, \widehat{\beta}_{6s+l}\right)$, $l \in \{0, ..., 5\}$, *are determined according to (C.3)–(C.4) when* $|\sigma| \neq 1$ *and according to (C.5) when* $|\sigma| = 1$.

Remark 2.1. *It is worth noting that there exist many classical mathematical sequences, such as the Chebyshev, Legendre, and Jacobi sequences, as well as sequences derived from wavelet functions, that play a fundamental role in approximation theory, spectral analysis, and signal representation. However, the general Fibonacci sequence (d-Fibonacci) possesses distinctive analytical properties that make it particularly well-suited for studying systems of nonlinear difference equations. While orthogonal sequences are typically defined through recurrence relations involving polynomial weights or orthogonality conditions, the general Fibonacci sequence is purely algebraic in nature and readily integrates with nonlinear recurrence structures. Its primary strength lies in its scalar flexibility, governed by the parameter d, which enables the modeling of growth, decay, and correlated interactions among sequences in multidimensional systems. This flexibility also allows it to be seamlessly combined with functional transformations such as* f*, yielding explicit closed-form analytical expressions that are often difficult to obtain using other classical sequences. Nevertheless, one of its main limitations is the absence of the orthogonality property that characterizes the Chebyshev or Legendre sequences, which makes it less suitable for applications relying on spectral decomposition or projection-based methods. Despite this limitation, within the scope of the present study, focused on explicit solvability and nonlinear coupling between sequences, the general Fibonacci sequence provides the most appropriate analytical framework for achieving these research objectives.*

Example 2.1. *Consider the nonlinear difference system*

$$\alpha_m = \frac{\alpha_{m-2}\beta_{m-3}\beta_{m-4}}{\alpha_{m-1}\left(0.2\beta_{m-5} + 0.3\beta_{m-3}\beta_{m-4}\right)}, \quad \beta_m = \frac{\beta_{m-2}\alpha_{m-3}\alpha_{m-4}}{\beta_{m-1}\left(0.2\alpha_{m-5} + 0.3\alpha_{m-3}\alpha_{m-4}\right)}, m \in \mathbb{N}. \tag{B.1}$$

with the parameter values $d = 1$, $\sigma = 0.2$, $\rho = 0.3$, *and initial values* $\{\alpha_{-5}, \alpha_{-4}, \alpha_{-3}, \alpha_{-2}, \alpha_{-1}\} = \{1.2, 1.3, 1.1, 1.4, 1.2\}$, $\{\beta_{-5}, \beta_{-4}, \beta_{-3}, \beta_{-2}, \beta_{-1}\} = \{1.1, 0.9, 1.2, 1.0, 1.3\}$. *Figure 1 below illustrates the corresponding dynamical evolution of the sequences* α_m *and* β_m.

Figure 1 illustrates that the system initially exhibits an oscillatory phase characterized by value fluctuations, which is then followed by a gradual stabilization phase where the values converge toward a stable equilibrium point. This behavior suggests that, under the specified conditions, the system possesses a stable final attractor. Moreover, the nonlinear interactions between the variables do not give rise to chaotic dynamics in this case but instead lead to long-term stability. To confirm the correctness of the closed-form solution derived in Theorem 2.1, we performed a numerical verification by comparing the values generated by direct iteration of system (A.2) and those computed using the closed-form expressions. The results are presented in Table 1 below.

This numerical comparison further confirms the validity of the analytical approach and demonstrates the consistency between the recursive and closed-form representations.

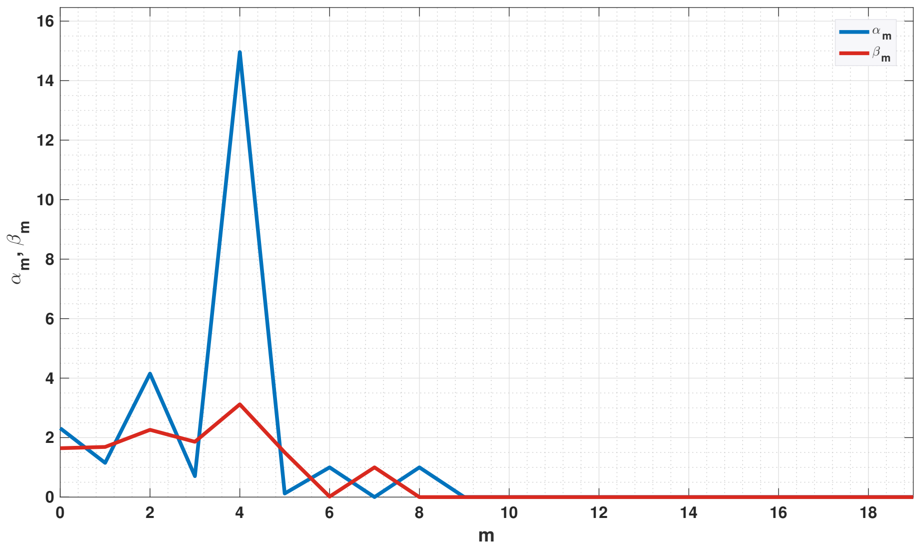

Figure 1. Numerical solution of system (B.1).

Table 1. Numerical verification of the closed-form solution (Theorem 2.1) compared with direct iteration of system (A.2).

m	α_m (by iteration)	α_m (by closed form)	β_m (by iteration)	β_m (by closed form)
0	2.316176	2.316159	1.644245	1.644225
1	1.151323	1.151300	1.686399	1.686426
2	4.151235	4.151252	2.262440	2.262445
3	0.704696	0.704698	1.860030	1.860028
4	14.960194	14.960229	3.118836	3.118840
5	0.121973	0.121980	1.502522	1.502541
6	1.000000	1.000021	0.016988	0.017018
7	0.001382	0.001378	1.000000	1.000033
8	1.000000	1.000019	0.000024	0.000022
9	0.000056	0.000052	0.000100	0.000109
10	0.000100	0.000107	0.000100	0.000104
11	0.000100	0.000103	0.000100	0.000103
12	0.000100	0.000103	0.000100	0.000103
13	0.000100	0.000103	0.000100	0.000103
14	0.000100	0.000103	0.000100	0.000103
15	0.000100	0.000103	0.000100	0.000103
16	0.000100	0.000103	0.000100	0.000103
17	0.000100	0.000103	0.000100	0.000103
18	0.000100	0.000103	0.000100	0.000103
19	0.000100	0.000103	0.000100	0.000103
20	0.000100	0.000103	0.000100	0.000103
21	0.000100	0.000103	0.000100	0.000103

Remark 2.2. *The results obtained in this section clearly highlight the fundamental advantages of the present study compared with previous works on nonlinear difference systems. Unlike conventional approaches that are often restricted to numerical investigations or approximate analytical methods, the current work provides explicit closed-form analytical derivations of the system (A.2). This is achieved through a set of appropriate transformations that link the original nonlinear system to a symmetric linear representation, thereby enabling an accurate characterization of the long-term behavior of its solutions. The model also provides a parameter-dependent control mechanism, where the qualitative nature of the solutions, whether linear, exponential, or oscillatory, can be adjusted through the choice of coefficients. In addition, the adopted formulation successfully reveals the* 12*th-order periodicity inherent in the system's dynamics and illustrates how the Fibonacci sequence can be effectively employed to capture the recursive and evolutionary behavior of the solutions. In summary, this approach constitutes a powerful analytical tool for the study of complex nonlinear structures, enriching the theoretical understanding of nonlinear difference systems. By combining exact analytical tractability with the potential for generalization to broader classes of models, it significantly extends the applicability and interpretive depth of current nonlinear system theory.*

For the functional system (A.3), the recursive nature of the relations poses significant challenges, since each term depends nonlinearly on multiple delayed components of both sequences. To ensure that the system is well-posed, the initial conditions are specified as (α_{-j},β_{-j}), $j = 1,\ldots,5$, which provide the necessary starting values to uniquely determine all subsequent iterates. The closed-form solution of this system not only reveals the explicit dependence of the sequences on the initial values and parameters (d,σ,ρ) but also highlights intrinsic connections with generalized Fibonacci numbers F_m. Indeed, the formulas obtained involve products of structured terms $N_{h,r}(s,j)$, which encode the layered interactions among delayed states through the transformation $f:\mathbb{R}\longrightarrow\mathbb{R}$. This representation captures the interplay between the nonlinear transformation, the scaling factor d, and the coupling symmetry of the two sequences. In what follows, we present the explicit closed-form relations satisfied by the iterates of system (A.3). These results demonstrate how the nonlinear recurrence can be unraveled into analytic expressions that depend systematically on the initial values, the parameters, and the Fibonacci structure underlying the dynamics.

Theorem 2.2. *Consider the nonlinear system (A.3) where the initial values are given by* $\left(\alpha_{-j},\beta_{-j}\right)$, $j = 1,\ldots 5$. *Then the sequences* (α_m) *and* (β_m) *satisfy the following closed-form relations:*

$$\begin{aligned}
\alpha_{12s} &= f^{-1}\left(\frac{(f(\alpha_{-4}))^{F_{12s+2}}}{(f(\alpha_{-3}))^{dF_{12s+3}}(f(\widehat{\alpha}_{12s}))^{F_0}}\left\{\prod_{j=0}^{s}N_{\widehat{\alpha},0}(s,j)\right\}\left\{\prod_{j=0}^{s-1}\widehat{N}_{\widehat{\alpha},0}(s,j)\right\}\right),\\
\alpha_{12s+1} &= f^{-1}\left(\frac{(f(\alpha_{-3}))^{F_{12s+4}}}{(f(\alpha_{-4}))^{dF_{12s+3}}}\left\{\prod_{j=0}^{s}\widetilde{N}_{\widehat{\alpha},1}(s,j)\right\}\left\{\prod_{j=0}^{s-1}\overline{N}_{\widehat{\alpha},1}(s,j)\right\}\right),\\
\alpha_{12s+2} &= f^{-1}\left(\frac{(f(\alpha_{-4}))^{F_{12s+4}}}{(f(\alpha_{-3}))^{dF_{12s+5}}(f(\widehat{\alpha}_{12s+2}))^{F_0}}\left\{\prod_{j=0}^{s}N_{\widehat{\alpha},1}(s,j)\right\}\left\{\prod_{j=0}^{s-1}\widehat{N}_{\widehat{\alpha},1}(s,j)\right\}\right),\\
\alpha_{12s+3} &= f^{-1}\left(\frac{(f(\alpha_{-3}))^{F_{12s+6}}}{(f(\alpha_{-4}))^{dF_{12s+5}}}\left\{\prod_{j=0}^{s}\widetilde{N}_{\widehat{\alpha},2}(s,j)\right\}\left\{\prod_{j=0}^{s-1}\overline{N}_{\widehat{\alpha},2}(s,j)\right\}\right),\\
\alpha_{12s+4} &= f^{-1}\left(\frac{(f(\alpha_{-4}))^{F_{12s+6}}}{(f(\alpha_{-3}))^{dF_{12s+7}}(f(\widehat{\alpha}_{12s+4}))^{F_0}}\left\{\prod_{j=0}^{s}N_{\widehat{\alpha},2}(s,j)\right\}\left\{\prod_{j=0}^{s-1}\widehat{N}_{\widehat{\alpha},2}(s,j)\right\}\right),\\
\alpha_{12s+5} &= f^{-1}\left(\frac{(f(\alpha_{-3}))^{F_{12s+8}}}{(f(\alpha_{-4}))^{dF_{12s+7}}}\left\{\prod_{j=0}^{s}\widetilde{N}_{\widehat{\alpha},3}(s,j)\right\}\left\{\prod_{j=0}^{s-1}\overline{N}_{\widehat{\alpha},3}(s,j)\right\}\right),\\
\alpha_{12s+6} &= f^{-1}\left(\frac{(f(\alpha_{-4}))^{F_{12s+8}}}{(f(\alpha_{-3}))^{dF_{12s+9}}(f(\widehat{\alpha}_{12s+6}))^{F_0}}\left\{\prod_{j=0}^{s}N_{\widehat{\alpha},3}(s,j)\right\}\left\{\prod_{j=0}^{s-1}\widehat{N}_{\widehat{\alpha},3}(s,j)\right\}\right),
\end{aligned}$$

$$\begin{aligned}
\alpha_{12s+7} &= f^{-1}\left(\frac{(f(\alpha_{-3}))^{F_{12s+10}}}{(f(\alpha_{-4}))^{dF_{12s+9}}}\left\{\prod_{j=0}^{s}\widetilde{N}_{\widehat{\alpha},4}(s,j)\right\}\left\{\prod_{j=0}^{s-1}\overline{N}_{\widehat{\alpha},3}(s,j)\right\}\right),\\
\alpha_{12s+8} &= f^{-1}\left(\frac{(f(\alpha_{-4}))^{F_{12s+10}}}{(f(\alpha_{-3}))^{dF_{12s+11}}(f(\widehat{\alpha}_{12s+8}))^{F_0}}\left\{\prod_{j=0}^{s}N_{\widehat{\alpha},4}(s,j)\right\}\left\{\prod_{j=0}^{s-1}\widehat{N}_{\widehat{\alpha},4}(s,j)\right\}\right),\\
\alpha_{12s+9} &= f^{-1}\left(\frac{(f(\alpha_{-3}))^{F_{12(s+1)}}}{(f(\alpha_{-4}))^{dF_{12s+11}}}\left\{\prod_{j=0}^{s}\widetilde{N}_{\widehat{\alpha},5}(s,j)\right\}\right),
\end{aligned}$$

$$\alpha_{12s+10} = f^{-1}\left(\frac{(f(\alpha_{-4}))^{F_{12(s+1)}}}{(f(\alpha_{-3}))^{dF_{12s+13}}(f(\widehat{\alpha}_{12s+10}))^{F_0}}\left\{\prod_{j=0}^{s}N_{\widehat{\alpha},5}(s,j)\right\}\right),$$

$$\alpha_{12s+11} = f^{-1}\left(\frac{(f(\alpha_{-3}))^{F_{12(s+1)+2}}}{(f(\alpha_{-4}))^{dF_{12(s+1)+1}}}\left\{\prod_{j=0}^{s+1}\widetilde{N}_{\widehat{\alpha},0}(s+1,j)\right\}\left\{\prod_{j=0}^{s}\overline{N}_{\widehat{\alpha},0}(s+1,j)\right\}\right),$$

and

$$\beta_{12s} = f^{-1}\left(\frac{(f(\beta_{-4}))^{F_{12s+2}}}{(f(\beta_{-3}))^{dF_{12s+3}}\left(f\left(\widehat{\beta}_{12s}\right)\right)^{F_0}}\left\{\prod_{j=0}^{s}N_{\widehat{\beta},0}(s,j)\right\}\left\{\prod_{j=0}^{s-1}\widehat{N}_{\widehat{\beta},0}(s,j)\right\}\right),$$

$$\beta_{12s+1} = f^{-1}\left(\frac{(f(\beta_{-3}))^{F_{12s+4}}}{(f(\beta_{-4}))^{dF_{12s+3}}}\left\{\prod_{j=0}^{s}\widetilde{N}_{\widehat{\beta},1}(s,j)\right\}\left\{\prod_{j=0}^{s-1}\overline{N}_{\widehat{\beta},1}(s,j)\right\}\right),$$

$$\beta_{12s+2} = f^{-1}\left(\frac{(f(\beta_{-4}))^{F_{12s+4}}}{(f(\beta_{-3}))^{dF_{12s+5}}\left(f\left(\widehat{\beta}_{12s+2}\right)\right)^{F_0}}\left\{\prod_{j=0}^{s}N_{\widehat{\beta},1}(s,j)\right\}\left\{\prod_{j=0}^{s-1}\widehat{N}_{\widehat{\beta},1}(s,j)\right\}\right),$$

$$\beta_{12s+3} = f^{-1}\left(\frac{(f(\beta_{-3}))^{F_{12s+6}}}{(f(\beta_{-4}))^{dF_{12s+5}}}\left\{\prod_{j=0}^{s}\widetilde{N}_{\widehat{\beta},2}(s,j)\right\}\left\{\prod_{j=0}^{s-1}\overline{N}_{\widehat{\beta},2}(s,j)\right\}\right),$$

$$\beta_{12s+4} = f^{-1}\left(\frac{(f(\beta_{-4}))^{F_{12s+6}}}{(f(\beta_{-3}))^{dF_{12s+7}}\left(f\left(\widehat{\beta}_{12s+4}\right)\right)^{F_0}}\left\{\prod_{j=0}^{s}N_{\widehat{\beta},2}(s,j)\right\}\left\{\prod_{j=0}^{s-1}\widehat{N}_{\widehat{\beta},2}(s,j)\right\}\right),$$

$$\beta_{12s+5} = f^{-1}\left(\frac{(f(\beta_{-3}))^{F_{12s+8}}}{(f(\beta_{-4}))^{dF_{12s+7}}}\left\{\prod_{j=0}^{s}\widetilde{N}_{\widehat{\beta},3}(s,j)\right\}\left\{\prod_{j=0}^{s-1}\overline{N}_{\widehat{\beta},3}(s,j)\right\}\right),$$

$$\beta_{12s+6} = f^{-1}\left(\frac{(f(\beta_{-4}))^{F_{12s+8}}}{(f(\beta_{-3}))^{dF_{12s+9}}\left(f\left(\widehat{\beta}_{12s+6}\right)\right)^{F_0}}\left\{\prod_{j=0}^{s}N_{\widehat{\beta},3}(s,j)\right\}\left\{\prod_{j=0}^{s-1}\widehat{N}_{\widehat{\beta},3}(s,j)\right\}\right),$$

$$\beta_{12s+7} = f^{-1}\left(\frac{(f(\beta_{-3}))^{F_{12s+10}}}{(f(\beta_{-4}))^{dF_{12s+9}}}\left\{\prod_{j=0}^{s}\widetilde{N}_{\widehat{\beta},4}(s,j)\right\}\left\{\prod_{j=0}^{s-1}\overline{N}_{\widehat{\beta},3}(s,j)\right\}\right),$$

$$\beta_{12s+8} = f^{-1}\left(\frac{(f(\beta_{-4}))^{F_{12s+10}}}{(f(\beta_{-3}))^{dF_{12s+11}}\left(f\left(\widehat{\beta}_{12s+8}\right)\right)^{F_0}}\left\{\prod_{j=0}^{s}N_{\widehat{\beta},4}(s,j)\right\}\left\{\prod_{j=0}^{s-1}\widehat{N}_{\widehat{\beta},4}(s,j)\right\}\right),$$

and

$$\beta_{12s+9} = f^{-1}\left(\frac{(f(\beta_{-3}))^{F_{12(s+1)}}}{(f(\beta_{-4}))^{dF_{12s+11}}}\left\{\prod_{j=0}^{s}\widetilde{N}_{\widehat{\beta},5}(s,j)\right\}\right),$$

$$\beta_{12s+10} = f^{-1}\left(\frac{(f(\beta_{-4}))^{F_{12(s+1)}}}{(f(\beta_{-3}))^{dF_{12s+13}}\left(f\left(\widehat{\beta}_{12s+10}\right)\right)^{F_0}}\left\{\prod_{j=0}^{s}N_{\widehat{\beta},5}(s,j)\right\}\right),$$

$$\beta_{12s+11} = f^{-1}\left(\frac{(f(\beta_{-3}))^{F_{12(s+1)+2}}}{(f(\beta_{-4}))^{dF_{12(s+1)+1}}}\left\{\prod_{j=0}^{s+1}\widetilde{N}_{\widehat{\beta},0}(s+1,j)\right\}\left\{\prod_{j=0}^{s}\overline{N}_{\widehat{\beta},0}(s+1,j)\right\}\right),$$

where the auxiliary factors are defined by:

$$N_{h,r}(s,j)=\prod_{l=0}^{r}\frac{\left(f\left(h_{6(2j)+2l-1}\right)\right)^{F_{12(s-j)+2(r-l)+1}}}{\left(f\left(h_{6(2j)+2l-2}\right)\right)^{F_{12(s-j)+2(r-l+1)}}},\qquad \widetilde{N}_{h,r}(s,j)=\prod_{l=0}^{r}\frac{\left(f\left(h_{6(2j)+2l-2}\right)\right)^{F_{12(s-j)+2(r-l)+1}}}{\left(f\left(h_{6(2j)+2l-1}\right)\right)^{F_{12(s-j)+2(r-l)}}},$$

$$\widehat{N}_{h,r}(s,j)=\prod_{l=1}^{5-r}\frac{\left(f\left(h_{6(2j)+2(l+r)+1}\right)\right)^{F_{12(s-j)-2l+1}}}{\left(f\left(h_{6(2j)+2(l+r-1)}\right)\right)^{F_{12(s-j)-2(l-1)}}},\qquad \overline{N}_{h,r}(s,j)=\prod_{l=1}^{5-r}\frac{\left(f\left(h_{6(2j)+2(l+r-1)}\right)\right)^{F_{12(s-j)-2l+1}}}{\left(f\left(h_{6(2j)+2(l+r)+1}\right)\right)^{F_{12(s-j)-2l}}}.$$

Moreover, the intermediate values $\left(\widehat{\alpha}_{6s+l},\widehat{\beta}_{6s+l}\right)$, $l\in\{0,...,5\}$, *are determined as follows: If* $|\sigma|\neq 1$, *then*

$$\begin{aligned}
\widehat{\alpha}_{6m-3}&=f^{-1}\left(\frac{\sigma^{2m}\left(f\left(\alpha_{-5}\right)\right)^{d}}{f\left(\alpha_{-3}\right)f\left(\alpha_{-4}\right)}+\rho\frac{\sigma^{2m}-1}{\sigma-1}\right), & \widehat{\beta}_{6m-3}&=f^{-1}\left(\frac{\sigma^{2m}\left(f\left(\beta_{-5}\right)\right)^{d}}{f\left(\beta_{-3}\right)f\left(\beta_{-4}\right)}+\rho\frac{\sigma^{2m}-1}{\sigma-1}\right),\\
\widehat{\alpha}_{6m-2}&=f^{-1}\left(\frac{\sigma^{2m}\left(f\left(\alpha_{-4}\right)\right)^{d}}{f\left(\alpha_{-2}\right)f\left(\alpha_{-3}\right)}+\rho\frac{\sigma^{2m}-1}{\sigma-1}\right), & \widehat{\beta}_{6m-2}&=f^{-1}\left(\frac{\sigma^{2m}\left(f\left(\beta_{-4}\right)\right)^{d}}{f\left(\beta_{-2}\right)f\left(\beta_{-3}\right)}+\rho\frac{\sigma^{2m}-1}{\sigma-1}\right),\\
\widehat{\alpha}_{6m-1}&=f^{-1}\left(\frac{\sigma^{2m}\left(f\left(\alpha_{-3}\right)\right)^{d}}{f\left(\alpha_{-1}\right)f\left(\alpha_{-2}\right)}+\rho\frac{\sigma^{2m}-1}{\sigma-1}\right), & \widehat{\beta}_{6m-1}&=f^{-1}\left(\frac{\sigma^{2m}\left(f\left(\beta_{-3}\right)\right)^{d}}{f\left(\beta_{-1}\right)f\left(\beta_{-2}\right)}+\rho\frac{\sigma^{2m}-1}{\sigma-1}\right),\\
\widehat{\alpha}_{6m}&=f^{-1}\left(\frac{\sigma^{2m}\left(f\left(\alpha_{-2}\right)\right)^{d}}{f\left(\alpha_{0}\right)f\left(\alpha_{-1}\right)}+\rho\frac{\sigma^{2m}-1}{\sigma-1}\right), & \widehat{\beta}_{6m}&=f^{-1}\left(\frac{\sigma^{2m}\left(f\left(\beta_{-2}\right)\right)^{d}}{f\left(\beta_{0}\right)f\left(\beta_{-1}\right)}+\rho\frac{\sigma^{2m}-1}{\sigma-1}\right),\\
\widehat{\alpha}_{6m+1}&=f^{-1}\left(\sigma^{2m}\left(\frac{\sigma\left(f\left(\beta_{-4}^{d}\right)\right)}{f\left(\beta_{-2}\right)f\left(\beta_{-3}\right)}+\rho\right)+\rho\frac{\sigma^{2m}-1}{\sigma-1}\right), & \widehat{\beta}_{6m+1}&=f^{-1}\left(\sigma^{2m}\left(\frac{\sigma\left(f\left(\alpha_{-4}^{d}\right)\right)}{f\left(\alpha_{-2}\right)f\left(\alpha_{-3}\right)}+\rho\right)+\rho\frac{\sigma^{2m}-1}{\sigma-1}\right),\\
\widehat{\alpha}_{6m+2}&=f^{-1}\left(\sigma^{2m}\left(\frac{\sigma\left(f\left(\beta_{-3}\right)\right)^{d}}{f\left(\beta_{-1}\right)f\left(\beta_{-2}\right)}+\rho\right)+\rho\frac{\sigma^{2m}-1}{\sigma-1}\right), & \widehat{\beta}_{6m+2}&=f^{-1}\left(\sigma^{2m}\left(\frac{\sigma\left(f\left(\alpha_{-3}\right)\right)^{d}}{f\left(\alpha_{-1}\right)f\left(\alpha_{-2}\right)}+\rho\right)+\rho\frac{\sigma^{2m}-1}{\sigma-1}\right),
\end{aligned}$$

and if $|\sigma|=1$ *then*

$$\begin{aligned}
\widehat{\alpha}_{6m-3}&=f^{-1}\left(\frac{\left(f\left(\alpha_{-5}\right)\right)^{d}}{f\left(\alpha_{-3}\right)f\left(\alpha_{-4}\right)}+m(\sigma+1)\rho\right), & \widehat{\beta}_{6m-3}&=f^{-1}\left(\frac{\sigma^{2m}\left(f\left(\beta_{-5}\right)\right)^{d}}{f\left(\beta_{-3}\right)f\left(\beta_{-4}\right)}+m(\sigma+1)\rho\right),\\
\widehat{\alpha}_{6m-2}&=f^{-1}\left(\frac{\left(f\left(\alpha_{-4}\right)\right)^{d}}{f\left(\alpha_{-2}\right)f\left(\alpha_{-3}\right)}+m(\sigma+1)\rho\right), & \widehat{\beta}_{6m-2}&=f^{-1}\left(\frac{\sigma^{2m}\left(f\left(\beta_{-4}\right)\right)^{d}}{f\left(\beta_{-2}\right)f\left(\beta_{-3}\right)}+m(\sigma+1)\rho\right),\\
\widehat{\alpha}_{6m-1}&=f^{-1}\left(\frac{\left(f\left(\alpha_{-3}\right)\right)^{d}}{f\left(\alpha_{-1}\right)f\left(\alpha_{-2}\right)}+m(\sigma+1)\rho\right), & \widehat{\beta}_{6m-1}&=f^{-1}\left(\frac{\sigma^{2m}\left(f\left(\beta_{-3}\right)\right)^{d}}{f\left(\beta_{-1}\right)f\left(\beta_{-2}\right)}+m(\sigma+1)\rho\right),\\
\widehat{\alpha}_{6m}&=f^{-1}\left(\frac{\left(f\left(\alpha_{-2}\right)\right)^{d}}{f\left(\alpha_{0}\right)f\left(\alpha_{-1}\right)}+m(\sigma+1)\rho\right), & \widehat{\beta}_{6m}&=f^{-1}\left(\frac{\sigma^{2m}\left(f\left(\beta_{-2}\right)\right)^{d}}{f\left(\beta_{0}\right)f\left(\beta_{-1}\right)}+m(\sigma+1)\rho\right),\\
\widehat{\alpha}_{6m+1}&=f^{-1}\left(\frac{\sigma\left(f\left(\beta_{-4}^{d}\right)\right)}{f\left(\beta_{-2}\right)f\left(\beta_{-3}\right)}+\rho+m(\sigma+1)\rho\right), & \widehat{\beta}_{6m+1}&=f^{-1}\left(\sigma^{2m}\left(\frac{\sigma\left(f\left(\alpha_{-4}^{d}\right)\right)}{f\left(\alpha_{-2}\right)f\left(\alpha_{-3}\right)}+\rho\right)+m(\sigma+1)\rho\right),\\
\widehat{\alpha}_{6m+2}&=f^{-1}\left(\frac{\sigma\left(f\left(\beta_{-3}\right)\right)^{d}}{f\left(\beta_{-1}\right)f\left(\beta_{-2}\right)}+\rho+m(\sigma+1)\rho\right), & \widehat{\beta}_{6m+2}&=f^{-1}\left(\sigma^{2m}\left(\frac{\sigma\left(f\left(\alpha_{-3}\right)\right)^{d}}{f\left(\alpha_{-1}\right)f\left(\alpha_{-2}\right)}+\rho\right)+m(\sigma+1)\rho\right).
\end{aligned}$$

Example 2.2. *Consider the transformed system*

$$\begin{aligned}
\alpha_m&=\exp\left(\frac{\left(\log\left(\alpha_{m-2}\right)\right)^{3}\log\left(\beta_{m-3}\right)\log\left(\beta_{m-4}\right)}{\log\left(\alpha_{m-1}\right)\left(5\left(\log\left(\beta_{m-5}\right)\right)^{3}+0.8\log\left(\beta_{m-3}\right)\log\left(\beta_{m-4}\right)\right)}\right),\\
\beta_m&=\exp\left(\frac{\left(\log\left(\beta_{m-2}\right)\right)^{3}\log\left(\alpha_{m-3}\right)\log\left(\alpha_{m-4}\right)}{\log\left(\beta_{m-1}\right)\left(5\left(\log\left(\alpha_{m-5}\right)\right)^{3}+0.8\log\left(\alpha_{m-3}\right)\log\left(\alpha_{m-4}\right)\right)}\right),\ m\in\mathbb{N},
\end{aligned}\tag{B.2}$$

with the parameter values $d = 3$, $\sigma = 5$, $\rho = 0.8$, $f(x) = \log(x)$, *and initial values* $\{\alpha_{-5}, \alpha_{-4}, \alpha_{-3}, \alpha_{-2}, \alpha_{-1}\} = \{1.2, 0.9, 1.5, 0.8, 1.3\}$, $\{\beta_{-5}, \beta_{-4}, \beta_{-3}, \beta_{-2}, \beta_{-1}\} = \{1.1, 1.4, 0.95, 1.2, 0.7\}$. *Figure 2 below illustrates the corresponding dynamical evolution of the sequences* α_m *and* β_m.

Figure 2 demonstrates that both sequences remain strictly positive and fluctuate within a narrow interval over time. The logarithmic transformation smooths these oscillations and stabilizes the trajectories, thereby preventing divergence. This result underscores the important role of nonlinear transformations in broadening the system's stability range.

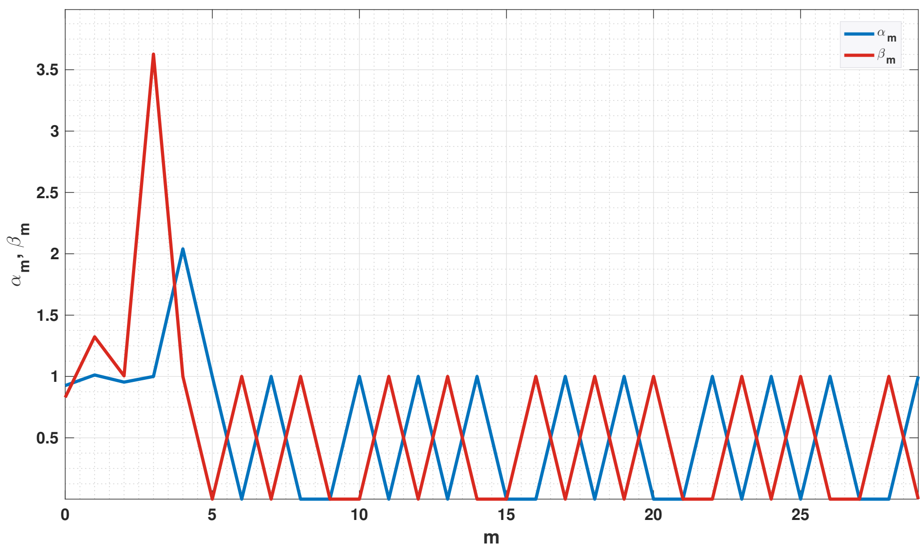

Figure 2. Dynamics of the logarithmic system (B.2).

Example 2.3. *Consider the inverse-logarithmic system*

$$\begin{aligned} \alpha_m &= \log\left(\frac{(\exp(\alpha_{m-2}))\exp(\beta_{m-3})\exp(\beta_{m-4})}{\exp(\alpha_{m-1})(2(\exp(\beta_{m-5})) + 0.5\exp(\beta_{m-3})\exp(\beta_{m-4}))}\right), \\ \beta_m &= \log\left(\frac{(\exp(\beta_{m-2}))\exp(\alpha_{m-3})\exp(\alpha_{m-4})}{\exp(\beta_{m-1})(2(\exp(\alpha_{m-5})) + 0.5\exp(\alpha_{m-3})\exp(\alpha_{m-4}))}\right), m \in \mathbb{N}, \end{aligned} \tag{B.3}$$

with the parameter values $d = 1$, $\sigma = 2$, $\rho = 0.5$, $f(x) = \exp(x)$, *and initial values* $\{\alpha_{-5}, \alpha_{-4}, \alpha_{-3}, \alpha_{-2}, \alpha_{-1}\} = \{0.2, 0.9, 0.5, 0.3, 1.0\}$, $\{\beta_{-5}, \beta_{-4}, \beta_{-3}, \beta_{-2}, \beta_{-1}\} = \{0.5, 0.4, 0.7, 0.2, 0.9\}$. *Figure 3 below illustrates the corresponding dynamical evolution of the sequences* α_m *and* β_m.

Figure 3 illustrates a pronounced oscillatory dynamic, where both sequences alternate between negative and positive values. The trajectories display large-amplitude fluctuations, suggesting that the system can exhibit chaotic or irregular oscillations. However, the system ultimately converges to an equilibrium state. This behavior indicates its capacity to absorb initial disturbances and return to stability, a key property in dynamic models.

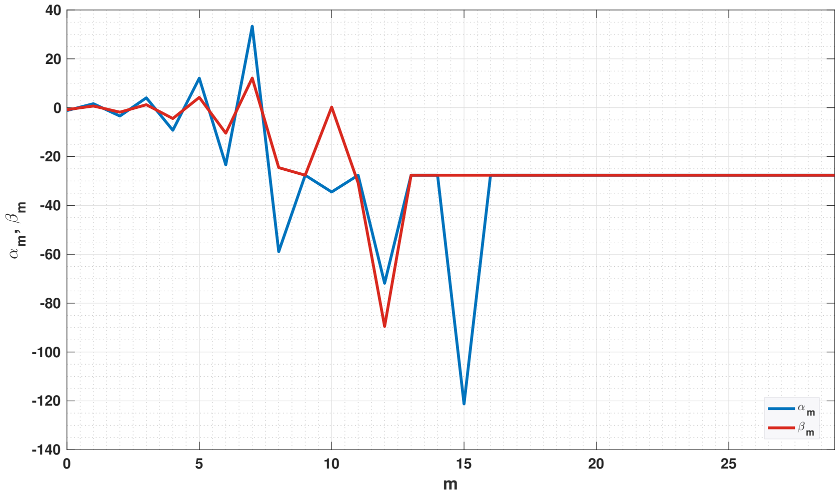

Figure 3. Dynamics of the inverse-logarithmic system (B.3).

Example 2.4. *Consider the generalized power-type system*

$$\begin{aligned} \alpha_m &= \left(\frac{(\alpha_{m-2})^{20}(\beta_{m-3})^{10}(\beta_{m-4})^{10}}{(\alpha_{m-1})^{10}\left(3(\beta_{m-5})^{20}+(\beta_{m-3})^{10}(\beta_{m-4})^{10}\right)}\right)^{1/10}, \\ \beta_m &= \left(\frac{(\beta_{m-2})^{20}(\alpha_{m-3})^{10}(\alpha_{m-4})^{10}}{(\beta_{m-1})^{10}\left(3(\alpha_{m-5})^{20}+(\alpha_{m-3})^{10}(\alpha_{m-4})^{10}\right)}\right)^{1/10}, m \in \mathbb{N}, \end{aligned} \tag{B.4}$$

with the parameter values $d = 2$*,* $\sigma = 3$*,* $\rho = 1$*,* $f(x) = x^{10}$*, and initial values* $\{\alpha_{-5}, \alpha_{-4}, \alpha_{-3}, \alpha_{-2}, \alpha_{-1}\} = \{0.2, 0.9, 1.5, 0.8, 1.3\}$, $\{\beta_{-5}, \beta_{-4}, \beta_{-3}, \beta_{-2}, \beta_{-1}\} = \{0.1, 1.4, 0.95, 1.2, 0.7\}$*. Figure 4 below illustrates the corresponding dynamical evolution of the sequences* α_m *and* β_m*.*

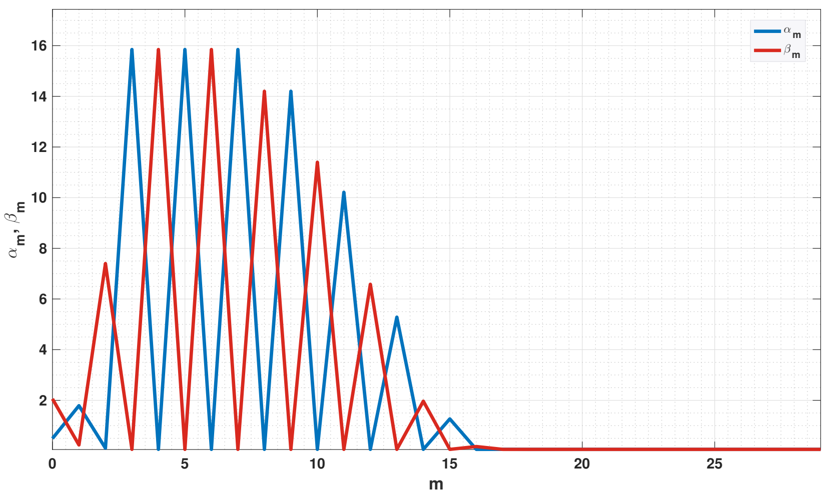

Figure 4. Dynamics of the power-type system (B.4).

Figure 4 shows that both trajectories exhibit rapid growth followed by stabilization. While the high-degree polynomial structure amplifies the effects of the initial conditions, the radical transformation counteracts this tendency and prevents divergence.

Remark 2.3. *The system (A.3) is distinguished by the presence of a scaling parameter d, which plays a central role in regulating the degree of nonlinearity in the recursive relationships. Varying the value of d controls the growth or decay behavior of the sequences* (α_m) *and* (β_m)*, as well as the nature of their interaction. This parameter is therefore a key element in shaping the long-term dynamical structure of the system. In addition, the transformation function* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *provides a broader analytical framework for the model. Selecting an appropriate function f reformulates the system in a way that uncovers clearer dynamical structures and facilitates the study of its properties. The choice of f is not arbitrary; rather, it directly influences the type of results that can be obtained. For example, choosing* $f(x) = \log(x)$ *emphasizes the multiplicative and exponential aspects of the solutions, aiding in the analysis of periodicity and oscillatory patterns. Alternatively, taking* $f(x) = x^r$ *(with* $r \in \mathbb{R}$*) provides another perspective. Thus, the transformation f serves as a powerful analytical tool, enabling the same system to be examined from multiple viewpoints and revealing hidden properties that may remain obscured in its original formulation.*

Remark 2.4. *Expressing the closed-form solutions of systems (A.2) and (A.3) in terms of the generalized Fibonacci sequence carries a twofold significance. On the one hand, it provides an exact numerical representation of the solutions, enabling us to track their patterns and interpret their long-term behavior. On the other hand, it establishes a direct connection between the studied recursive systems and classical numerical sequences, which are known to possess well-developed algebraic and analytical properties. Recent studies show that representing solutions through special numerical sequences opens new avenues for understanding the stability and dynamics inherent in such systems. For example, Ghezal et al. [20] linked solutions to the Pell sequence, and this formulation highlighted the global stability property of the system. In another work [21] by the same team, Pell coefficients were employed to establish the stability of solutions to a higher-order multidimensional system. Furthermore, solutions of other systems have been associated with the Jacobsthal and Jacobsthal–Lucas sequences [22], as well as with the Mersenne and Mersenne–Lucas sequences [23]. Building on these results, the use of the generalized Fibonacci sequence in formulating solutions is not merely a symbolic choice, but rather a natural and powerful approach for uncovering the deeper structure of the system. It lays the foundation for an integrated line of research that bridges recursive systems with number theory, enriching both fields.*

Remark 2.5. *It should be emphasized that the nonlinear character of the function f plays a decisive role in shaping the dynamic behavior of the system, influencing both its stability properties and the emergence of complex oscillatory patterns. Numerical investigations reveal that logarithmic-type functions, as illustrated in Figure 2, may induce instability under certain parameter configurations. Conversely, rapidly increasing functions, including exponential or high-order polynomial types (Figures 3 and 4), exhibit the potential to generate wide-amplitude oscillations or quasi-chaotic dynamics, particularly when coupled with large parameter values or varying initial conditions.*

3. Auxiliary lemmas and proofs

This section introduces two auxiliary problems that lay the mathematical groundwork for proving the main theorems presented earlier. These results play a fundamental role in simplifying the analytical structure of the considered system and in clarifying the essential properties of its solutions under well-defined admissibility conditions. Moreover, the section provides the necessary proofs that support the validity of the theoretical results. To ensure the internal consistency of the proposed model, we begin by verifying that the solution of system (A.2) is well-defined and free from singularities. Specifically, the following conditions must hold to avoid division by zero or undefined expressions:

$$\alpha_{m-1}\left(\sigma\beta_{m-5}^{d}+\rho\beta_{m-3}\beta_{m-4}\right)\neq 0,$$
$$\beta_{m-1}\left(\sigma\alpha_{m-5}^{d}+\rho\alpha_{m-3}\alpha_{m-4}\right)\neq 0,\ m\in\mathbb{N}.$$

Violation of these conditions may result in undefined values appearing in the system. Accordingly, the system is rewritten as follows:

$$\frac{\alpha_{m-2}^{d}}{\alpha_m\alpha_{m-1}}=\frac{\sigma\beta_{m-5}^{d}+\rho\beta_{m-3}\beta_{m-4}}{\beta_{m-3}\beta_{m-4}},\quad \frac{\beta_{m-2}^{d}}{\beta_m\beta_{m-1}}=\frac{\sigma\alpha_{m-5}^{d}+\rho\alpha_{m-3}\alpha_{m-4}}{\alpha_{m-3}\alpha_{m-4}}.$$

To simplify the system, we apply an appropriate transformation of the variables, resulting in a symmetric linear system as follows:

$$\widehat{\alpha}_m=\sigma\widehat{\beta}_{m-3}+\rho,\quad \widehat{\beta}_m=\sigma\widehat{\alpha}_{m-3}+\rho,\ m\in\mathbb{N}. \tag{C.1}$$

Equation (C.1) represents a symmetric linearized formulation, which facilitates the analysis of the system's periodic behavior. From this form, we can separate the system for each variable, resulting in the following system:

$$\widehat{\alpha}_m=\sigma^2\widehat{\alpha}_{m-6}+\sigma\rho+\rho,\quad \widehat{\beta}_m=\sigma^2\widehat{\beta}_{m-6}+\sigma\rho+\rho. \tag{C.2}$$

Elaydi's Lemma [19] provides an important result regarding the general solutions to this class of linear equations. Using this result, we can analyze the possible solutions of the system (C.2) and examine their behavior at different values of the coefficients σ and ρ.

Lemma 3.1. *The closed-form solution to the following difference equation:*

$$\widehat{\alpha}_m=\sigma\widehat{\alpha}_{m-k}+\rho,\ m\geq k,\ k,\ m\in\mathbb{N},$$

where $\widehat{\alpha}_{-l}$, $l\in\{0,...,k-1\}$,

- *If* $\sigma\neq 1$:
$$\widehat{\alpha}_{km-l}=\sigma^m\widehat{\alpha}_{-l}+(\sigma^m-1)(\sigma-1)^{-1}\rho,\ l\in\{0,...,k-1\},\ m\in\mathbb{N}.$$
This solution exhibits exponential behavior, where the compound effect of the variable σ amplifies or shrinks the current value $\widehat{\alpha}_m$ as m changes based on the initial values. Additionally, a constant component, ρ, is added, which represents the effect of the constant transformation over time.

- *If* $\sigma = 1$:

$$\widehat{\alpha}_{km-l} = \widehat{\alpha}_{-l} + m\rho,\ l \in \{0, ..., k-1\},\ m \in \mathbb{N}.$$

In this case, the solution is linear, where $\widehat{\alpha}_m$ *increases with m in a constant manner according to the value of* ρ.

Proof. The proof is based on partitioning the sequence $(\widehat{\alpha}_m)$ into k-subsequences, determined by the remainder obtained when dividing by k. For each $l \in \{0, ..., k-1\}$, we consider the subsequence $(\widehat{\alpha}_{km-l})_{m\geq 0}$, which then reduces to a first-order linear difference relation. By solving each subsequence independently, we derive explicit closed-form expressions that describe the temporal evolution of the values for the cases $\sigma \neq 1$ and $\sigma = 1$. The core idea is to apply the principle of mathematical induction to verify the validity of the general formula for all $m \in \mathbb{N}$. We first check the initial condition, and then demonstrate that if the relation holds for some positive integer m, it consequently holds for $m + 1$. This process ultimately leads to the final formulas presented in the text. This approach is consistent with standard analytical techniques commonly found in the literature on linear difference equations. A comparable proof can be found in the classical work of [19], which remains a fundamental reference in this field. □

We now present the closed-form solution of the system (C.2), which describes the evolution of the variables $\widehat{\alpha}_m$ and $\widehat{\beta}_m$ over time, depending on the system parameters and the initial condition. The solutions are determined by the value of the parameter σ, leading to different behaviors when $|\sigma| \neq 1$ compared to the special case $|\sigma| = 1$, as shown in the following lemma.

Lemma 3.2. *The closed-form periodic solution of the system (C.1), which exhibits a periodicity of* 6*, is determined by the following two cases:*

When $|\sigma| \neq 1$ *: The evolution is exponential, influenced by the scale factor* σ*, meaning that the values can grow or decay depending on the value of* σ*:*

$$\begin{aligned}
\widehat{\alpha}_{6m-3} &= \sigma^{2m}\frac{\alpha_{-5}^{d}}{\alpha_{-3}\alpha_{-4}} + \left(\sigma^{2m}-1\right)(\sigma-1)^{-1}\rho,\\
\widehat{\alpha}_{6m-2} &= \sigma^{2m}\frac{\alpha_{-4}^{d}}{\alpha_{-2}\alpha_{-3}} + \left(\sigma^{2m}-1\right)(\sigma-1)^{-1}\rho,\\
\widehat{\alpha}_{6m-1} &= \sigma^{2m}\frac{\alpha_{-3}^{d}}{\alpha_{-1}\alpha_{-2}} + \left(\sigma^{2m}-1\right)(\sigma-1)^{-1}\rho,\\
\widehat{\alpha}_{6m} &= \sigma^{2m}\frac{\alpha_{-2}^{d}}{\alpha_{0}\alpha_{-1}} + \left(\sigma^{2m}-1\right)(\sigma-1)^{-1}\rho,\\
\widehat{\alpha}_{6m+1} &= \sigma^{2m}\left(\sigma\frac{\beta_{-4}^{d}}{\beta_{-2}\beta_{-3}} + \rho\right) + \left(\sigma^{2m}-1\right)(\sigma-1)^{-1}\rho,\\
\widehat{\alpha}_{6m+2} &= \sigma^{2m}\left(\sigma\frac{\beta_{-3}^{d}}{\beta_{-1}\beta_{-2}} + \rho\right) + \left(\sigma^{2m}-1\right)(\sigma-1)^{-1}\rho,
\end{aligned} \tag{C.3}$$

$$\begin{aligned}
\widehat{\beta}_{6m-3} &= \sigma^{2m}\frac{\beta_{-5}^{d}}{\beta_{-3}\beta_{-4}}+\left(\sigma^{2m}-1\right)(\sigma-1)^{-1}\rho,\\
\widehat{\beta}_{6m-2} &= \sigma^{2m}\frac{\beta_{-4}^{d}}{\beta_{-2}\beta_{-3}}+\left(\sigma^{2m}-1\right)(\sigma-1)^{-1}\rho,\\
\widehat{\beta}_{6m-1} &= \sigma^{2m}\frac{\beta_{-3}^{d}}{\beta_{-1}\beta_{-2}}+\left(\sigma^{2m}-1\right)(\sigma-1)^{-1}\rho,\\
\widehat{\beta}_{6m} &= \sigma^{2m}\frac{\beta_{-2}^{d}}{\beta_{0}\beta_{-1}}+\left(\sigma^{2m}-1\right)(\sigma-1)^{-1}\rho,\\
\widehat{\beta}_{6m+1} &= \sigma^{2m}\left(\sigma\frac{\alpha_{-4}^{d}}{\alpha_{-2}\alpha_{-3}}+\rho\right)+\left(\sigma^{2m}-1\right)(\sigma-1)^{-1}\rho,\\
\widehat{\beta}_{6m+2} &= \sigma^{2m}\left(\sigma\frac{\alpha_{-3}^{d}}{\alpha_{-1}\alpha_{-2}}+\rho\right)+\left(\sigma^{2m}-1\right)(\sigma-1)^{-1}\rho.
\end{aligned} \tag{C.4}$$

In contrast, when $|\sigma| = 1$*: The evolution becomes linear in m, leading to a more stable pattern without exponential amplification or decay:*

$$\begin{aligned}
\widehat{\alpha}_{6m-3} &= \frac{\alpha_{-5}^{d}}{\alpha_{-3}\alpha_{-4}}+m(\sigma+1)\rho, & \widehat{\beta}_{6m-3} &= \frac{\beta_{-5}^{d}}{\beta_{-3}\beta_{-4}}+m(\sigma+1)\rho,\\
\widehat{\alpha}_{6m-2} &= \frac{\alpha_{-4}^{d}}{\alpha_{-2}\alpha_{-3}}+m(\sigma+1)\rho, & \widehat{\beta}_{6m-2} &= \frac{\beta_{-4}^{d}}{\beta_{-2}\beta_{-3}}+m(\sigma+1)\rho,\\
\widehat{\alpha}_{6m-1} &= \frac{\alpha_{-3}^{d}}{\alpha_{-1}\alpha_{-2}}+m(\sigma+1)\rho, & \widehat{\beta}_{6m-1} &= \frac{\beta_{-3}^{d}}{\beta_{-1}\beta_{-2}}+m(\sigma+1)\rho,\\
\widehat{\alpha}_{6m} &= \frac{\alpha_{-2}^{d}}{\alpha_{0}\alpha_{-1}}+m(\sigma+1)\rho, & \widehat{\beta}_{6m} &= \frac{\beta_{-2}^{d}}{\beta_{0}\beta_{-1}}+m(\sigma+1)\rho,\\
\widehat{\alpha}_{6m+1} &= \sigma\frac{\beta_{-4}^{d}}{\beta_{-2}\beta_{-3}}+m(\sigma+1)\rho+\rho, & \widehat{\beta}_{6m+1} &= \sigma\frac{\alpha_{-4}^{d}}{\alpha_{-2}\alpha_{-3}}+m(\sigma+1)\rho+\rho,\\
\widehat{\alpha}_{6m+2} &= \sigma\frac{\beta_{-3}^{d}}{\beta_{-1}\beta_{-2}}+m(\sigma+1)\rho+\rho, & \widehat{\beta}_{6m+2} &= \sigma\frac{\alpha_{-3}^{d}}{\alpha_{-1}\alpha_{-2}}+m(\sigma+1)\rho+\rho.
\end{aligned} \tag{C.5}$$

Proof. The proof of this result builds upon the analytical approach employed in Lemma 3.1, while accounting for the binary structure of system (C.1), which involves the sequences $\widehat{\alpha}_m$ and $\widehat{\beta}_m$. The system is initially partitioned into six independent subsequences, determined by the remainders obtained upon division by 6. Each subsequence thus represents a recurring periodic component within the overall system. Subsequently, each of the six relations is expressed as a first-order linear difference equation, which is solved using iterative techniques analogous to those utilized in Lemma 3.1. Through this procedure, closed-form expressions are derived for the two cases $|\sigma| \neq 1$ and $|\sigma| = 1$. In the first case, exponential behavior naturally emerges due to the accumulated influence of the factor σ over successive periodic intervals, leading to a formula that depends on σ^{2m}. In contrast, when $|\sigma| = 1$, this influence becomes constant, resulting in a linear evolution of the variable m and producing a stable periodic behavior without amplification or decay. The sixth-order periodicity directly follows from the coupled structure of the sequences $\widehat{\alpha}_m$ and $\widehat{\beta}_m$, where the same configuration reappears every six iterations. This confirms that the solutions obtained in the lemma fully satisfy the required periodicity conditions. Consequently, the closed-form formulas presented constitute the complete analytical solutions of the system under the two specified conditions. □

3.1. Proof of Theorem 2.1

To derive the closed-form solution of the system, we start by applying the inverse variable transformation, which establishes the relationship between successive variables as follows:

$$\alpha_m = \frac{\alpha_{m-2}^d}{\widehat{\alpha}_m \alpha_{m-1}}, \quad \beta_m = \frac{\beta_{m-2}^d}{\widehat{\beta}_m \beta_{m-1}}, \text{ for all } m.$$

By applying recursive substitution, each term can be represented as a multiplicative composition of the preceding terms, whose exponents follow a generalized Fibonacci pattern, as illustrated below.

$$\begin{aligned}
\alpha_{-2} &= \frac{\alpha_{-4}^d}{\widehat{\alpha}_{-2}\alpha_{-3}} &&= \frac{\alpha_{-4}^{dF_0}}{\widehat{\alpha}_{-2}^{F_0}\alpha_{-3}^{F_1}},\\
\alpha_{-1} &= \frac{\alpha_{-3}^d}{\widehat{\alpha}_{-1}\alpha_{-2}} &&= \frac{\alpha_{-3}^{d+1}\widehat{\alpha}_{-2}}{\widehat{\alpha}_{-1}\alpha_{-4}^d} = \frac{\alpha_{-3}^{F_2}\widehat{\alpha}_{-2}^{F_1}}{\widehat{\alpha}_{-1}^{F_0}\alpha_{-4}^{dF_1}},\\
\alpha_0 &= \frac{\alpha_{-2}^d}{\widehat{\alpha}_0\alpha_{-1}} &&= \frac{\alpha_{-4}^{d+d^2}\widehat{\alpha}_{-1}}{\widehat{\alpha}_0\widehat{\alpha}_{-2}^{d+1}\alpha_{-3}^{2d+1}} = \frac{\alpha_{-4}^{dF_2}\widehat{\alpha}_{-1}^{F_1}}{\widehat{\alpha}_0^{F_0}\widehat{\alpha}_{-2}^{F_2}\alpha_{-3}^{F_3}},\\
\alpha_1 &= \frac{\alpha_{-1}^d}{\widehat{\alpha}_1\alpha_0} &&= \frac{\alpha_{-3}^{1+3d+d^2}\widehat{\alpha}_{-2}^{2d+1}\widehat{\alpha}_0}{\widehat{\alpha}_1\widehat{\alpha}_{-1}^{d+1}\alpha_{-4}^{d+2d^2}} = \frac{\alpha_{-3}^{F_4}\widehat{\alpha}_{-2}^{F_3}\widehat{\alpha}_0^{F_1}}{\widehat{\alpha}_1^{F_0}\widehat{\alpha}_{-1}^{F_2}\alpha_{-4}^{dF_3}},\\
\alpha_2 &= \frac{\alpha_0^d}{\widehat{\alpha}_2\alpha_1} &&= \frac{\alpha_{-4}^{\left(1+3d+d^2\right)d}\widehat{\alpha}_{-1}^{2d+1}\widehat{\alpha}_1}{\widehat{\alpha}_2\widehat{\alpha}_0^{1+d}\widehat{\alpha}_{-2}^{1+3d+d^2}\alpha_{-3}^{1+4d+3d^2}} = \frac{\alpha_{-4}^{dF_4}\widehat{\alpha}_{-1}^{F_3}\widehat{\alpha}_1^{F_1}}{\widehat{\alpha}_2^{F_0}\widehat{\alpha}_0^{F_2}\widehat{\alpha}_{-2}^{F_4}\alpha_{-3}^{F_5}},\\
\alpha_3 &= \frac{\alpha_1^d}{\widehat{\alpha}_3\alpha_2} &&= \frac{\alpha_{-3}^{1+5d+6d^2+d^3}\widehat{\alpha}_{-2}^{1+4d+3d^2}\widehat{\alpha}_0^{2d+1}\widehat{\alpha}_2}{\widehat{\alpha}_3\widehat{\alpha}_1^{1+d}\widehat{\alpha}_{-1}^{1+3d+d^2}\alpha_{-4}^{d\left(1+4d+3d^2\right)}} = \frac{\alpha_{-3}^{F_6}\widehat{\alpha}_{-2}^{F_5}\widehat{\alpha}_0^{F_3}\widehat{\alpha}_2^{F_1}}{\widehat{\alpha}_3^{F_0}\widehat{\alpha}_1^{F_2}\widehat{\alpha}_{-1}^{F_4}\alpha_{-4}^{dF_5}},
\end{aligned}$$

$$\begin{aligned}
\beta_{-2} &= \frac{\beta_{-4}^d}{\widehat{\beta}_{-2}\beta_{-3}} &&= \frac{\beta_{-4}^{dF_0}}{\widehat{\beta}_{-2}^{F_0}\beta_{-3}^{F_1}},\\
\beta_{-1} &= \frac{\beta_{-3}^d}{\widehat{\beta}_{-1}\beta_{-2}} &&= \frac{\beta_{-3}^{d+1}\widehat{\beta}_{-2}}{\widehat{\beta}_{-1}\beta_{-4}^d} = \frac{\beta_{-3}^{F_2}\widehat{\beta}_{-2}^{F_1}}{\widehat{\beta}_{-1}^{F_0}\beta_{-4}^{dF_1}},\\
\beta_0 &= \frac{\beta_{-2}^d}{\widehat{\beta}_0\beta_{-1}} &&= \frac{\beta_{-4}^{d+d^2}\widehat{\beta}_{-1}}{\widehat{\beta}_0\widehat{\beta}_{-2}^{d+1}\beta_{-3}^{2d+1}} = \frac{\beta_{-4}^{dF_2}\widehat{\beta}_{-1}^{F_1}}{\widehat{\beta}_0^{F_0}\widehat{\beta}_{-2}^{F_2}\beta_{-3}^{F_3}},\\
\beta_1 &= \frac{\beta_{-1}^d}{\widehat{\beta}_1\beta_0} &&= \frac{\beta_{-3}^{1+3d+d^2}\widehat{\beta}_{-2}^{2d+1}\widehat{\beta}_0}{\widehat{\beta}_1\widehat{\beta}_{-1}^{d+1}\beta_{-4}^{d+2d^2}} = \frac{\beta_{-3}^{F_4}\widehat{\beta}_{-2}^{F_3}\widehat{\beta}_0^{F_1}}{\widehat{\beta}_1^{F_0}\widehat{\beta}_{-1}^{F_2}\beta_{-4}^{dF_3}},\\
\beta_2 &= \frac{\beta_0^d}{\widehat{\beta}_2\beta_1} &&= \frac{\beta_{-4}^{\left(1+3d+d^2\right)d}\widehat{\beta}_{-1}^{2d+1}\widehat{\beta}_1}{\widehat{\beta}_2\widehat{\beta}_0^{1+d}\widehat{\beta}_{-2}^{1+3d+d^2}\beta_{-3}^{1+4d+3d^2}} = \frac{\beta_{-4}^{dF_4}\widehat{\beta}_{-1}^{F_3}\widehat{\beta}_1^{F_1}}{\widehat{\beta}_2^{F_0}\widehat{\beta}_0^{F_2}\widehat{\beta}_{-2}^{F_4}\beta_{-3}^{F_5}},\\
\beta_3 &= \frac{\beta_1^d}{\widehat{\beta}_3\beta_2} &&= \frac{\beta_{-3}^{1+5d+6d^2+d^3}\widehat{\beta}_{-2}^{1+4d+3d^2}\widehat{\beta}_0^{2d+1}\widehat{\beta}_2}{\widehat{\beta}_3\widehat{\beta}_1^{1+d}\widehat{\beta}_{-1}^{1+3d+d^2}\beta_{-4}^{d\left(1+4d+3d^2\right)}} = \frac{\beta_{-3}^{F_6}\widehat{\beta}_{-2}^{F_5}\widehat{\beta}_0^{F_3}\widehat{\beta}_2^{F_1}}{\widehat{\beta}_3^{F_0}\widehat{\beta}_1^{F_2}\widehat{\beta}_{-1}^{F_4}\beta_{-4}^{dF_5}}.
\end{aligned}$$

Hence, the appearance of Fibonacci numbers F_m in the system expressions arises naturally from the iterative structure of the sequences (α_m) and (β_m). Using the principle of mathematical induction, we can easily prove the following general relationship for all $m \geq -1$:

$$\alpha_{2m} = \frac{\alpha_{-4}^{F_{2m+2}}}{\alpha_{-3}^{dF_{2m+3}}\widehat{\alpha}_{2m}^{F_0}} \prod_{j=1}^{m+1} \left(\widehat{\alpha}_{2j-3}^{F_{2m-2j+3}} \Big/ \widehat{\alpha}_{2j-4}^{F_{2m-2j+4}} \right), \quad \beta_{2m} = \frac{\beta_{-4}^{F_{2m+2}}}{\beta_{-3}^{dF_{2m+3}}\widehat{\beta}_{2m}^{F_0}} \prod_{j=1}^{m+1} \left(\widehat{\beta}_{2j-3}^{F_{2m-2j+3}} \Big/ \widehat{\beta}_{2j-4}^{F_{2m-2j+4}} \right),$$

$$\alpha_{2m+1} = \frac{\alpha_{-3}^{F_{2m+4}}}{\alpha_{-4}^{dF_{2m+3}}} \prod_{j=1}^{m+2} \left(\widehat{\alpha}_{2j-4}^{F_{2m-2j+5}} \Big/ \widehat{\alpha}_{2j-3}^{F_{2m-2j+4}} \right), \quad \beta_{2m+1} = \frac{\beta_{-3}^{F_{2m+4}}}{\beta_{-4}^{dF_{2m+3}}} \prod_{j=1}^{m+2} \left(\widehat{\beta}_{2j-4}^{F_{2m-2j+5}} \Big/ \widehat{\beta}_{2j-3}^{F_{2m-2j+4}} \right).$$

Since the system exhibits a periodicity of degree 6 for the variables $\left(\widehat{\alpha}_m, \widehat{\beta}_m\right)$, applying the result of the Lemma 3.2 allows us to conclude that the closed-form solution follows a periodicity of degree 12. This means that the values of the variables repeat after 12 steps,

$$\alpha_{12s} = \frac{\alpha_{-4}^{F_{12s+2}}}{\alpha_{-3}^{dF_{12s+3}}\widehat{\alpha}_{12s}^{F_0}} \prod_{j=1}^{6s+1} \left(\widehat{\alpha}_{2j-3}^{F_{12s-2j+3}} \Big/ \widehat{\alpha}_{2j-4}^{F_{12s-2j+4}} \right), \qquad \alpha_{12s+1} = \frac{\alpha_{-3}^{F_{12s+4}}}{\alpha_{-4}^{dF_{12s+3}}} \prod_{j=1}^{6s+2} \left(\widehat{\alpha}_{2j-4}^{F_{12s-2j+5}} \Big/ \widehat{\alpha}_{2j-3}^{F_{12s-2j+4}} \right),$$

$$\alpha_{12s+2} = \frac{\alpha_{-4}^{F_{12s+4}}}{\alpha_{-3}^{dF_{12s+5}}\widehat{\alpha}_{12s+2}^{F_0}} \prod_{j=1}^{6s+2} \left(\widehat{\alpha}_{2j-3}^{F_{12s-2j+5}} \Big/ \widehat{\alpha}_{2j-4}^{F_{12s-2j+6}} \right), \qquad \alpha_{12s+3} = \frac{\alpha_{-3}^{F_{12s+6}}}{\alpha_{-4}^{dF_{12s+5}}} \prod_{j=1}^{6s+3} \left(\widehat{\alpha}_{2j-4}^{F_{12s-2j+7}} \Big/ \widehat{\alpha}_{2j-3}^{F_{12s-2j+6}} \right),$$

$$\alpha_{12s+4} = \frac{\alpha_{-4}^{F_{12s+6}}}{\alpha_{-3}^{dF_{12s+7}}\widehat{\alpha}_{12s+4}^{F_0}} \prod_{j=1}^{6s+3} \left(\widehat{\alpha}_{2j-3}^{F_{12s-2j+7}} \Big/ \widehat{\alpha}_{2j-4}^{F_{12s-2j+8}} \right), \qquad \alpha_{12s+5} = \frac{\alpha_{-3}^{F_{12s+8}}}{\alpha_{-4}^{dF_{12s+7}}} \prod_{j=1}^{6s+4} \left(\widehat{\alpha}_{2j-4}^{F_{12s-2j+9}} \Big/ \widehat{\alpha}_{2j-3}^{F_{12s-2j+8}} \right),$$

$$\alpha_{12s+6} = \frac{\alpha_{-4}^{F_{12s+8}}}{\alpha_{-3}^{dF_{12s+9}}\widehat{\alpha}_{12s+6}^{F_0}} \prod_{j=1}^{6s+4} \left(\widehat{\alpha}_{2j-3}^{F_{12s-2j+9}} \Big/ \widehat{\alpha}_{2j-4}^{F_{12s-2j+10}} \right), \qquad \alpha_{12s+7} = \frac{\alpha_{-3}^{F_{12s+10}}}{\alpha_{-4}^{dF_{12s+9}}} \prod_{j=1}^{6s+5} \left(\widehat{\alpha}_{2j-4}^{F_{12s-2j+11}} \Big/ \widehat{\alpha}_{2j-3}^{F_{12s-2j+10}} \right),$$

$$\alpha_{12s+8} = \frac{\alpha_{-4}^{F_{12s+10}}}{\alpha_{-3}^{dF_{12s+11}}\widehat{\alpha}_{12s+8}^{F_0}} \prod_{j=1}^{6s+5} \left(\widehat{\alpha}_{2j-3}^{F_{12s-2j+11}} \Big/ \widehat{\alpha}_{2j-4}^{F_{12(s+1)-2j}} \right), \quad \alpha_{12s+9} = \frac{\alpha_{-3}^{F_{12(s+1)}}}{\alpha_{-4}^{dF_{12s+11}}} \prod_{j=1}^{6s+6} \left(\widehat{\alpha}_{2j-4}^{F_{12(s+1)-2j+1}} \Big/ \widehat{\alpha}_{2j-3}^{F_{12(s+1)-2j}} \right),$$

$$\alpha_{12s+10} = \frac{\alpha_{-4}^{F_{12(s+1)}}}{\alpha_{-3}^{dF_{12s+13}}\widehat{\alpha}_{12s+10}^{F_0}} \prod_{j=1}^{6s+6} \left(\widehat{\alpha}_{2j-3}^{F_{12(s+1)-2j+1}} \Big/ \widehat{\alpha}_{2j-4}^{F_{12(s+1)-2j+2}} \right),$$

$$\alpha_{12s+11} = \frac{\alpha_{-3}^{F_{12(s+1)+2}}}{\alpha_{-4}^{dF_{12(s+1)+1}}} \prod_{j=1}^{6s+7} \left(\widehat{\alpha}_{2j-4}^{F_{12(s+1)-2j+3}} \Big/ \widehat{\alpha}_{2j-3}^{F_{12(s+1)-2j+2}} \right),$$

$$\beta_{12s}=\frac{\beta_{-4}^{F_{12s+2}}}{\beta_{-3}^{dF_{12s+3}}\widetilde{\beta}_{12s}^{F_0}}\prod_{j=1}^{6s+1}\left(\widetilde{\beta}_{2j-3}^{F_{12s-2j+3}}\Big/\widetilde{\beta}_{2j-4}^{F_{12s-2j+4}}\right),\qquad \beta_{12s+1}=\frac{\beta_{-3}^{F_{12s+4}}}{\beta_{-4}^{dF_{12s+3}}}\prod_{j=1}^{6s+2}\left(\widetilde{\beta}_{2j-4}^{F_{12s-2j+5}}\Big/\widetilde{\beta}_{2j-3}^{F_{12s-2j+4}}\right),$$

$$\beta_{12s+2}=\frac{\beta_{-4}^{F_{12s+4}}}{\beta_{-3}^{dF_{12s+5}}\widetilde{\beta}_{12s+2}^{F_0}}\prod_{j=1}^{6s+2}\left(\widetilde{\beta}_{2j-3}^{F_{12s-2j+5}}\Big/\widetilde{\beta}_{2j-4}^{F_{12s-2j+6}}\right),\qquad \beta\alpha_{12s+3}=\frac{\beta_{-3}^{F_{12s+6}}}{\beta_{-4}^{dF_{12s+5}}}\prod_{j=1}^{6s+3}\left(\widetilde{\beta}_{2j-4}^{F_{12s-2j+7}}\Big/\widetilde{\beta}_{2j-3}^{F_{12s-2j+6}}\right),$$

$$\beta_{12s+4}=\frac{\beta_{-4}^{F_{12s+6}}}{\beta_{-3}^{dF_{12s+7}}\widetilde{\beta}_{12s+4}^{F_0}}\prod_{j=1}^{6s+3}\left(\widetilde{\beta}_{2j-3}^{F_{12s-2j+7}}\Big/\widetilde{\beta}_{2j-4}^{F_{12s-2j+8}}\right),\qquad \beta_{12s+5}=\frac{\beta_{-3}^{F_{12s+8}}}{\beta_{-4}^{dF_{12s+7}}}\prod_{j=1}^{6s+4}\left(\widetilde{\beta}_{2j-4}^{F_{12s-2j+9}}\Big/\widetilde{\beta}_{2j-3}^{F_{12s-2j+8}}\right),$$

$$\beta_{12s+6}=\frac{\beta_{-4}^{F_{12s+8}}}{\beta_{-3}^{dF_{12s+9}}\widetilde{\beta}_{12s+6}^{F_0}}\prod_{j=1}^{6s+4}\left(\widetilde{\beta}_{2j-3}^{F_{12s-2j+9}}\Big/\widetilde{\beta}_{2j-4}^{F_{12s-2j+10}}\right),\qquad \beta_{12s+7}=\frac{\beta_{-3}^{F_{12s+10}}}{\beta_{-4}^{dF_{12s+9}}}\prod_{j=1}^{6s+5}\left(\widetilde{\beta}_{2j-4}^{F_{12s-2j+11}}\Big/\widetilde{\beta}_{2j-3}^{F_{12s-2j+10}}\right),$$

$$\beta_{12s+8}=\frac{\beta_{-4}^{F_{12s+10}}}{\beta_{-3}^{dF_{12s+11}}\widetilde{\beta}_{12s+8}^{F_0}}\prod_{j=1}^{6s+5}\left(\widetilde{\beta}_{2j-3}^{F_{12s-2j+11}}\Big/\widetilde{\beta}_{2j-4}^{F_{12(s+1)-2j}}\right),\quad \beta_{12s+9}=\frac{\beta_{-3}^{F_{12(s+1)}}}{\beta_{-4}^{dF_{12s+11}}}\prod_{j=1}^{6s+6}\left(\widetilde{\beta}_{2j-4}^{F_{12(s+1)-2j+1}}\Big/\widetilde{\beta}_{2j-3}^{F_{12(s+1)-2j}}\right),$$

$$\beta_{12s+10}=\frac{\beta_{-4}^{F_{12(s+1)}}}{\beta_{-3}^{dF_{12s+13}}\widetilde{\beta}_{12s+10}^{F_0}}\prod_{j=1}^{6s+6}\left(\widetilde{\beta}_{2j-3}^{F_{12(s+1)-2j+1}}\Big/\widetilde{\beta}_{2j-4}^{F_{12(s+1)-2j+2}}\right),$$

$$\beta_{12s+11}=\frac{\beta_{-3}^{F_{12(s+1)+2}}}{\beta_{-4}^{dF_{12(s+1)+1}}}\prod_{j=1}^{6s+7}\left(\widetilde{\beta}_{2j-4}^{F_{12(s+1)-2j+3}}\Big/\widetilde{\beta}_{2j-3}^{F_{12(s+1)-2j+2}}\right).$$

3.2. Proof of Theorem 2.2

To establish the result, we introduce a new change of variables defined by $\widetilde{\alpha}_m = f(\alpha_m)$ and $\widetilde{\beta}_m = f(\beta_m)$ for all m. Under this transformation, the system (A.3) is rewritten in the following equivalent form

$$\widetilde{\alpha}_m=\frac{\widetilde{\alpha}_{m-2}^{d}\widetilde{\beta}_{m-3}\widetilde{\beta}_{m-4}}{\widetilde{\alpha}_{m-1}\left(\sigma\widetilde{\beta}_{m-5}^{d}+\rho\widetilde{\beta}_{m-3}\widetilde{\beta}_{m-4}\right)},\quad \widetilde{\beta}_m=\frac{\widetilde{\beta}_{m-2}^{d}\widetilde{\alpha}_{m-3}\widetilde{\alpha}_{m-4}}{\widetilde{\beta}_{m-1}\left(\sigma\widetilde{\alpha}_{m-5}^{d}+\rho\widetilde{\alpha}_{m-3}\widetilde{\alpha}_{m-4}\right)},\ d,m\in\mathbb{N}. \tag{C.6}$$

It is immediate to observe that this new system is exactly of the same type as system (A.2), but expressed in terms of the transformed variables $\widetilde{\alpha}_m$ and $\widetilde{\beta}_m$. Hence, by applying Theorem 2.1, the closed-form solution of system (C.6) is obtained directly, with $\widetilde{\alpha}_m$ and $\widetilde{\beta}_m$ taking the role of the original variables. Finally, since the mapping f is assumed to be a one-to-one continuous function, the inverse function f^{-1} exists and is continuous as well. Therefore, by applying the inverse transformation, we immediately deduce the closed-form solution of system (A.3) in the form $\alpha_m = f^{-1}(\widetilde{\alpha}_m)$ and $\beta_m = f^{-1}\left(\widetilde{\beta}_m\right)$ for all m. This completes the proof.

4. Conclusions

In this paper, we present a new model for two-dimensional nonlinear difference equation systems, derive their closed-form solutions, and establish connections with the generalized d-Fibonacci sequence. Our analysis shows that the proposed model exhibits a rich dynamical structure, with the potential to generate either regular or complex periodic behaviors depending on parameter values and initial conditions. The study also emphasizes the pivotal role of the transformation function f

in broadening the scope of such systems and providing a unified framework that bridges difference equations with integer sequences.

Despite the encouraging results, this study has several limitations that merit further discussion. First, the analytical results depend on specific functional forms of the transfer function f, which may restrict the general applicability of the proposed framework when extended to larger or more complex nonlinear systems. Additionally, obtaining explicit closed-form solutions becomes increasingly challenging as the system dimensionality grows, introducing significant computational complexity in higher-order models. Moreover, the system's high sensitivity to slight variations in parameters or initial conditions may complicate both numerical validation and practical implementation. To address these challenges, future research could explore alternative transfer functions or hybrid methodologies that integrate numerical approximations with exact analytical techniques, enabling the derivation of approximate solutions for more intricate nonlinear settings. In addition, a comprehensive analysis of bifurcation and chaotic behaviors could yield a deeper insight into the long-term dynamics and stability characteristics of the system.

Use of Generative-AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Funding

This work was supported by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia [KFU254049].

Conflict of interest

The author declares no conflicts of interest in this paper.

References

1. H. Althagafi, A. Ghezal, Stability analysis of biological rhythms using three-dimensional systems of difference equations with squared terms, *J. Appl. Math. Comput.*, **71** (2025), 3211–3232. https://doi.org/10.1007/s12190-024-02363-2
2. H. Althagafi, A. Ghezal, Solving a system of nonlinear difference equations with bilinear dynamics, *AIMS Math.*, **9** (2024), 34067–34089. https://doi.org/10.3934/math.20241624
3. H. Althagafi, Dynamics of difference systems: A mathematical study with applications to neural systems, *AIMS Math.*, **10** (2025), 2869–2890. https://doi.org/10.3934/math.2025134
4. A. Ghezal, O. Alzeley, Probabilistic properties and estimation methods for periodic threshold autoregressive stochastic volatility, *AIMS Math.*, **9** (2024), 11805–11832. https://doi.org/10.3934/math.2024578
5. O. Alzeley, A. Ghezal, On an asymmetric multivariate stochastic difference volatility: Structure and estimation, *AIMS Math.*, **9** (2024), 18528–18552. https://doi.org/10.3934/math.2024902

6. R. Abo-Zeid, C. Cinar, Global behavior of the difference equation $x_{n+1} = Ax_{n-1}/B - Cx_nx_{n-2}$, *Bol. Soc. Paran. Mat.*, **31** (2013), 43–49.

7. E. M. Elsayed, On the solutions and periodicity of some rational systems of difference equations, *B. Math. Soc. Sci. Math.*, **60** (2017), 159–171.

8. E. M. Elsayed, On a max type recursive sequence of order three, *Miskolc Math. Notes*, **17** (2016), 837–859. https://doi.org/10.18514/MMN.2017.534

9. E. M. Elsayed, Expression and behavior of the solutions of some rational recursive sequences, *Math. Method. Appl. Sci.*, **39** (2016), 5682–5694. https://doi.org/10.1002/mma.3953

10. M. Gümüş, Global asymptotic behavior of a discrete system of difference equations with delays, *Filomat*, **37** (2023), 251–264. https://doi.org/10.2298/FIL2301251G

11. M. Gümüş, R. Abo-Zeid, An explicit formula and forbidden set for a higher order difference equation, *J. Appl. Math. Comput.*, **63** (2020), 133–142. https://doi.org/10.5553/IISL/2020063002005

12. M. Gümüş, R. Abo-Zeid, Global behavior of a rational second order difference equation, *J. Appl. Math. Comput.*, **62** (2020), 119–133. https://doi.org/10.1007/s12190-019-01276-9

13. M. Gümüş, The periodic character in a higher order difference equation with delays, *Math. Method. Appl. Sci.*, **43** (2020), 1112–1123. https://doi.org/10.1002/mma.5915

14. T. H. Tran, A. D. Nguyen, T. A. Pham, Global dynamics of some system of second-order difference equations, *Electron. Res. Arch.*, **29** (2021), 4159–4175. http://dx.doi.org/10.3934/era.2021077

15. A. Ghezal, N. Attia, Closed-form solutions of a new class of three-dimensional nonlinear difference equations, *AIMS Math.*, **10** (2025), 23518–23533. http://dx.doi.org/2010.3934/math.20251044

16. M. Balegh, A. Ghezal, Dynamical analysis of a system of fuzzy difference equations with power terms, *Int. J. Dynam. Control*, **13** (2025), 364. https://doi.org/10.1007/s40435-025-01876-3

17. E. M. Elsayed, M. T. Alharthi, Analysis and qualitative behaviour of a tenth-order rational difference equation, *Bol. Soc. Paran. Mat.*, **42** (2024), 1–13. https://doi.org/10.5269/bspm.64858

18. M. Kara, Y. Yazlik, N. Touafek, On a difference equation whose solution is related to Fibonacci numbers, *Filomat*, **38** (2024), 7199–7207. https://doi.org/10.2298/FIL2420199K

19. S. Elaydi, *An introduction to difference equations*, Springer, New York, 2005.

20. A. Ghezal, K. Zerari, I. Zemmouri, On solutions of a two-dimensional (m + 1)-order system of difference equations via Pell numbers, *Bol. Soc. Paran. Mat.*, **43** (2025), 1–10. Available from: `https://periodicos.uem.br/ojs/index.php/BSocParanMat/article/view/67451`.

21. A. Ghezal, I. Zemmouri, Global stability of a multi-dimensional system of rational difference equations of higher-order with Pell-coefficients, *Bol. Soc. Paran. Mat.*, **43** (2025), 1–9. Available from: `https://periodicos.uem.br/ojs/index.php/BSocParanMat/article/view/67343`.

22. A. Ghezal, I. Zemmouri, On a solvable bidimensional system of rational difference equations via Jacobsthal numbers, *Bol. Soc. Paran. Mat.*, **43** (2025), 1–11. Available from: `https://periodicos.uem.br/ojs/index.php/BSocParanMat/article/view/68245`.

23. A. Ghezal, I. Zemmouri, Solvability of a bidimensional system of rational difference equations via Mersenne numbers, *Palest. J. Math.*, **13** (2024), 84–93.

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